

Unusual Hilbert or Hölder space frames for the elementary particles transport (Vlasov) equation

to (potentially) enable a proof of Landau damping in a nonlinear context
with only appropriate physical relevant regularity assumptions

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Abstract

The Boltzmann equation is a (non-linear) integrodifferential equation which forms the basis for the kinetic theory of gases. This not only covers classical gases, but also electron /neutron /photon transport in solids & plasmas / in nuclear reactors / in superfluids and radiative transfer in planetary and stellar atmospheres ([CeC]). The Boltzmann equation is derived from the Liouville equation for a gas of rigid spheres, without the assumption of “molecular chaos”; the basic properties of the Boltzmann equation are then expounded and the idea of model equations introduced. Related equations are e.g. the Boltzmann equations for polyatomic gases, mixtures, neutrons, radiative transfer as well as the Fokker-Planck and Vlasov equations. The treatment of corresponding boundary conditions leads to the discussion of the phenomena of gas-surface interactions and the related role played by proof of the Boltzmann H-theorem.

In [MoC] an exponential Landau damping is proven based on analytical regularity assumptions and corresponding analytical norms having up to 5 parameters (which is far away from any physical meaning). This note is about a proposed modified frame based on (distributional Hilbert scale) functional spaces and related functional inequalities for the nonlinear transport equation. Our alternative approach is based on the results of [CoA], incorporating the Hilbert transform concept to define appropriate Hilbert space (resp. Hölder space) norms. The Galerkin-Ritz method is proposed to calculate corresponding (quasi-optimal) approximation solutions, e.g. with underlying boundary elements approximation spaces enabled by the wavelet analysis tool. We note that the Hilbert transform is also applied in [DeP] for a spectral theory of the linearized Vlasov-poisson equation.

The Vlasov-Poisson equation (the collisionless Boltzmann equation) is time-reversible (for short periods of time due to the Landau damping, [MoC]). However, for long times the deformation of the distribution function approaches increasingly shorter scales which at some point in time go along any reasonable plasma physical length scale. At this point in time changes must be considered irreversible. Landau predicted this irreversible behavior on the analysis of the solution of the Cauchy problem for the linearized Vlasov equation around a spatially homogeneous Maxwellian (Gaussian) equilibrium. Landau formally solved the equation by means of Fourier and Laplace transforms. This phenomenon prevents instability from developing, and creates a region of stability in the parameter space.

In [GIR], [GIR1], it is shown that a solution of the linearized Vlasov equation in the whole space (linearized around a homogeneous equilibrium $f_0 := f(0)$ of infinite mass) decays at best like modulo logarithmic corrections, for

$$f_0(v) = \frac{c}{1+2|v|}$$

and like $O(\log t - \alpha)$ if $f_0(v)$ is a Gaussian. In order to get an answer to the question, if convergence holds in infinite time for the solution of the “full” nonlinear equation there is a mechanism required that would keep the distribution function f close to the original equilibrium.

We propose to apply the distributional Hilbert space concept of this paper to derive model adequate a priori estimate for the transport equation. The objective is, that the appropriately defined (distributional Hilbert space) norms enable appropriate Landau damping estimates, based on “realistic” physical modelling assumptions:

Following the ideas from [BrK1], [BrK3], [BrK5], this first leads to a change from

$$\|f\|_{(0)}^2 := \|f\|_{L_\infty}^2 + \left\| \frac{\tilde{f}}{x} \right\|_0^2 + \|f\|_0^2$$

to

$$\|f\|_{(-1/2)}^2 := \|f\|_{L_\infty}^2 + \left\| \frac{\tilde{f}}{x} \right\|_{-1/2}^2 + \|f\|_{(-1/2)}^2 \cong \|f\|_{L_\infty}^2 + \left\| \frac{\tilde{f}}{\sqrt{x}} \right\|_0^2 + \|f\|_{1/2}^2$$

to anticipate the non-linear character of the transport equation and the compactness results from [LiP], [LiP1], we suggest the Hilbert scale norms

$$\|f\|_\alpha^2 := \|f\|_{C^{\alpha\alpha}}^2 + \|x^\alpha \tilde{f}\|_0^2 + \|f\|_\alpha^2, \quad \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

and an analysis in a weak $H_{-1/2}$ Hilbert space with correspondingly defined appropriate wavelets. The simple rationale for this approach is, that an (classical or variational) integral equation model requires less regularity assumptions as the corresponding PDE representation, while, at the same time, the wavelet analysis tool requires much less regularity requirements than the regular (smoothing) cut-off functioning approach. By co-incidence or by chance it all meets to a $H_{1/2}$ (energy, variational) Hilbert space framework. The alternatively proposed Schrödinger (Calderon) momentum operator [BrK5] hasn't been included into the revisited CLM vorticity equation model with viscosity term (appendix). In [DaD] a Galerkin analysis for Schrödinger equation by wavelets is provided.

1. The Boltzmann equation

The Boltzmann equation with Maxwellian (Gaussian) molecules is given by

$$\frac{\partial f}{\partial t}(t, v) = Q(f, f)(t, v)$$

with the non-linear collision operator $Q(f, f)$. The Boltzmann's equation is a nonlinear integro-differential equation with a linear first-order operator. The nonlinearity comes from the quadratic integral (collision) operator that is decomposed into two parts (usually called the gain and the loss terms). In [LiP] it is proven that the gain term enjoys striking compactness properties. The Boltzmann equation and the Fokker-Planck (Landau) equation are concerned with the Kullback information, which is about a differential entropy defined by

$$H(X) = \int h(x) \log \frac{1}{h(x)} dx \cdot$$

It plays a key role in the mathematical expression of the entropy principle. The Boltzmann entropy is given by

$$S = - \iint h \cdot \log h dv dx \cdot$$

The Fokker-Planck (Landau) equation is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = \frac{1}{2\pi} \frac{\log \Lambda}{\Lambda} Q_L(f, f)$$

with

$$F[f](t, x) := - \iint \nabla W(x - y) f(t, y, w) dw dy \cdot$$

The existence of global solutions of the Boltzmann and Landau equations depends heavily on the structure of the collision operators ([LiP1]). The corresponding variational representation of $B = A + K$ with a H_α - coercive operator A and a compact disturbance K fulfills a coerciveness condition (Garding type inequality) in the form (see also [KaY])

$$(Bu, v) \geq c \cdot \|u\|_\alpha \cdot \|v\|_\alpha - (Ku, v) \quad \text{or} \quad (Bu, v) \geq c_1 \cdot \|u\|_\alpha^2 - c_2 \cdot \|u\|_\beta^2$$

with $H_\alpha \subset H_\beta$ compactly embedded. The corresponding (Hilbert scale) approximation theory is e.g. given in [BrK], [AzA], lemma 4.2.

The idea of this paper, is, to propose an appropriate Hilbert scale frame for analyzing the nonlinear (collisionless) Vlasov equation concerning global stability (avoiding analytical norms, which are "hybrid" and "gliding" [MoC]), applying

- compactness properties enjoyed by global solutions (which can be interpreted as compact disturbance to the linear case)
- Hölder norm estimates (to anticipate the Landau damping effect in the nonlinear context; ([NiJ], ([NiJ1], ([NiJ2]))
- the wavelet analysis tool analyzing very large time scales to anticipate dissipative phenomena.

In [Vil] the existence and uniqueness of nonnegative eigenfunction is analyzed. In [MoB] the eigenvalue spectrum of the linear neutron transport (Boltzmann) operator has been studied. The spectrum turns out to be quite different from that obtained according to the classical theory. The two theories about related physical aspects have one aspect in common: namely that there exists a region of the spectral plane which filled up by the spectrum.

The Gibbs principle states that the maximum entropy of a thermodynamic system is achieved, when all considered macroscopic parameters of the described systems yield stationary values. The entropy concept is applied in (non-linear) partial differential evolution equations to analyze well-posedness of those equations providing qualitative descriptions of the behavior of its solutions, especially in the long-term regime.

The (negative) entropy

$$h(t) := E[f] := \int_{R^3} f \cdot \log f dv$$

fulfills the so-called H-theorem, i.e. $\dot{h} \leq 0$. In the sub-space of the probability distribution function space, consisting of distribution functions with momentum equal zero and (normed) temperature equal 1 the strictly convex entropy functional $E[f]$ is minimized by the 3-D Maxwell distribution

$$f_{\infty}(v) := \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}$$

Therefore, the corresponding “relative entropy” defined by

$$g(t) := E_r[f] := E[f] - E[f_{\infty}]$$

fulfills $E_r[f] \geq 0$, $E_r[f_{\infty}] = 0$.

The Gronwall lemma dilemma

In order to analyze the convergence of the entropy generation resp. its dissipation \dot{g} , the term is tried to be estimated to the below by the relative entropy itself. In case it would hold $\dot{g}(t) \geq \mu \cdot g(t)$ for some constant $\mu > 0$ it follows $\dot{g}(t) \leq -\mu g(t)$. From the Gronwall lemma then it follows the exponential convergence to zero.

Generalized lemma of Gronwall: let $\varphi, \vartheta \in C[0, T]$ and $\vartheta \geq 0$ fulfilling

$$\varphi(t) \leq c_1 + \int_0^t \vartheta(\tau) \cdot \varphi(\tau) d\tau \text{ for all } t \in [0, T]$$

then it holds

$$\varphi(t) \leq c_1 \cdot e^{\int_0^t \vartheta(\tau) d\tau} .$$

Stronger forms of Gronwall's lemma are produced, replacing the above assumption with more general inequalities, which usually fit the form (see [WiD] and the corresponding references, e.g. C. E. Langenhop, F. Brauer, V. Lakshmikantham)

$$\varphi(t) \leq c_1 + h \left(\int_0^t g(t, \tau, \varphi(\tau)) d\tau \right) .$$

From [LaC] we recall the following upper and lower bounds on the norm of a solution of the equation

$$\frac{dz}{dx} = F(x, z) \quad \text{with} \quad |F(x, z)| \leq v(x)g(|z|)$$

given by

$$G^{-1} \left[G(|z(a)|) - \int_a^x v(s) ds \right] \leq |z(x)| \leq G^{-1} \left[G(|z(a)|) + \int_a^x v(s) ds \right]$$

where

$$G(u) := \int_{u_0}^u \frac{dt}{g(t)} .$$

We note that special classes of the Riccati equations

$$\dot{y}(t) + p(t)y(t) - y^2(t) = q(t)$$

play a key role in the (long time regime) analysis of the non-stationary, non-linear Navier-Stokes equations. The assumptions of corresponding "general solution" theorems are based on Gronwall-type integral values ([BuK]). In [BaA] special solutions of the Riccati equations are provided applicable to the multidimensional Gross-Pitaevskii equation of Bose-Einstein condensates.

The lemma of Gronwall is also sometime applied in finite element approximation analysis of parabolic or hyperbolic PDE enabled by the Ritz/Galerkin approximation theory. In all those cases there might be quasi-optimal error estimates derived with respect to the expected convergence factor. However, applying the lemma of Gronwall to derive those estimates always require purely additional regularity assumptions, just for technical reasons. This is already be the case for the most simple linear parabolic PDE, the heat equation. From a mathematical point of view the underlying handicap is just about the not appropriately used norms. Overcoming this handicap in the case of the heat equation this lead to the "appropriate" norm in the form ([NiJ4])

$$\| \| u \| \|_{\alpha}^2(t) := \int_0^t \| u \|_{\alpha}^2(\tau) d\tau .$$

It results into required optimal shift theorems, enabling truly quasi-optimal finite element error estimates. In order to enable an “entropy” analysis in an appropriate Hilbert scale framework two things need to be adapted:

- the strictly convex entropy functional $E[f]$ minimizing by the 3-D Maxwell distribution needs to be represented as a energy or operator minimization problem
- the proposed distributional Hilbert scale framework for the NSE in [BrK1] needs to be acknowledged leading to a correspondingly defined distributional time-dependent norm for the considered cases in this paper.

The Hilbert transform of $\log(|f|)$ is analyzed in [MaJ]. A very first idea could be a modified $\log_{(x)}(f)$ operator in the form $\log_{(x;\tau)}(f)$.

Alternatively, in line with the proposed distributional $H_{-\frac{1}{2}}$ -Hilbert space concept of this paper, we suggest to define “continuous entropy” in a weak $H_{-\frac{1}{2}}$ -frame in the form

$$h(X) := (f, \log \frac{1}{f})_{-1/2},$$

where X denotes a continuous random variable with density $f(x)$. In this case it can be derived from a Shannon (discrete) entropy in the limit of n , the number of symbols in distribution $P(x)$ of a discrete random variable X ([MaC]):

$$H(X) := \sum_i P(x_i) \log\left(\frac{1}{P(x_i)}\right).$$

This distribution $P(x)$ can be derived from a set of axioms. This is not the case, in case of the standard entropy in the form

$$h(X) := (f, \log \frac{1}{f})_0.$$

Regarding the second topic above we mention the ergodic mean definition in the well established theory of asymptotic behavior of evolution systems ([BrH]).

The alternative Schrödinger (Calderòn) momentum operator and the $H_{-\frac{1}{2}}$ – Hilbert space

Following the idea of [BrK5] distributional Hilbert scales can be defined based on the eigenpairs (λ_i, φ_i) of the Laplace operator in the form

$$(x, y)_\alpha := \sum_i \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i \lambda_i^\alpha x_i y_i \quad , \quad \|x\|_\alpha^2 := (x, x)_\alpha .$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined with “exponential decay” $e^{-\sqrt{\lambda_i}t}$ in the form

$$(x, y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i}t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 := (x, x)_{(t)}^2$$

In [BrK5] the distributional $H_{-1/2}$ – Hilbert space is proposed to model quantum states, alternatively to the Hilbert space H_0 . For

$$x = x_0 + x_0^\perp \in H_0 \otimes H_0^\perp$$

with

$$\|x_0\|_0 = 1 \quad , \quad \sigma := \|x_0^\perp\|_{-1/2}^2$$

the following inequality is valid for any $x \in H_0$

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1} e^{1-\sqrt{\lambda_i}\sigma} x_i^2 .$$

In the appendix we provide the relationship to the statistical thermodynamics concept of E. Schrödinger leading to a kind of “Schrödinger-heat-bath-room” concept, given by H_1^\perp whereby

$$H_1 \underset{(c)}{\subset} H_{1/2} = H_1 \otimes H_1^\perp .$$

Tauberian theorems are usually assumed to connect the asymptotic behavior of a generalized function (or a distribution) in the neighbourhood of zero with that of its Fourier transform, Laplace transform, or some other integral transform at infinity. The inverse theorems are usually called “Abelian”. In [VIV] (quasi-) asymptotic properties of solutions of convolution equations are analyzed considering several cases, like the wave equation, the Klein-Gordon equation, the telegraph equation and the Cauchy problem for the (generalized) heat equation.

The distribution function of trapped electrons at the point x corresponding to the potential $\varphi(x)$ can be described by the solution of an integral equation [Bel]. The entire ion distribution $f_+(E)$ and the distribution of untrapped electrons $f_-(E)$ are integral equations of the convolution type for the distribution function of the trapped electrons. It can be readily solved by Laplace transformation.

For Laplace’s asymptotic formula ([EsR] 3.5) the following corresponding formula for the Gaussian function play a key role ([EsR] p. 107):

$$e^{-((\sqrt{\mu}x)^2)} \approx \sum_{n=0}^{\infty} \frac{\text{Gamma}(\frac{2n+1}{2}) \cdot \delta^{(2n)}(x)}{(2n)! \cdot \mu^{(2n+1)/2}} \quad \text{as } \mu \rightarrow \infty .$$

The plasma dispersion function, the Dawson function and the $H_{-\frac{1}{2}}$ – Hilbert space

In [BrK3] the relationship to a corresponding newly proposed ground state energy model is provided enabled by Pseudo-Differential Operator theory (e.g. [EsG]). The relationship to corresponding Hölder space and corresponding Schauder estimates are e.g. given in [NiJ1], [NiJ2]. With respect to the wavelet analysis tool we note that the wavelet admissibility condition corresponds to the distributional $H_{-1/2}$ – Hilbert space norm.

We further note that the plasma dispersion function $Z(\zeta)$ defined by

$$Z(\zeta) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{s - \zeta} ds$$

is identical to the Hilbert transform of the Gaussian function, the Dawson function, defined by

$$Z(\zeta) = F(\zeta) := e^{-\zeta^2} \int_0^{\zeta} e^{t^2} dt.$$

This identity is called the Hermite weight formula ([GaW], section 4 below).

One of the key properties of the Dawson functions are ([AbM] 7.1.15/16/21, 7.3.25), [GrI] 3.896, 3.952):

- i) $F(x) = x {}_1F_1(1, \frac{3}{2}, -x^2) = x \cdot e^{-x^2} {}_1F_1(\frac{1}{2}, \frac{3}{2}, x^2)$
- ii) ${}_1F_1(\frac{1}{2}, \frac{3}{2}, -x^2) = \frac{\sqrt{\pi}}{2x} \operatorname{erf}(x)$, $x {}_1F_1(\frac{1}{2}, \frac{3}{2}, i\frac{\pi}{2}x^2) = C(c) + iS(x)$
- iii) $F(x) = \sqrt{\pi} \int_0^{\infty} \sin(2\sqrt{\pi}xt) \cdot e^{-t^2} dt$
- iv) $F'(x) + 2xF(x) = 1$ **resp.** $F(x) = \frac{1 - F'(x)}{2x}$
- v) $F'(0) = 1$, $\lim_{x \rightarrow \infty} 2xF(x) = 1$
- vi) maximum and inflection points:

$$F(0.9241388730\dots) = 0.5410442246\dots \quad , \quad F(1.5019752682\dots) = 0.4276866160\dots$$

$$\text{vii) } -\frac{1}{\sqrt{\pi}} I(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{x-t} dt = 2F(x) = \frac{1}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \frac{H_k^{(n)}}{x - x_k^{(n)}}$$

with $x_k^{(n)}$, $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials (see also [GrF], [PaJ]).

Let

$$f(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , \quad f_H(x) := H[f](x)$$

then it holds

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx = 1 \quad , \quad \int_{-\infty}^{\infty} f^2(x) dx = \int_{-\infty}^{\infty} f_H^2(x) dx = \frac{1}{2\sqrt{\pi}}$$

and therefore

$$\int_{-\infty}^{\infty} [2F(x)]^2 dx = \frac{1}{\sqrt{2\pi}}.$$

In [KoV] the asymptotic expansion for the Kummer function obtained in the study of the linear response of magnetized Bose plasma at are presented for large and small values of its parameters. For the theory of nuclear fusion, space physics, nonlinear plasma theory (plasmas as fluids, single-particle motions, waves in plasmas) we refer to [ChF]. With respect to the below in the context of estimates from the theory of concentration of measure the logarithmic Sobolev inequality plays a key role in some mean-field problems. This is about the inequality

$$\int g \log g d\mu \leq \frac{1}{2\lambda} \int \frac{|\nabla g|^2}{g} d\mu$$

for all $g \in L_1(d\mu)$ such that $g \geq 0$ and $\int fg d\mu = 1$.

From the above Dawson function (resp. the dispersion function) property ii) it follows

$$F'(x)F(x) + 2xF^2(x) = F(x)$$

resp.

$$d\tilde{\mu}(x) := dF^2(x) = 2F(x)(1 - 2xF(x))dx \approx F'(x)d(\log x) \quad .$$

As

$$\lim_{x \rightarrow \infty} 2F(x) = \lim_{x \rightarrow \infty} \frac{1}{x} = \lim_{x \rightarrow 0} 2F(x) = 0$$

and

$$\lim_{x \rightarrow \infty} (1 - 2xF(x)) = \lim_{x \rightarrow 0} (1 - 2xF(x)) = 1$$

one could also define

$$\mu(x) := \begin{cases} 1 - 2xF(x) & x \leq x_0 \\ 2F(x) & x \geq x_0 \end{cases}$$

where

$$2F(x_0) = 1 - 2x_0F(x_0) \quad \text{i.e.} \quad 2F(x_0) = (1 + x_0)^{-1}.$$

For a product theorem for Hilbert transforms we refer to ([BeE]).

In the context of the Clausen function/integral of the following section we recall from [BoJ], p.56, the formula

$$\frac{1}{2\pi} \int_0^\pi (\pi - \vartheta)^2 \log^2 \left(2 \sin \left(\frac{\vartheta}{2} \right) \right) d\vartheta = \sum_{n=1}^\infty \frac{\left(\sum_{k=1}^n \frac{1}{k} \right)^2}{n+1}.$$

The Rutherford α – particle cross section scattering model

The same “trick” as above to build a modified density function by replacing

$$d\tilde{\mu} \rightarrow d\mu$$

can be applied to Rutherford’s (statistical) cross section concept for the α –particle scattering model:

putting

$$C := \pi \left[\frac{2e^2 \cdot Z}{m_\alpha \cdot v^2} \right]^2$$

whereby $e \cdot Z$ denotes the charge of the kernel of the atom, m_α its mass and v the velocity of the α – particle, Rutherford’s cross section differential is given by

$$d\sigma := -C \frac{\cot\left(\frac{\vartheta}{2}\right)}{\sin^2\left(\frac{\vartheta}{2}\right)} = C \frac{d}{d\vartheta} \left[\cot^2\left(\frac{\vartheta}{2}\right) \right] .$$

Referring to [BrK4], the cross section differential is related to the fractional function $\rho(\vartheta)$ resp. its corresponding Hilbert transform $H[\rho](\vartheta) := \rho_H(\vartheta)$ by the following identities:

$$\rho(\vartheta) := \frac{1}{2\pi} (\vartheta - [\vartheta]) = \frac{1}{2\pi} \left[\pi - 2 \sum_{n=1}^{\infty} \frac{\sin(n\vartheta)}{n} \right] \in L_2^\#(0, 2\pi)$$

$$g(x) := \rho_H(\vartheta) = \sum_{n=1}^{\infty} \frac{\cos(n\vartheta)}{n} = -\log\left(2 \sin\left(\frac{\vartheta}{2}\right)\right) \in L_2^\#(0, 2\pi).$$

Then it holds

$$f(\vartheta) := \rho'_H(\vartheta) = -\frac{1}{2} \cdot \cot\left(\frac{\vartheta}{2}\right) = -\log'\left(2 \sin\left(\frac{\vartheta}{2}\right)\right) \in H_{-1}^\#(0, 2\pi) \quad \text{i.e. weakly } \in H_{-1/2}^\#(0, 2\pi)$$

$$f'(\vartheta) = \rho''_H(\vartheta) = \frac{1}{4 \cdot \sin^2\left(\frac{\vartheta}{2}\right)} \in H_{-2}^\#(0, 2\pi) \quad \text{i.e. weakly } \in H_{-1}^\#(0, 2\pi)$$

resp.

$$\cot^2\left(\frac{\vartheta}{2}\right) = 4 \cdot \rho_H'^2(\vartheta)$$

and

$$\frac{f'(\vartheta)}{f(\vartheta)} = \frac{\rho''_H(\vartheta)}{\rho'_H(\vartheta)} = -\frac{\cot\left(\frac{\vartheta}{2}\right)}{2 \cdot \sin^2\left(\frac{\vartheta}{2}\right)} = \frac{1}{2} \frac{d}{d\vartheta} \left[\cot^2\left(\frac{\vartheta}{2}\right) \right] = \log' f(\vartheta) = \log'(\rho'_H(\vartheta)) .$$

As it holds

$$((f, v))_{-\frac{1}{2}} < \infty \quad \forall v \in H_{-1/2}^\#(0, 2\pi) \quad \text{resp.} \quad (f, g)_{-1/2} \leq \|f\|_{-1} \|g\|_0 < \infty$$

with the Clausen integral (e.g. [BrK4])

$$h(\vartheta) := \int_0^\vartheta g(t) dt \quad , \quad \vartheta \in [0, \pi]$$

one could replace

$$\|f\|_{-1}^2 \rightarrow (f, h')_{-\frac{1}{2};(0, \pi)} + (f, g)_{-\frac{1}{2};(\pi, 2\pi)} = (f, h)_{0;(0, \pi)} + (f, g)_{-\frac{1}{2};(\pi, 2\pi)} .$$

The Vlasov equation and the $H_{-\frac{1}{2}}$ – Hilbert space

A plasma consists of heavy, positively charged ions and smaller negatively charged electrons. Each electron possesses a random velocity v , which are distributed according to a distribution function f . Then the Vlasov equation describes the time evolution of the distribution function for a collisionless physical system of the plasma. Together with the Poisson equation, which yields the potential, it forms the Vlasov-Poisson system. According to the Landau damping phenomenon the density ρ of the plasma converges to its mean value, while the interaction force F converges to 0. Following the notation of [MoC] the Vlasov equation is given by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0 \cdot$$

Putting

$$\hat{\rho}_k(t) := \int \hat{f}(t, k, v) dv$$

we define

$$\|\rho\|_{-1}^2(t) := \int \frac{\hat{\rho}_k^2(t)}{|k|^2} dk \quad , \quad \|f\|_0^2(t) := \iint \hat{f}_k^2(t, v) dv dk \quad , \quad \|f\|_{-1/2}^2(t) := \iint \frac{\hat{f}_k^2(t, v)}{|k \cdot v|} dv dk \cong \iint \frac{\hat{g}^2(t, \omega)}{|\omega|} d\omega \cdot$$

Multiplying the corresponding Fourier coefficient equation of the Vlasov-Poisson equation

$$\hat{f}_k + 2\pi i \cdot (v \cdot k) \cdot \hat{f} = 2\pi i \cdot (k \hat{W} \cdot \int \hat{f}(\cdot, w) dw) \cdot \nabla_v f_0(v)$$

with \hat{f} / k leads to

$$\frac{1}{2} \frac{d}{dt} \left(\frac{1}{v} \hat{f}_k^2 \right) + 2\pi i \cdot \hat{f}_k^2 = 2\pi i \cdot \hat{W}_k \cdot \hat{f}_k \cdot \hat{\rho}_k \cdot \frac{\nabla_v f_0(v)}{v} \cdot$$

After integration with respect to the parameters k, v , one gets

$$\frac{1}{4\pi i} \frac{d}{dt} \|f\|_{-1/2}^2 + \|f\|_0^2 = \iint \hat{W}_k \cdot \hat{f}_k \cdot \hat{\rho}_k \cdot \frac{\nabla_v f_0(v)}{v} dv dk \cdot$$

Alternatively to the Landau damping criterion

$$\int \frac{\nabla_v f_0(v)}{v} dv < 0$$

we propose the condition

$$\left| \frac{\nabla_v f_0(v)}{v} \right| \leq c_1 < \infty \cdot$$

In case of a Coloumb potential fulfilling the inequality

$$\hat{W}_k \leq \frac{c_2}{|k|^2}$$

this leads to the a priori estimate

$$\frac{1}{4\pi i} \frac{d}{dt} \|f\|_{-1/2}^2 + \|f\|_0^2 \leq c_1 c_2 \iint \frac{\hat{\rho}_k^2}{|k|^2} dk = c_1 c_2 \|\rho\|_{-1}^2(t) \cdot$$

For an alternative analysis in Hölder spaces frame (with appropriately defined parameter(s) $(\alpha, \beta = \alpha)$) dealing with Volterra integrals based on Gaussian kernel functions we refer to [HöK]. Related a priori estimates with respect to time-weighted Hilbert space norms are provided in [Ni3]. One of the applied auxiliary inequality is

$$a^{1/4} \cdot b^{1/4} \cdot c \leq \frac{1}{4} a + \frac{1}{4} a + \frac{1}{2} c^2$$

to govern corresponding integral integral inequalities (see also the proof technique in [GIR].

The collision operator of the Fokker-Planck (Landau) equation and the Leray-Hopf (Helmholtz-Weyl) operator in the $H_{-\frac{1}{2}}$ – Hilbert space framework

Following the notation of [LiP1] in case of space dimension $n = 3$ the symmetric, non-negative, even in z matrix of the Fokker-Planck collision operator is given by

$$a_{i,j}(z) = \frac{a(z)}{|z|} \left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right)$$

where $a(z)$ is even, smooth and positive.

In [LeN] the Oseen kernel is analyzed with respect to the kernel of corresponding multipliers, which involves the incomplete gamma function and the confluent hypergeometric functions of the first kind. This explicit expression provides directly the classical decay estimates with sharp bounds. In this context the action of the Leray projector on Gaussian functions is analyzed.

In a weak $H_{-\frac{1}{2}}$ –Hilbert space frame the action on the modified Leray projector on the Hilbert transformed Gaussian is (re-) producing the Gaussian function, due to the skew-symmetry property of the Hilbert resp. the Riesz operators ([BrK1], [BrK5]).

The Leray-Hopf (Helmholtz-Weyl) operator is the matrix valued Fourier multiplier given by

$$P(z) = \left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right)_{1 \leq i, j \leq n}$$

It is not a classical pseudodifferential operator, but a Fourier multiplier with same continuity properties as those of the Riesz operators $R(z)$. Related to this there is the matrix of operators given by

$$Q = (R_i R_j)_{1 \leq i, j \leq n} = Q^2$$

Comparing both matrix multipliers, the Fokker-Planck collision operator (FPCO) and the Leray-Hopf projection operator (LHPO) their relationship can be interpreted in that way that the FPCO is a compact disturbance of the LHPO in a $H_{-\frac{1}{2}}$ –variational framework, i.e. a corresponding Garding type inequality is given. Then standard (Hilbert scale) approximation theory can be applied, e.g. given in [BrK], [AZA], lemma 4.2.

We further note the relationship between the BMO and the $H_{-\frac{1}{2}}$, “function” spaces, whereby the latter one is proposed alternatively for a generalization of the logarithmic Sobolev inequality ([IbH]).

We further mention the ergodic mean definition in the well established theory of asymptotic behavior of evolution systems ([BrH]).

2. Landau damping for the linearized Vlasov Poisson equation in a weakly collisional regime ([Trl])

This section recalls the result of [Trl] in order to support further Fourier wave/ Calderon wavelet analysis in an appropriately defined Hilbert (Sobolev) frame. The Garding type inequality of the Boltzmann and Landau equations indicates an underlying selfadjoint, positive definite operator with corresponding energy norm $\|u\|_\alpha$. Its counterpart in the context of the Navier-Stokes equations is the Stokes operator. With respect to the Vlasov equation this is about the Boltzmann operator or the Fokker-Planck operator

$$C(h) := \rho M - h = \Delta_v h + \text{div}_v(vh)$$

where M is the Maxwellian distribution, h is a mean-zero perturbation of the solution of the Vlasov equation in the form

$$\partial_t f + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = \varepsilon \cdot C(f)$$

defined by

$$f(t, x, v) := M(v) + h(t, x, v).$$

In the case of the collisional Vlasov equation (w/o the term coming from the mean-field interaction between particles)

$$\partial_t h + v \cdot \nabla_x h = \varepsilon \cdot C(h)$$

for fixed $\varepsilon > 0$, it is clear that hypocoercive effect is dominant for large times. Applying the results of [DoJ] obtains a result of decay at infinity with a rate of type $e^{-\varepsilon \lambda t}$ for some $\lambda > 0$ in some Hilbert space. With respect to further Fourier analysis we note that the Fourier transform of $e^{-\varepsilon \lambda t}$ is given by

$$\frac{\varepsilon \lambda}{(|\xi|^2 + (\varepsilon \lambda)^2)^{n/2}}.$$

In [Trl] Sobolev spaces H_l with parameter l are considered for fixed $l > n/2$

$$\|h\|_{L_t^2 H_x^n}^2 := \sum_{|\alpha| \leq n} \int_{\mathbb{R}^n / (2\pi\mathbb{Z})^2 \times \mathbb{R}^2} (1 + |v|^2)^{2l} |\partial_v^\alpha f|^2 dx dv$$

in order to ensure continuous functions based on the Sobolev embedding theorem. For the same reason the regularity of the Dirac function depends also from the space dimension, i.e. $\delta \in H_{-n/2-\varepsilon}$.

With respect to the decay rate at infinity $e^{-\varepsilon \lambda t}$ for some $\lambda > 0$ above (and the proposed distributional Hilbert space frame in the following section) we note that the Fourier transform of the Poisson kernel $P(\xi, \rho)$ given by

$$P(\xi, \rho) = \frac{1}{(2\pi)^n} \text{Fourier}(e^{-\rho|\cdot|}) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{\rho}{(|\xi|^2 + \rho^2)^{(n+1)/2}}.$$

The corresponding equality for the Gauss-Weierstrass kernel is given by

$$P(\xi, \rho) = \frac{1}{(2\pi)^n} \text{Fourier}(e^{-\rho|\cdot|^2}) = \frac{1}{(4\pi\rho)^{n/2}} e^{-|\xi|^2/(4\rho)}.$$

With respect to time-weighted Hilbert space norms we note the following identity ([StE])

$$e^{-b} = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t} e^{-b^2/t} dt = \sqrt{\frac{\rho}{\pi}} \int_0^\infty \tau^{-1/2} e^{-\rho\tau} e^{-b^2/(\rho\tau)} d\tau.$$

The Fourier-Hermite expansion of the ordinary one-dimensional Boltzmann equation (in natural units) for the single-particle distribution function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} a_{\alpha\beta} e^{i\alpha x} H_{\beta}(v) \cdot e^{-v^2/2}$$

where

$$H_{\beta}(v) = (-1)^{\beta} e^{-v^2/2} \left(\frac{d}{dv}\right)^{\beta} e^{-v^2/2} \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(v) H_m(v) e^{-v^2/2} dv = \delta_{nm} n! \quad .$$

Regarding the Landau damping for the linearized Vlasov Poisson equation in a weakly collisional regime the main result of [Trl] is

Theorem 1.1.: Consider a mean-zero distribution $h_i \in H_n := L_x^2 H_v^n$ (i.e. the wavelet admissibility condition is fulfilled). There exist $\varepsilon_0 > 0, \lambda_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ the density $\rho = \rho(t, x)$ of the solution $h = h(t, x, v)$ satisfies the following estimates

$$(1) \quad \|\rho(t, 0)\|_{L_x^2} \leq \frac{c}{(1+|t|^2)^{n/2}}$$

$$(2) \quad |\hat{h}(t, 0, \xi)| \leq c \cdot e^{-\varepsilon \lambda_0 t} \quad \forall \xi \in \mathbb{R}^n$$

$$(3) \quad |\hat{h}(t, k, \xi)| \leq c \cdot \frac{1}{(1+|\xi|^2 + |t|^2)^{(n-2)/2}} \quad \forall k \in \mathbb{Z}^n - \{0\}, \quad \forall \xi \in \mathbb{R}^n$$

These estimates are valid for

- i) the model associated to the linear Boltzmann collision operator :
for all times $t \geq 0$
- ii) for the model with Fokker-Planck collision : for $t \in [0, 1/\varepsilon]$.

For related a priori estimates with respect to time-weighted Hilbert space norms we refer to e.g. [Ni3] and the related references.

Following the idea of [BrK5] the above L_{∞} – norm based inequalities are proposed to be replaced by corresponding (distributional) Hilbert norm based estimates. The same idea is proposed with respect to the following section.

We further note that in the context of estimates from the theory of concentration of measure the logarithmic Sobolev inequality plays a key role in some mean-field problems. This is about the inequality

$$\int f \log f d\mu \leq \frac{1}{2\lambda} \int \frac{|\nabla f|^2}{f} d\mu$$

for all $f \in L_1(d\mu)$ such that $f \geq 0$ and $\int f d\mu = 1$.

For a generalization of a logarithmic Sobolev inequality to the Hölder class we refer to [lbH].

3. Time decay for solutions to one dimensional two component plasma equations ([GIR])

In [GIR] the time decay of solutions to one-dimensional two component plasma equations for the Vlasov-Poisson equation (VP) and the relativistic Vlasov-Poisson (RVP) are provided. Let f denote the number density in phase space of particles with mass one and positive unit charge, while g is the number density of particles with mass $m > 0$ and negative unit charge, i.e. both are solutions of the Vlasov equation. Let further denote

$$E(t, x) := \frac{1}{2} \cdot \left[\int_{-\infty}^x \rho(t, y) dy - \int_x^{\infty} \rho(t, y) dy \right]$$

the electric field. Then it holds

$$E_x(t, x) = \rho(t, x) := \int_{-\infty}^{\infty} (f(t, x, v) - g(t, x, v)) dv \quad \text{and} \quad E_t(t, x) = -j(t, x) = - \int_{-\infty}^{\infty} v(f(t, x, v) - g(t, x, v)) dv \cdot$$

For the solutions of both equations, the Vlasov-Poisson system (VP) and relativistic Vlasov-Poisson system (RVP), it holds

$$\lim_{t \rightarrow \infty} \|E(t, \cdot)\|_{L^\infty} = 0 \cdot$$

This result depends on the space dimension $n = 1$ due to the Sobolev embedding theorem state, i.e.

$$H_k \subset C^0 \quad \text{for} \quad k > n/2$$

i.e.

$$H_1 \subset H_{1/2+\varepsilon} \subset C^0$$

Following the idea of [BrK5], replacing the Dirac function $\delta \in H_{-1/2-\varepsilon}$ by the distributional Hilbert space $H_{-1/2}$ this leads to a replacement of

$$\|E(t, \cdot)\|_{L^\infty} \quad \rightarrow \quad \|E(t, \cdot)\|_{H_{1/2}} = \|E'(t, \cdot)\|_{H_{-1/2}} = \|\rho(t, \cdot)\|_{H_{-1/2}} \cdot$$

4. The plasma dispersion function and ion waves and their damping

The first part of this section recalls the corresponding sections of ([ChF] 7.9.1). The distribution of thermal kinetic energies for a gas in the Maxwellian state is e.g. given in [BiJ] 7.3:

The Vlasov equation is used to derive the dispersion relation for electron plasma oscillations. In zeroth order one assumes a uniform plasma with distribution $f_0(v)$. In first order, one denotes the perturbation $f(r, v, t)$ by $f_1(r, v, t)$ in the form

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t)$$

The plasma dispersion function $Z(\zeta)$ defined by

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{s - \zeta} ds, \quad \text{Im}(\zeta) > 0$$

is applied to calculate the ion Landau damping of ion acoustic waves in the absence of magnetic fields. It is derived from the solution

$$f_{1j} = -\frac{iq_j E}{m_j} \cdot \frac{\partial f_{0j} / \partial v_j}{\omega - kv_j}$$

(the j th species has charge q_j , mass m_j , and particle velocity v_j) of the Vlasov equation.

The density perturbation of the j th species is given by

$$n_{ij} = \int_{-\infty}^{\infty} f_{1j}(v_j) dv_j = -\frac{iq_j E}{m_j} \int_{-\infty}^{\infty} \frac{\partial f_{0j} / \partial v_j}{\omega - kv_j} dv_j.$$

Setting

$$\Omega_{pj} = (n_{0j} Z_j^2 e^2 / \epsilon_0 M_j)^{1/2}$$

one gets the dispersion relation

$$k^2 = \frac{\omega_p^2}{v_{the}^2} Z'(\zeta_e) + \sum_j \frac{\Omega_{pj}^2}{v_{mj}^2 Z'(\zeta_j)}$$

from which electron plasma waves can be obtained setting $\Omega_{pj} = 0$ (infinitely massive ions).

Putting

$$k_D^2 := 2 \frac{\omega_p^2}{v_{me}^2} = \frac{1}{\lambda_D^2}, \quad \zeta_j := \omega / (kv_{mj})$$

one obtains

$$\frac{k^2}{k_D^2} = \frac{1}{2} Z'(\zeta_e),$$

which is the same as ([BiJ] (6.2)):

$$1 = \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} \frac{\partial f_{0e} / \partial v}{v - (\omega/k)} dv$$

when f_{0e} is Maxwellian. The term $k^2 \cdot \lambda_D^2$ represents the deviation from quasi-neutrality.

For the special case of a single ion species ($k^2 \cdot \lambda_D^2 \ll 1$) the dispersion relation becomes

$$Z'\left(\frac{\omega}{kv_{ni}}\right) = \frac{2T_i}{ZT_e} \cdot$$

“Solving this equation is a nontrivial problem” ([ChF] (7.128)). Considering the limit $\zeta_j \gg 1$ the asymptotic expression

$$Z'(\zeta_i) = -2i\sqrt{\pi}\zeta_i e^{-\zeta_i^2} + \zeta_i^{-2} + \frac{3}{2}\zeta_i^{-4} + \dots$$

is applied to calculate the approximate damping rate.

With respect to the following sections (and [BrK1,3,4,5]) we note, that the plasma dispersion function $Z(\zeta)$ defined by

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-s^2}}{s - \zeta} ds$$

is up to a constant for real ζ the Hilbert transform of the Gaussian function

$$H[e^{-x^2}] := -\frac{1}{\pi} I(x) := \frac{1}{\pi} I(-x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{x-t} dt$$

leading to the Dawson function

$$F(x) := e^{-x^2} \int_0^x e^{t^2} dt$$

by the Hermite weight formula ([GaW])

$$-\frac{1}{\pi} I(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{x-t} dt = \frac{1}{\pi} 2\sqrt{\pi} F(x) \cdot$$

5. Exact nonlinear plasma oscillation

In [Bel] the problem of a one-dimensional stationary nonlinear electrostatic wave in a plasma free from inter-particle collisions is solved exactly by elementary means. It is demonstrated that, by adding appropriate numbers of particles trapped in the potential-energy troughs, essentially arbitrary traveling wave solutions can be constructed.

When one passes to the limit of small-amplitude waves it turns out that the distribution function does not possess an expansion whose first term is linear in the amplitude, as is conventionally assumed. This disparity is associated with trapped particles. *It is possible, however, to salvage the usual linearized theory by admitting singular distribution functions.* These, of course, do not exhibit Landau damping, which is associated with the restriction to well-behaved distribution functions.

The possible existence of such waves in an actual plasma will depend on factors, which is ignored in [Bel].

6. The Hilbert transform applied to a nonlocal transport equation

For $0 < \beta < 1$ inner products are defined by ([CoA])

$$((u, v))_\beta := \int_{-\infty}^{\infty} |x|^{-2\beta} (-\bar{u}(x) \cdot v(x)) \frac{dx}{x} \geq 0 \quad \text{for even } u \in H_\beta^1(\mathbb{R})$$

$$((u, v))_\beta := \int_{-\infty}^{\infty} |x|^{2\beta} u(x) \cdot (-v_H(x)) \frac{dx}{x} \geq 0 \quad \text{for even } u \in H_{-\beta}^1(\mathbb{R}).$$

Then the central a priori estimates are given by ([CoA], theorem 1.1/1.4)

A. For $f \in C^1(\mathbb{R}) \cap H_\beta^1(\mathbb{R})$ and

A1: $0 < \beta < 1$, f even, it holds

$$\int \frac{(f(x) - f(0))^2}{|x|^{1+2\beta}} dx \leq -c_\beta \int \frac{f'(x) f_H(x)}{|x|^{2\beta}} dx$$

A2: $1/2 < \beta < 1$ and f nonnegative (or nonpositive), it holds

$$\int \frac{(f(x) - f(0))^2}{|x|^{1+2\beta}} dx \leq -c_\beta \int \frac{f'(x) \bar{f}_H(x)}{|x|^{2\beta}} dx$$

B. For $f \in C^1(\mathbb{R}) \cap H_\beta^1(\mathbb{R})$ and

B1: For $0 < \beta < 1$, f even, it holds

$$\int \frac{f^2(x)}{|x|^{1-2\beta}} dx \leq -c_\beta \int \frac{f'(x) f_H(x)}{|x|^{-2\beta}} dx$$

B2: For $1/2 < \beta < 1$, f nonnegative (or nonpositive), it holds

$$\int \frac{f^2(x)}{|x|^{1-2\beta}} dx \leq -c_\beta \int \frac{f'(x) f_H(x)}{|x|^{-2\beta}} dx \quad \cdot$$

7. The reduced (semi-infinite & finite) Hilbert transform

The reduced (semi-infinite & finite) Hilbert transform, Stieltjes integral, Plemelj formula, and its related diagonalizing operator are analyzed in ([ShE]):

The Hilbert transform operator

$$H[u](x) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{u(y)}{y-x} dy$$

acting on functions $u \in L_2(-\infty, \infty)$ defines an unitary, symmetric operator on $L_2(-\infty, \infty)$. Its spectrum consists just of the points ± 1 . Hilbert transforms on R^+ are defined by ($x > 0$)

$$H^{\pm}[u](x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{u(y)}{y \pm i\varepsilon - x} dy \cdot$$

By Plemelj's formulas we have the relations

$$H^{\pm}[u] = \frac{u}{2} \pm H[u] \cdot$$

For $t \in (-\infty, \infty)$, $\sigma \in (-1/2, 1/2)$ and

$$M[u](t) := \int_0^{\infty} u(x) x^{\frac{1}{2}-it} \frac{dx}{x}, \quad \rho_{\sigma}(t) := \left[e^{2\pi i(\sigma - \frac{1}{2} + it)} - 1 \right]^{-1} \cdot$$

it holds for certain (fastly decreasing) functions α ([ShE] Theorem 1.1)

$$H^{\pm}[\alpha](x) = x^{-\sigma} H^{\pm}[x^{\sigma} \alpha](x) \quad \text{i.e.} \quad [x^{\sigma} H^{\pm} - H^{\pm} x^{\sigma}][\alpha](x) = 0$$

where

$$MH_{\sigma}^{+}[\alpha] = \rho_{\sigma} \alpha, \quad MH_{\sigma}^{-}[\alpha] = (1 + \rho_{\sigma}) M[\alpha]$$

yielding a spectral decomposition of the isometry M ([ShE] Remark 1.2).

The reduced finite Hilbert transform operator and the Schrödinger differential operator are analyzed in ([KoW]):

The finite Hilbert transform operators

$$T_{(a,b)}[u](x) := \frac{1}{\pi i} \int_a^b \frac{u(y)}{y-x} dy, \quad a < b, \quad a < x < b$$

are bounded in $L_2(a, b)$ with norm $\|T_{(a,b)}\| \leq 1$. The self-adjoint Schrödinger differential operator $i \frac{d}{dx}$ defined for $u \in L_2(-\infty, \infty)$ is isometrically equivalent to

$$D_{(a,b)} := \frac{1}{\pi i} \sqrt{i(x-a)(b-x)} \frac{d}{dx} \sqrt{i(x-a)(b-x)}, \quad g \in L_2(a, b)$$

which is isometrically equivalent to the multiplicative operator

$$Q[g](\xi) := -\left(\frac{b-a}{2\pi} \log \frac{1+\xi}{1-\xi} \right) g(\xi) = -\frac{b-a}{2\pi} \log \frac{1+T_{(a,b)}}{1-T_{(a,b)}},$$

defined for all $g \in L_2(-1, 1)$ such that

$$\int_{-1}^1 \left| \log \frac{1+\xi}{1-\xi} g(\xi) \right|^2 d\xi < \infty \cdot$$

8. The Landau damping, integral inequalities for the Hilbert transform applied to a nonlocal (Burgers type) transport equation in one space variable

The Vlasov-Poisson equation (the collisionless Boltzmann equation) is time-reversible (for short periods of time due to the Landau damping, [MoC]). However, for long times the deformation of the distribution function approaches increasingly shorter scales which at some point in time go beyond any reasonable plasma physical length scale. At this point in time changes must be considered irreversible. Landau predicted this irreversible behavior on the analysis of the solution of the Cauchy problem for the linearized Vlasov equation around a spatially homogeneous Maxwellian (Gaussian) equilibrium. Landau formally solved the equation by means of Fourier and Laplace transforms. This phenomenon prevents instability from developing, and creates a region of stability in the parameter space.

In [GIR], [GIR1], it is shown that a solution of the linearized Vlasov equation in the whole space (linearized around a homogeneous equilibrium $f_0 := f(0)$ of infinite mass) decays at best like modulo logarithmic corrections, for $f_0(v) = c \cdot (1 + 2|v|)^{-1}$; and like $O(\log t - \alpha)$ if f_0 is a Gaussian. In order to get an answer to the question, if convergence holds in infinite time for the solution of the "full" nonlinear equation there is a mechanism required that would keep the distribution function f close to the original equilibrium.

This note is about a new proposed Landau theory, based on (distributional Hilbert scale) functional spaces and related functional inequalities for the nonlinear transport equation, alternatively to the approach in [MoC] establishing exponential Landau damping in analytical regularity built on analytical norms having up to 5 parameters (which is far away from any physical meaning). Our alternative approach is based on the results of [CoA], incorporating the Hilbert transform concept to define appropriate Hilbert space norms. The Galerkin-Ritz method is proposed to calculate corresponding (quasi-optimal) approximation solutions, e.g. with underlying boundary elements approximation spaces or trigonometric functions approximation spaces ([BrK]). We note that the Hilbert transform is also applied in [DeP] for a spectral theory of the linearized Vlasov-poisson equation.

In [CoA] the existence of finite-time singularities for a Burgers type equation

$$f_t - f_H \cdot f_x = 0$$

with nonlocal velocity in one space variable is shown. The motivation for the study of that equation is its analogy for the 3D Euler equation in vorticity form, having its origin in the CLM-model (Constatin-Lax-Majda), see also [BrK1]). The proposed function space $H_\beta^1(\mathbb{R})$, is the closure of $C_0^1(\mathbb{R})$ under the norm

$$\|f\|_\beta^2 = \|f\|_{L^\infty}^2 + \int \frac{(f(x) - f(0))^2}{|x|^{2\beta+1}} dx + \int \frac{(f'(x))^2}{|x|^{2\beta-1}} dx, \quad 0 < \beta < 1.$$

It is straightforward to obtain the following a priori estimate

$$\frac{d}{dt} \left\{ \|f\|_0^2 + \|f''\|_0^2 \right\} \leq c \left\{ \|f\|_0^2 + \|f''\|_0^2 \right\}^{3/2}$$

which implies local (in time) existence of the Cauchy problem with initial data in Sobolev space $H_1(\mathbb{R})$, which also cannot be justified by corresponding physical requirements/meanings.

The above is about a particle dynamics given by the ordinary differential equation

$$X'(t) = f_H(X(t), t)$$

and the equation implies that f is constant along the trajectories.

If one changes coordinates to a system of reference in which the maximum is stationary, i.e. if one defines $x_M(t)$ to be the trajectory where f reaches its maximum, and

$$y = x - x_M(t) \quad , \quad \tau = t$$

one obtains from the equation above the equation

$$\bar{f}_\tau - x'_M(t)\bar{f}_y - \bar{f}_H(y, \tau)\bar{f}_y = 0$$

resp.

$$\bar{f}_\tau - (\bar{f}_H(y) - \bar{f}_H(0))\bar{f}_y = 0$$

where

$$\bar{f}(y, \tau) = f(y + x_M(t), t) \quad .$$

We note the

Theorem of Privalov ([BuP], p. 20):

For $\alpha \in (0,1)$ let $f \in Lip(\alpha)$ be a periodic function, then $f_H \in Lip(\alpha)$.

We propose to apply the distributional Hilbert space concept of this paper to derive model adequate a priori estimate for the transport equation. The objective is, that the appropriately defined (distributional Hilbert space) norms enable appropriate Landau damping estimates, based on “realistic” physical modelling assumptions:

Following the ideas from [BrK1] [BrK3] this first leads to a change from

$$\|f\|_{(0)}^2 := \|f\|_{L_\infty}^2 + \left\| \frac{\bar{f}}{x} \right\|_0^2 + \|f\|_0^2$$

to

$$\|f\|_{(-1/2)}^2 := \|f\|_{L_\infty}^2 + \left\| \frac{\bar{f}}{x} \right\|_{-1/2}^2 + \|f\|_{-1/2}^2 \cong \|f\|_{L_\infty}^2 + \left\| \frac{\bar{f}}{\sqrt{x}} \right\|_0^2 + \|f\|_{1/2}^2 \quad .$$

We suggest the slightly “weaker” norm

$$\|f\|_\alpha^2 := \|f\|_{C^{0,\alpha}}^2 + \|x^\alpha \bar{f}\|_0^2 + \|f\|_\alpha^2 \quad , \quad \alpha \in (-\frac{1}{2}, \frac{1}{2})$$

and a corresponding analysis of a weak $H(-1/2)$ -Hilbert space by the wavelet analysis tool with (physical) problem adequately defined wavelet related to

$$\hat{f}_i(x) = \frac{1}{\sqrt{t}} f_i\left(\frac{x}{2\pi t}\right) \quad , \quad H[f_i](x) = 2 \int_0^\infty f_i(\xi) \sin(2\pi \xi x) d\xi \quad .$$

With respect to Hölder regularity and fractional diffusion transport equation we refer to [ChD].

Appendix: Boltzmann equation related topics

a. An alternative (Maxwell-) Boltzmann statistics

In quantum statistics the function

$$\omega(x) := \frac{1}{e^x - 1} = \sum_{n=0}^{\infty} e^{-nx}$$

plays a key role Bose-Einstein statistic, which is about bosons, liquid Helium and Bose-Einstein condensate. For large energy E (whereby $x = \beta(E - \mu)$) the distribution converge to the Boltzmann statistics. The Zeta function representation in the form

$$\zeta(s)\Gamma(s) = \int_0^{\infty} x^s \omega(x) \frac{dx}{x}$$

builds the relationship to the Planck black body radiation law (whereby the total radiation and its spectral density is identical). Putting, for instance,

$$\phi(x) := -x {}_1F_1\left(\frac{1}{2}; \frac{3}{2}, -x\right) \quad \text{and} \quad \omega^*(x) := \sum_{n=0}^{\infty} \phi(nx)$$

this leads to an alternative distribution in the form

$$\frac{s}{2s-1} \zeta(s)\Gamma(s) = \int_0^{\infty} x^s \omega^*(x) \frac{dx}{x} \cdot$$

The Fermi-Dirac statistics and the Bose-Einstein statistics converge for large energies resp. large temperatures to the (Maxwell-) Boltzmann statistics. Its density function (see also [AnJ] 2.2) is given by ($t \geq 0$)

$$\beta_t(x) := \frac{1}{\sqrt{\pi}} \frac{x^2}{t} e^{-\frac{x^2}{4t}} = \frac{1}{x} \frac{1}{\sqrt{\pi}} \left(\frac{x^2}{t}\right)^{3/2} e^{-\frac{x^2}{4t}} \cdot$$

The cummulative distribution function (which also enables an integral representation of the Navier-Stokes equations, [PeR]) is given by

$$\operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) - \frac{x}{\sqrt{\pi}} e^{-\frac{x^2}{4t}} \cdot$$

The accumulative Boltzmann distribution can be represented in the form

$$\frac{x}{\sqrt{\pi}} \left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}, \frac{x^2}{8t}\right) - e^{-\frac{x^2}{4t}} \right] \cdot$$

The Dawson function

$$F(x) = x {}_1F_1\left(1, \frac{3}{2}, -x^2\right) = \sqrt{\pi} x e^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, x^2\right) = \frac{\sqrt{\pi}}{2} f_H\left(\frac{x}{\sqrt{\pi}}\right) = \sqrt{\pi} \int_0^{\infty} f(t) \sin(2\sqrt{\pi}xt) dt \approx \frac{1}{2x} \arctan\left(\frac{\pi}{2}x\right)$$

in the form

$$F\left(\frac{x}{2\sqrt{t}}\right) \approx \beta_t(x)$$

is proposed as alternative “Boltzmann” density.

The corresponding entropy for the alternatively proposed Boltzmann density then is given by

$$\int_0^{\infty} F(x) \log F(x) dx = \pi \int_0^{\infty} \left[e^{-\pi^2} \right] \int_0^{\infty} \sin(2\sqrt{\pi}xt) \log \sin(2\sqrt{\pi}xt) dx dt \cdot$$

We note the identities [Grl] 4.384,

$$\int_0^1 \log(\sin(\pi x)) \sin(2\pi x) dx = 0 \quad , \quad \int_0^1 \log(\sin(\pi x)) \sin(2\pi x) dx = 0 \quad , \quad \int_0^1 \log(\sin(ax)) \sin(x) dx = -\frac{1}{a} [\gamma + \log a - ci(a)]$$

b. Thermodynamics, Boltzmann thermodynamics and absolute zero

[FeE] 31: A thermodynamical state of a system is not a sharply defined state of the system, because it corresponds to a large number of dynamical states. This consideration led to the Boltzmann relation

$$S = k \log p$$

where p is the (infinite) number of dynamical states that correspond to the given thermodynamical state. The value of p , and therefore the value of the entropy also, depends on the arbitrarily chosen size of the cells by which the phase space is divided of which having the same hyper-volume τ . If the volume of the cells is made vanishing small, both p and S become infinite. It can be shown, however, that if we change τ , p is altered by a factor. But from the Boltzmann relation it follows that an undetermined factor in p gives rise to an undetermined additive constant in S . Therefore the classical statistical mechanics cannot lead to a determination of the entropy constant. This arbitrariness associated with p can be removed by making use of the principles of quantum theory (providing discrete quantum state without making use of the arbitrary division of the phase space into cells). According to the Boltzmann relation, the value of p which corresponds to $S = 0$ is $p = 1$.

Statistically interpreted, therefore, *Nernst's theorem* (the third law of thermodynamics) states that "to the thermodynamic state of a system at absolute zero there corresponds only one dynamical state, namely, the dynamical state of lowest energy compatible with the given crystalline structure or state of aggregation of the system".

Nernst's theorem applied to solids leads to the entropy of the body at the temperature T in the form

$$S = \int_0^T \frac{C(T)}{T} dT$$

where the thermal capacity at absolute zero $C(0)$ needs to be zero, otherwise the integral would diverges.

The understanding that the zero state energy is uniquely determined is a miss understanding ([BrK3]). The value is just determined by the chosen mathematical model, i.e a purely mathematical requirement to ensure convergent series and integrals (note: the GRT requires differentiable manifolds, whereby only continuous manifolds are required from a physical modelling perspective). The Debye "temperature" constant θ for the specific heat of solids elements is an example in the context of above, leading to the theoretical formula

$$C(T) = 3RD\left(\frac{T}{\theta}\right)$$

We claim that as a consequence of the alternative harmonic quantum oscillator model (i.e. the alternative "plasma"/"wave package" state/energy spaces and corresponding continuous spectra) there is a challenge on the 3rd thermodynamical law:

"The entropy of every system at absolute zero can always be taken to zero".

Only all orthogonal projection of those states (resp. the corresponding eigenvalues of the projection operator) onto the test space H_0 **are** zero.

We propose an alternative entropy definition and a related closed absolute zero (Hilbert) state space in the form

$$h(p) := (p, \log p) \rightarrow (p, p)_{-1/2}, H_0^{\perp, \|\cdot\|_{-1/2}}.$$

c. The Bhatnagar-Gross-Krook (BGK) collision model

[CeC] II, 10: One of the major shortcomings in dealing with the Boltzmann equation is the complicated structure of the collision integral. The idea behind a replacement by a collision (kinetic) model is that a large amount of detail of the two-body interaction is not likely to influence significantly the values of many experimentally measured quantities. That is that the fine structure of the collision operator $Q(f, f)$ can be replaced by a blurred image, based upon a simpler operator $J(f)$, which retains only the qualitative and average properties of the true collision operator. The most widely known collision model is usually called the BGK model. The idea behind the BGK model is that the essential features of a collision operator are:

the collision model must satisfy

- i) $\int \psi_\alpha J(f) d\xi = 0$, $\alpha = 0, 1, 2, 3, 4$, ψ_α collision invariants
- ii) $\int \log f \cdot J(f) d\xi \leq 0$ (with equality holding iff f is a Maxwellian).

The second property expresses the tendency of the gas to a Maxwellian distribution. The simplest way of taking this feature into account seems to assume that the average effect of collisions is the change the distribution function $f(\xi)$ by an amount proportional to the departure of f from a Maxwellian $\Phi(\xi)$. so, if ν is a constant with respect to ξ , we introduce the following collision model

$$J(f) := \nu[\Phi(\xi) - f(\xi)].$$

The Maxwellian $\Phi(\xi)$ has five disposable scalar parameters (ρ, ν, T) according to the equations

$$f(\xi) = A \cdot e^{-\alpha(\xi-\nu)^2}$$

$$\alpha = 3 \cdot (4e)^{-1} = (2RT)^{-1} \quad , \quad A = \rho \left(\frac{4}{3}\pi e\right)^{-\frac{3}{2}} = \rho(2\pi RT)^{-\frac{3}{2}}.$$

The BGK model satisfy ii) and equality applies iff f is a Maxwellian.

The nonlinearity of $J(f)$ is much worse than the nonlinearity of the collision term. In fact the latter is simply quadratic in f , while the former contains f in both the numerator and the denominator of an exponential (the ν and T appearing in Φ are functionals of f). In other words, the collision term can be interpreted as a compact disturbance of the $J(f)$ model.

The main advantage in using the BGK model collision term is that for any given problem one can deduce integral equations for the macroscopic variables ν, ν, T .

If P is a probability density ($\int P d\mu = 1$), then

$$-H(P) = \int \log P = \int P \cdot \log P d\mu$$

(where $d\mu$ is the volume element in the space M of the events, whose probability density is P) is a suitable measure of the likelihood of P . In other words, if we take several P 's "at random", provided positive and normalized, most of them will be close to the probability density P for which $-H(P)$ is minimum.

The latter one enables variational theory concepts. In ([CeC] IV, 10, linear transport) a variational principle for the linearized Boltzmann equation is provided, based on an appropriately defined self-adjoint operator with respect to a certain scalar product.

**d. A wavelet based turbulent $H_{-1/2}$ -signal analysis
for elementary particles transport kinetics**

Plasma is the fourth state of matter, where from general relativity and quantum theory its known that all of them are fakes resp. interim specific mathematical model items. Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas are not. As a consequence the quantitative fluid/gas behavior as it is described by the Euler or the Navier-Stokes equations can not be applied as adequate mathematical model. Even this would be possible there is no linkage to the quantitative fluid/gas/plasma behavior and its corresponding turbulence behavior as it is described by the Euler or the Navier-Stokes equations. The approach in statistical turbulence is about low- and high-pass filtering Fourier coefficients analysis which is about a “local Fourier spectrum” analysis. As pointed out in [FaM] this is a contradiction in itself, as, either it is non-Fourier, or it is nonlocal. The proposal in [BrK1] is about a combination of the wavelet based solution concept of [FaM], [FaM1], with a revisited CLM equation model in a physical $H_{-1/2}$ -Hilbert space framework. The intension is, that this approach enables a turbulent $H_{-1/2}$ -signal which can be split into two components: coherent bursts and incoherent noise. Additionally the model enables a localized Heisenberg uncertainty inequality in the closed (“noise”/“wave packages”) subspace $H_{-1/2}-H_0$, while the momentum-location commutator vanishes in the (coherent bursts) test space H_0 . As a first trial we propose the Morlet wavelet, which is a sin wave that is windowed (i.e. multiplied point by point) by a Gaussian, having a mean value of zero.

We recall a few central symbols/formulas/equalities:

$F(v) = (2\pi)^{-1/2} e^{-v^2/2}$	Maxwellian velocity distribution
E	electric density
$f(x)$	distribution function of electrons
$f_0 = \hat{f}(0)$	Fourier transform of initial perturbation in $f(x)$
$C(f)$	collision term (functional of f)
H_β	Hermite polynomials of order β
n	electron density, Fourier-Laplace transform of electron density

The ordinary one-dimensional Boltzmann equation (in natural units) for the single-particle distribution function $f(x)$ of the electrons is given by

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} - E \frac{\partial f}{\partial v} = C(f)$$

$$\frac{\partial E}{\partial x} = \int_{-\infty}^{\infty} f dv - 1 \quad \cdot$$

The Fourier-Hermite expansion is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{\alpha=-\infty}^{\infty} \sum_{\beta=-\infty}^{\infty} a_{\alpha\beta} e^{i\alpha x} H_\beta(v) \cdot e^{-v^2/2}$$

where

$$H_\beta(v) = (-1)^\beta e^{-v^2/2} \left(\frac{d}{dv}\right)^\beta e^{-v^2/2} \quad \text{and} \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(v) H_m(v) e^{-v^2/2} dv = \delta_{nm} n! \quad \cdot$$

A Galerkin analysis for Schrödinger equation by wavelets is provided in [DaD].

e. Superconductivity, superfluids and condensates

[AnJ]: A superconductor is a charged fluid, which is a Bose-condensed state of interacting bosons. The degree of freedom of the Ginzburg-Landau theory (varying smoothly in space) are two fields

1. complex-valued „order parameter“ field $\psi(\mathcal{F}) = |\psi(\mathcal{F})| \cdot e^{i\theta(\mathcal{F})}$, defining superconducting order, as a function of the “superfluid density $n_s(\mathcal{F})$ and a complex phase angle $\theta(\mathcal{F})$
2. vector potential, representing the electromagnetic degrees of freedom.

The existence of the “order parameter” is postulated by the GLAG theory. It characterizes the superconducting state, in the same way as the magnetization does in ferromagnet. It is assumed to be some (unspecific) physical quantity, which characterizes the state of the system. In the normal state above the critical temperature of the superconductor it is zero, below this state it is nonzero.

Referring to the “rotating fluids” concept of [BrK1], and, at the same time, in line with the alternative “ground state energy model of the harmonic quantum oscillator” in [BrK3], we propose (with test space H_0 , state space $H_{-1/2}$ and energy space $H_{1/2}$)

$$\psi(\mathcal{F}) := P\phi(r), \quad P: H_{-1/2} \rightarrow H_0^\perp, \quad \phi = \chi + \psi \in H_{-1/2}, \quad \chi \in H_0, \quad \psi \in H_0^\perp$$

as an appropriately related “order parameter” projection operator. The closed space $H_0^\perp \subset H_{-1/2}$ could be interpreted as the “plasma (quantum) field” state space (where the Heisenberg uncertainty inequality is valid), while H_0 remains to be the “test space” of (measurable) observations, governed by the two self-adjoint ladder operators ([AnJ] 5.2).

**f. A simple one-dimensional turbulent flow model
based on the revisited CLM vorticity equation with viscosity term**

The trilinear form of the non-linear NSE term is antisymmetric. Therefore the energy inequality of the NSE with respect to the physical H^0 space does not take into account any contribution from the non-linear term. At the same time the regularity of the non-linear term cannot be smoother than the linear term. An alternative physical space $H^{-1/2}$ is the baseline of the unique 3D-NSE solution of this page.

Kolmogorov's turbulence theory is a purely statistical model, based on Brownian motion, which describes the qualitative behavior of turbulent flows. There is no linkage to the quantitative model of fluid behavior, as it is described by the Euler or Navier-Stokes equations.

Kolmogorov's famous 4/5 law is based on an analysis of low- and high-pass filtering Fourier coefficients. The physical counterpart to this is about a "local Fourier spectrum" which is (according to ([FaM]) nonsensical because, as, either it is non-Fourier, or it is nonlocal.

In Kolmogorov's spectral theory the two central concepts of a turbulent flow are homogeneous and isotropic flows (unfortunately they never encounter in nature). A flow is homogeneous if there is no "space" gradient in any averaged quantity, i.e. the statistics of turbulent flow is not a function of space. A flow is isotropic, if rotation and buoyancy are not relevant (they can be neglected) and there is no mean flow.

[FaM1] *"The definition of the appropriate "object" that composes a turbulent field is still missing. It would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. If the appropriate "object" has been defined that composes a turbulent field it would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. ...*

Turbulent flows have non-zero vorticity and are characterized by a strong three-dimensional vortex generation mechanism (vortex stretching). Brownian motion describes the random motion of particles suspended in a fluid resulting from their collision with quick atoms or molecules in a gas or a liquid. In mathematics it is described by the Wiener process. It is related to the normal density function. A Brownian (=red) noise is produced by a Brownian motion (i.e. a random walk noise). It is obtained as the integral of a white noise signal.

[FaM1] *"The notion of "local spectrum" is antinomic and paradoxical when we consider the spectrum as decomposition in terms of wave numbers for as they cannot be defined locally. Therefore a "local Fourier spectrum" is nonsensical because, either it is non-Fourier, or it is nonlocal. There is no paradox if instead we think in terms of scales rather than wave numbers. Using wavelet transform then there can be a space-scale energy be defined with a correspondingly defined scale decomposition in the vicinity of location x and a correspondingly defined local wavelet energy spectrum. By integration this defines a local energy density and a global wavelet energy spectrum. The global wavelet spectrum can be expressed in terms of Fourier energy spectrum. It shows that the global wavelet energy spectrum corresponds to the Fourier spectrum smoothed by the wavelet spectrum at each scale. The concept enables the definition of a space-scale Reynolds number, where the average velocity is being replaced by the characteristics root mean square velocity $Re(l, x)$ at scale l and location x. At large scale (i.e. $l \approx L$) $Re(L)$ coincides with the usual large-scale Reynolds number, where $Re(L)$ is defined as*

$$Re(L) = \iint_{R^n} Re(L, x) dx \cdot "$$

A wavelet series of a function $g(x)$ converges locally to $g(x)$, even if $g(x)$ is a distribution as long as the order of the distribution does not exceed the regularity of the analyzing wavelet. The admissibility condition ensures the validity of the inverse wavelet transform which then is valid for all Hilbert scale values.

Based on the re-revisited generalized CLM equation with viscosity term ([MaA] 5.2) we propose a turbulent flow model which allows non-stationary random functions with finite variance and related spectrum ([FrU] (4.54)) with respect to the $H_{1/2}$ – energy norm.

If the solution of the Euler equation is smooth then the solution to the slightly viscous NSE with same initial data is also smooth. Adding diffusion to the CLM model it makes the solution less regular [MuA]. As a consequence of this the CLM model lost most of the interest in the context of NSE analysis. In [MuA] a nonlocal diffusion term is proposed removing this drawback. The modification goes along with a reduced regularity of the “dissipation” term resulting in a reduced “energy” Hilbert scale of Hilbert scale factor $-1/2$. As this modification did not modify in same manner the non-linear term this leads to an unbalanced energy equation. As the non-linear term governs the dissipative term in case of turbulence, this is an argument to reject current revisited CLM model with viscosity term ([DeS], [MuA], [OkH], [SaT], [SaT1]). At the same time those suggested modifications being applied in same manner to the linear term would fit to the Stieltjes integral based Kolmogorov theory [ShA], as well as to the conceptual idea of this paper (i.e. an $H_{1/2}$ – energy inner product enabling an energy inequality which does not exclude any information from the non-linear term). Combining both conceptual ideas provides a functional analytical common framework ([BrK], [BrK3]) for a statistical fluid mechanics theory ([MoA]), a statistics of gases and highly turbulent fluid flows [HoE].

The building concept of the revisited generalized CLM model is therefore as follows: we consider periodic boundary condition and assume that the initial condition of ω is symmetric with respect to the origin ([SaT1]). We propose a weak $H_{-1/2}$ – variation representation of the extended Schochet-CLM model ([ScS]) in the form

$$(\mathcal{A}v)_{-1/2} - \varepsilon(\omega_{xx}, v)_{-1/2} = (\omega H[\omega], v)_{-1/2}, \quad \forall v \in H_{-1/2}.$$

With the notation of [BrK] this representation is equivalent to

$$(A\mathcal{A}v)_0 + \varepsilon(H[\omega], v)_{1/2} = (A[\omega H[\omega]], v)_0, \quad \forall v = H[w] \in H_{-1/2}.$$

Taken into account that the Hilbert transform is an isometry on all Hilbert scales and that $H^2[v] = -v$ and putting $\omega_H := H[\omega]$ this can be reformulation in the form

$$(\mathcal{A}_H^2, w)_{-1/2} + \varepsilon(\omega, w)_{1/2} = (H[\omega H[\omega]], w)_{-1/2}, \quad \forall w \in H_{-1/2}.$$

From [MaA] we recall the identity

$$2H[\omega H[\omega]] = \omega_H^2 - \omega^2$$

leading to

$$(\mathcal{A}_H^2 - \varepsilon\omega', w)_{-1/2} = \frac{1}{2}(\omega_H^2 - \omega^2, w)_{-1/2}, \quad \forall w \in H_{-1/2}.$$

The left hand side of the variation representations above is reflecting to current revisited proposals of the CLM model, while now the right hand side of the variation equation shows a modified non-linear CLM model operator (as the domain has changed).

In [SaT1] for periodic boundary conditions the Fourier (spectral) representation of the non-linear term $\omega H[\omega] = \omega A[\omega_x]$ is given, whereby ω denotes the vorticity and H the Hilbert transform operator. The spectral method analysis of the equation above follows the same way leading to:

$$\hat{\omega}_n \approx n(\alpha \omega_n + \sum_{k=1}^{n-1} \omega_k \omega_{n-k}) \quad , \quad \omega_n(0) = \frac{A_n}{2} \quad , \quad \hat{\omega}_0 = 0$$

whereby

$$\omega(x,0) = \sum_1^{\infty} A_n \sin(nx) \cdot$$

Following the concept of [FaM] the turbulent $H_{-1/2}$ -signal can be split into two contributions: coherent bursts, corresponding to that part of the signal which can be compressed in a H_0 -wavelet basis, plus incoherent noise, corresponding to that part of the signal which cannot be compressed in a H_0 -wavelet basis, but in the $H_{-1/2}$ -wavelet basis. For the $n=1$ periodic case the later one corresponds to the alternative zero-state energy model of the harmonic quantum oscillator.

The spectral analysis above is also linked to the solution framework of [BrK4]. The Hilbert transform of the Gaussian is the Dawson function, which is norm equivalent to the Gaussian due to the related property of the Hilbert transform. Therefore a Dawson basis function based Hilbert space framework enables an alternative statistical hydromechanics:

Let $f(x)$ resp. $f_H(x)$ denote the Gaussian function resp. its Hilbert transform $f_H(x) = H[f](x)$ with

$$H[v](x) := \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{v(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{v(y)}{x-y} dy \quad .$$

Then it holds $\hat{f}(0) = 1$ resp. $\hat{f}_H(0) = 0$, $f_H(x)$ is up to a constant identical with the Dawson function and $f(x)$ and $f_H(x)$ are norm-equivalent, i.e.

$$(f, w) = (f_H, w) \quad \forall w \in H_0(-\infty, \infty) .$$

We further note the following properties of the Hilbert transform:

$$[xH - Hx][u](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy = \hat{u}(0) \quad \text{and} \quad (u, Hu)_0 = 0 \quad \text{for} \quad u, Hu \in L_2 .$$

It is proven by the insertion of a new variable $z = x - y$ for the term

$$H[xu(x)](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z))}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z) dz = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy \quad .$$

The Gaussian function is not a wavelet, as it does not fulfill the admissibility condition, but its first derivative and $f_H(x)$ do. When considering $f'(x)$ in a $H_{-1/2}$ – (physical) Hilbert space framework the following identities hold true:

$$(f', w)_{-1/2} = (Af', w)_0 = (f_H', w)_0 = (f, w)_0 \quad \forall w \in H_0(-\infty, \infty).$$

In other words, a Fourier synthesis (and a probability analysis) in a H_0 – Hilbert space framework can be transformed into a wavelet analysis (“white noise” signal appropriate, ([VoR])) $H_{-1/2}$ – Hilbert space framework.

The corresponding central “distributional” functions in case of the 2π – periodic function are given by the identities [BrK4]:

$$\frac{1}{2} \cot\left(\frac{x}{2}\right) = \frac{d}{dx} \ln\left(2 \sin\left(\frac{x}{2}\right)\right) = -\frac{d}{dx} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} = \sum_{n=1}^{\infty} \sin(nx) \in H_{-1}^{\#}(0, 2\pi).$$

We note that the Hilbert transform of a wavelet is again a wavelet ([WeJ]) and that the Hilbert transform is an isomorphism on any Hilbert scale H_{β} .

[FaM]: *“The turbulent regime develops when the non-linear term of the NSE strongly dominates the linear term. Superposition principle holds no more for non-linear phenomena. Therefore turbulent flows cannot be decomposed as a sum of independent subsystems that can be separately studied. A wavelet representation allows analyzing the dynamics in both space and scale, retaining those degrees of freedom which are essential to compute the flow evolution”.*

[MeM]: *“Methods based on wavelet (Galerkin) expansions in L_2 – framework face the issue that in Galerkin methods the degrees of freedom are the expansion coefficients of a set of basis functions and these expansion coefficients are not in physical space (means in wavelet space). First map wavelet space to physical space, compute non-linear term in physical space and then back to wavelet space, is not very practical”.*

The Galerkin method based on wavelet expansion requires (ongoing) mappings between wavelet and physical space (during computing process) in case both spaces are different. This is the case for most of current (weak) variation PDE representations in a L_2 – Hilbert space framework ([MeM]).

The proposed $H_{-1/2}$ – (weak) physical space concept enables identical wavelet and physical Hilbert spaces, while at the same time enabling the full power of Galerkin method computing non-linear terms in this (newly) physical space. This leads back to the "solution" section of this page with the weak NSE solution of the corresponding weak NSE representation in the $H_{-1/2}$ – Hilbert space and the (H_0 – based) physical principles of quantum theory [HeW].

g. Statistical Thermodynamics, E. Schrödinger's view [ScE]

There is, essentially, only one problem in statistical thermodynamics: the distribution of a given amount of energy E over N identical systems. Or perhaps better: to determine the distribution of an assembly of N identical systems over the possible states in which this assembly can find itself, given that the energy of the assembly is a constant E . The idea is that there is weak interaction between them, so weak that the energy of interaction can be disregarded, that one can speak of the "private" energy of every one of them and that the sum of their "private" energies has to equal E

"To determine the distribution" .. mean in principle to make oneself familiar with any possible distribution-of-the-energy (or state-of-the-assembly) is (always the same) the mathematical problem; we shall (soon) present its general solution, from which in the case of every particular kind of system every particular classification that may be desirable can be found as a special case:

But there are two different attitudes as regards the physical application of the mathematical result. ...

The older and more naïve application is to N actually existing physical systems in actual physical interaction with each other, e.g. gas molecules or electrons or Planck oscillators or degrees of freedom ("ether oscillators") of a "hohlraum". The N of them together represent the actual physical system under consideration. This original point of view is associated with the names of Maxwell, Boltzmann and others.

But it suffices only dealing with a very restricted class of physical systems – virtually only with gases. It is not applicable to a system which does not consist of a great number of identical constituents with "private" energies. ...

Hence a second point of view ... has been developed. It has a particular beauty of its own, is applicable quite generally to every physical system, and has some advantages to be mentioned forthwith. Here the N identical systems are mental copies of the one system under consideration – of the one macroscopic device that is actually erected on our laboratory table. Now what on earth could it mean, physically, to distribute a given amount of energy E over these N mental copies? The idea is, in my view, that you can, of course, imagine that you really had N copies of your system, that they really were in "weak interaction" with each other, but isolated from the rest of the world. Fixing your attention on one of them, you find it in a peculiar kind of "heat-bath" which consists of the $N-1$ others.

Now you have, on the one hand, the experience that in thermodynamical equilibrium the behavior of a physical which you place in a heat-bath is always the same whatever be the nature of the heat-bath that keeps it at constant temperature, provided, of course, that the bath is chemically neutral towards your system, i.e. that there is nothing else but heat exchange between them. On the other hand, the statistical calculations do not refer to the mechanism of interaction: they only assume that it is "purely mechanical", that it does not affect the nature of the single systems (e.g. that it never blows them to pieces), but merely transfers energy from one to the other.

These considerations suggest that we may regard the behavior of any one of those N systems as describing the one actually existing system when placed in a heat-bath of given temperature. Moreover, since N systems are alike and number similar conditions, we can

then obviously, from their simultaneous statistics, judge of the probability of finding our system, when placed in a heat-bath of given temperature, in one or other of its private states. Hence all questions concerning the system in a heat-bath can be answered. ...

The advantage consists not only in the general applicability, but also in the following two points:

- i) N can be made arbitrarily large. In fact, in case of doubt, we always mean $\lim N = \infty$ (... Stirling's formula for $N!$)
- ii) No question about the individuality of the members of the assembly can ever arise – as it does, according to the “new statistics”, with particles. Our systems are macroscopic systems, which we could, in principle, furnish with labels. Thus two states of the assembly differing by system No. 6 and system No. 13 having exchanged their roles are, of course, to be counted as different states, while the same may not be true when two similar atoms within system No. 6 have exchanged their roles; ...

Remark: The approach in [BrK5] with the proposed quantum state Hilbert space given by

$$x = x_0 + x_0^\perp \in H_{-1/2} = H_0 \otimes H_0^\perp$$

fulfills the same “advantages” points i), ii) above, while the (compactly embedded, (!)) subspace $H_0 \subset H_{-1/2}$ “covers” the statistically relevant (measurable) macroscopic world and its related orthogonal space H_0^\perp “covers” the “particle-interaction-world. The “heat-bath-room” of given temperature is in line with the corresponding domain of the alternatively proposed Schrödinger operator, given by H_1^\perp of the corresponding energy space

$$H_1 \subset H_{1/2} = H_1 \otimes H_1^\perp.$$

The second section of [ScE] is concerned with the method of the most probable distribution, allowing infinite numbers of identical systems over their energy levels. We briefly sketch the central mathematical idea of chapter II:

for an assembly of N identical systems the nature of any of them is described by its possible state by enumerating them with labels $1,2,3,...l,...$. In a quantum-mechanical system those states are to be described by the eigenvalues of a complete set of commuting variables. The eigenvalues of the energy in these states are called $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l, \dots$ so that $\varepsilon_{l+1} \geq \varepsilon_l$. In a “classical system” the schema can also be applied, when the states will have to be described as cells in phase-space (p_k, q_k) of equal volume – whether infinitesimal in all directions or not – at any rate such that the energy does not vary appreciable within the cell. The considered model parameters are

State No.	$1,2,3...l,...$
Energy	$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_l, \dots$
Occupation No.	$\alpha_1, \alpha_2, \dots, \alpha_l, \dots$

The number of single states belonging to this class is

$$P = \frac{N!}{\alpha_1! \alpha_2! \dots \alpha_l! \dots}.$$

The set of numbers must comply with the conditions

$$N = \sum \alpha_s, \quad E = \sum \varepsilon_l \alpha_s.$$

“In this form the result is wholly unsurveyable. ... For N large, but finite, the assumption is only approximately true. Indeed, in the application to the Boltzmann case, the distributions with occupation numbers deviating from the “maximum set” must not be entirely disregarded. They give information on the thermodynamic fluctuations of the Boltzmann system, when kept at constant energy E , i.e. in perfect heat isolation. ...

Now the fluctuations of a system in a heat bath at constant temperature are much more easily obtained directly from the Gibbs point of view.”
 We choose the logarithm of

$$P = \frac{N!}{\alpha_1! \alpha_2! \dots \alpha_l! \dots}$$

as the function whose maximum we are determine, taking care of the accessory conditions in the usual way by Lagrange multipliers, λ and μ , seeking the unconditional maximum of

$$\log P - \lambda \sum_l \alpha_l - \mu \sum_l \varepsilon_l \alpha_l$$

leading to

$$N = \sum_l e^{-\lambda - \mu \varepsilon_l} \quad , \quad E = \sum_l \varepsilon_l e^{-\lambda - \mu \varepsilon_l} .$$

Calling $U := E/N$ the average share of energy of one system, the whole result is given by

$$U = \frac{E}{N} = \frac{\sum_l \varepsilon_l e^{-\mu \varepsilon_l}}{\sum_l e^{-\mu \varepsilon_l}} = - \frac{\partial}{\partial \mu} \log \left(\sum_l e^{-\mu \varepsilon_l} \right)$$

whereby

$$\alpha_l = \frac{e^{-\mu \varepsilon_l}}{\sum_l e^{-\mu \varepsilon_l}} = - \frac{N}{\mu} \frac{\partial}{\partial \varepsilon_l} \log \left(\sum_l e^{-\mu \varepsilon_l} \right)$$

indicates the distribution of the N systems over their energy levels.

Chapter IV provides three examples, which are

- i) Free mass-point (ideal monatomic gas)

$$\Psi = k \log Z = kL \log V + \frac{3}{2} kL \log T + \text{constant} \quad \text{for } L \text{ atoms}$$

$$U = T^2 \frac{\partial \Psi}{\partial T} = \frac{3}{2} kLT \quad , \quad p = T \frac{kL}{V}$$

- ii) Planck oscillator

$$\Psi = k \log Z = k \log \left(\sum_l e^{-\mu h \nu (l+1/2)} \right) = -k \log \left(\sinh \left(\frac{x}{2} \right) \right) - k \log 2 ,$$

$$\text{with } x := (h\nu)/(kT) = \mu h \nu =: \mu \varepsilon$$

$$U = \frac{\varepsilon}{2} + \frac{\varepsilon}{e^{\varepsilon/(kT)} - 1}$$

- iii) Fermi oscillator

$$\Psi = k \log Z = k \log (1 + e^{-\varepsilon/kT})$$

$$U = \frac{\varepsilon}{e^{\varepsilon/(kT)} + 1} .$$

We also briefly sketch the corresponding mathematical approach for the n-particle problem ([ScE] chapter VII):

The sum-over-state of the considered n-particle problem is given by

$$Z = \sum_{n_s} e^{\mu \sum n_s \alpha_s}$$

where $n_s, s = 1, 2, \dots$ denote the numbers of particles on level $\alpha_s, s = 1, 2, \dots$, $\mu := 1/kT$ and the levels ε_j is given by

$$\varepsilon_j = \sum n_s \alpha_s \cdot$$

For the Bose-Einstein gas and Fermi-Dirac gas the values admitted for every n_s are

i) $n_s = 0, 1, 2, 3, 4, \dots$ Bose-Einstein gas

ii) $n_s = 0, 1$ Fermi-Dirac gas

Putting

$$z_s = e^{-\mu \alpha_s}$$

thus

$$Z = \sum_{n_s} z_1^{n_1} \cdot z_2^{n_2} \cdot z_3^{n_3} \cdot \dots \cdot z_s^{n_s} \cdot \dots$$

This results into

i) $Z = \prod_s (1 - z_s)^{-1}$ Bose-Einstein gas

ii) $Z = \prod_s (1 + z_s)$ Fermi-Dirac gas

which is combined into the following formula

$$Z = \prod_s (1 \pm z_s)^{\mu 1} \cdot$$

In case the condition

$$n = \sum n_s$$

is imposed, this formula is not yet the final result. For a glance at the original form

$$Z = \sum_{n_s} z_1^{n_1} \cdot z_2^{n_2} \cdot z_3^{n_3} \cdot \dots \cdot z_s^{n_s} \cdot \dots$$

indicates that one have to select from

$$Z = \sum_{n_s} z_1^{n_1} \cdot z_2^{n_2} \cdot z_3^{n_3} \cdot \dots \cdot z_s^{n_s} \cdot \dots$$

only the terms homogeneous of order n in all the z_s . That is most conveniently done by the method of the residue integral.

Putting

$$f(\zeta) := \prod_s (1 + \mu \zeta \cdot z_s)^{\mu_1}$$

the correct is rigorously represented by the following integral:

$$Z = \frac{1}{2\pi i} \oint \frac{f(\zeta) d\zeta}{\zeta^n \zeta}$$

The corresponding analysis leads to

- i) $N = \sum \left(\frac{1}{\zeta} e^{\mu \alpha_s} \mu_1 \right)^{-1}$
- ii) $\log Z = -n \log \zeta + \log f(\zeta) = -n \log \zeta + \mu \sum_s \log(1 + \mu \zeta e^{\mu \alpha_s})$
- iii) $n_s = -\frac{1}{\mu} \frac{\partial \log Z}{\partial \alpha_s} = \left(\frac{1}{\zeta} e^{\mu \alpha_s} \mu_1 \right)^{-1}$.

The related thermodynamic parameters are given by

$$n = \frac{4\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty \left(\frac{1}{\zeta} e^{x^2} \mu_1 \right)^{-1} x^2 dx$$

$$\Psi = k \log Z = -nk \log(\zeta) + \mu \frac{4\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty \log(1 + \mu \zeta e^{x^2}) x^2 dx$$

$$U = kT \frac{4\pi V}{h^3} (2mkT)^{3/2} \int_0^\infty \left(\frac{1}{\zeta} e^{x^2} \mu_1 \right)^{-1} x^4 dx \cdot$$

At the end this leads to a "thermodynamic potential" in the form

$$nkT \log(\zeta) = U - TS + pV = T^2 \frac{\partial \Psi}{\partial T} - TS + VT \frac{\partial \Psi}{\partial V} = T^2 \frac{\partial \Psi}{\partial T} - TS + \frac{2}{3} U$$

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