An unusual shift theorem enabling $L_\infty -$ boundedness of the Ritz-Galerkin operator for hyperbolic problems

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Abstract

In [NiJ], [NiJ1] $L_\infty -$boundedness of the Ritz-Galerkin operator is proven for elliptic and parabolic problems in case of finite element approximation spaces. The common conceptual two elements are so-called weighted functions with corresponding appropriate super-approximation properties for finite element approximation spaces providing appropriate relationships between weighted norms $\|\Phi\|_\alpha^2 := (\Phi, \mu^{-\alpha}\Phi)$ and the $\|\Phi\|_{L_\infty} -$norm, as well as a duality argument (Nitsche trick) in the form $A^* w = \mu^{-\alpha-1}\Phi$ whereby an 'optimal order' shift theorem is required in the form

$$\|w\|_{H_{k+2}} \le c \cdot \|Aw\|_{H_k} \quad \text{resp.} \quad \|w\|_{L_2(H_{k+2})} \le c \cdot \|Aw\|_{L_2(H_k)} .$$

We present a counter example that same 'optimal order' shift theorem for hyperbolic equation is not necessarily fulfilled. In addition we propose for $t > 0$ an alternative problem adequate inner product resp. norm with exponential decay in the form

$$(u,v)_{(t)} := \sum e^{-\sqrt{\lambda_i} t} (u, \varphi_i)(v, \varphi_i) \; .$$

It enables an 'optimal order' shift theorem in the form

$$\|w\|_{L_2(H_{k+2,t})} \le c \cdot \|Aw\|_{L_2(H_{k,t})} .$$

As a consequence the technique from [NiJ], [NiJ1] can be applied to prove the $L_\infty -$boundedness of the Ritz-Galerkin operator for hyperbolic problems.

§1 Hilbert scale approximation theory for hyperbolic problems

In [NiJ], [NiJ1] $L_\infty -$boundedness of the Ritz-Galerkin operator is proven for elliptic and parabolic problems in case of finite element approximation spaces. Both papers basically apply the same two central concepts. This is about

i) A weighted functions concept with corresponding appropriate super-approximation properties for finite element approximation spaces and appropriate relationship of weighted norms $\|\Phi\|_\alpha^2 := (\Phi, \mu^{-\alpha}\Phi)$ and the $\|\Phi\|_{L_\infty}$ $-$norm, with (for $t_0 > 0$ fixed and $0 < t \le t_0$)

$$\mu(x) := \rho^2 + |x - x_0|^2 \quad \text{resp.} \quad \mu(x,t) := \rho^2 + |x - x_0|^2 + t_0 - t .$$

ii) A duality argument (Nitsche trick) in the form

$$A^* w = \mu^{-\alpha-1}\Phi$$

whereby an ‘optimal order’ shift theorem is required in the form

(*) $$\|w\|_{H_{k+2}} \le c \cdot \|Aw\|_{H_k} \quad \text{resp.} \quad \|w\|_{L_2(H_{k+2})} \le c \cdot \|Aw\|_{L_2(H_k)} .$$

In order to apply same concept to hyperbolic problems the corresponding weighted function would be

$$\mu(x,t) := \rho^2 + |x - x_0|^2 + (t_0 - t)^2 .$$

In case of space dimension $n = 1$ (and therefore $\alpha = 1$) the connection of weighted norms and the $L_\infty -$ norm is given by the inequality

$$\|v\|_\alpha \le ch^{-1/2}\|v\|_\infty$$

and

$$\|\chi\|_{L_\infty} \le ch^{1/2} \sup\{\|\chi\|_\alpha \,|\, x_0 \in [a,b]\} \quad \text{for} \quad \chi \in \dot{S}_h .$$

Unfortunately an ‘optimal order’ shift theorem in the sense of (*) is not necessarily ensured for hyperbolic problems. In §4 below we give a counter example for (*) and propose an alternative Hilbert space framework enabling an ‘optimal order’ shift theorem.

§ 2 The Huygens principle and distortion-free, progressive waves

The weighted functions are related to the Poisson kernel by ([PeB] II, §3):

$$\frac{1}{(2\pi)^n}\int e^{-i(x,\xi)}e^{-\rho x}dx = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{\rho}{(\rho^2+|\xi|^2)^{\frac{n+1}{2}}} .$$

The Fourier transforms of the uniform distributions μ resp. ν of unit mass over the unit ball resp. of unit mass over the unit sphere with center at the origin are given by ([PeB] II, §2):

$$\hat{\mu}(\xi) = 2^{n/2}|\xi|^{-n/2}\Gamma(\frac{n+2}{2})J_{n/2}(|\xi|) \quad , \quad \hat{\nu}(\xi) = 2^{(n-2)/2}|\xi|^{(2-n)/2}\Gamma(\frac{n}{2})J_{(n-2)/2}(|\xi|)$$

whereby J_σ denotes the Bessel function of the first kind of order σ.

In the context of the wave equation radiation problem and its relationship to the Huygens principle and to distortion-free, progressive waves ([CoR] VI §10.3) we give a characterization of the 4-dimensional Minkowski space.

It is proposed to answer the Courant-Hilbert conjecture ([CoR] VI §10.3) that families of distortion-free, progressive waves only exist if the Huygens principle is valid and that those families only exist for $n=2$ and $n=4$.

Theorem 1: The 4-dimensional Minkowski space is characterized by the fact that the differential of the Fourier transform of the uniform distribution of unit mass over the unit sphere at the origin vanishes, i.e. $d\hat{\nu}_{n=4}(0)=0$.

Proof: From [WaG] III, 3.2, we recall

$$\frac{d}{dr}\left[r^\rho J_\rho(r)\right] = r^\rho J_{\rho-1}(r) .$$

As it holds $J_{\rho-1}(0)=0$ iff $\rho=1$ this proves the theorem.

§3 Approximation theory in Hilbert scales

The finite element error analysis in the framework of Sobolev spaces can be addressed by Hilbert scale approximation theory ([BrK]). Given an operator A fulfilling

i) A is self-adjoint and positive definite
ii) A^{-1} is compact

the corresponding eigen-pairs of $A\varphi_i = \lambda_i \varphi_i$ enable the definition of an inner product resp. norm in the form

$$(u,v)_\beta := \sum \lambda_i^\beta (u,\varphi_i)(v,\varphi_i) \quad \text{resp.} \quad \|u\|_\beta^2 := (u,u)_\beta := \sum \lambda_i^\beta (u,\varphi_i)^2$$

defining the related Hilbert space H_β. Because of $\lambda_i \to \infty$ for $i \to \infty$ there is a polynomial decay of the factors λ_i^β.

The main idea of this paper is to apply for $t > 0$ an additional inner product resp. norm with exponential decay by ([BrK])

$$(u,v)_{(t)} := \sum e^{-\sqrt{\lambda_i} t} (u,\varphi_i)(v,\varphi_i)$$

resp.

$$\|u\|_{(t)}^2 := (u,u)_{(t)} \ .$$

Obviously it holds $\|u\|_{(t)} \le c(\alpha,t) \cdot \|u\|_\alpha$. On the other side any negative norm, i.e. $\|u\|_\beta$ with $\beta < 0$, is bounded by the $0-$norm and the new $(t)-$norm, i.e. it holds that for $\alpha > 0$ be fixed the $(-\alpha)-$norm of any $u \in H_0$ is bounded by

$$\|u\|_{-\alpha}^2 \le \delta^{2\alpha} \|u\|_0^2 + e^{t/\delta} \|u\|_{(t)}^2$$

with $\delta > 0$ being arbitrary. For $t, \delta > 0$ be fixed to any $u \in H_0$ there is an $v \in H_1$ according to

$$\|u - v\| \le \|u\| \ , \ \|v\|_1 \le \delta^{-1} \|u\| \ , \ \|u - v\|_{(t)} \le e^{-t/\delta} \|v\| \ .$$

and therefore

$$\inf_{\xi \in S} \left\{ e^{-t/2\kappa} \|x - \xi\|_0 + \|x - \xi\|_{(t)} \right\} \le 4 e^{-t/2\kappa} \|x\| \ .$$

§4 An unusual 'optimal order' shift theorem for hyperbolic problems

We consider the hyperbolic model problem

$$\begin{aligned} \ddot{u} - \Delta u = f \quad & \text{in } \Omega \times (0,T] \\ u = 0 \quad & \text{on } \partial\Omega \times (0,T] \\ u(0) = u_0,\ \dot{u}(0) = u_1 \quad & \text{in } \Omega \ . \end{aligned}$$

An 'optimal order' shift theorem in the sense of (*) is not necessarily ensured for hyperbolic problems. Let

$$\Psi(x,t) := e^{-(\frac{1}{2}-(x-t))^2} \quad \text{and} \quad f(x,t) := 2\Psi(x,t) - 4t\Psi'(x,t)$$

then because of $\dot{\Psi} + \Psi' = 0$ and $\ddot{\Psi} = \Psi''$ those functions build a counter example in the following way

Lemma: The function $u(x,t) := t^2 \cdot \Psi(x,t)$ is a solution of the wave equation with $u_0 = u_1 = 0$ fulfilling $\|u''\|_{L_2(L_2)} \approx \|\Psi''\|_{L_2(L_2)}$ but at the same time it holds $\|f\|_{L_2(L_2)} = \|Au\|_{L_2(L_2)} \approx \|\Psi'\|_{L_2(L_2)}$.

In the framework of the Hilbert spaces $H_{k.t}$ we prove the following

Theorem 2: The hyperbolic self-adjoint, positive definite wave equation operator fulfills an 'optimal order' shift theorem in the form

$$\|u\|_{L_2(H_{k+2.t})} \le c \cdot \|Au\|_{L_2(H_{k.t})} .$$

Proof: The Fourier coefficients of the wave equation are given by

$$u_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin \sqrt{\lambda_i}(t-\tau) f_i(\tau) d\tau .$$

For later use we recall from [Grl] 2.663

$$\int e^{ax} \sin^2(bx) dx = \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4b^2} \left\{ \frac{a}{2} \cos(2bx) + b \sin(2bx) \right\} .$$

With this one gets

$$\int_0^T e^{-\sqrt{\lambda_i}t} u_i^2(t) dt = \frac{1}{\lambda_i} \int_0^T e^{-\sqrt{\lambda_i}t} (\int_0^t \sin \sqrt{\lambda_i}(t-\tau) f_i(\tau) d\tau)^2 dt$$

$$\le \frac{1}{\lambda_i} \int_0^T e^{-\frac{1}{2}\sqrt{\lambda_i}t} (\int_0^t e^{-\frac{1}{2}\sqrt{\lambda_i}\tau} \sin^2 \sqrt{\lambda_i}(t-\tau) d\tau) \cdot (\int_0^t e^{-\frac{1}{2}\sqrt{\lambda_i}\tau} f_i^2(\tau) d\tau)) dt .$$

From the above it follows

$$\int_0^t e^{-\frac{1}{2}\sqrt{\lambda_i}\tau} \sin^2 \sqrt{\lambda_i}(t-\tau)d\tau \le \frac{c}{\sqrt{\lambda_i}}$$

and therefore by changing the order of integration

$$\int_0^T e^{-\sqrt{\lambda_i}t} u_i^2(t)dt \le \lambda_i^{3/2} \int_0^T e^{-\frac{1}{2}\sqrt{\lambda_i}t} \int_0^t e^{-\frac{1}{2}\sqrt{\lambda_i}\tau} f_i^2(\tau)d\tau dt$$

$$= \lambda_i^{3/2} \int_0^T \int_\tau^T e^{-\frac{1}{2}\sqrt{\lambda_i}\tau} f_i^2(\tau) e^{-\frac{1}{2}\sqrt{\lambda_i}t} dt d\tau$$

$$\le \lambda_i^{-2} \int_0^T e^{-\sqrt{\lambda_i}\tau} f_i^2(\tau)d\tau \cdot$$

§5 L_∞ – Boundedness of the FEM for hyperbolic problems

For finite element approximation spaces $S_h \subset \overset{\circ}{W}{}^1_2$ the standard Ritz-Galerkin approximation for our hyperbolic model problem is defined by

$$(\ddot{u}_h, \chi) + D(u_h, \chi) = (f, \chi) \text{ for } \chi \in S_h \text{ and } t \in (0, T]$$

with $u_h(0) = P_h u_0$ and $\dot{u}_h(0) = P_h u_1$, whereby P_h denotes the L_2- projection operator ([NiJ1]).

The 'optimal order' shift theorem enables an analog proof as applied for the parabolic (heat) equation problem in a (now hyperbolic) problem adequate corresponding Hilbert space framework:

for $\Phi := u_h - P_h u \in S_h$ it follows $\Phi(0) = 0$. Putting $e = \varepsilon - \Phi := (u - P_h u) - (u_h - P_h u)$ one gets the defining relation for Φ in the form

$$(\ddot{\Phi}, \chi) + D(\Phi, \chi) = D(\varepsilon, \chi) \text{ for } \chi \in S_h \text{ and } t \in (0, T].$$

The objective is to find a bound for

$$\frac{d}{dt}\|\dot{\Phi}\|_\alpha^2 \approx 2(\ddot{\Phi}, \mu^{-\alpha}\Phi) + c\|\dot{\Phi}\|_{\alpha+1}^2 .$$

Analog as in [NiJ1] one puts $\chi := P_h(\mu^{-\alpha}\Phi) \in S_h$ and starts from the defining relation for $\Phi \in S_h$ with the following identity

$$(\ddot{\Phi}, \mu^{-\alpha}\Phi) = (\ddot{\Phi}, \chi) = -D(\Phi, \mu^{-\alpha}\Phi) + D(\varepsilon, \mu^{-\alpha}\Phi) + D(\Phi, \mu^{-\alpha}\Phi - \chi) - D(\varepsilon, \mu^{-\alpha}\Phi - \chi) .$$

It results into an estimate in the form

$$\frac{d}{dt}\|\dot{\Phi}\|_\alpha^2 + 2\|\nabla\Phi\|_\alpha^2 \le c\|\dot{\Phi}\|_{\alpha+1}^2 + c\left\{h^{-1}|\varepsilon|_\alpha^{'\,2} + h^{-2}\|\varepsilon\|_\alpha^2\right\} .$$

Then the term $\|\dot{\Phi}\|_{\alpha+1}^2$ is estimated with a duality argument in the form

$$\begin{aligned} \ddot{w} - \Delta w &= \mu^{-\alpha-1}\tilde{\dot{\Phi}} && \text{in } \Omega \times (0, t_0] \\ w &= 0 && \text{on } \partial\Omega \times (0, t_0] \\ w(t = t_0) &= \dot{w}(t = t_0) = 0 && \text{in } \Omega . \end{aligned}$$

Analog as in [NiJ1] (but now operating with the Hilbert space norms $H_{k.t}$) this leads to the estimate

$$\|\dot{\Phi}(t_0)\|_\alpha^2 + \int_0^{t_0}(\|\nabla\Phi(t)\|_\alpha^2 + \|\dot{\Phi}(t)\|_{\alpha+1}^2)dt \le c\int_0^{t_0}\left\{h^{-1}|\varepsilon|_\alpha^{'\,2} + h^{-2}\|\varepsilon\|_\alpha^2\right\}dt \cdot$$

Choosing $\alpha > n/2 + 1$ and applying $\|\varepsilon\|_\alpha^2 \le c \cdot \rho^{n+2-2\alpha}\|\varepsilon\|_{L_\infty(L_\infty)}^2$ one finally gets

$$\|\dot{\Phi}(t_0)\|_\alpha^2 \le c \cdot (\frac{\rho}{h})^2 \rho^{n-2\alpha}\|\varepsilon\|_{L_\infty(L_\infty)}^2 .$$

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