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Remarks on Weil's quadratic functional in the theory of prime numbers, I

Memoria (*) di Enrico Bombieri

Abstract. — This *Memoir* studies Weil's well-known Explicit Formula in the theory of prime numbers and its associated quadratic functional, which is positive semidefinite if and only if the Riemann Hypothesis is true. We prove that this quadratic functional attains its minimum in the unit ball of the L^2-space of functions with support in a given interval $[-t, t]$, and prove again Yoshida's theorem that it is positive definite if t is sufficiently small. The Fourier transform of the functional gives rise to a quadratic form in infinitely many variables and we then study its finite truncations and corresponding eigenvalues. In particular, if the Riemann Hypothesis is false but only with finitely many non-trivial zeros off the critical line we show that the number of negative eigenvalues is precisely one-half of the number of zeros failing to satisfy the Riemann Hypothesis, provided the truncation is big enough.

Key words: Prime number theory; Riemann Hypothesis; Explicit Formula.

Riassunto. — *Osservazioni sul funzionale quadratico di Weil nella teoria dei numeri primi, I.* Questa *Memoria* studia la nota Formula Esplicita di Weil nella teoria dei numeri primi e il funzionale quadratico associato ad essa. Questo funzionale è positivo semidefinito se e solo se l'Ipotesi di Riemann è valida. Dimostriamo qui che il minimo di questo funzionale nello spazio delle funzioni L^2 con supporto compatto nell'intervallo $[-t, t]$ è raggiunto, e dimostriamo nuovamente il risultato di Yoshida che dà la positività per t sufficientemente piccolo. La trasformata di Fourier del funzionale dà luogo ad una forma quadratica in un numero infinito di variabili, e ne studiamo i suoi troncamenti finiti e gli autovalori corrispondenti. In particolare, se l'Ipotesi di Riemann è falsa ma solamente con un numero finito di eccezioni, si dimostra che il numero di autovalori negativi è la metà del numero di eccezioni all'Ipotesi di Riemann, purché il troncamento sia abbastanza grande.

1. Introduction

The Explicit Formula in the theory of prime numbers is a generalization of Riemann's famous exact formula of 1859 expressing the number of primes up to a given limit in terms of a sum of a certain Mellin transform, evaluated at the zeros of the Riemann zeta function. Since then, it has found wide use in analytic number theory. However, most applications found in the literature involve the use of well-chosen explicit test functions and use only approximations, not exact evaluations.

In 1942, A.P. Guinand [2] studied for the first time the Explicit Formula in a general setting, viewing it as a transformation formula not unlike Poisson Summation Formula.

In 1952, A. Weil [6] (1) put forward a more general Explicit Formula which had

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(1) It appears that Weil was either unaware of Guinand's work or dismissed it as uninteresting. Certainly Weil's aim in his paper was entirely different from Guinand's.

an identical formulation both in the classical case and the so-called function field case. The main point of Weil's paper was to highlight the deep analogies between classical zeta functions and congruence zeta functions arising from function fields of curves over a finite field of positive characteristic.

Since Weil had already proved the analogue of the Riemann Hypothesis in the function field case, it was natural to ask if the classical Riemann Hypothesis could also be interpreted in the same light. Thus in the same paper Weil formulated the Riemann Hypothesis as the positivity of a certain quadratic functional arising from the Explicit Formula.

Even if Weil's work has since then been vastly generalized and reinterpreted in the framework of adeles, nothing of consequence in prime number theory has emerged so far from a direct study of Weil's functional, which unfortunately appears to be as intractable as the Riemann Hypothesis itself. A first study of Weil's functional was done by H. Yoshida [5] in 1992. In his paper, Yoshida studies the behaviour of the associated hermitian form in certain Hilbert spaces of functions supported in an interval $[-t, t]$, obtaining several interesting results. Besides reproving Weil's criterion for the validity of the Riemann hypothesis, he shows that the positivity of the functional in the class of smooth even functions with compact support is equivalent to the validity of the Riemann hypothesis excluding real zeros. Moreover, he shows how the positivity of this functional for functions supported in a fixed interval $[-t, t]$ can be reduced to a finite calculation (depending on t), and verifies this positivity for $t = (\log 2)/2$.

In this paper, we study the Weil functional from a variational point of view, restricted to two classes of Hilbert spaces each of which is sufficiently wide for testing the validity of the Riemann Hypothesis. We verify that the infimum of the Weil functional in the unit sphere of spaces in the second class is always attained, and also in spaces of the first class if the infimum in question is negative. It is noteworthy that the proof of the first result hinges on the special structure of the term for the «prime at infinity» in the Explicit Formula.

Next, we examine the Fourier transform of the functional, which is easily interpreted as a quadratic form in infinitely many variables. This leads to the study of the eigenvalues of certain infinite matrices, as well as of their finite dimensional truncations. In view of the special structure of these matrices, it turns out that the eigenvalue 0 cannot occur for the finite dimensional approximations. From this result, which appears to be new, we deduce the invariance of the number of negative eigenvalues under certain deformations of the matrices, thereby determining the exact number of negative eigenvalues.

The results obtained for the finite dimensional case carry over to the infinite dimensional case, provided we deal with negative eigenvalues bounded away from 0 and provided their number remains bounded. For the applications we have in mind, the latter condition follows from the assumption that there are only finitely many non-trivial zeros of $\zeta(s)$ not on the critical line. The existence of a negative eigenvalue follows again from our stability result in the finite dimensional case, and the main question here is to decide whether or not this negative eigenvalue stays bounded away from zero, as we approximate the infinite dimensional matrix by its finite dimensional truncations.

The upshot is that if the negative eigenvalue stays bounded away from zero then the Weil functional, restricted to the Hilbert space determining the infinite matrix, is not positive definite there, while on the other hand if the negative eigenvalue tends to 0 then we obtain in the limit a non-trivial linear relation among monomials $x^{-\rho}$, where ρ ranges over the non-trivial zeros of the Riemann zeta function.

Rather than giving in this introduction a summary of every result, the following corollary of one of our theorems best illustrates what is obtained here in the end. We show that one of the following statements holds:

(i) *the Riemann Hypothesis is true*;
(ii) *there are infinitely many complex zeros ρ of the Riemann zeta function with $\Re(\rho) \neq \frac{1}{2}$*;
(iii) *there is a linear combination*

$$\sum_{\rho} \frac{c_\rho}{\rho(1-\rho)} x^{-\rho} + A + B x^{-1}$$

with $\sum |c_\rho|^2 = 1$ and vanishing identically for $1 \leq x \leq M_0$, where $M_0 > 1$ is an explicitly computable constant. Moreover, at least $1/2$ of the ℓ^2-mass of the coefficients is supported on the non-trivial zeros of $\zeta(s)$ off the critical line.

More precise results of this type are contained in §10, Theorem 10 and §11, Theorem 11.

The question of linear independence which arises in (iii) is of some interest and, although probably quite difficult, deserves study. Linear relations such as in (iii) above do occur, but what we obtain here are well-determined relations arising from a specific limiting process, and they may carry interesting information about consequences of a hypothetical failure of the Riemann Hypothesis.

The content of this paper is as follows. In Sections 2 and 3 we restate the Guinand-Weil Explicit Formula and Weil's formulation of the Riemann Hypothesis as the positivity of a certain quadratic functional. Section 4 introduces two variational eigenvalue problems associated to Weil's quadratic functional and proves the existence of extremals for the second problem. Sections 5 and 6 study the structure of extremals and a reformulation of the eigenvalue problems, obtained by taking Fourier transforms. Section 7 shows that the first eigenvalue problem admits a resolvent and determines some of its properties, although results in this section are not used anywhere else in this paper. Section 8 introduces finite approximations, odd and even eigenfunctions and proves a key result (Theorem 8) about eigenvalues of finite matrices in the approximation of the first eigenvalue problem. The main result of this section is quite general and does not depend on properties of zeta functions or arithmetic. Section 9 obtains corresponding results for the second eigenvalue problem. Section 10 shows how to pass to the limit from the finite approximations to the case of interest, namely zeta functions. However, the possibility that in the limiting process a negative eigenvalue may have limit 0 gives rise to linear dependence relations, as in alternative (iii) above. Section 11 carries the same limiting process for the second eigenvalue problem and gives an example of how linear relations may arise in the case of zeta functions. Finally, Section 12 proves again Yoshida's result of the positivity of Weil's quadratic functional for test functions with sufficiently small compact support.

In this paper we have not attempted to achieve maximum generality or the sharpest possible statements. All results of this paper extend, *mutatis mutandis*, to Dedekind zeta functions of number fields. It should also be possible to replace the assumption that there are only finitely many non-trivial zeros of $\zeta(s)$ off the critical line, which is used in several places, by a suitable density hypothesis.

2. The Guinand-Weil Explicit Formula

We state the Explicit Formula, in the special case of the Riemann zeta function, in the following form.

For a function $f(x)$ on $(0,\infty)$ we define f^* by the formula

$$f^*(x) = \frac{1}{x} f\left(\frac{1}{x}\right),$$

and say that f is even if $f = f^*$ and odd if $f = -f^*$.

Explicit Formula. *Let* $f(x) \in C_0^\infty((0,\infty))$ *be a smooth complex-valued function with compact support in* $(0,\infty)$.

Let

$$\widetilde{f}(s) = \int_0^\infty f(x)\, x^{s-1}\, \mathrm{d}x$$

be the Mellin transform of f. *Then we have* (2)

$$\sum_\rho \widetilde{f}(\rho) = \int_0^\infty f(x)\,\mathrm{d}x + \int_0^\infty f^*(x)\,\mathrm{d}x - \sum_{n=1}^\infty \Lambda(n)\{f(n) + f^*(n)\} -$$
$$- (\log 4\pi + \gamma) f(1) - \int_1^\infty \left\{ f(x) + f^*(x) - \frac{2}{x} f(1) \right\} \frac{x\,\mathrm{d}x}{x^2 - 1}$$

where the first sum ranges over all complex zeros of the Riemann zeta function.

Moreover, the last two terms in the right-hand side of the Explicit Formula can be written as

$$-(\log \pi) f(1) + \frac{1}{2\pi i} \int_{(\frac{1}{2})} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right)\right] \widetilde{f}(w)\,\mathrm{d}w.$$

Proof. The proof is an application of the calculus of residues. An elementary treatment is as follows.

We write as usual $\sigma = \Re(s)$ and $t = \Im(s)$ for the real and imaginary part of the complex variable s. Let

$$Z(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

(2) Here γ is Euler's constant $\gamma = 0.5772156649\ldots$.

and consider the integral

$$I(f) = \frac{1}{2\pi i} \int_{(c)} \frac{Z'}{Z}(w) \widetilde{f}(w) \, \mathrm{d}w \tag{2.1}$$

where the integration is over the line $(c - i\infty\,,\, c + i\infty)$ and $c > 1$. Since $f(x)$ is smooth with compact support, its Mellin transform $\widetilde{f}(s)$ is an entire function of s of order 1 and exponential type, rapidly decreasing in every fixed vertical strip.

The logarithmic derivative of $Z(s)$ is holomorphic for $\sigma > 1$ and has logarithmic growth on any vertical line $(c - i\infty\,,\, c + i\infty)$ with $c > 1$, hence the above integral is absolutely convergent.

For $\Re(w) > 1$ we have

$$\frac{Z'}{Z}(w) = -\frac{1}{2}(\log \pi) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right) - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^w}\,,$$

whence

$$I(f) = -\frac{1}{2}(\log \pi) \frac{1}{2\pi i} \int_{(c)} \widetilde{f}(w) \, \mathrm{d}w + \frac{1}{4\pi i} \int_{(c)} \frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right) \widetilde{f}(w) \, \mathrm{d}w - \\ - \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{(c)} \widetilde{f}(w)\, n^{-w} \, \mathrm{d}w,$$

because term-by-term integration is justified by total convergence. The inverse Mellin transform formula is

$$f(x) = \frac{1}{2\pi i} \int_{(c)} \widetilde{f}(w)\, x^{-w} \, \mathrm{d}w$$

and the formula for $I(f)$ becomes

$$I(f) = -\frac{1}{2}(\log \pi) f(1) + \frac{1}{4\pi i} \int_{(c)} \frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right) \widetilde{f}(w) \, \mathrm{d}w - \sum_{n=1}^{\infty} \Lambda(n) f(n). \tag{2.2}$$

Now we compute $I(f)$ in another way. In (2.1), we move the line of integration to the left, to a line $(c' - i\infty\,,\, c' + i\infty)$ with $c' < 0$. This requires some justification, namely integrating over a rectangle with vertices at $(-c' - iT, c - iT, c + iT, c' + iT)$ and showing that the integral over the horizontal sides tends to 0 if we let T go to infinity along a well-chosen sequence $\{T_\nu\}$. For this step, which is very classical, we refer to Ingham's excellent Cambridge Tract [3].

In moving the line of integration to the left we encounter the residues of $Z'/Z(w)$ due to the simple poles of $Z(w)$ at $w = 0$ and $w = 1$ and to the zeros of $Z(w)$ inside the critical strip $0 \le \Re(w) \le 1$. It follows that

$$I(f) = -\widetilde{f}(0) - \widetilde{f}(1) + \sum_{\rho} \widetilde{f}(\rho) + \frac{1}{2\pi i} \int_{(c')} \frac{Z'}{Z}(w) \widetilde{f}(w) \, \mathrm{d}w. \tag{2.3}$$

Now we use the functional equation $Z(w) = Z(1 - w)$, which for $\Re(w) < 0$ gives us

$$\frac{Z'}{Z}(w) = -\frac{Z'}{Z}(1 - w) = \frac{1}{2}(\log \pi) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1 - w}{2}\right) + \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-w}}\,.$$

We substitute into (2.3) and obtain

$$
\begin{aligned}
I(f) = {} & -\widetilde{f}(0) - \widetilde{f}(1) + \sum_{\rho} \widetilde{f}(\rho) + \\
& + \frac{1}{2}(\log \pi) \frac{1}{2\pi i} \int_{(c')} \widetilde{f}(w)\, \mathrm{d}w - \frac{1}{4\pi i} \int_{(c')} \frac{\Gamma'}{\Gamma}\left(\frac{1-w}{2}\right) \widetilde{f}(w)\, \mathrm{d}w + \\
& + \sum_{n=1}^{\infty} \Lambda(n) \frac{1}{2\pi i} \int_{(c')} \widetilde{f}(w)\, n^{-1+w}\, \mathrm{d}w = \\
= {} & -\widetilde{f}(0) - \widetilde{f}(1) + \sum_{\rho} \widetilde{f}(\rho) + \\
& + \frac{1}{2}(\log \pi) f(1) - \frac{1}{4\pi i} \int_{(c')} \frac{\Gamma'}{\Gamma}\left(\frac{1-w}{2}\right) \widetilde{f}(w)\, \mathrm{d}w + \sum_{n=1}^{\infty} \Lambda(n) f^{*}(n),
\end{aligned}
$$

again because term-by-term integration is justified by total convergence.

We compare this formula with formula (2.2) for $I(f)$ and find

$$
\begin{aligned}
\sum_{\rho} \widetilde{f}(\rho) = {} & \widetilde{f}(0) + \widetilde{f}(1) - \sum_{n=1}^{\infty} \Lambda(n) \{f(n) + f^{*}(n)\} - (\log \pi) f(1) + \\
& + \frac{1}{4\pi i} \int_{(c)} \frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right) \widetilde{f}(w)\, \mathrm{d}w + \frac{1}{4\pi i} \int_{(c')} \frac{\Gamma'}{\Gamma}\left(\frac{1-w}{2}\right) \widetilde{f}(w)\, \mathrm{d}w.
\end{aligned}
\tag{2.4}
$$

In order to obtain the explicit formula we compute the last two integrals as follows. First of all, we move the line of integration in the last two integrals to $c = \frac{1}{2}$ and $c' = \frac{1}{2}$, which we may without encountering any pole of the integrand. Thus the sum of the two integrals becomes

$$
\frac{1}{2\pi} \int_{-\infty}^{\infty} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i}{2} v\right)\right] \widetilde{f}\left(\frac{1}{2} + iv\right) \mathrm{d}v.
$$

This already proves the Explicit Formula with the last two terms expressed in the alternative way by means of a complex integral.

Now we recall that [7, Ch. XII, §12.16, p. 241]

$$
\frac{\Gamma'}{\Gamma}(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \left\{\frac{1}{n} - \frac{1}{n+z}\right\}
$$

and

$$
1 + \frac{1}{2} + \ldots + \frac{1}{N} = \log N + \gamma + O\left(\frac{1}{N}\right),
$$

thus

$$\frac{\Gamma'}{\Gamma}(z) = \log N - \sum_{n=0}^{N} \frac{1}{n+z} + O\Big(\frac{1+|z|}{N}\Big)$$

uniformly for $\Re(z) \geq -N/2$, z not a negative integer or 0. This gives

$$\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right] = \log N - \sum_{n=0}^{N} \frac{4n+1}{(2n+\frac{1}{2})^2+v^2} + O\Big(\frac{1+|v|}{N}\Big)$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right] \widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\log N - \sum_{n=0}^{N}\frac{4n+1}{\left(2n+\frac{1}{2}\right)^2+v^2}\right)\widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v +$$

$$+ O\Big(\int_{-\infty}^{\infty} \frac{1+|v|}{N}\left|\widetilde{f}\left(\frac{1}{2}+iv\right)\right|\mathrm{d}v\Big).$$

Since $\widetilde{f}$ is rapidly decreasing on any vertical line, the last integral converges and the $O(\ )$ term is indeed $O(1/N)$. Also

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{f}(\frac{1}{2}+iv)\,\mathrm{d}v = f(1)\,,$$

whence

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right] \widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v =$$

(2.5)

$$= (\log N) f(1) - \sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4n+1}{(2n+\frac{1}{2})^2+v^2}\widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v + O\Big(\frac{1}{N}\Big).$$

We have by Fubini's theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+v^2}\widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2+v^2}\int_0^{\infty} f(x)x^{-1/2+iv}\,\mathrm{d}x\,\mathrm{d}v =$$

(2.6)

$$= \int_0^{\infty} f(x)x^{-1/2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{v-ia}-\frac{1}{v+ia}\right)x^{iv}\,\mathrm{d}v\,\mathrm{d}x.$$

An easy application of the calculus of residues [7, Ch. VI, §6.22, pp. 113-114] shows that for $\Re(a) > 0$ we have

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{v-ia}-\frac{1}{v+ia}\right)x^{iv}\,\mathrm{d}v = \min(x, 1/x)^a$$

hence by (2.5) and (2.6) we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right]\widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v=$$

$$(2.7)\qquad \begin{aligned}&=-\int_1^{\infty}\left(\sum_{n=0}^{N}x^{-2n}\right)f(x)\,\frac{\mathrm{d}x}{x}-\int_0^1\left(\sum_{n=0}^{N}x^{2n}\right)f(x)\,\mathrm{d}x+\\&\quad+(\log N)f(1)+O\Big(\frac{1}{N}\Big)=\\&=-\int_1^{\infty}\left(\sum_{n=0}^{N}x^{-2n}\right)\big(f(x)+f^*(x)\big)\frac{\mathrm{d}x}{x}+(\log N)f(1)+O\Big(\frac{1}{N}\Big).\end{aligned}$$

Finally, we have

$$\int_1^{\infty}\left(\sum_{n=0}^{N}x^{-2n}\right)\big(f(x)+f^*(x)\big)\frac{\mathrm{d}x}{x}=\int_1^{\infty}\left(\sum_{n=0}^{N}x^{-2n}\right)\left\{f(x)+f^*(x)-\frac{2}{x^2}f(1)\right\}\frac{\mathrm{d}x}{x}+$$
$$+\left(1+\frac{1}{2}+\ldots+\frac{1}{N+1}\right)f(1).$$

We substitute into (2.7), getting

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right]\widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v=$$
$$=-\int_1^{\infty}\frac{1-x^{-2N-2}}{1-x^{-2}}\left\{f(x)+f^*(x)-\frac{2}{x^2}f(1)\right\}\frac{\mathrm{d}x}{x}+$$
$$+\left(\log N-\sum_{n=1}^{N+1}\frac{1}{n}\right)f(1)+O\Big(\frac{1}{N}\Big).$$

Now we take the limit for $N\to\infty$ and deduce

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right]\widetilde{f}\left(\frac{1}{2}+iv\right)\,\mathrm{d}v=$$

$$(2.8)\qquad \begin{aligned}&=-\int_1^{\infty}\left\{f(x)+f^*(x)-\frac{2}{x^2}f(1)\right\}\frac{x\,\mathrm{d}x}{x^2-1}-\gamma f(1)=\\&=-(\log 4+\gamma)f(1)-\int_1^{\infty}\left\{f(x)+f^*(x)-\frac{2}{x}f(1)\right\}\frac{x\,\mathrm{d}x}{x^2-1}.\end{aligned}$$

If we substitute into (2.4) and note that

$$\widetilde{f}(0)=\int_0^{\infty}f^*(x)\,\mathrm{d}x,\qquad \widetilde{f}(1)=\int_0^{\infty}f(x)\,\mathrm{d}x$$

we get the Explicit Formula in the form stated here.

3. A necessary and sufficient condition for the Riemann Hypothesis

As Weil [6] showed, one obtains from the Explicit Formula a necessary and sufficient condition for the validity of the Riemann Hypothesis, expressed through the positivity of a certain quadratic functional. Let us consider test functions $f(x) \in C_0^\infty((0\,,\infty))$ which are the multiplicative convolution of a function g and its transpose conjugate $\overline{g}^*$, hence

$$f(x) = \int_0^\infty g(x/y)\,\overline{g}^*(y)\,\frac{\mathrm{d}y}{y} = \int_0^\infty g(xy)\,\overline{g(y)}\,\mathrm{d}y.$$

We have

$$\widetilde{f}(s) = \widetilde{g}(s)\ \widetilde{\overline{g}}(1-s).$$

We have the following strengthening of Weil's criterion.

Theorem 1. *The Riemann Hypothesis holds if and only if*

$$\sum_\rho \widetilde{g}(\rho)\ \widetilde{\overline{g}}(1-\rho) > 0 \tag{3.1}$$

for every complex-valued $g(x) \in C_0^\infty((0\,,\infty))$, *not identically* 0.

Proof. The Riemann Hypothesis is the statement that $1-\rho = \overline{\rho}$ for every non-trivial zero ρ of $\zeta(s)$. Thus on the Riemann Hypothesis we have

$$\sum_\rho \widetilde{g}(\rho)\ \widetilde{\overline{g}}(1-\rho) = \sum_\rho \widetilde{g}(\rho)\ \widetilde{\overline{g}}(\overline{\rho}) = \sum_\rho |\widetilde{g}(\rho)|^2 \geq 0. \tag{3.2}$$

It is also easy to show that equality holds only if $g(x)$ is identically 0. In fact equality can hold only if $\widetilde{g}(\rho) = 0$ for every ρ, whence $\widetilde{g}(s)$ has at least $(1/\pi + o(1))R\log R$ zeros in a disk $|s| \leq R$. On the other hand, $\widetilde{g}(s)$ is an entire function of exponential type, thus if $\widetilde{g}$ is not identically 0 it can have at most $O(R)$ zeros in the disk $|s| \leq R$. This proves our claim about equality in (3.2) and shows that the Riemann Hypothesis implies (3.1).

The proof of the converse statement can be obtained by a modification of Weil's proof. We shall sketch here another argument in [1], based on a criterion due to X.-J. Li [4], namely that the Riemann Hypothesis is equivalent to the statement

$$\sum_\rho \Big\{ \big[1-(1-1/\rho)^n\big] + \big[1-(1-1/(1-\rho))^n\big] \Big\} > 0 \tag{3.3}$$

for $n = 1\,,2\,,3\,,\dots$. A quick proof is as follows [1]. Let $\beta = \Re(\rho)$ and notice that

$$|1-1/\rho|^2 = 1 + (1-2\beta)/|\rho|^2.$$

This proves that (3.3) holds if $\Re(\rho) = \frac{1}{2}$ for every ρ.

Conversely, suppose there is ρ with $\Re(\rho) < \frac{1}{2}$. Since $(1-2\beta)/|1-\rho|^2$ tends to 0 as $|\rho| \to \infty$, the maximum of this quantity as ρ varies is attained and there are finitely many zeros ρ_k, $k = 1\,,\dots\,,K$ such that $|1-1/\rho| = 1+t = \max > 1$. For any other

ρ we have $|1-1/\rho| \le 1+t-\delta$ for a fixed small $\delta > 0$. Let ϕ_k be the argument of $1-1/\rho_k$. Then

$$1-(1-1/\rho_k)^n = 1-(1+t)^n e^{in\phi_k}.$$

For $\rho \ne \rho_k$ we have $|1-1/\rho|^n = O\big((1+t-\delta)^n\big)$ and also

$$1-(1-1/\rho)^n = n/\rho + O(n^2/|\rho|^2)$$

as soon as $|\rho| > n$, as an easy calculation shows. Hence the sum over $|\rho| > n$ is $O(n^2)$, because $\sum 1/|\rho|^2$ is convergent. The number of zeros with $|\rho| \le n$ is $O(n^2)$, again because $\sum 1/|\rho|^2$ is convergent. Hence the contribution of terms in (3.3) other than those arising from ρ_k, $k=1,\ldots,K$ is $O\big(n^2(1+t-\delta)^n\big)$. The contribution of the zeros ρ_k, $k=1,\ldots,K$ is $O(K)-2(1+t)^n\sum\cos(n\phi_k)$, and we have shown that

$$\sum_\rho \Big\{\big[1-(1-1/\rho)^n\big] + \big[1-(1-1/(1-\rho))^n\big]\Big\} =$$

$$= -2(1+t)^n \sum_{k=1}^{K} \cos(n\phi_k) + O(K) + O\big(n^2(1+t-\delta)^n\big).$$

By Dirichlet's theorem on simultaneous diophantine approximations we can find arbitrarily large values of n such that the sum of cosines is arbitrarily close to K, making it plain that the sum in (3.3) takes negative values infinitely often if the Riemann Hypothesis does not hold.

Now we have the identity

$$\big[1-(1-1/s)^n\big] + \big[1-(1-1/(1-s))^n\big] = \big[1-(1-1/s)^n\big]\cdot\big[1-(1-1/(1-s))^n\big].$$

Hence if $g_n(x)$ is the inverse Mellin transform of $1-(1-1/s)^n$, the sum in (3.1) is simply $\sum \widetilde{g}_n(\rho)\,\widetilde{g}_n(1-\rho)$, which, at least formally, is the left-hand side of the Explicit Formula for $g_n * \overline{g}_n^*$, because g_n is real.

The function g_n is

$$g_n(x) = \begin{cases} P_n(\log x) & \text{if } \; 0<x<1 \\ n/2 & \text{if } \; x=1 \\ 0 & \text{if } \; x>1 \end{cases}$$

where $P_n(x)$ is the polynomial

$$P_n(x) = \sum_{j=1}^{n} \binom{n}{j} \frac{x^{j-1}}{(j-1)!}.$$

Since g_n is not a smooth function with compact support, we cannot apply the Explicit Formula directly. Thus we replace g_n by its truncation

$$g_{n,\varepsilon}(x) = \begin{cases} g_n(x) & \text{if } \; \varepsilon < x \le \infty \\ \frac{1}{2}g_n(\varepsilon) & \text{if } \; x = \varepsilon \\ 0 & \text{if } \; x<\varepsilon \end{cases}$$

where $\varepsilon > 0$ and note that

$$\lim_{\varepsilon\to+0}\sum_{\rho}\widetilde{g}_{n,\varepsilon}(\rho)\,\widetilde{g}_{n,\varepsilon}(1-\rho)=\sum_{\rho}\widetilde{g}_{n}(\rho)\,\widetilde{g}_{n}(1-\rho). \tag{3.4}$$

This can be proved easily using a zero-free region for $\zeta(s)$, because

$$\left|\widetilde{g}_{n,\varepsilon}(s)\,\widetilde{g}_{n,\varepsilon}(1-s)-\widetilde{g}_{n}(s)\,\widetilde{g}_{n}(1-s)\right|=O\big(\varepsilon^{\min(\Re(s)\,,\,\Re(1-s))}(\log 1/\varepsilon)^{n-1}/|s|^{2}\big)$$

for $0\le\Re(s)\le 1$ and $|s|\ge 1$. Then using De la Vallée-Poussin's zero-free region

$$\frac{c}{\log(|\rho|+2)}\le\Re(\rho)\le 1-\frac{c}{\log(|\rho|+2)}$$

it is easy to verify (3.4).

Finally, if n is such that $\sum\widetilde{g}_{n}(\rho)\,\widetilde{g}_{n}(1-\rho)<0$, by (3.4) we see that we still have $\sum\widetilde{g}_{n,\varepsilon}(\rho)\,\widetilde{g}_{n,\varepsilon}(1-\rho)<0$ for $\varepsilon>0$ small enough. The function $g_{n,\varepsilon}$ is real-valued with compact support, and the proof of Theorem 1 is completed by replacing $g_{n,\varepsilon}$ by a multiplicative convolution with a smooth approximation with compact support to a Dirac at 1.

By the Explicit Formula and Theorem 1 we have

THEOREM 2. *Let $\mathcal{T}[f]$ be the linear functional* (3)

$$\mathcal{T}[f]=\int_0^\infty f(x)\,\mathrm{d}x+\int_0^\infty f^*(x)\,\mathrm{d}x-\sum_{n=1}^{\infty}\Lambda(n)\,\{f(n)+f^*(n)\}-$$

$$-(\log 4\pi+\gamma)f(1)-\int_1^\infty\left\{f(x)+f^*(x)-\frac{2}{x}f(1)\right\}\frac{x\,\mathrm{d}x}{x^2-1}$$

on the space $C_0^\infty((0\,,\infty))$ of complex-valued smooth functions with compact support in $(0\,,\infty)$. Then we have

$$\mathcal{T}[f]=\mathcal{T}[f^*]=\sum_{\rho}\widetilde{f}(\rho)\,,$$

where the sum ranges over all complex zeros of the Riemann zeta function.

Moreover, the Riemann Hypothesis is equivalent to the statement that

$$\mathcal{T}[f*\overline{f}^*]\ge 0$$

on $C_0^\infty((0\,,\infty))$, with equality only if f is identically 0.

We have in fact proved a little more, namely that the positivity of the Weil quadratic functional is equivalent to the statement that $\mathcal{T}[g_{n,\varepsilon}]>0$ for every positive integer n and $\varepsilon>0$.

(3) We use the symbol $\mathcal{T}$ because we regard $\mathcal{T}[f]$ as an analogue of Weil's definition of trace of a correspondence.

4. The variational equation

It is easily seen that the functional $T[f*\overline{f}^*]$ is invariant by the translation operator (4) $\tau_a f(x) = a^{-1/2} f(x/a)$. This means however that if we try to minimize this functional in the unit sphere of a translation invariant Hilbert space we obtain too many extremals. Thus our Hilbert space cannot be translation invariant by τ_a.

On the other hand, we want to keep the definition of the norm in our Hilbert space as simple as possible. Hence a first attempt consists in killing the translations τ_a by fixing the support of f. This works fine to some extent, and one readily shows that $T[f*\overline{f}^*]$ is bounded below in the unit sphere of the space $L^2(\mathcal{E})$ of square-integrable functions with compact support in $\mathcal{E}$, where $\mathcal{E}$ is a finite union of bounded closed intervals in $(0, \infty)$.

However, one may also consider the infinitesimal translation given by the differential operator $D = x(\mathrm{d}/\mathrm{d}x)$, and one way to kill it is to work in the space of functions f for which Df is square-integrable with compact support in $[M^{-1}, M]$, because in general Df will have a jump at $M^{\pm}1$ and $D^2 f$ will have a Dirac point mass there, so it will no longer be square-integrable.

These considerations motivate the two problems below.

Problem 1. *Minimize* $\mathcal{T}[f*\overline{f}^*]$ *in the unit sphere of the Hilbert space* $\mathcal{W}_0 = W_0^{1,2}([M^{-1}, M])$ *of functions* f *with* f, Df *in* $L^2((0,\infty))$ *and compact support in* $[M^{-1}, M]$, *with norm*

$$\|f\|^2_{\mathcal{W}_0} = \int_{M^{-1}}^{M} |Df(x)|^2 \,\mathrm{d}x,$$

with D *the translation invariant differential operator* $D = x(\mathrm{d}/\mathrm{d}x)$.

Problem 2. *Let* $\mathcal{E}$ *be a finite union of intervals in* $(0, \infty)$. *Minimize* $\mathcal{T}[f*\overline{f}^*]$ *in the unit sphere of the space* $L^2(\mathcal{E})$ *of functions* f *with compact support in* $\mathcal{E}$, *with norm*

$$\|f\|^2 = \int_{\mathcal{E}} |f(x)|^2 \,\mathrm{d}x.$$

Remark. There is some evidence that it may also be of interest to consider a variant of Problem 1 in which instead of a Dirichlet condition $f(1/M) = f(M) = 0$ one works with odd functions $f(x) = -f^*(x)$ and imposes a Neumann condition

$$(Df + \frac{1}{2}f)(1/M) = (Df + \frac{1}{2}f)(M) = 0.$$

The analysis of this problem will not be done in this paper.

Remark. It may be useful, in studying regularity of solutions in Problem 2, to consider also viscosity solutions, namely minimizers of the regularized functional $\mathcal{T}[f*\overline{f}^*] + \varepsilon\|Df\|^2$ for $\varepsilon > 0$, and their limits in $L^2(\mathcal{E})$ as $\varepsilon \to 0$.

(4) We view this as the analogue of a normalized Frobenius operator.

LEMMA 1. *The variational equation for Problem* 1 *is*

$$\lambda \mathcal{D} f(a) - \mathcal{L}[f](a) = 0 \quad \text{for } a \in (M^{-1}, M) \tag{4.1}$$

where $\mathcal{D}$ *is the Laplacian* $\mathcal{D} = -D - D^2$ *and* $\mathcal{L}[f]$ *is the Euler-Lagrange linear operator*

$$\mathcal{L}[f](a) = \int_0^\infty f(ay)\,\mathrm{d}y + \int_0^\infty \frac{1}{y} f\left(\frac{a}{y}\right)\mathrm{d}y - \sum_{n=1}^\infty \Lambda(n)\left\{f(an) + \frac{1}{n} f\left(\frac{a}{n}\right)\right\} -$$
$$- (\log 4\pi + \gamma) f(a) - \int_1^\infty \left\{f(ax) + \frac{1}{x} f\left(\frac{a}{x}\right) - \frac{2}{x} f(a)\right\} \frac{x\,\mathrm{d}x}{x^2 - 1}.$$

The eigenvalue λ *is given by*

$$\lambda = \frac{\mathcal{T}[f * \overline{f}^*]}{\|Df\|^2}.$$

The variational equation for Problem 2 *is*

$$\lambda f(a) - \mathcal{L}[f](a) = 0 \quad \text{for } a \in \mathcal{E} \tag{4.2}$$

and the eigenvalue λ *is given by*

$$\lambda = \frac{\mathcal{T}[f * \overline{f}^*]}{\|f\|^2}.$$

PROOF. Let

$$\langle u, v\rangle = \int_0^\infty u(x)\,\overline{v(x)}\,\mathrm{d}x$$

be the scalar product in $L^2((0,\infty))$, so that $\langle v, u\rangle = \overline{\langle u, v\rangle}$. Let us make a variation $f + \varepsilon\varphi$ of f, where φ is a complex-valued smooth function with compact support. We have

$$\mathcal{T}[(f + \varepsilon\varphi) * \overline{(f + \varepsilon\varphi)}^*] = \mathcal{T}[f * \overline{f}^*] + 2\varepsilon\,\Re[\langle \mathcal{L}[f], \varphi\rangle] + \varepsilon^2\,\mathcal{T}[\varphi * \overline{\varphi}^*]$$

and

$$\|D(f + \varepsilon\varphi)\|^2 = \|Df\|^2 + 2\varepsilon\,\Re[\langle \mathcal{D}f, \varphi\rangle] + \varepsilon^2\|D\varphi\|^2.$$

Hence the vanishing of the first variation of $\mathcal{T}[f * \overline{f}^*]/\|Df\|^2$ gives the equation

$$\Re[\langle \mathcal{L}[f], \varphi\rangle - \lambda\langle \mathcal{D}f, \varphi\rangle] = 0$$

for every complex-valued smooth φ with compact support in $(0,\infty)$, with $\lambda = \mathcal{T}[f * \overline{f}^*]/\|Df\|^2$. This completes the proof for the first assertion of the lemma. The proof of the second assertion is identical, *mutatis mutandis.*

We shall prove that the infimum for the preceding two variational problems is attained. We need some simple preliminary results.

LEMMA 2. *If* $f \in \mathcal{W}_0$ *then* f *is Hölder continuous with exponent* $1/2$, *and verifies the pointwise inequality*

$$|f(y) - f(x)| \le \|Df\| \left|\frac{1}{x} - \frac{1}{y}\right|^{1/2}$$

for $0 < x < y < \infty$. *In particular,* f *is bounded as* $|f(x)| \le M^{1/2}\|Df\|$ *and* $\|f\| \le$ $\le (2\log M)^{1/2}\|Df\|$.

Moreover, for any $f, g \in L^2((0\,,\infty))$ *we have the pointwise bound*

$$\left|(f * g^*)(x)\right| \le \frac{\|f\| \cdot \|g\|}{\sqrt{x}}.$$

Proof. We have

$$f(y) - f(x) = \int_x^y (Df)(t)\,\frac{\mathrm{d}t}{t}$$

and the first statement of the lemma follows from Cauchy's inequality.

The second statement is also clear, because

$$(f * g^*)(x) = \int_0^\infty f(xy)\,g(y)\,\mathrm{d}y$$

and $\int |f(xy)|^2\,\mathrm{d}y = x^{-1}\|f\|^2$.

Lemma 3. *For* $f \in L^2$ *with compact support in* $[M^{-1}\,, M]$ *we have*

$$\mathcal{T}[f * \overline{f}^*] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i}{2}v\right)\right] \left|\widetilde{f}\left(\frac{1}{2} + iv\right)\right|^2 \mathrm{d}v + O(M)\,\|f\|^2.$$

The constant involved in the $O(\quad)$ *symbol is absolute.*

Proof. Since f is supported in $[M^{-1}\,, M]$ we see that $F = f * \overline{f}^*$ is supported in $[M^{-2}\,, M^2]$. Hence by the last estimate of Lemma 2 we have

$$\left|\int_0^\infty F(x)\,\frac{\mathrm{d}x}{x}\right| = \left|\int_0^\infty F(x)\,\mathrm{d}x\right| \le \int_{M^{-2}}^{M^2} \|f\|^2\,\frac{\mathrm{d}x}{\sqrt{x}} < 2M\,\|f\|^2.$$

Again by Lemma 2, we get in a similar way

$$\left|2\sum_{n=1}^{\infty} \Lambda(n)F(n)\right| \le 2\,\|f\|^2 \sum_{n=1}^{M^2} \frac{\Lambda(n)}{\sqrt{n}} = O(M)\,\|f\|^2.$$

Also,

$$F(1) = \|f\|^2.$$

Finally, by the Explicit Formula the last two terms in $\mathcal{T}[f * \overline{f}^*]$ are

$$-(\log\pi)F(1) + \frac{1}{2\pi}\int_{-\infty}^{\infty} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{i}{2}v\right)\right] \left|\widetilde{f}\left(\frac{1}{2} + iv\right)\right|^2 \mathrm{d}v.$$

This proves Lemma 3.

Lemma 4. *Let* $f \in L^2$ *be with compact support in* $[M^{-1}\,, M]$. *Then we have the pointwise bound*

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \widetilde{f}\left(\frac{1}{2} + it\right)\right| \le \sqrt{2}\,(\log M)^{k+1/2}\,\|f\|.$$

PROOF. We have

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \widetilde{f}\left(\frac{1}{2}+it\right)\right| = \left| i^k \int_{1/M}^{M} f(x)\,(\log x)^k x^{-\frac{1}{2}+it}\,\mathrm{d}x\right| \leq$$

$$\leq (\log M)^k \int_{1/M}^{M} \frac{|f(x)|}{\sqrt{x}}\,\mathrm{d}x \leq (\log M)^k \left(\int_{1/M}^{M} \frac{\mathrm{d}x}{x}\right)^{\frac{1}{2}} \|f\|.$$

This proves the lemma.

THEOREM 3. *Let $\mathcal{E}$ be a finite union of closed finite intervals in $(0\,,\infty)$. Then the infimum of $\mathcal{T}[f*\overline{f}^*]$ in the unit sphere of the space $L^2(\mathcal{E})$ of L^2-functions with compact support in $\mathcal{E}$ is attained.*

PROOF. Let $\{f_\nu\}$ be a minimizing sequence for this problem and let $\lambda = \inf \mathcal{T}[f*\overline{f}^*]$. Let $\mathcal{E} \subset [M^{-1}\,, M]$.

For $0 < \Re(w)$ we have

$$\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right)\right] = \log|w| + O(1) \tag{4.3}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\widetilde{f}\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v = \|f\|^2$$

by Plancherel's formula. Hence Lemma 3 shows that

$$\mathcal{T}[f_\nu * \overline{f}_\nu^*] \geq -C_1 M\,\|f_\nu\|^2$$

and in particular the functional $\mathcal{T}[f*\overline{f}^*]$ is bounded below in $L^2(\mathcal{E})$. Since this functional is automatically bounded above along a minimizing sequence, Lemma 3 and (4.3) also shows that

$$\int_{-\infty}^{\infty}(1+\log^+|v|)\left|\widetilde{f}_\nu\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v = O(1) \tag{4.4}$$

uniformly in ν as $\nu\to\infty$.

By Lemma 4, the coefficients $a_{\nu\,,m}$ of the Taylor expansion with center $\frac{1}{2}+it$ of the entire function $\widetilde{f}_\nu(s)$ are bounded by $C(\delta)\delta^m$, $m = 0\,,1\,,\ldots,$ for any $\delta>0$, for a suitable constant $C(\delta)$ depending only on δ. Hence by a standard diagonal selection process there is a subsequence of the functions f_ν, which we still denote by $\{f_\nu\}$ for simplicity, such that the limits

$$\lim_{\nu\to\infty} a_{\nu\,,m} = a_m$$

exist and again $|a_m| \leq C(\delta)\delta^m$. Therefore, setting

$$\widetilde{f}(s) = \sum_{m=0}^{\infty} a_m s^m$$

we see that $\widetilde{f}(s)$ is an entire function and that the sequence $\{\widetilde{f}_\nu(s)\}$ converges to $\widetilde{f}(s)$ uniformly on compact subsets of $\mathbb{C}$. It follows that for any fixed T we have

$$\lim_{\nu\to\infty}\int_{-T}^{T}\left|\widetilde{f}_\nu\left(\frac{1}{2}+iv\right)-\widetilde{f}\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v=0. \tag{4.5}$$

For the remaining integral over $|v|>T$, we note that by lower semicontinuity we have

$$\int_{-\infty}^{\infty}(1+\log^+|v|)\left|\widetilde{f}\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v\le \liminf_{\nu\to\infty}\int_{-\infty}^{\infty}(1+\log^+|v|)\left|\widetilde{f}_\nu\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v. \tag{4.6}$$

In conjunction with (4.4), this proves

$$\int_{|v|>T}\left|\widetilde{f}_\nu\left(\frac{1}{2}+iv\right)-\widetilde{f}\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v=O\left(\frac{1}{\log T}\right)$$

and by (4.5) it follows that

$$\lim_{\nu\to\infty}\int_{-\infty}^{\infty}\left|\widetilde{f}_\nu\left(\frac{1}{2}+iv\right)-\widetilde{f}\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v=0.$$

Therefore, by Plancherel's formula we get $\lim_\nu \|f_\nu - f\| = 0$ and the sequence $\{f_\nu\}$ converges to f strongly in $L^2(\mathcal{E})$.

Finally, we have

$$\lim_{\nu\to\infty}\mathcal{T}[f_\nu * \overline{f}^*_\nu]=\mathcal{T}[f*\overline{f}^*].$$

This is verified as follows. Let us abbreviate $F_\nu = f_\nu * \overline{f}^*_\nu$ and $F = f * \overline{f}^*$. Then by Lemma 2 we see that

$$\begin{aligned}
\left|F_\nu(x)-F(x)\right| &= \left|\int_0^\infty f_\nu(xy)\overline{f_\nu(y)}\,\mathrm{d}y-\int_0^\infty f(xy)\overline{f(y)}\,\mathrm{d}y\right| \le \\
&\le \left|\int_0^\infty \big(f_\nu(xy)-f(xy)\big)\,\overline{f_\nu(y)}\,\mathrm{d}y\right| + \left|\int_0^\infty f(xy)\big(\overline{f_\nu(y)}-\overline{f(y)}\big)\,\mathrm{d}y\right| \le \\
&\le \|f_\nu\|\left(\int_0^\infty |f_\nu(xy)-f(xy)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} + \|f_\nu - f\|\left(\int_0^\infty |f(xy)|^2\,\mathrm{d}y\right)^{\frac{1}{2}} = \\
&= \frac{2}{\sqrt{x}}\,\|f_\nu - f\|.
\end{aligned}$$

Since $\{f_\nu\}$ converges strongly to f in L^2, this proves that $\{F_\nu(x)\}$ converges uniformly to $F(x)$ as $\nu\to\infty$. This suffices to show that in the right-hand side of the Explicit Formula for $\mathcal{T}[f_\nu * \overline{f}^*_\nu]$ all terms converge to the corresponding terms for $\mathcal{T}[f*\overline{f}^*]$, except possibly for the last term. However, by (4.3), (4.5) and lower semicontinuity we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right]\left|\widetilde{f}\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v \le \\
&\qquad \le \liminf_{\nu\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i}{2}v\right)\right]\left|\widetilde{f}_\nu\left(\frac{1}{2}+iv\right)\right|^2 \mathrm{d}v.
\end{aligned}$$

Therefore, we obtain

$$\mathcal{T}[f * \overline{f}^*] \le \lim_{\nu\to\infty} \mathcal{T}[f_\nu * \overline{f}^*_\nu]$$

and our assertion follows because $\{f_\nu\}$ is a minimizing sequence.

This completes the proof of Theorem 3.

We can prove the corresponding result for Problem 1 only in the hypothetical case in which $\mathcal{T}[f * \overline{f}^*]$ is negative for some $f \in \mathcal{W}_0$.

THEOREM 4. *Let* λ *be the infimum of* $\mathcal{T}[f * \overline{f}^*]$ *in the unit sphere of the space* $\mathcal{W}_0 = W_0^{1,2}([M^{-1}, M])$.

If $\lambda < 0$, *then this infimum is attained.*

PROOF. The proof follows the same pattern as for Theorem 3. Let again $\{f_\nu\}$ be a minimizing sequence and f a weak limit in $\mathcal{W}_0$ of a subsequence of $\{f_\nu\}$.

This time, the normalization $\|Df_\nu\| = 1$ means that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{1}{4} + v^2\right)\left|\widetilde{f}_\nu\left(\frac{1}{2} + iv\right)\right|^2 \mathrm{d}v = 1\,,$$

and the proof that $\mathcal{T}[f_\nu * \overline{f}^*_\nu] \to \mathcal{T}[f * \overline{f}^*]$ is immediate.

The trouble here is that there is no immediate reason why $\|Df\|$ should continue to have norm 1, and we have only $\|Df\| \le 1$ by semicontinuity.

However, suppose that $\lambda < 0$. Then $\mathcal{T}[f * \overline{f}^*] = \lambda < 0$ and in particular f cannot be identically 0. Thus $0 < \|Df\| \le 1$. Now $f_0 = f/\|Df\|$ has $\|Df_0\| = 1$ and $\mathcal{T}[f_0 * \overline{f}^*_0] = \lambda/\|Df\|^2$. If we had $\|Df\| < 1$ then the assumption $\lambda < 0$ shows that $\lambda/\|Df\|^2 < \lambda$, contradicting the fact that λ is the infimum of $\mathcal{T}[f * \overline{f}^*]$ in $\mathcal{W}_0$. It follows that $\|Df\| = 1$ and, by well-known results, f_ν converges strongly to f in $\mathcal{W}_0$, proving the theorem.

We conclude this section with

THEOREM 5. *Let* $\mu^+(M)$ *and* $\mu^-(M)$ *be the infimum of* $\mathcal{T}[f * f^*]$ *in the class of even and odd functions in* $L^2([M^{-1}, M])$ *of norm* 1. *Then* $\mu^+(M)$ *and* $\mu^-(M)$ *are continuous decreasing functions of* M.

PROOF. It is clear that the functions $\mu^{\pm}(M)$ are decreasing functions of M, so we need to prove continuity.

Let $f(x)$ be even or odd and a minimizer for $\mathcal{T}[f * f^*]$. Consider, for $\varepsilon > 0$, the variation

$$f_\varepsilon(x) = (1+\varepsilon)^{1/2} x^{\frac{\varepsilon}{2}} f(x^{1+\varepsilon}).$$

Then $f_\varepsilon(x) \in L^2([M^{-1/(1+\varepsilon)}, M^{1/(1+\varepsilon)}])$, has norm 1 and has the same parity as f.

It follows that, with the sign $\pm$ according to the parity of f:

$$\liminf_{\varepsilon\to 0} \mu^{\pm}(M^{1/(1+\varepsilon)}) \le \liminf_{\varepsilon\to 0} \mathcal{T}[f_\varepsilon * f^*_\varepsilon] = \mathcal{T}[f * f^*] = \mu^{\pm}(M).$$

Here the equality $\lim \mathcal{T}[f_\varepsilon * f_\varepsilon^*] = \mathcal{T}[f * f^*]$ can be verified directly without much trouble, because f_ε converges strongly in $L^2([M^{-1}, M])$ to f and because

$$\widetilde{f}_\varepsilon(s) = (1+\varepsilon)^{-\frac{1}{2}} \widetilde{f}\left(s + \frac{\varepsilon}{2(1+\varepsilon)}\right),$$

this formula being used to show that

$$\int_{-\infty}^{\infty} (1 + \log^+ |v|) \left| \widetilde{f}_\varepsilon\left(\frac{1}{2} + iv\right)\right|^2 \mathrm{d}v$$

is a continuous function of ε as $\varepsilon \to 0$. This completes the proof.

5. The first eigenvalue problem

In this section we do a formal calculation which will suggest the structure of solutions of the eigenvalue problem of the preceding sections.

We assume *a priori* throughout this section that we have $f(x) \in \mathcal{W}_0$ such that

$$\lambda \mathcal{D} f(x) = \mathcal{L}[f](x), \quad x \in (M^{-1}, M). \tag{5.1}$$

Lemma 5. *The distribution $D^2 f$ is the sum of a bounded function and two Dirac distributions at $x = M^{\pm 1}$. Hence the Mellin transform $\widetilde{f}(s)$ satisfies a bound $\widetilde{f}(s) = O\big((|s|+1)^{-2}\big)$ in any fixed vertical strip $a \le \Re(s) \le b$. The constant implied in the $O(\,)$ symbol depends on M and a, b.*

We also have

$$\mathcal{L}[f](x) = O(M) x^{-\frac{1}{2}} \|Df\| \quad \textit{for } x \in (M^{-1}, M).$$

Proof. Consider $\mathcal{L}[f](x)$ for $x \in (M^{-1}, M)$. By Lemma 2, $|f(x)| \le x^{-\frac{1}{2}} \|Df\|$. For $x \in (M^{-1}, M)$, the functions $f(xy)$ and $f(x/y)$ are supported in $y \in (M^{-2}, M^2)$. Hence

$$\int_0^\infty f(xy)\,\mathrm{d}y + \int_0^\infty \frac{1}{y} f\left(\frac{x}{y}\right) \mathrm{d}y = O(M) x^{-\frac{1}{2}} \|Df\|.$$

The next term in $\mathcal{L}[f]$ is

$$-\sum_{n=1}^{\infty} \Lambda(n) \left\{ f(nx) + \frac{1}{n} f\left(\frac{x}{n}\right)\right\}$$

and, by the same argument as before, it is majorized by $O(M) x^{-\frac{1}{2}} \|Df\|$.

Also it is clear that $-(\log \pi) f(x) = O(x^{-\frac{1}{2}} \|Df\|)$. The remaining contribution to $\mathcal{L}[f]$ is

$$\frac{1}{2\pi i} \int_{(\frac{1}{2})} \Re\left[\frac{\Gamma'}{\Gamma}\left(\frac{w}{2}\right)\right] x^{-w} \widetilde{f}(w)\,\mathrm{d}w.$$

Since $f \in W^{1,2}$ we have by Plancherel's formula

$$\frac{1}{2\pi i}\int_{(\frac{1}{2})} |w|^2 |\widetilde{f}(w)|^2 \, \mathrm{d}w = \|Df\|^2 < \infty ,$$

and since $\Gamma'/\Gamma = O(\log|w|)$ we see from Cauchy's inequality that the contribution coming from this last term is again $O(x^{-1/2}\|Df\|)$. This proves the last statement of Lemma 5.

By (5.1), we see that

$$\mathcal{D}f(x) = -(D + D^2)f(x) = -(x^2 f'(x))'$$

is bounded in (M^{-1}, M). It follows that $f(x)$ is of differentiability class $C^{1,1}$ in (M^{-1}, M). Since Df is 0 outside $[M^{-1}, M]$ we see that $Df(x)$ has right and left limits for $x \to M^{\pm 1}$ and the first statement of the lemma follows. The second statement now is clear from

$$s^2 \widetilde{f}(s) = \int_0^\infty D^2 f(x) x^{s-1} \, \mathrm{d}x.$$

This completes the proof.

Lemma 6. *For a solution* $f(x)$ *we have*

$$\mathcal{L}[f](x) = \sum \widetilde{f}(\rho) x^{-\rho}$$

where the sum ranges over all non-trivial zeros of $\zeta(s)$. *In particular, the sum is absolutely convergent because* $\widetilde{f}(\rho) = O(|\rho|^{-2})$.

Proof. Since $\mathcal{L}[f](a) = \mathcal{T}[f(ax)]$ and the Mellin transform of $f(ax)$ is $a^{-s}\widetilde{f}(s)$, Lemma 6 follows from Theorem 2 and Lemma 5.

We write the equation $\lambda \mathcal{D}f = \mathcal{L}[f]$ in the form

$$-\lambda(x^2 f'(x))' = \sum_\rho \widetilde{f}(\rho) x^{-\rho} \tag{5.2}$$

for $x \in (M^{-1}, M)$, with the boundary condition $f(M) = f(1/M) = 0$.

Lemma 7. *Suppose* $\lambda \neq 0$. *Then for* $x \in (1/M, M)$ *we have*

$$\lambda f(x) = \sum_\rho \frac{\widetilde{f}(\rho)}{\rho(1-\rho)} x^{-\rho} - A - Bx^{-1}$$

where the constants A *and* B *are determined by the boundary condition* $f(1/M) = f(M) = 0$.

Moreover, for every $\alpha < 1$ *the function* f *is of differentiability class* $C^{2,\alpha}$ *in the open interval* (M^{-1}, M).

Proof. We recall that by Lemma 3 we have $\widetilde{f}(\rho) = O(|\rho|^{-2})$, where the constant implied in the $O(\ )$ symbol depends only on M. It follows that the function

$$f_0(x) = \lambda^{-1} \sum_\rho \frac{\widetilde{f}(\rho)}{\rho(1-\rho)} x^{-\rho}$$

is for $\alpha < 1$ a $C^{2,\alpha}((0\,,\infty))$ solution of

$$-\lambda(x^2 f_0'(x))' = \sum_\rho \widetilde{f}(\rho)x^{-\rho}\,, \qquad x \in (0\,,\infty)\,,$$

as one verifies directly.

The associated homogeneous equation $-(x^2u')' = 0$ has general solution $u(x) = A + B/x$. Thus we have $f(x) = f_0(x) + u(x)$, and A, B are determined by the boundary condition. This completes the proof of the lemma.

6. The dual eigenvalue problem

Let $\phi(x)$ be the characteristic function of $(1/M, M)$. Consider complex-valued functions $f \in \mathcal{W}_0$ given by

$$f(x) = \sum_\rho X_\rho \phi(x)x^{-\rho} + X_0\phi(x) + X_1\phi(x)x^{-1} \quad \textit{if} \quad x \in (M^{-1}\,, M) \tag{6.1}$$

and $f(x) = 0$ if $x \notin (M^{-1}\,, M)$. Here the sum runs over all complex zeros ρ of $\zeta(s)$, repeated according to their multiplicity $m(\rho)$.

The coefficients X_0 and X_1 are determined by the condition $f(M) = f(1/M) = 0$. An easy calculation shows that

$$\begin{aligned} X_0 &= -\sum_\rho \frac{M^{1-\rho} - M^{\rho-1}}{M - M^{-1}} X_\rho\,, \\ X_1 &= -\sum_\rho \frac{M^{\rho} - M^{-\rho}}{M - M^{-1}} X_\rho. \end{aligned} \tag{6.2}$$

We have

$$\widetilde{f}(s) = \sum_\rho X_\rho \widetilde{\phi}(s-\rho) + X_0\widetilde{\phi}(s) + X_1\widetilde{\phi}(s-1). \tag{6.3}$$

By Lemma 7, for $f(x)$ to be a formal solution of our eigenvalue problem it will be sufficient (5)

$$X_\rho = \frac{\widetilde{f}(\rho)}{\lambda\rho(1-\rho)}\,.$$

In view of (6.3), this yields the eigenvalue equation

$$\begin{aligned} &\lambda\rho(1-\rho)X_\rho = \\ &= \sum_{\rho'} \left[\widetilde{\phi}(\rho-\rho') - \frac{M^{1-\rho'} - M^{\rho'-1}}{M - M^{-1}}\,\widetilde{\phi}(\rho) - \frac{M^{\rho'} - M^{-\rho'}}{M - M^{-1}}\,\widetilde{\phi}(\rho-1)\right] X_{\rho'}. \end{aligned} \tag{6.4}$$

(5) This condition is only sufficient because there may be lack of uniqueness in the expansion (6.1).

7. The resolvent

We investigate here directly the linear system (6.4). We make a change of variables by setting $\rho = \frac{1}{2} + i\gamma$, $M = e^t$ with $t > 0$, $\Lambda = 1/\lambda$ and defining the multiplicity $m(\gamma)$ of γ as $m(\gamma) = m(\rho)$. We also abbreviate

$$K(x) = \frac{\sin(x)}{x}$$

and

$$\begin{aligned} K^*(x, y, t) = K\big(t(x-y)\big) &- \frac{t}{\sinh(t)}\left(\frac{1}{2} + iy\right) K\left(t\left(\frac{i}{2} - y\right)\right) K\left(t\left(\frac{i}{2} + x\right)\right) - \\ &- \frac{t}{\sinh(t)}\left(\frac{1}{2} - iy\right) K\left(t\left(\frac{i}{2} + y\right)\right) K\left(t\left(\frac{i}{2} - x\right)\right). \end{aligned}$$

It is easily verified that

$$\begin{aligned} &\left(\frac{1}{4} + x^2\right) K^*(x, y, t) = \left(\frac{1}{4} + y^2\right) K^*(y, x, t) = \\ &= \left(\frac{1}{4} + xy\right) K\big(t(x-y)\big) - \frac{\cosh(t)\cos\big(t(x-y)\big) - \cos\big(t(x+y)\big)}{2t\sinh(t)}. \end{aligned} \tag{7.1}$$

After the change of variables and setting

$$z_\gamma = X_\rho, \quad w_\gamma = \left(\frac{1}{4} + \gamma^2\right) z_\gamma, \quad \Lambda = 1/\lambda \tag{7.2}$$

and

$$H(x, y, t) = \frac{2t\, K^*(x, y, t)}{(\frac{1}{4} + y^2)} \tag{7.3}$$

equation (6.4) becomes

$$w_\gamma = \Lambda \sum_{\gamma'} H(\gamma, \gamma', t)\, w_{\gamma'}. \tag{7.4}$$

We write

$$\mathcal{H}(\Gamma; t) = \Big[H(\gamma, \gamma', t)\Big]_{\gamma, \gamma' \in \Gamma}$$

and $D(\Lambda, t)$ for the resolvent determinant

$$D(\Lambda, t) = \det\big[I - \Lambda \mathcal{H}(\Gamma; t)\big]. \tag{7.5}$$

We have

Theorem 6. *The resolvent* (7.5) *is well defined and is an entire function of the complex variable* Λ, *at most of order* 1 *and finite exponential type.*

Moreover,

$$D(\Lambda, t) = 1 + \sum_{n=1}^{\infty} (-1)^n \Delta_n(t) \frac{\Lambda^n}{n!}$$

where

$$\Delta_n(t) = \sum_{\gamma_1, \dots, \gamma_n \in \Gamma} \det \Big[H(\gamma_j, \gamma_k, t) \Big]_{j,k=1,\dots,n}.$$

Finally, as $N \to \infty$ *the truncations* $D_N(\Lambda, t)$ *given by*

$$\det \Big[\delta_{\gamma,\gamma'} - \Lambda H(\gamma, \gamma', t) \Big]_{\gamma,\gamma' \in \Gamma_N}$$

with $\Gamma_N = \Gamma \bigcap \{|\gamma| \le N\}$, *converge to* $D(\Lambda, t)$ *uniformly for* Λ *in compact subsets of* $\mathbb{C}$.

Proof. We claim that

$$\sum_{\gamma,\gamma'} \big| H(\gamma, \gamma', t) \big| < +\infty. \tag{7.6}$$

To see this, we start with the inequality

$$H(x, y, t) \ll \frac{1}{(1+|x|)(1+|y|)} \min \left(1, \frac{1}{|x-y|} \right) \tag{7.7}$$

valid for fixed t and x, y in a fixed horizontal strip excluding a neighborhood of $x = \pm i/2$ (this last restriction is inconsequential in what follows). Thus (7.6) is implied by the convergence of the series

$$\sum_{\gamma,\gamma'} \frac{1}{(1+|\gamma|)(1+|\gamma'|)} \frac{1}{1+|\gamma-\gamma'|}.$$

We split the summation over γ' into subsums where γ' ranges over $n \le |\gamma' - \gamma| < n+1$, for $n = 0, 1, 2, \dots$. Hence the series is majorized by

$$\sum_{\gamma} \frac{1}{1+|\gamma|} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{n-1 \le |\gamma'-\gamma| < n} \frac{1}{1+|\gamma'|}. \tag{7.8}$$

In (7.8), we split the sum over n into the three ranges $n \le [|\gamma|/2]$, $[|\gamma|/2] < n < [3|\gamma|/2]$ and $n \ge [3|\gamma|/2]$. In the first range, γ' is of order γ and [6]

$$\sum_{n \le |\gamma'-\gamma| < n+1} 1 \ll \log(|\gamma|+2).$$

Thus the sum over the first range contributes to (7.8), up to a constant factor, an amount

$$\sum_{\gamma} \frac{1}{1+|\gamma|} \left(\sum_{n=1}^{[|\gamma|/2]} \frac{\log(|\gamma|+2)}{(1+|\gamma|)n} \right) \ll \sum_{\gamma} \frac{(\log(|\gamma|+2))^2}{(1+|\gamma|)^2} < +\infty.$$

The contribution of the sum over the third range is computed in a similar way. This time γ' is of precise order n and the number of points γ' in the range $n-1 \le |\gamma'-\gamma| < n$

[6] We use Vinogradov's notation $a \ll b$ to indicate an inequality $|a| \le Cb$ for some unspecified positive constant C.

is $O(\log(n+2))$. Thus the sum over the third range contributes to (7.8), up to a constant factor, an amount

$$\sum_{\gamma} \frac{1}{1+|\gamma|}\Bigg(\sum_{n>[3|\gamma|/2]} \frac{\log(n+2)}{n^2}\Bigg) \ll \sum_{\gamma} \frac{\log(|\gamma|+2)}{(1+|\gamma|)^2} <+\infty.$$

In the second range $|\gamma'|$, up to a constant factor, is at least $1+\big|n-|\gamma|\big|$ and the number of points γ' in the range $n-1 \le |\gamma'-\gamma| < n$ is $O(\log(|\gamma|+2))$. Thus the sum over the second range contributes to (7.8), up to a constant factor, an amount

$$\sum_{\gamma} \frac{\log(|\gamma|+2)}{(1+|\gamma|)^2} \sum_{[|\gamma|/2]<n<[3|\gamma|/2]} \frac{1}{1+\big|n-|\gamma|\big|} \ll \sum_{\gamma} \frac{\log(|\gamma|+2)^2}{(1+|\gamma|)^2} <+\infty.$$

This completes the proof of (7.8) and, with it, of (7.6).

Once we have (7.6), the convergence of $D_N(\Lambda, t)$ to a limit is an application of von Koch's criterion [7, Ch. II, §2.81, p. 36] for the convergence of an infinite determinant.

We have a Taylor expansion

$$D_N(\Lambda, t) = 1 + \sum_{n=1}^{J+1} (-1)^n \Delta_{n,N}(t)\, \frac{\Lambda^n}{n!}$$

where $J+1$ is the cardinality of Γ_N and

$$\Delta_{n,N}(t) = \sum_{\gamma_1, \ldots, \gamma_n \in \Gamma_N} \det\Big[H(\gamma_j, \gamma_k, t)\Big]_{j,k=1,\ldots,n}. \tag{7.9}$$

Now, using (7.7), we apply Lord Kelvin's inequality (also more widely known as Hadamard's inequality) to each determinant and find

$$\sum_{k=1}^{n} \big|H(\gamma_j, \gamma_k, t)\big|^2 \ll \sum_{k=1}^{n} \frac{1}{1+(|\gamma_j|-|\gamma_k|)^2}\, \frac{1}{(1+|\gamma_j|)^2(1+|\gamma_k|)^2}. \tag{7.10}$$

Since the number of γ_k in an interval $(m, m+1)$ is $O(\log(m+2))$, the right-hand side of (7.10) is majorized, up to a constant factor, by

$$\frac{1}{(1+|\gamma_j|)^2} \sum_{m=1}^{\infty} \frac{\log(m+2)}{m^2(1+\big|m-|\gamma|\big|)^2} \ll \frac{\log(|\gamma_j|+2)}{(1+|\gamma_j|)^4}.$$

Thus the application of Lord Kelvin's inequality yields

$$\Big|\det\Big[H(\gamma_j, \gamma_k, t)\Big]_{j,k=1,\ldots,n}\Big| \le A(t)^n \prod_{j=1}^{n} \left(\frac{\log(|\gamma_j|+2)}{(1+|\gamma_j|)^4}\right)^{1/2}$$

for some constant $A(t)$ depending only on t. We also have

$$\sum_{\gamma_1,\ldots,\gamma_n} \prod_{j=1}^{n} \left(\frac{\log(|\gamma_j|+2)}{(1+|\gamma_j|)^4}\right)^{1/2} = \left(\sum_{\gamma} \frac{\sqrt{\log(|\gamma|+2)}}{(1+|\gamma|)^2}\right)^n.$$

This proves that

$$|\Delta_{n,N}(t)| \le \big(A(t)B\big)^n$$

with $B = \sum \sqrt{\log(|\gamma|+2)}\,(1+|\gamma|)^{-2} <+\infty$. This bound, uniform in N, continues to hold for $\Delta_n(t)$.

It is now obvious that $D(\Lambda\,,t)$ is an entire function of Λ, at most of order 1 and exponential type bounded by $\log(A(t)B)$.

This completes the proof of Theorem 6.

Our next result shows that zeros of the resolvent give rise to solutions $f\in\mathcal{W}_0$ of the eigenvalue problem $\lambda\mathcal{D}f = \mathcal{L}[f]$ for $x\in(1/M, M)$.

Theorem 7. *Let* Λ_0 *be a zero of* $D(\Lambda\,,t)=0$, *of multiplicity* m. *Define*

$$D(\gamma\,,\gamma_0;\Lambda\,,t) = \Lambda\sum_{n=0}^{\infty}(-1)^n\Delta_n(\gamma\,,\gamma_0;t)\,\frac{\Lambda^n}{n!}$$

where $\Delta_0(\gamma\,,\gamma_0) = H(\gamma\,,\gamma_0\,,t)$ *and in general*

$$\Delta_n(\gamma\,,\gamma_0;t) = \sum_{\gamma_1,\dots,\gamma_n\in\Gamma}\det\begin{bmatrix} H(\gamma\,,\gamma_0\,,t) & H(\gamma\,,\gamma_k\,,t)\\ H(\gamma_j\,,\gamma_0\,,t) & H(\gamma_j\,,\gamma_k\,,t)\end{bmatrix}_{j,k=1,\dots,n}.$$

Then $D(\gamma\,,\gamma_0;\Lambda\,,t)$ *is an entire function of* Λ, *at most of order* 1 *and finite exponential type.*

There exist γ_0 *and an integer* l, $0\le l\le m-1$, *such that the vector* $\{w_\gamma\}$ *with components*

$$w_\gamma = \frac{1}{l!}(\partial/\partial\Lambda)^l D(\gamma\,,\gamma_0;\Lambda_0\,,t)$$

is a solution, not identically 0, *of the linear system* (7.6).

We also have

$$w_\gamma \ll \sqrt{\log(|\gamma|+2)}/(1+|\gamma|)^2\,.$$

Moreover, the function $f(x)$ *defined by* (6.1), (6.2), $M=e^t$ *and* $X_\rho = z_\gamma$ *is a solution in* $\mathcal{W}_0$ *of*

$$\mathcal{D}f(x) = \Lambda_0\,\mathcal{L}[f](x)\,,\qquad x\in(1/M, M)$$

with Dirichlet boundary conditions $f(1/M)=f(M)=0$.

Proof. This is obtained (see the discussion in the continuous case in [7, Ch. XI, §11.23, pp. 219-220]) by solving first the finite linear system (7.4) restricted to Γ_N and then going to the limit as $N\to\infty$.

Let $J+1$ be the number of points $\gamma\in\Gamma_N$ and define

$$\Delta_{n,N}(\gamma\,,\gamma_0;t) = \sum_{\gamma_1,\dots,\gamma_n\in\Gamma_N}\det\begin{bmatrix} H(\gamma\,,\gamma_0\,,t) & H(\gamma\,,\gamma_k\,,t)\\ H(\gamma_j\,,\gamma_0\,,t) & H(\gamma_j\,,\gamma_k\,,t)\end{bmatrix}_{j,k=1,\dots,n}.$$

Then we set

$$D_N(\gamma, \gamma_0; \Lambda, t) = \Lambda \sum_{n=0}^{J} (-1)^n \Delta_{n,N}(\gamma, \gamma_0; t) \frac{\Lambda^n}{n!}.$$

Consider the $(J+1) \times (J+1)$ matrix

$$\Big[\delta_{jk} - \Lambda H(\gamma_j, \gamma_k, t)\Big]_{j,k=0,1,\dots,J}$$

where δ_{jk} is Kronecker's delta. Then for $j = 1, \dots, J$ the cofactor of the $(j+1)$-th element of the first row of this matrix turns out to be $D_N(\gamma, \gamma_0; \Lambda, t)$. If $j = 0$ the cofactor is given by a modified formula, namely

$$D_N(\gamma_0, \gamma_0; \Lambda, t) + D_N(\Lambda, t) - (-1)^{J+1} \Delta_{J+1,N}(t) \frac{\Lambda^{J+1}}{(J+1)!}.$$

By Laplace's identity, it follows that if

$$W_{\gamma,N} = D_N(\gamma, \gamma_0; \Lambda, t) + \left\{ D_N(\Lambda, t) - (-1)^{J+1} \Delta_{J+1,N}(t) \frac{\Lambda^{J+1}}{(J+1)!} \right\} \delta_{\gamma,\gamma_0}, \tag{7.11}$$

where δ_{γ,γ_0} is Kronecker's delta, then

$$W_{\gamma,N} = \Lambda D_N(\Lambda, t) + \Lambda \sum_{\gamma' \in \Gamma_N} H(\gamma, \gamma', t) W_{\gamma',N}. \tag{7.12}$$

Now we let $N \to \infty$. By Theorem 5, $D_N(\Lambda, t)$ converges uniformly in compact subsets to the entire function $D(\Lambda, t)$. Moreover, the same proof as in Theorem 5 shows that $D_N(\gamma, \gamma_0; \Lambda, t) \to D(\gamma, \gamma_0; \Lambda, t)$, an entire function of Λ of order at most 1 and finite exponential type. Thus we can pass to the limit as $N \to \infty$ in the above equation, and notationwise its effect is to drop the suffix N. In the limit as $N \to \infty$, we modify the definition of W_{γ_0} by omitting the term $-(-1)^{J+1}\Delta_{J+1,N}(t)\Lambda^{J+1}/(J+1)!$, because it tends to 0 as $N \to \infty$.

We have the Fredholm identity

$$\sum_{\gamma} D(\gamma, \gamma; \Lambda, t) = -\Lambda \frac{\partial D(\Lambda, t)}{\partial \Lambda}. \tag{7.13}$$

Since $D(\Lambda, t)$ has a zero of multiplicity m at $\Lambda = \Lambda_0$, the above formula shows that there is γ_0 such that $D(\gamma_0, \gamma_0; \Lambda, t)$ has a zero of multiplicity exactly $l \le m-1$ at $\Lambda = \Lambda_0$ and every $D(\gamma, \gamma_0; \Lambda, t)$ vanishes at $\Lambda = \Lambda_0$ with multiplicity not less than l. By differentiating (7.13) l times and setting

$$w_\gamma = \frac{1}{l!} \left(\frac{\partial}{\partial \Lambda}\right)^l D(\gamma, \gamma_0; \Lambda_0, t)$$

we get

$$w_\gamma = \Lambda_0 \sum_{\gamma' \in \Gamma} H(\gamma, \gamma', t)\, w_\gamma$$

and $w_{\gamma_0} \neq 0$.

The application of Lord Kelvin's inequality shows that

$$\frac{1}{l!}\Big(\frac{\partial}{\partial\Lambda}\Big)^l D(\gamma\,,\gamma_0;\Lambda_0\,,t) \ll \frac{\sqrt{\log(|\gamma|+2)}}{(1+|\gamma|)^2}\,.$$

This proves that $X_\rho = z_\gamma \ll \sqrt{\log(|\gamma|+2)}/(1+|\gamma|)^4$; thus the series for $f(x)$ and $Df(x)$ are both absolutely convergent. Since $f(1/M) = f(M) = 0$ is assured by the structure of the linear system (6.4), we conclude that $f \in \mathcal{W}_0$ and with it the proof of Theorem 7.

8. The eigenvalues of finite approximations

In this section we analyse more closely the case in which the set of points ρ is a finite set with the same symmetries as the set of zeros of the Riemann zeta function. The main result of this section, namely Theorem 8, shows that the matrix $\big[H(\gamma\,,\gamma'\,,t)\big]$ has only real eigenvalues and determines the number of negative eigenvalues and the multiplicity of 0 as an eigenvalue.

It proves to be notationally convenient to state all our results in terms of the complex Fourier transform rather than the Mellin transform.

Let $\mathcal{Z}$ be a finite multiset (7) of complex numbers, repeated according to their multiplicity. We assume that if $\rho \in \mathcal{Z}$ then $\overline{\rho}$ and $1-\rho$ are again elements of $\mathcal{Z}$, with the same multiplicity as ρ. For $\rho \in \mathcal{Z}$ we write $\rho = \frac{1}{2} + i\gamma$ and denote by Γ the corresponding set of points γ.

We denote by $\Phi(x)$ the characteristic function of the closed interval $[-t, t]$ and let V be the $\mathbb{C}$-vector space generated by linear combinations of the functions $\Phi(x)e^{x/2}$, $\Phi(x)e^{-x/2}$ and $\Phi(x)e^{-i\gamma x}$ for $\gamma \in \Gamma$. We also denote by V° the subspace of codimension 2 consisting of functions $F(x) \in V$ satisfying the additional condition

$$F(-t) = F(t) = 0.$$

Finally by (P, Q) we denote the usual inner product in $L^2(\mathbb{R})$ given by

$$(P,\, Q) = \int_{-\infty}^{\infty} P(x)\,\overline{Q(x)}\,\mathrm{d}x.$$

Consider the linear functional on V given by (8)

$$L[F](x) = \sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} F(x+u)\,e^{i\gamma u}\,\mathrm{d}u. \tag{8.1}$$

We have

Lemma 8. *If $F \in V$ then the inner product $(L[F]\,,F)$ is real.*

(7) By a multiset we mean a set whose elements have positive integral multiplicity.

(8) Here and in what follows the sum is understood to be over the multiset Γ, so that each term appears according to its proper multiplicity.

PROOF. For $F \in V$ we have, using $\Gamma = -\overline{\Gamma}$,

$$(L[F]\,,\,F) = \int_{-\infty}^{\infty}\left(\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty} F(u)\,e^{i\gamma(u-x)}\,\mathrm{d}u\right)\overline{F(x)}\,\mathrm{d}x =$$
$$= \sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u)\overline{F(x)}\,e^{i\gamma(u-x)}\,\mathrm{d}x\,\mathrm{d}u =$$
$$= \sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{F(u)}F(x)\,e^{i\gamma(x-u)}\,\mathrm{d}x\,\mathrm{d}u =$$
$$= \sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{F(u)}F(x)\,e^{-i\overline{\gamma}(u-x)}\,\mathrm{d}x\,\mathrm{d}u =$$
$$= \overline{\sum_{\gamma\in\Gamma}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u)\overline{F(x)}\,e^{i\gamma(u-x)}\,\mathrm{d}x\,\mathrm{d}u} = \overline{(L[F]\,,\,F)}.$$

All steps are justified by Fubini's theorem, because F is bounded with compact support, concluding the proof.

Let $\Delta = -(\mathrm{d}/\mathrm{d}x)^2$. If $P(x)$ is a smooth function on $[-t, t]$ and $P(-t) = P(t) = 0$ we have by integration by parts

$$\int_{-t}^{t} \Delta P(x)\,\overline{P}(x)\,\mathrm{d}x = \int_{-t}^{t} \left|P'(x)\right|^2 \mathrm{d}x. \tag{8.2}$$

A first consequence is the following result.

LEMMA 9. *Suppose that* $F(x) \in V^\circ$ *is not identically* 0 *and satisfies the eigenvalue equation*

$$\lambda\left(\frac{1}{4} + \Delta\right) F(x) = L[F](x) \quad \textit{for } -t < x < t. \tag{8.3}$$

Then λ *is real.*

PROOF. We take the inner product of (8.3) with F. Since $F \in V^\circ$ we verify by integration by parts that

$$\left(\left(\frac{1}{4} + \Delta\right) F, F\right) = \frac{1}{4}\int_{-\infty}^{\infty} |F(x)|^2\,\mathrm{d}x + \int_{-\infty}^{\infty} |F'(x)|^2\,\mathrm{d}x > 0.$$

The lemma follows because $(L[F]\,,\,F)$ is real.

COROLLARY. *With the notation of the preceding section, the resolvent*

$$D_N(\Lambda\,,\,t) = \det\left[I - \Lambda\,\mathcal{H}(\Gamma_N; t)\right] \tag{8.4}$$

has only real roots.

LEMMA 10. *The matrix* $\mathcal{H}(\Gamma; t)$ *admits the eigenvalue* 0 *if and only if there is* $\gamma \in \Gamma$ *with multiplicity greater than* 1, *in which case the multiplicity of* 0 *as an eigenvalue is exactly* $\sum'[m(\gamma) - 1]$, *where* $\sum'$ *ranges over all distinct* $\gamma \in \Gamma$.

Moreover if every γ *is real all eigenvalues of* $\mathcal{H}(\Gamma; t)$ *are non-negative.*

PROOF. We have

$$\left(\frac{1}{4}+y^2\right) H(x,y,t) = \int_{-t}^{t} e^{ixu}\, e^{-iyu}\, \mathrm{d}u - \frac{e^{(\frac{1}{2}-iy)t} - e^{-(\frac{1}{2}-iy)t}}{e^t - e^{-t}}\, \frac{e^{(\frac{1}{2}+ix)t} - e^{-(\frac{1}{2}+ix)t}}{\frac{1}{2}+ix} - \\ - \frac{e^{(\frac{1}{2}+iy)t} - e^{-(\frac{1}{2}+iy)t}}{e^t - e^{-t}}\, \frac{e^{(\frac{1}{2}-ix)t} - e^{-(\frac{1}{2}-ix)t}}{\frac{1}{2}-ix}.$$

Therefore, the eigenvalue equation $\det\left[I\lambda - \mathcal{H}(\Gamma;t)\right] = 0$ is equivalent to the solubility of the linear system

$$\lambda\left(\frac{1}{4}+\gamma^2\right) z_\gamma = \int_{-t}^{t} e^{i\gamma u}\left(\sum_{\gamma'} e^{-i\gamma' u} z_{\gamma'}\right)\mathrm{d}u - \\ - \frac{e^{(\frac{1}{2}+i\gamma)t} - e^{-(\frac{1}{2}+i\gamma)t}}{(\frac{1}{2}+i\gamma)(e^t-e^{-t})}\left[\sum_{\gamma'}\left(e^{(\frac{1}{2}-i\gamma')t} - e^{-(\frac{1}{2}-i\gamma')t}\right) z_{\gamma'}\right] - \\ - \frac{e^{(\frac{1}{2}-i\gamma)t} - e^{-(\frac{1}{2}-i\gamma)t}}{(\frac{1}{2}-i\gamma)(e^t-e^{-t})}\left[\sum_{\gamma'}\left(e^{(\frac{1}{2}+i\gamma')t} - e^{-(\frac{1}{2}+i\gamma')t}\right) z_{\gamma'}\right].$$

We write for simplicity

$$Z(u) = \sum_{\gamma} e^{-i\gamma u} z_\gamma \tag{8.5}$$

and then the above linear system takes the form

$$\lambda\left(\frac{1}{4}+\gamma^2\right) z_\gamma = \int_{-t}^{t} e^{i\gamma u} Z(u)\,\mathrm{d}u - \frac{e^{(\frac{1}{2}+i\gamma)t} - e^{-(\frac{1}{2}+i\gamma)t}}{(\frac{1}{2}+i\gamma)(e^t-e^{-t})}\left[e^{t/2}Z(t) - e^{-t/2}Z(-t)\right] - \\ - \frac{e^{(\frac{1}{2}-i\gamma)t} - e^{-(\frac{1}{2}-i\gamma)t}}{(\frac{1}{2}-i\gamma)(e^t-e^{-t})}\left[e^{t/2}Z(-t) - e^{-t/2}Z(t)\right]. \tag{8.6}$$

We note that since $\Gamma = \overline{\Gamma}$ we have

$$\sum_{\gamma\in\Gamma} e^{i\gamma u}\,\overline{(z_{\overline{\gamma}})} = \sum_{\gamma\in\Gamma} e^{i\overline{\gamma} u}\,\overline{(z_\gamma)} = \overline{\left(\sum_{\gamma\in\Gamma} e^{-i\gamma u} z_\gamma\right)} = \overline{Z(u)}. \tag{8.7}$$

By differentiating (8.7), we see that

$$\sum_{\gamma\in\Gamma} i\gamma\, e^{i\gamma u}\,\overline{(z_{\overline{\gamma}})} = \overline{Z'(u)} \tag{8.8}$$

and

$$\sum_{\gamma\in\Gamma} \gamma^2\, e^{i\gamma u}\,\overline{(z_{\overline{\gamma}})} = \overline{\Delta Z(u)}. \tag{8.9}$$

We multiply both sides of (8.6) by $(\frac{1}{4}+\gamma^2)\overline{(z_{\overline{\gamma}})}$ and sum over γ. In view of equations (8.7), (8.8) and (8.9) we find

$$
\begin{aligned}
\lambda \sum_{\gamma\in\Gamma} \left(\frac{1}{4}+\gamma^2\right)^2 z_\gamma \overline{(z_{\overline{\gamma}})} = \int_{-t}^{t} Z(u) \overline{\left(\frac{1}{4}+\Delta\right) Z(u)}\,\mathrm{d}u \,+ \\
+ \overline{Z'(t)}\, Z(t) - \overline{Z'(-t)}\, Z(-t) - \\
- \frac{e^t + e^{-t}}{2(e^t - e^{-t})}\left[|Z(t)|^2 + |Z(-t)|^2\right] + \\
+ \frac{1}{e^t - e^{-t}}\left[\overline{Z(t)}\, Z(-t) - Z(t)\,\overline{Z(-t)}\right].
\end{aligned}
\tag{8.10}
$$

Integration by parts shows that

$$
\int_{-t}^{t} Z(u)\,\overline{\Delta Z(u)}\,\mathrm{d}u = -\overline{Z'(t)}\, Z(t) + \overline{Z'(-t)}\, Z(-t) + \int_{-t}^{t} \left|Z'(u)\right|^2 \mathrm{d}u
$$

and (8.10) becomes

$$
\begin{aligned}
\lambda \sum_{\gamma\in\Gamma} \left(\frac{1}{4}+\gamma^2\right)^2 z_\gamma \overline{(z_{\overline{\gamma}})} = \frac{1}{4}\int_{-t}^{t} \left|Z(u)\right|^2 \mathrm{d}u + \int_{-t}^{t} \left|Z'(u)\right|^2 \mathrm{d}u - \\
- \frac{e^t + e^{-t}}{2(e^t - e^{-t})}\left[\left|Z(t)\right|^2 + \left|Z(-t)\right|^2\right] + \\
+ \frac{1}{e^t - e^{-t}}\left[\overline{Z(t)}\, Z(-t) - Z(t)\,\overline{Z(-t)}\right].
\end{aligned}
\tag{8.11}
$$

A further simplification is achieved if we decompose $Z(u)$ into its even and odd parts, namely

$$
Z^{\pm}(u) = \frac{1}{2}\left[Z(u) \pm Z(-u)\right].
$$

Then we verify that

$$
\begin{aligned}
\frac{e^t + e^{-t}}{2(e^t - e^{-t})}\left[\left|Z(t)\right|^2 + \left|Z(-t)\right|^2\right] - \frac{1}{e^t - e^{-t}}\left[\overline{Z(t)}\, Z(-t) - Z(t)\,\overline{Z(-t)}\right] = \\
= \frac{e^{t/2} - e^{-t/2}}{e^{t/2} + e^{-t/2}} \left|Z^+(t)\right|^2 + \frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}} \left|Z^-(t)\right|^2.
\end{aligned}
$$

Also, for any function g we have

$$
\int_{-t}^{t} |g(u)|^2\,\mathrm{d}u = \int_{-t}^{t} |g^+(u)|^2\,\mathrm{d}u + \int_{-t}^{t} |g^-(u)|^2\,\mathrm{d}u.
$$

Finally, since

$$
\left(\frac{1}{4}+\gamma^2\right)^2 z_\gamma \overline{(z_{\overline{\gamma}})} = w_\gamma \overline{(w_{\overline{\gamma}})}\,,
$$

we can transform (8.11) into

$$\lambda\sum_{\gamma\in\Gamma} w_\gamma\,\overline{(w_{\overline{\gamma}})} = \sum_{\pm}\left\{\frac{1}{4}\int_{-t}^{t}\left|Z^{\pm}(u)\right|^2 \mathrm{d}u + \int_{-t}^{t}\left|(Z^{\pm})'(u)\right|^2 \mathrm{d}u\right\} - \tag{8.12}$$
$$- \frac{e^{t/2}-e^{-t/2}}{e^{t/2}+e^{-t/2}}\left|Z^{+}(t)\right|^2 - \frac{e^{t/2}+e^{-t/2}}{e^{t/2}-e^{-t/2}}\left|Z^{-}(t)\right|^2.$$

Now we prove that

$$\frac{e^{t/2}\mp e^{-t/2}}{e^{t/2}\pm e^{-t/2}}\left|Z^{\pm}(t)\right|^2 \le \frac{1}{4}\int_{-t}^{t}\left|Z^{\pm}(u)\right|^2 \mathrm{d}u + \int_{-t}^{t}\left|(Z^{\pm})'(u)\right|^2 \mathrm{d}u \tag{8.13}$$

for any continuously differentiable even or odd function $Z^{\pm}(u)$. Furthermore, equality holds if and only if

$$Z^{\pm}(u) = Z^{\pm}(t)\,\frac{e^{u/2}\pm e^{-u/2}}{e^{t/2}\pm e^{-t/2}}\,.$$

We verify this statement as follows. By direct calculation, equality holds in (8.13) if $Z^{\pm}(u) = e^{u/2}\pm e^{-u/2}$. Let

$$F^{\pm}(u) = Z^{\pm}(u) - Z^{\pm}(t)\,\frac{e^{u/2}\pm e^{-u/2}}{e^{t/2}\pm e^{-t/2}}\,.$$

Then using the fact that $e^{u/2}\pm e^{-u/2}$ yields equality in (8.13), and that $F^{\pm}(u)$ is even or odd we verify by integration by parts the equation

$$\frac{1}{4}\int_{-t}^{t}\left|F^{\pm}(u)\right|^2 \mathrm{d}u + \int_{-t}^{t}\left|(F^{\pm})'(u)\right|^2 \mathrm{d}u =$$
$$= \frac{1}{4}\int_{-t}^{t}\left|Z^{\pm}(u)\right|^2 \mathrm{d}u + \int_{-t}^{t}\left|(Z^{\pm})'(u)\right|^2 \mathrm{d}u - \frac{e^{t/2}\mp e^{-t/2}}{e^{t/2}\pm e^{-t/2}}\left|Z^{\pm}(t)\right|^2.$$

In conclusion, we have

$$\lambda\sum_{\gamma\in\Gamma} w_\gamma\,\overline{(w_{\overline{\gamma}})} = \frac{1}{4}\int_{-t}^{t}\left|F(u)\right|^2 \mathrm{d}u + \int_{-t}^{t}\left|F'(u)\right|^2 \mathrm{d}u \tag{8.14}$$

with

$$F(u) = Z(u) - Ae^{u/2} - Be^{-u/2} \tag{8.15}$$

and A, B determined by the condition $F(t) = F(-t) = 0$.

This proves our statement.

The proof of Lemma 10 follows from (8.14). Suppose first that every γ has multiplicity 1. If we had $\lambda = 0$ then (8.14) and (8.15) show that $F(u)$ vanishes identically in the interval $(-t, t)$. On the other hand, the functions $e^{u/2}$, $e^{-u/2}$ and $e^{-i\gamma u}$ are linearly independent over any finite interval because here Γ is a finite set; hence, we would have $z_\gamma = 0$ for every γ, a contradiction.

The general case can be treated much in the same way. It is easy to show that the spectrum of $\mathcal{H}(\Gamma; t)$ consists of the point 0 with multiplicity $\sum'[m(\gamma)-1]$ together with the spectrum of the matrix

$$\left[H(\gamma_j\,,\gamma_k\,,t)m(\gamma_k)\right]$$

where now $\{\gamma_j\}$ is the set of all distinct $\gamma \in \Gamma$. The same argument as before goes through, provided we count each z_γ with its multiplicity $m(\gamma)$.

The last conclusion of Lemma 10 is obvious from (8.14), because if every γ is real then (8.14) becomes

$$\lambda \sum_{\gamma\in\Gamma} |w_\gamma|^2 = \frac{1}{4}\int_{-t}^{t} |F(u)|^2\,\mathrm{d}u + \int_{-t}^{t} |F'(u)|^2\,\mathrm{d}u.$$

THEOREM 8. *The number of negative eigenvalues of the matrix* $\mathcal{H}(\Gamma; t)$ *equals the number of distinct complex conjugate pairs* $(\gamma\,,\overline{\gamma})$ *in* Γ.

PROOF. Suppose first that all elements of Γ are distinct and consider a continuous deformation of Γ into a new set $\widetilde{\Gamma}$, such that during the deformation the elements of the set remain distinct and the invariance of the set by complex conjugation and multiplication by -1 remains preserved. By Lemma 10 all eigenvalues remain real and not 0, while moving continuously during the deformation; in particular, the number of negative and positive eigenvalues remains constant.

Thus Theorem 8 for a multiset Γ with distinct elements will follow, once it has been verified for any other set $\widetilde{\Gamma}$ with distinct elements, invariant by complex conjugation and multiplication by -1, with the same number of complex conjugate pairs of elements as Γ. The general case then is an immediate consequence of Lemma 10.

Now we verify Theorem 8, for a multiset Γ with distinct elements, by induction on the number of complex conjugate pairs. Note that, according to our preceding discussion, to complete the induction step it suffices to verify it adding to Γ some new element γ_0 and its transforms by complex conjugation and by multiplication by -1. This element γ_0 is otherwise at our disposal.

By Lemma 10, Theorem 8 is true if every γ is real. We examine separately the case in which we increase our multiset by $\{\gamma_0\,,\overline{\gamma}_0\,,-\gamma_0\,,-\overline{\gamma}_0\}$ with γ_0 complex and not purely imaginary, or by $\{\gamma_0\,,\overline{\gamma}_0\}$ with γ_0 purely imaginary.

Suppose Theorem 8 holds for Γ and let us show that it remains true for the new set $\Gamma\bigcup\{\gamma_0\,,\overline{\gamma}_0\,,-\gamma_0\,,-\overline{\gamma}_0\}$ where $\gamma_0 = a+ib$, provided a and b are large enough.

In what follows, we keep Γ fixed and consider γ_0 as a variable quantity, so our estimates will be uniform only with respect to γ_0. In the end, we let

$$a/b \to 0\,, \qquad b \to +\infty.$$

A simple asymptotic calculation shows that with the above choice of γ_0 the matrix

$\left[H(\gamma, \gamma', t)\right]$ for $\gamma, \gamma' \in \{\gamma_0, \overline{\gamma}_0, -\gamma_0, -\overline{\gamma}_0\}$ is (9)

$$\begin{bmatrix} o(B) & B + o(B) & o(B) & o(B) \\ B + o(B) & o(B) & o(B) & o(B) \\ o(B) & o(B) & o(B) & B + o(B) \\ o(B) & o(B) & B + o(B) & o(B) \end{bmatrix}$$

where for simplicity we have written

$$B = \frac{\sinh(2tb)}{2tb\,|\gamma_0|^2}\,.$$

The eigenvalue equation is obtained by equating to 0 the determinant of

$$\begin{bmatrix} o(B) - x & B + o(B) & o(B) & o(B) & \widetilde{H}(\gamma_0, \gamma', t) \\ B + o(B) & o(B) - x & o(B) & o(B) & \widetilde{H}(\overline{\gamma}_0, \gamma', t) \\ o(B) & o(B) & o(B) - x & B + o(B) & \widetilde{H}(-\gamma_0, \gamma', t) \\ o(B) & o(B) & B + o(B) & o(B) - x & \widetilde{H}(-\overline{\gamma}_0, \gamma', t) \\ H(\gamma, \gamma_0, t) & \widetilde{H}(\gamma, \overline{\gamma}_0, t) & H(\gamma, -\gamma_0, t) & H(\gamma, -\overline{\gamma}_0, t) & H(\gamma, \gamma', t) - \delta_{\gamma, \gamma'} x \end{bmatrix}.$$

We know already that this matrix has only real eigenvalues.

Let $D(x)$ denote this determinant. We expand $D(x)$ according to a block Laplace expansion using the first four rows of the matrix. The entries of the matrix (save for the term x) are of order

(8.16)
- $O(B)$ *for the entries in the* 4×4 *upper-left corner* ;
- $o(B)$ *for the other entries in the first four rows or columns* ;
- $O(1)$ *for all other entries.*

If $M \times M$ is the size of our matrix, we obtain using (8.16) the approximation

$$D(x) = (x^2 - B^2)^2 \det\left[Ix - \mathcal{H}(\Gamma; t)\right] + o\big((B + |x|)^4 (1 + |x|)^{M-4}\big). \tag{8.17}$$

Suppose first that $x = O(1)$. Then (8.17) shows that

$$B^{-4} D(x) = \det\left[Ix - \mathcal{H}(\Gamma; t)\right] + o(1).$$

It follows that $D(x)$ has, for $a/b \to 0$, $b \to +\infty$, $M - 4$ real roots arbitrarily close to the eigenvalues of $\mathcal{H}(\Gamma; t)$, and in particular of the same sign.

If instead x is of order B we have

$$\det\left[Ix - \mathcal{H}(\Gamma; t)\right] \sim x^{M-4}$$

and setting $x = By$ we get

$$B^{-4} x^{-M+4} D(x) = (y^2 - 1)^2 + o(1).$$

Therefore, $D(x)$ has, for $a/b \to 0$, $b \to +\infty$, two real roots asymptotic to $-B$ and two real roots asymptotic to B. Since we have accounted for all roots of $D(x)$, the set

(9) We write $o(B)$ to indicate suitable entries of that order of magnitude.

$\Gamma \bigcup \{\gamma_0, \overline{\gamma}_0, -\gamma_0, -\overline{\gamma}_0\}$ again satisfies the conclusion of Theorem 8, completing the induction step.

It remains for consideration the case in which $\gamma_0 = ib$ is purely imaginary. This time the determinant equation is

$$D(x) = \det \begin{bmatrix} o(B) - x & B + o(B) & H(\gamma_0, \gamma', t) \\ B + o(B) & o(B) - x & H(\overline{\gamma}_0, \gamma', t) \\ H(\gamma, \gamma_0, t) & H(\gamma, \overline{\gamma}_0, t) & H(\gamma, \gamma', t) - \delta_{\gamma,\gamma'} x \end{bmatrix} = 0.$$

The same argument as before shows that, as $a/b \to 0$, $b \to +\infty$, besides the roots asymptotic to the eigenvalues of $\mathcal{H}(\Gamma; t)$ we acquire one real root asymptotic to B and another real root asymptotic to $-B$, completing the proof of Theorem 8.

9. The second eigenvalue problem: even and odd eigenfunctions

We will need the analogue of Theorem 8 for the second eigenvalue problem. Our problem is

$$\lambda z_\gamma = \sum_{\gamma'} \left(\int_E e^{i(\gamma-\gamma')u} \, du \right) z_{\gamma'}. \tag{9.1}$$

We shall assume that E is a finite union of bounded closed intervals. We say that E is symmetric if

$$E = -E. \tag{9.2}$$

The associated eigenfunction is

$$F(u) = \sum_{\gamma} z_\gamma e^{-i\gamma u} \quad \text{for } u \in E \tag{9.3}$$

and $F(u) = 0$ for $u \notin E$.

If E is symmetric then

$$\int_E e^{i(\gamma-\gamma')u} \, du = \int_E e^{i(\gamma'-\gamma)u} \, du,$$

and since $\Gamma = -\Gamma$ we see that if $\{z_\gamma\}$ is a solution of the linear system (9.1) then $\{z_{-\gamma}\}$ is another solution, for the same eigenvalue. Therefore, by considering $\{\frac{1}{2}(z_\gamma \pm z_{-\gamma})\}$ we see that we have a basis of solutions which are either even or odd, namely such that

$$z_\gamma = \pm z_{-\gamma} \quad \text{for } \gamma \in \Gamma. \tag{9.4}$$

Accordingly, the corresponding eigenfunction is even or odd in the usual sense

$$F(-u) = \pm F(u). \tag{9.5}$$

Hence

Lemma 11. *If E is symmetric there is a basis of solutions of (9.1) consisting of even or odd eigenvectors $\{z_\gamma\}$ satisfying $z_\gamma = \pm z_{-\gamma}$. The associated eigenfunctions satisfy $F(-u) = \pm F(u)$.*

We can separate the space of solutions of (9.1) into its even and odd parts as follows. Let us define

(9.6) $$K_E^{\pm}(x,y) = \int_E e^{i(x-y)u}\,\mathrm{d}u \pm \int_E e^{i(x+y)u}\,\mathrm{d}u$$

and define

(9.7) $$\Gamma_0 = \{\gamma : \gamma \in \Gamma,\ \Re(\gamma) > 0\} \bigcup \{\gamma : \gamma \in \Gamma,\ \Re(\gamma) = 0,\ \Im(\gamma) > 0\}.$$

Then $\Gamma - \{0\} = \Gamma_0 \cup -\Gamma_0$.

If $0 \notin \Gamma$ the linear system (9.1) splits into two new linear systems

(9.8) $$\lambda z_\gamma = \sum_{\gamma' \in \Gamma_0} K_E^{\pm}(\gamma,\gamma') z_{\gamma'}, \qquad \gamma \in \Gamma_0,$$

where if $0 \in \Gamma$ one should add a term $\frac{1}{2} K_E^{+}(\gamma,0) z_0$ to the right-hand side of the equation in the $+$ case.

We have the following analogue and strengthening of Lemma 10 for the second eigenvalue problem.

Lemma 12. *Suppose that* E *is symmetric and suppose for simplicity that* $0 \notin \Gamma$. *Then the matrix* $\mathcal{K}_E(\Gamma)$ *is equivalent to the direct sum of the matrices*

$$\mathcal{K}_E^{\pm}(\Gamma) = \Big[K_E^{\pm}(\gamma,\gamma')\Big]_{\gamma,\gamma' \in \Gamma_0}.$$

A matrix $\mathcal{K}_E^{\pm}(\Gamma)$ *admits the eigenvalue* 0 *if and only if there is* $\gamma \in \Gamma_0$ *with multiplicity greater than* 1, *in which case the multiplicity of* 0 *as an eigenvalue is exactly* $\sum'[m(\gamma) - 1]$, *where* $\sum'$ *ranges over all distinct* $\gamma \in \Gamma_0$.

Moreover if every γ *is real all eigenvalues of* $\mathcal{K}_E^{\pm}(\Gamma)$ *are non-negative.*

Remark. If $0 \in \Gamma$ the result continues to hold, modifying the definition of $\mathcal{K}_E^{+}(\Gamma)$ when γ or γ' equal 0.

Proof. Consider first the even case. Let $\{z_\gamma\}$ be an even solution of (9.1), hence

$$z_{-\gamma} = z_\gamma.$$

For $\gamma \in \Gamma_0$ we have

$$\begin{aligned}
\lambda z_\gamma &= \sum_{\gamma' \in \Gamma} K_E(\gamma,\gamma')\, z_{\gamma'} = \\
&= \sum_{\gamma' \in \Gamma_0} K_E(\gamma,\gamma')\, z_{\gamma'} + \sum_{\gamma' \in -\Gamma_0} K_E(\gamma,\gamma')\, z_{\gamma'} = \\
&= \sum_{\gamma' \in \Gamma_0} \left[K_E(\gamma,\gamma')\, z_{\gamma'} + K_E(\gamma,-\gamma')\, z_{-\gamma'}\right] = \\
&= \sum_{\gamma' \in \Gamma_0} \left[K_E(\gamma,\gamma')\, z_{\gamma'} + K_E(\gamma,-\gamma')\, z_{\gamma'}\right] = \sum_{\gamma' \in \Gamma_0} K_E^{+}(\gamma,\gamma')\, z_{\gamma'}.
\end{aligned}$$

Exactly the same argument works in the odd case. This proves the first part of the lemma.

The proof of the second part is equally easy. For $\gamma \in \Gamma$ we have

$$\lambda \sum_{\gamma \in \Gamma} z_\gamma \, \overline{(z_{\overline{\gamma}})} = \int_E \Big(\sum_\gamma \overline{(z_{\overline{\gamma}})} \, e^{i\gamma u} \Big) F(u) \, \mathrm{d}u =$$

$$= \int_E \overline{\Big(\sum_\gamma z_{\overline{\gamma}} \, e^{-i\overline{\gamma} u} \Big)} F(u) \, \mathrm{d}u =$$

$$= \int_E \overline{\Big(\sum_\gamma z_\gamma \, e^{-i\gamma u} \Big)} F(u) \, \mathrm{d}u = \int_E \left| F(u) \right|^2 \mathrm{d}u \, .$$

If $\{z_\gamma\}$ is either even or odd we have

$$\sum_{\gamma \in \Gamma} z_\gamma \, \overline{(z_{\overline{\gamma}})} = 2 \sum_{\gamma \in \Gamma_0} z_\gamma \, \overline{(z_{\overline{\gamma}})}.$$

Hence

$$2\lambda \sum_{\gamma \in \Gamma_0} z_\gamma \, \overline{(z_{\overline{\gamma}})} = \int_E \left| F(u) \right|^2 \mathrm{d}u$$

and we conclude as in Lemma 10.

Theorem 9. *Let E be a finite union of bounded closed intervals. Then the number of negative eigenvalues of $\mathcal{K}_E(\Gamma)$ equals the number of distinct complex conjugate pairs $\{\gamma\,,\overline{\gamma}\}$ in* Γ.

Suppose also that E is symmetric. Then the number of negative eigenvalues of $\mathcal{K}_E^+(\Gamma)$ equals the number of distinct complex conjugate pairs $(\gamma\,,\overline{\gamma})$ with $\Re(\gamma) > 0$, and the number of negative eigenvalues of $\mathcal{K}_E^-(\Gamma)$ equals the number of distinct complex conjugate pairs $(\gamma\,,\overline{\gamma})$ with $\Re(\gamma) \geq 0$.

Proof. The proof follows the proof of Theorem 8, with some minor modifications. It suffices to prove it in the case in which all γ's are distinct, proceeding by induction on the number of complex conjugate pairs in Γ, or Γ_0 if we are dealing with the symmetric case. We give some details only in the symmetric case.

Suppose Theorem 9 holds for Γ_0 and let us show that it continues to hold for the new set $\Gamma_0 \bigcup \{\gamma_0\,,\overline{\gamma}_0\}$, for some $\gamma_0 = a + ib$ with $a \geq 0$ and $b > 0$.

Suppose first that $a > 0$. As in the proof of Theorem 8, it suffices to study the zeros of the determinant

$$D^\pm(x) = \det \begin{bmatrix} K_E^\pm(\gamma_0\,,\gamma_0) - x & K_E^\pm(\gamma_0\,,\overline{\gamma}_0) & K_E^\pm(\gamma_0\,,\gamma') \\ K_E^\pm(\overline{\gamma}_0\,,\gamma_0) & K_E^\pm(\overline{\gamma}_0\,,\overline{\gamma}_0) - x & K_E^\pm(\overline{\gamma}_0\,,\gamma') \\ K_E^\pm(\gamma\,,\gamma_0) & K_E^\pm(\gamma\,,\overline{\gamma}_0) & K_E^\pm(\gamma\,,\gamma') - \delta_{\gamma,\gamma'}x \end{bmatrix}.$$

We already know that the zeros of $D^\pm(x)$ are all real.

Let t be the largest element in E. We let $b \to +\infty$ and $a/b \to 0$. Then

$$K_E^{\pm}(\gamma_0, \gamma_0) = \int_E \left(1 \pm e^{(2ia-2b)u}\right) \mathrm{d}u = o(e^{2bt}/b)$$

$$K_E^{\pm}(\gamma_0, \overline{\gamma}_0) = \int_E \left(e^{-2bu} \pm e^{2iau}\right) \mathrm{d}u \sim \frac{e^{2bt}}{2b}$$

$$K_E^{\pm}(\overline{\gamma}_0, \gamma_0) = \int_E \left(e^{2bu} \pm e^{2iau}\right) \mathrm{d}u \sim \frac{e^{2bt}}{2b}$$

$$K_E^{\pm}(\overline{\gamma}_0, \overline{\gamma}_0) = \int_E \left(1 \pm e^{(2ia+2b)u}\right) \mathrm{d}u = o(e^{2bt}/b).$$

If we write $B = e^{2bt}/(2b)$ and M is the cardinality of Γ_0 we see that

$$D^{\pm}(x) = (x^2 - B^2)\,\det[Ix - \mathcal{K}_E^{\pm}(\Gamma)] + o\big((B+|x|)^2(1+|x|)^{M-2}\big)$$

and we argue as in the proof of Theorem 8.

If however $\gamma_0 = ib$ is purely imaginary we have

$$K^{\pm}(\gamma_0, \gamma_0) = \int_E \left(1 \pm e^{-2bu}\right) \mathrm{d}u \sim \pm e^{2bt}/(2b) = \pm B.$$

The same argument shows that $D^{+}(x)$ has a real root asymptotic to B while $D^{-}(x)$ has a real root asymptotic to $-B$; all other roots are asymptotic to the roots of $\det[Ix - \mathcal{K}_E^{\pm}(\Gamma)]$. This completes the proof.

10. The passage to the limit

In this section we pass to the limit from the finite sets Γ_N to infinite sets Γ. The simplest situation is when the set of complex pairs of Γ_N is finite.

Thus let Γ be an infinite multiset of complex numbers, satisfying as always the convergenge condition $\sum 1/(1+|\gamma|)^{1+\varepsilon} < +\infty$ for every $\varepsilon > 0$ and the symmetry conditions $\Gamma = \overline{\Gamma}$, $\Gamma = -\Gamma$. By Γ_N we denote the truncations of Γ at $|\gamma| \le N$.

Suppose that Γ has finitely many, but at least one, complex pairs of elements and let N be so large that Γ_N contains all complex pairs of Γ. Then $\mathcal{H}(\Gamma_N; t)$ has at least one negative eigenvalue $\lambda_N < 0$, which we may choose to be the largest in absolute value.

Let $\{w_{\gamma,N}\}$ be a solution of the corresponding eigenvalue equation

$$\lambda_N w_{\gamma,N} = \sum_{\gamma' \in \Gamma_N} H(\gamma, \gamma', t)\, w_{\gamma',N}$$

normalized so that

$$\sum_{\gamma \in \Gamma} |w_{\gamma,N}|^2 = 1. \tag{10.1}$$

Then equation (8.14) shows that

$$\lambda_N \sum_{\gamma\in\Gamma} w_{\gamma,N}\,\overline{(w_{\overline{\gamma},N})} = \frac{1}{4}\int_{-t}^{t} |F_N(u)|^2\,\mathrm{d}u + \int_{-t}^{t} |F_N'(u)|^2\,\mathrm{d}u \tag{10.2}$$

where

$$F_N(u) = \sum_{\gamma\in\Gamma_N} z_{\gamma,N}e^{-i\gamma u} - Ae^{u/2} - Be^{-u/2}, \tag{10.3}$$

$z_{\gamma,N} = w_{\gamma,N}/(\frac{1}{4}+\gamma^2)$ and $F_N(t) = F_N(-t) = 0$.

Now we note that, since $\lambda_N < 0$, we have

$$\lambda_N \sum_{\gamma\in\mathbb{R}} w_{\gamma,N}\,\overline{(w_{\overline{\gamma},N})} = \lambda_N \sum_{\gamma\in\mathbb{R}} |w_{\gamma,N}|^2 = -|\lambda_N| \sum_{\gamma\in\mathbb{R}} |w_{\gamma,N}|^2.$$

Therefore, after division by $|\lambda_N|$, (10.2) becomes

$$-\sum_{\gamma\notin\mathbb{R}} w_{\gamma,N}\,\overline{(w_{\overline{\gamma},N})} = \sum_{\gamma\in\mathbb{R}} |w_{\gamma,N}|^2 + \frac{1}{|\lambda_N|}\left\{\frac{1}{4}\int_{-t}^{t} \left|F_N(u)\right|^2\mathrm{d}u + \int_{-t}^{t} \left|F_N'(u)\right|^2\mathrm{d}u\right\}$$

and in particular

$$-\sum_{\gamma\notin\mathbb{R}} w_{\gamma,N}\,\overline{(w_{\overline{\gamma},N})} \geq \sum_{\gamma\in\mathbb{R}} |w_{\gamma,N}|^2.$$

Hence by Cauchy's inequality we infer

$$\sum_{\gamma\notin\mathbb{R}} |w_{\gamma,N}|^2 \geq \sum_{\gamma\in\mathbb{R}} |w_{\gamma,N}|^2$$

and conclude with the lower bound

$$2\sum_{\gamma\notin\mathbb{R}} |w_{\gamma,N}|^2 \geq \sum_{\gamma\in\Gamma} |w_{\gamma,N}|^2.$$

This inequality, in conjunction with (10.1), shows that

Lemma 13. *If $\lambda_N < 0$ we have*

$$\frac{1}{2} \leq \sum_{\gamma\notin\mathbb{R}} |w_{\gamma,N}|^2 \leq 1$$

and $|w_{\gamma,N}| \leq 1$ for every $\gamma \in \Gamma_N$.

Now we recall that $z_{\gamma,N} = w_{\gamma,N}/(\frac{1}{4}+\gamma^2)$ and that $\sum 1/(1+|\gamma|)^2 <+\infty$. It follows that the sum

$$Z_N(u) = \sum_{\gamma\in\Gamma_N} z_{\gamma,N}e^{-i\gamma u}$$

is uniformly bounded and majorized by an absolutely convergent series. In fact, more than this is true, because $\left|(\frac{1}{4}+\gamma^2)\,z_{\gamma,N}\right|$ is uniformly bounded by 1; therefore, there is a subsequence of $N \to \infty$ such that each $z_{\gamma,N}$ converges to a limit z_γ, again satisfying $\left|(\frac{1}{4}+\gamma^2)\,z_\gamma\right| \le 1$. It follows that $Z_N(u)$ converges pointwise and uniformly to the limit

$$Z(u) = \sum_{\gamma\in\Gamma} z_\gamma\, e^{-i\gamma u}$$

in compact subsets of $\mathbb{R}$. In particular, $Z(t)$ and $Z(-t)$ are bounded and the functions $F_N(u)$ converge pointwise and uniformly to a limit

$$F(u) = \sum_{\gamma\in\Gamma} z_\gamma\, e^{-i\gamma u} - Ae^{u/2} - Be^{-u/2}$$

with A and B determined by the condition $F(t) = F(-t) = 0$.

Now note that the eigenvalue λ_N is uniformly bounded from below.

This is seen as follows. By Lemma 13, if there are J complex $\gamma \in \Gamma$ then for one of them we have $|w_{\gamma,N}| \ge 1/\sqrt{2J}$. Thus the equation

$$\lambda_N w_{\gamma,N} = \sum_{\gamma'\in\Gamma_N} H(\gamma,\gamma',t)\, w_{\gamma',N}$$

shows that

$$|\lambda_N| \le \sqrt{2J}\, \max\left(\sum_{\gamma'\in\Gamma} \left|H(\gamma,\gamma',t)\right|^2\right)^{1/2}$$

where the maximum runs over the finitely many complex $\gamma\in\Gamma$, proving our claim.

The basic equation (10.2) also shows that

$$\text{(10.4)} \qquad \frac{1}{4}\int_{-t}^{t} \left|F_N(u)\right|^2 \mathrm{d}u + \int_{-t}^{t} \left|F_N'(u)\right|^2 \mathrm{d}u \le |\lambda_N| \sum_{\gamma\in\Gamma_N} |w_{\gamma,N}|^2 = |\lambda_N|.$$

By semicontinuity, from (10.4) we deduce

$$\text{(10.5)} \qquad \frac{1}{4}\int_{-t}^{t} |F(u)|^2 \mathrm{d}u + \int_{-t}^{t} |F'(u)|^2 \mathrm{d}u \le \liminf_{N\to\infty} |\lambda_N| < +\infty.$$

Suppose that $F(u)$ is not identically 0 in $(-t, t)$. Then $\lambda = \liminf_{N\to\infty} \lambda_N < 0$ and, keeping in mind the rapid decay of $H(\gamma,\gamma',t)$ with γ', we see that

$$\text{(10.6)} \qquad \lambda\, w_\gamma = \sum_{\gamma'\in\Gamma} H(\gamma,\gamma',t)\, w_{\gamma'}$$

with $w_\gamma = (\frac{1}{4}+\gamma^2)\,z_\gamma$.

We have verified already that a solution of (10.6) yields a solution of the eigenvalue problem

$$\lambda \mathcal{D} f(x) = \mathcal{L}[f](x)$$

for $x \in (M^{-1}, M)$, $M = e^t$, with $f \in \mathcal{W}_0$. Indeed, $F(u) = e^{u/2} f(e^u)$. Moreover, for any $\alpha < 1$ we have $\mathcal{D}f \in C^{1,\alpha}$ in the interval (M^{-1}, M) and $\mathcal{D}f$ is not identically 0 because $F(u)$ is not identically 0 by hypothesis. It follows that

$$\text{(10.7)} \qquad \mathcal{T}[f * \overline{f}^{\,*}] = \lambda\, \|\mathcal{D}f\|^2 < 0.$$

On the other hand, since this holds for any $t > 0$, we can choose $t < \frac{1}{2}\log 2$, hence $M < \sqrt{2}$. It follows that $f * \overline{f}^*$ has compact support in $(1/2\,,2)$ and in particular in Weil's Explicit Formula the terms involving $\Lambda(n)$ are absent.

It is possible [10] to prove directly the positivity of Weil's Explicit Formula in this restricted case where the support of f is $[M_0^{-1}\,, M_0]$, with $M_0 > 1$ an explicitly computable constant. This contradicts (10.7). Therefore, we conclude with

THEOREM 10. *Suppose $\zeta(s)$ has only finitely many non-trivial zeros $\frac{1}{2} + i\gamma$ with $\gamma \notin \mathbb{R}$, and at least one such zero.*

Then for $t > 0$ there are complex coefficients w_γ and $\lambda \le 0$ such that

$$\sum_{\gamma\in\Gamma} |w_\gamma|^2 = 1\,,$$

$$\sum_{\gamma\notin\mathbb{R}} |w_\gamma|^2 \ge -\sum_{\gamma\notin\mathbb{R}} w_\gamma\,\overline{(w_{\overline{\gamma}})} \ge \frac{1}{2}\,,$$

$$\lambda\, w_\gamma = \sum_{\gamma'\in\Gamma} H(\gamma\,,\gamma'\,,t)\, w_{\gamma'} \quad \textit{for every } \gamma \in \Gamma\,,$$

and with the following property. Let

$$Z(u) = \sum_{\gamma} \frac{w_\gamma}{\frac{1}{4} + \gamma^2}\, e^{-i\gamma u}$$

and

$$F(u) = Z(u) - \frac{e^{t/2} Z(t) - e^{-t/2} Z(-t)}{e^t - e^{-t}}\, e^{u/2} - \frac{e^{t/2} Z(-t) - e^{-t/2} Z(t)}{e^t - e^{-t}}\, e^{-u/2}.$$

Now define

$$f(x) = \phi(x)\, x^{-1/2} F(\log x)$$

where $\phi(x)$ is the characteristic function of $[e^{-t}\,, e^t]$. Then either

$$\mathcal{T}[f * \overline{f}^*] = \lambda\, \|Df\|^2 < 0$$

or $f(x)$ is identically 0. In addition, if $0 < t < t_0$ where t_0 is a suitable explicitly computable constant, the function $f(x)$ must be identically 0.

11. THE SECOND EIGENVALUE PROBLEM: LINEAR INDEPENDENCE

The considerations of the preceding two sections can also be done in the setting in which we minimize the quadratic functional $\mathcal{T}[f * \overline{f}^*]/\|f\|^2$ in $L^2(\mathcal{E})$ rather than with respect to the norm $\|Df\| = 1$.

Let $E \subset \mathbb{R}$ be a finite union of disjoint intervals and consider the eigenvalue problem

$$\lambda F(u) = L[F](u) \quad \text{for } u \in E. \tag{11.1}$$

(10) We defer the proof to §12, Theorem 12.

As in Section 8, we get the equation

$$F(u) = \sum_{\gamma\in\Gamma} z_\gamma e^{-i\gamma u}\Phi_E(u)$$

where Φ_E is the characteristic function of E, and the eigenvalue equation

$$\lambda z_\gamma = \sum_{\gamma'\in\Gamma} z_{\gamma'} \int_E e^{i\gamma u}\, e^{-i\gamma' u}\, \mathrm{d}u. \tag{11.2}$$

Again, this yields

$$\lambda \sum_{\gamma\in\Gamma} z_\gamma\, \overline{(z_{\overline{\gamma}})} = \int_E \left|F(u)\right|^2 \mathrm{d}u. \tag{11.3}$$

Thus $\lambda \neq 0$ if Γ is a finite set. Suppose $\lambda < 0$ and normalize $\{z_\gamma\}$ to have ℓ^2-norm equal to 1. Then, exactly as in the preceding section, we have

$$\sum_{\gamma\notin\mathbb{R}} |z_\gamma|^2 \geq \frac{1}{2}. \tag{11.4}$$

We apply this to the finite set Γ_N and the negative eigenvalue λ_N and let $N \to \infty$, keeping $\lambda_N < 0$ largest in absolute value. Let $\{z_{\gamma,N}\}$ be the corresponding eigenvector as in (11.2). We may assume by going to a subsequence that

$$z_\gamma = \lim_{N\to\infty} z_{\gamma,N}$$

exists for every γ, and now (11.4) continues to hold. The limit $F(u) = \sum z_\gamma e^{-i\gamma u}$ exists as an L^2-function and is a weak limit of the sequence

$$F_N(u) = \sum_{\gamma\in\Gamma_N} z_{\gamma,N} e^{-i\gamma u}.$$

Note that (11.3) implies $\|F_N\| \leq \sqrt{-\lambda_N}$, hence

$$\|F\|^2 \leq -\lambda\,, \tag{11.5}$$

with $\lambda = \lim \lambda_N$. Hence if $\lambda = 0$ we must have $F(u) = 0$ identically on E.

Next, we note that (11.2) continues to remain true in the limit. We have, for $K \geq 1$ and $N > (K+1)|\gamma|$, the bound

$$\begin{aligned}\left|\sum_{(K+1)|\gamma|<|\gamma'|\leq N} z_{\gamma',N} \int_E e^{i(\gamma-\gamma')u}\, \mathrm{d}u\right|^2 &\leq \left(\sum_{K|\gamma|<|\gamma'-\gamma|} |z_{\gamma',N}| \cdot \left|\int_E e^{i(\gamma-\gamma')u}\, \mathrm{d}u\right|\right)^2 \leq \\ &\leq \|\{z_{\gamma,N}\}\|_{\ell^2}^2 \cdot \sum_{|\gamma'-\gamma|>K|\gamma|} \left|\int_E e^{i(\gamma-\gamma')u}\, \mathrm{d}u\right|^2 \ll \\ &\ll \sum_{|\gamma'-\gamma|>K|\gamma|} \frac{1}{(1+|\gamma-\gamma'|)^2} \ll \\ &\ll \sum_{n>K|\gamma|} \frac{\log(n+1+|\gamma|)}{(n+1)^2} \ll \frac{\log(K|\gamma|+2)}{K|\gamma|+2}.\end{aligned}$$

Hence if $N > (K+1)|\gamma|$ we have

$$\begin{aligned} 0 &= \lambda_N z_{\gamma,N} - \sum_{|\gamma'|\le N} z_{\gamma',N} \int_E e^{i(\gamma-\gamma')u}\,\mathrm{d}u = \\ &= \lambda_N z_{\gamma,N} - \sum_{|\gamma'|\le (K+1)|\gamma|} z_{\gamma',N} \int_E e^{i(\gamma-\gamma')u}\,\mathrm{d}u + \\ &\qquad + O\Bigg(\Bigg|\sum_{(K+1)|\gamma|<|\gamma'|\le N} z_{\gamma',N} \int_E e^{i(\gamma-\gamma')u}\,\mathrm{d}u\Bigg|\Bigg) = \\ &= \lambda_N z_{\gamma,N} - \sum_{|\gamma'|\le (K+1)|\gamma|} z_{\gamma',N} \int_E e^{i(\gamma-\gamma')u}\,\mathrm{d}u + O\left(\left(\frac{\log(K|\gamma|+2)}{K|\gamma|+2}\right)^{1/2}\right). \end{aligned}$$

This estimate is uniform in N and the sum over $|\gamma'| \le (K+1)|\gamma|$ is a finite sum indexed independently of N. Thus for fixed γ and K we can pass to the limit as $N \to \infty$, obtaining

$$0 = \lambda\, z_\gamma - \sum_{|\gamma'|\le (K+1)|\gamma|} z_{\gamma'} \int_E e^{i(\gamma-\gamma')u}\,\mathrm{d}u + O\left(\left(\frac{\log(K|\gamma|+2)}{K|\gamma|+2}\right)^{1/2}\right).$$

Now we let $K \to \infty$ and conclude that

$$\lambda\, z_\gamma = \sum_{\gamma'} z_{\gamma'} \int_E e^{i\gamma u}\, e^{-i\gamma' u}\,\mathrm{d}u$$

for every γ, as asserted.

This equation can be rewritten as

$$\int_E e^{i\gamma u} F(u)\,\mathrm{d}u = \lambda z_\gamma\,,$$

hence noting that $|\Im(\gamma)| \le \frac{1}{2}$ we infer

$$\int_E e^{|u|/2}\,\big|F(u)\big|\,\mathrm{d}u \ge |\lambda|\,\max|z_\gamma|\,.$$

Therefore, by (11.4) we find

$$\int_E \big|F(u)\big|\,\mathrm{d}u \ge (2J)^{-1} e^{-m(E)/2}|\lambda| \tag{11.6}$$

where J is the number of complex zeros of $\zeta(s)$ off the critical line and $m(E) = \max_E |x|$. In particular, in view of (11.5) we have shown

Lemma 14. *We have* $F(u) = 0$ *identically in* E *if and only if* $\lambda = \liminf \lambda_N = 0$.

This gives the analogue of Theorem 8 in this setting.

Theorem 11. *Suppose that* $\zeta(s)$ *has only finitely many non-trivial zeros* $\frac{1}{2} + i\gamma$ *with* $\gamma \notin \mathbb{R}$*, and at least one such zero. Then for any finite union of intervals* E *there are complex* *coefficients* $z_\gamma = z_\gamma(E)$ *and* $\lambda = \lambda(E) \le 0$ *such that*

$$\sum_{\gamma\in\Gamma} |z_\gamma|^2 = 1\,,$$

$$\sum_{\gamma\notin\mathbb{R}} |z_\gamma|^2 \ge -\sum_{\gamma\notin\mathbb{R}} z_\gamma\,\overline{(z_{\overline{\gamma}})} \ge \frac{1}{2}$$

and with the following property. Let

$$F(u) = \sum_\gamma z_\gamma\, e^{-i\gamma u}.$$

Then $F(u)$ *is locally in* L^2 *and*

$$\lambda\, z_\gamma = \int_E e^{i\gamma u}\, F(u)\,\mathrm{d}u \quad \textit{for every } \gamma \in \Gamma.$$

Moreover, if

$$f(x) = \phi(x)\, x^{-1/2} F(\log x)$$

where $\phi(x)$ *is the characteristic function of* $\mathcal{E} = e^E$, *we have either*

$$\mathcal{T}[f * \overline{f}^*] = \lambda\, \|f\|^2 < 0$$

or $\lambda = 0$ *and* $f(x)$ *is identically* 0.

As before, we have

Corollary. *Let* $\mathcal{E} \subset (0\,,\infty)$ *be a finite union of bounded closed intervals and suppose that* $\mathcal{T}[f * \overline{f}^*] \ge 0$ *for every smooth function* f *with compact support in* $\mathcal{E}$.

Then either the Riemann Hypothesis is true, or $\zeta(s)$ *has infinitely many zeros off the critical line, or the functions* $x^{-\rho}$ *are linearly dependent over* $\mathcal{E}$, *for a suitable sequence of coefficients* $\{c_\rho\} \in \ell^2$ *such that at least half of its* ℓ^2*-mass is supported on the set of zeros* ρ *with* $\Re(\rho) \ne \frac{1}{2}$.

One may ask if linear dependence relations occur at all. The following example shows that they may occur for Dedekind zeta functions.

An Example. Consider two Dirichlet L-functions $L(s\,,\chi)$ and $L(s\,,\chi')$ for a same modulus $q > 1$ and two distinct primitive characters with the same parity, hence $\chi(-1) = \chi'(-1)$. Let also $p_0 \ge 2$ be the first prime for which $\chi(p_0) \ne \chi'(p_0)$.

These two Dirichlet L-functions have the same conductor and Gamma factors in their functional equations. Consequently, the Explicit Formula associated with these two functions will have the same contribution from the «prime at infinity». Moreover, if we evaluate the Explicit Formula for a function $f(x)$ with compact support in $(1/p_0\,, p_0)$, the contribution arising from the primes $p < p_0$ is the same in both cases, and it is 0 for the primes $p \ge p_0$ because of the condition imposed on the support of $f(x)$. By taking the difference, we obtain the relation

$$\sum_\rho \widetilde{f}(\rho) - \sum_{\rho'} \widetilde{f}(\rho') = 0$$

where ρ runs over the non-trivial zeros of $L(s, \chi)$ and ρ' runs over those of $L(s, \chi')$.

Now fix $f_0(x)$ with compact support in $(e^{-\varepsilon}, e^{\varepsilon})$ and let $f(x) = f_0(ax)$ with $e^{\varepsilon}/p_0 < < a < e^{-\varepsilon} p_0$. Then $\widetilde{f}(s) = a^{-s} \widetilde{f}_0(s)$ and we get the relation

$$\sum_{\rho} \widetilde{f}_0(\rho)\, a^{-\rho} - \sum_{\rho'} \widetilde{f}_0(\rho')\, a^{-\rho'} = 0$$

for $a \in (e^{\varepsilon}/p_0 , e^{-\varepsilon} p_0)$.

As pointed out by J. Bourgain, the existence of linear relations over intervals of arbitrary length also follows from the fact that the gap between consecutive γ's tends to 0 as $\gamma \to \infty$. In any case, it should be noted that the coefficients of the relations in question are obtained as limits of eigenvectors of reasonably well-behaved matrices. Computer experiments (11) suggest that if a negative eigenvalue λ_N for the problem above approaches 0 as $N \to \infty$, then it does so at an exponential or nearly exponential rate and also the corresponding eigenvector seems to converge rapidly to a limit. Thus a deeper study of the eigenvectors associated to finite approximations may shed some light on this question of linear independence.

We suggest that splitting the eigenvalue problem into its even and odd components and complexifying it with the help of Cramér's function

$$V(z) = \sum_{\Re(\gamma)>0} e^{i\gamma z}$$

may bring the problem to a form amenable to the extraordinarily powerful Riemann-Hilbert techniques recently introduced in other contexts. An examination of these possibilities may prove to be a valuable undertaking in studying the structure of the limiting linear relations.

12. Sets of positivity

In this section we prove the positivity statement needed for the proof of the Corollary to Theorem 8. Another proof can be found in Yoshida's paper [5]. Let $T[F]$ be the linear form

$$T[F] = \sum_{\gamma \in \Gamma} \widehat{F}(\gamma) \tag{12.1}$$

where $\widehat{F}$ denote the Fourier transform.

Definition. *We say that a closed subset* $E \subset \mathbb{R}$ *is a set of positivity for* T *if*

$$T[F(x) * \overline{F(-x)}] > 0 \quad \textit{for every smooth function } F \textit{ with compact support in } E$$

where

$$(F * G)(x) = \int_{-\infty}^{\infty} F(u)\, G(x-u)\, \mathrm{d}u$$

(11) The experiments were done by adding a «fake zero» ρ_0 off the critical line, together with its symmetric images, to the first k non-trivial zeros of $\zeta(s)$, with k up to 320.

is the additive convolution, hence

$$F(x) * \overline{F(-x)} = \int_{-\infty}^{\infty} F(x+u)\overline{F(u)}\,\mathrm{d}u. \tag{12.2}$$

In this setting, the Explicit Formula becomes

$$T[F] = \int_{-\infty}^{\infty} 2\cosh(x/2)\,F(x)\,\mathrm{d}x \;-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\big[F(\log n)+F(-\log n)\big]-$$
$$-\,(\log 4\pi+\gamma)\,F(0) - \int_0^{\infty}\Big[e^{x/2}\big(F(x)+F(-x)\big)-2F(0)\Big]\frac{\mathrm{d}x}{e^{x}-e^{-x}}\,.$$

Moreover, the last two terms in this formula can be written as

$$-(\log\pi)F(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\Big[\frac{\Gamma'}{\Gamma}\big(\frac14+i\frac{v}{2}\big)\Big]\,\widehat{F}(v)\,\mathrm{d}v\,,$$

as one verifies using Theorem 2 and a change of variables.

Theorem 12. *If* $F(x)$ *has compact support in an interval* I *of length* $|I|<\log 2$ *we have*

$$T[F(x)*\overline{F(-x)}] = \sum_{\gamma}\widehat{F}(\gamma)\,\overline{\widehat{F}(\overline{\gamma})} \ge \Big(\log\frac{1}{|I|} - \log^{+}\log\frac{1}{|I|} - O(1)\Big)\|F\|^2.$$

Proof. Since $G(x)=F(x)*\overline{F(-x)}$ remains unchanged if we replace $F(x)$ by a translation $F(x+c)$, we may assume that $F(x)$ is supported in the interval $[-a/2\,,\,a/2]$ with $a<\log 2$. Then $G(x)$ is supported in $[-a\,,\,a]\subset(-\log 2\,,\log 2)$. Thus in the Explicit Formula as above for $T[G]$, the contribution of the sum involving $\Lambda(n)$ vanishes. It follows that

$$T[G] = \int_{-\infty}^{\infty} 2\cosh(x/2)\,G(x)\,\mathrm{d}x -$$
$$-\,(\log\pi)\,G(0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\Big[\frac{\Gamma'}{\Gamma}\big(\frac14+i\frac{v}{2}\big)\Big]\,\widehat{G}(v)\,\mathrm{d}v. \tag{12.3}$$

We have

$$G(0)=\|F\|^2\,,\qquad \widehat{G}(v)=|\widehat{F}(u)|^2 \tag{12.4}$$

and by Plancherel's Formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{F}(v)|^2\,\mathrm{d}v = \|F\|^2. \tag{12.5}$$

Now

$$\Big|\int_{-a}^{a}2\cosh(x/2)\,G(x)\,\mathrm{d}x\Big| \le$$
$$\le 2\cosh(a/2)\int_{-a}^{a}\Big|\int_{-\infty}^{\infty}F(x+u)\overline{F(u)}\,\mathrm{d}u\Big|\,\mathrm{d}x \le 4a\cosh(a/2)\,\|F\|^2. \tag{12.6}$$

Since $a<\log 2$ and

$$\Re\Big[\frac{\Gamma'}{\Gamma}\big(\frac14+i\frac{v}{2}\big)\Big] = \log^{+}|v| + O(1)\,,$$

we infer from (12.3) to (12.6) that

$$T[G] \geq -O(1)\,\|F\|^2 + \frac{1}{2\pi}\int_{-\infty}^{\infty} (\log^+|v|)\,|\widehat{F}(v)|^2\,\mathrm{d}v. \tag{12.7}$$

For any $K > 1$ we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} (\log^+|v|)\,|\widehat{F}(v)|^2\,\mathrm{d}v &\geq \frac{\log K}{2\pi}\int_{|v|>K} |\widehat{F}(v)|^2\,\mathrm{d}v = \\ &= (\log K)\,\|F\|^2 - \frac{\log K}{2\pi}\int_{|v|\leq K} |\widehat{F}(v)|^2\,\mathrm{d}v \geq \\ &\geq (\log K)\,\|F\|^2 - (4K^2\log K)\,\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\sin(v/K)}{v}\,\widehat{F}(v)\right|^2\mathrm{d}v, \end{aligned} \tag{12.8}$$

because $|\sin(v/K)| \geq |v|/(2K)$ for $|v| \leq K$.

The function $(\sin(v/K)/v)\widehat{F}(v)$ is the Fourier Transform of the convolution

$$\int_{-1/K}^{1/K} F(x-y)\,\mathrm{d}y,$$

which is supported in the interval $I = [-\frac{a}{2} - \frac{1}{K}, \frac{a}{2} + \frac{1}{K}]$.

Therefore, by Plancherel's Formula and Cauchy's inequality we get

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\sin(v/K)}{v}\,\widehat{F}(v)\right|^2\mathrm{d}v &= \int_{-\infty}^{\infty}\left|\int_{-1/K}^{1/K} F(x-y)\,\mathrm{d}y\right|^2\mathrm{d}x \leq \\ &\leq \int_I\left(\int_{-a/2}^{a/2} |F(y)|\,\mathrm{d}y\right)^2\mathrm{d}x \leq (a + 2/K)a\,\|F\|^2. \end{aligned} \tag{12.9}$$

By (12.7), (12.8), (12.9) we infer

$$T[G] \geq \left[\log K - O(1) - 4(a+2/K)a\,K^2\log K\right]\cdot\|F\|^2.$$

Theorem 12 follows by choosing

$$K = \frac{1}{a}\left(1 + \log\frac{1}{a}\right)^{-1}.$$

13. Some numerical experiments

It is easy to follow numerically the behaviour of eigenvalues and eigenfunctions of finite approximations, as the number of zeros increases.

In this section we consider finite approximations to Problem 2, where the set of points consists of the first N zeros of $\zeta(s)$ with positive imaginary parts and a fictitious zero ρ_0 off the critical line, together with their images by complex conjugation and reflection about the point $1/2$.

The set E is the interval $[-t, t]$ with various ranges of t and the number N of zeros ranges up to $N = 160$. The fictitious zero off the line has been arbitrarily set at $\rho_0 = 0.52 + i3.14$.

In the symmetric case considered here there is a natural division of eigenvalues and eigenfunctions into even and odd eigenfunctions. The numerical evidence gathered here indicates in each case the existence of a critical value $t_c^{\pm} > 0$ such that the unique negative eigenvalue $\lambda_N^{\pm}(t)$ tends to 0 if $t < t_c^{\pm}$, as $N \to \infty$. The rate of convergenge is quite fast, suggesting an exponential rate. For $t > t_c^{\pm}$ this eigenvalue converges, albeit less rapidly so, to a strictly negative value. The behaviour of the corresponding eigenfunctions is markedly different for $t < t_c^{\pm}$ and $t > t_c^{\pm}$, although even and odd eigenfunctions behave rather similarly.

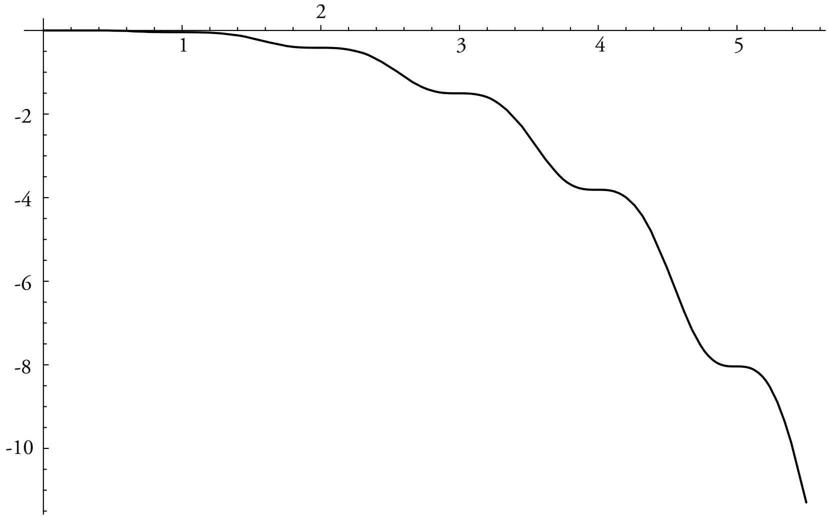

Fig. 1. – The negative even eigenvalue, ρ_0=.52+i3.14, N=10, 0<t<5.5.

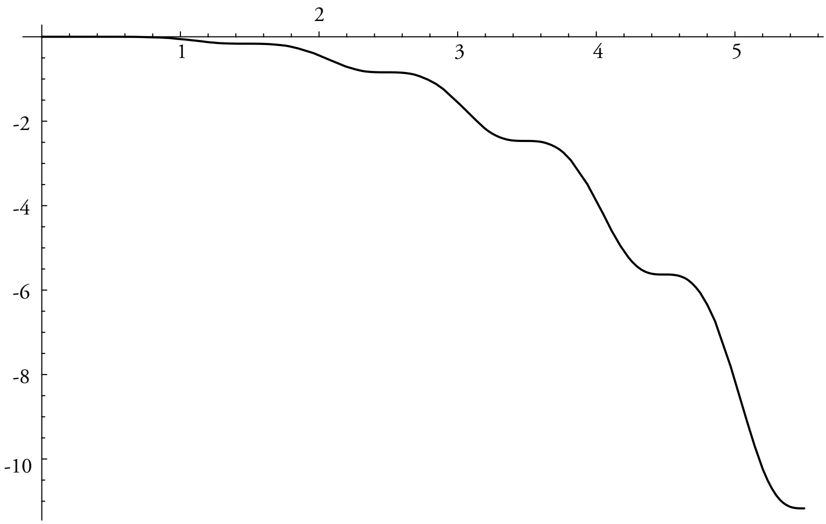

Fig. 2. – The negative odd eigenvalue, ρ_0=.52+i3.14, N=10, 0<t<0.55.

The two plots together:

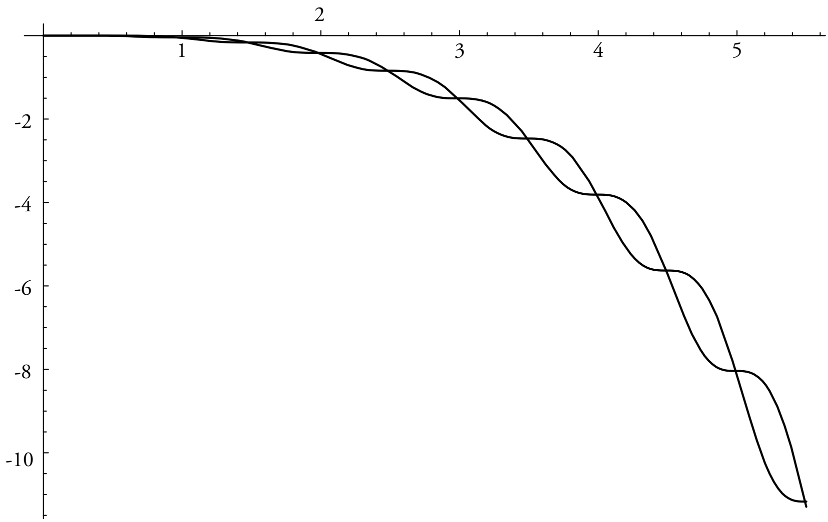

Fig. 3. – The two negative eigenvalues, ρ_0=.52+i3.14, N=10, 0<t<0.55.

Normalized eigenfunctions, $\|f\| = 1$, for $t < t_c^+$.

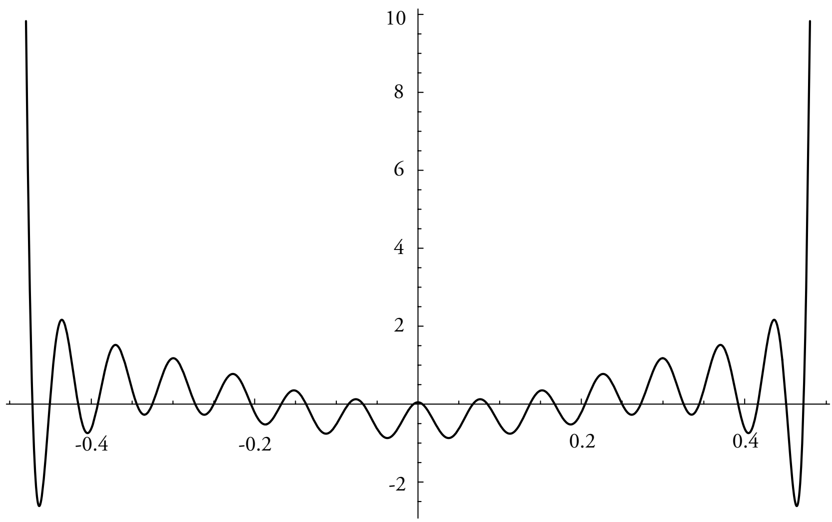

Fig. 4. – The eigenfunction for the negative even eigenvalue, ρ_0=.52+i3.14, N=20, t=0.48.

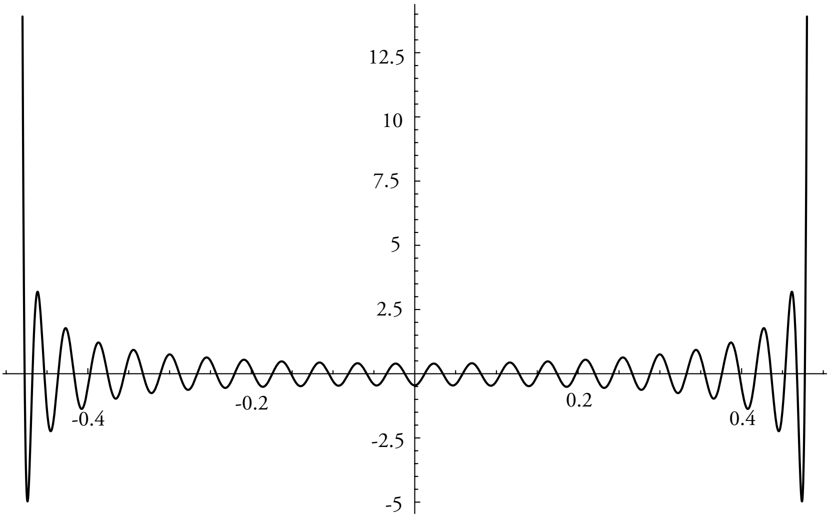

Fig. 5. – The eigenfunction for the negative even eigenvalue, $\rho_0 = .52 + i3.14$, $N = 40$, $t = 0.48$.

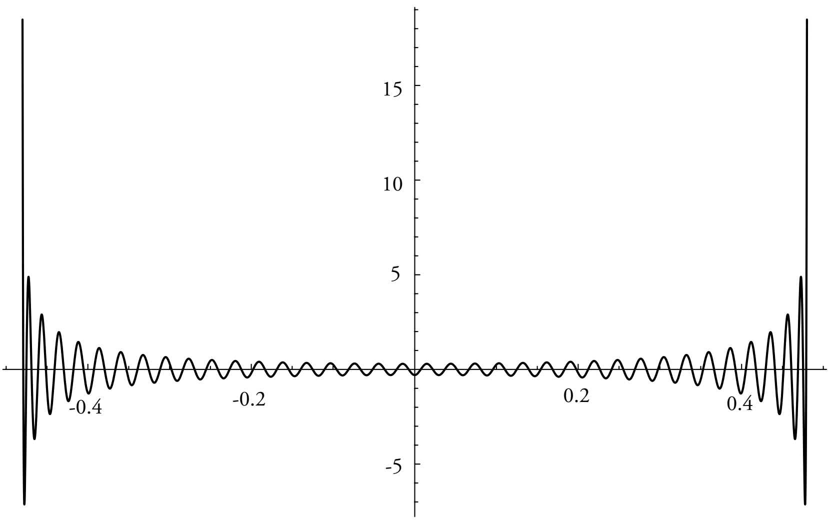

Fig. 6. – The eigenfunction for the negative even eigenvalue, $\rho_0 = .52 + i3.14$, $N = 80$, $t = 0.48$.

Normalized eigenfunctions, $\|f\| = 1$, for $t > t_c^+$.

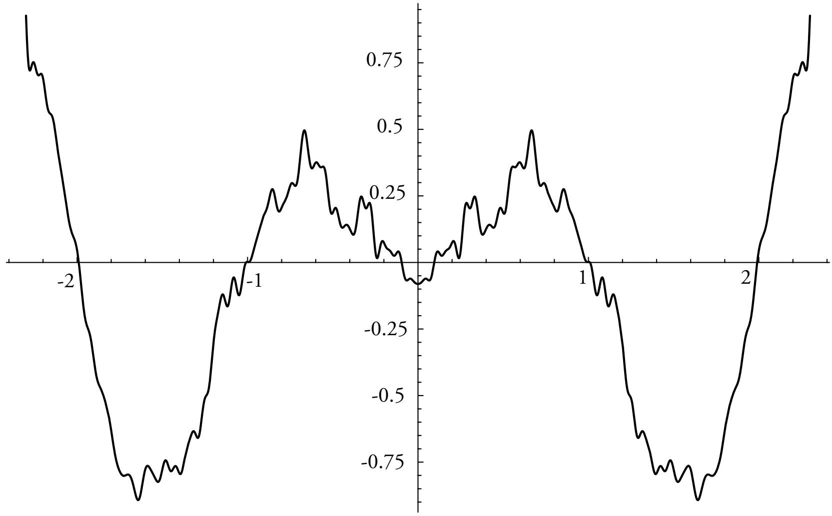

Fig. 7. – The eigenfunction for the negative even eigenvalue, ρ_0=.52+i3.14, N=40, t=2.3.

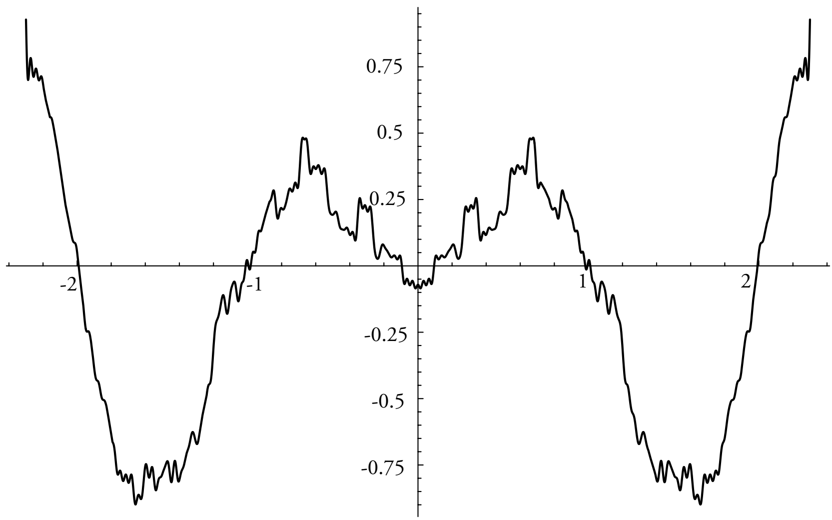

Fig. 8. – The eigenfunction for the negative even eigenvalue, ρ_0=.52+i3.14, N=80, t=2.3.

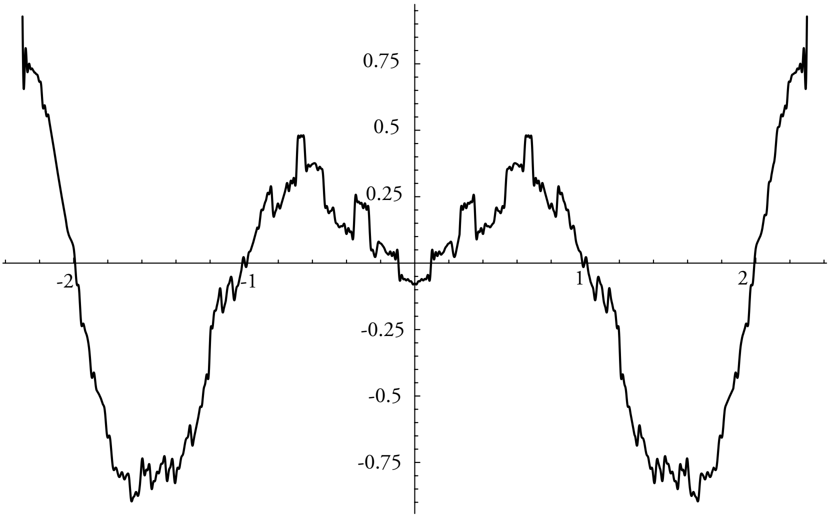

Fig. 9. – The eigenfunction for the negative even eigenvalue, $\rho_0 = .52 + i3.14$, $N = 160$, $t = 2.3$.

Further computer experiments for $t < t_c^+$ and large N, both for the negative eigenvalue and a sequence of positive eigenvalues tending to 0 as N increases, point out to a marked difference in behaviour for eigenvectors belonging to the negative eigenvalue compared with eigenvectors belonging to positive eigenvalues.

In the case $t < t_c^{\pm}$, normalized eigenfunctions $f(x)$ with $\|f\| = 1$ converge weakly to 0. The numerical experiments indicate that the L^2-mass of the function gets more and more concentrated at the boundary of the interval $(-t, t)$, as $N \to \infty$, and one should study the asymptotic behaviour of eigenfunctions at the boundary, after the appropriate rescaling. This should be of particular interest at the critical value $t = t_c^{\pm}$, which represent the transition from weak convergence to 0 to strong convergence in L^2.

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