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A characterization of conformal mappings in $\mathbb{R}^4$ by a formal differentiability condition

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Abstract

We show that conformal mappings in $\mathbb{R}^4$ can be characterized by a formal differentiability condition. The notion of differentiability described in this paper generalizes the classical concept of differentiability in the sense of putting the differential of a function into relation with variable differential forms of first order. This approach provides further an application of the use of those arbitrary orthonormal sets which are used in works of V. Kravchenko, M. Shapiro and N. Vasilevski on quaternionic analysis. However, it is crucial to consider variable orthonormal sets, so-called moving frames.

1 Introduction

In classical complex function theory the geometric property of preserving angles called conformality in the sense of Gauss is closely linked with differentiability and analyticity. Every conformal mapping in the sense of Gauss is either holomorphic or antiholomorphic.

Several approaches to generalize complex analyticity to hypercomplex analysis have been made in the past. G. Scheffers [24] (1893), A. S. Melijhzon [20] (1948) and A. Sudbery [26] (1979) provided important contributions on the discussion about the possibility of extending the concept of complex analyticity to quaternions by the approach considering differential quotients.
However, because of the non-commutativity of the quaternions, only the linear affine functions turn out to be quaternionic differentiable or so-called M-differentiable by generalizing differentiability in the strict sense of a differential quotient. Thus, M-differentiability which

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means a linear relation between the differential of the function and a fixed differential form of first order is too restrictive to develop a powerful function theory in quaternions.
A different and actually more efficient approach to generalize complex analyticity to quaternionic and Clifford analysis is the Cauchy-Riemann approach considering functions in the kernel of the generalized Cauchy-Riemann operator which are often called regular, monogenic or hypercomplex-analytic functions.

A. C. Dixon ([9]), R. Fueter ([10]), G. Moisil, N. Theodorescu ([21]), V. Iftimie ([14]) and R. Delanghe ([8]) are some of the most important creators of a function theory in quaternions and Clifford algebras built on this approach. In particular, in the period of 1932 - 1950 R. Fueter and some of his students managed to generalize many results of complex analysis to quaternionic analysis and also to Clifford analysis. A summary of the research of R. Fueter et al. on hypercomplex function theory can be found in [11] while a summary of the modern Clifford analysis endowed in particular with functional analytic tools and applications is presented in [7].

A. Sudbery showed in 1979 in [26] that quaternionic-analytic functions can also be endowed with a modified notion of differentiability described by relations between differential forms of second and third order. In 1999 K. Gürlebeck and H. Malonek extended A. Sudbery's description to Clifford analysis in their paper [12].

In contrast to the planar case, one observes that in $\mathbb{R}^n$ with $n \geq 3$ the set of conformal mappings which coincides with the set of Möbius transformations (cf. e.g. [18], [4], [6], [15]) is disjoint with the set of hypercomplex-analytic functions.
It is an essential observation that precisely those conformal mappings which are described by linear-affine Möbius transformations are M-differentiable. However, Möbius transformations composed by inversions are not M-differentiable. M-differentiability is actually a too restrictive notion to describe the complete set of conformal mappings.

One of the main concerns of this work is to illuminate in which way one has to weaken the classical condition of M-differentiability in order to describe precisely the whole set of Möbius transformations. This paper provides furthermore a correction and an extension of [3] as well as a complementary work to [26] and [12].

We observe that left M-differentiable functions are characterized by a system of differential equations of the form

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_0} e_k \qquad k = 1,2,3, \tag{1}$$

The crucial idea to extend (1) in order to obtain the complete set of Möbius transformations as solutions is to replace in (1) the set of the canonical imaginary units e_1, e_2, e_3 by a variable arbitrary orthonormal frame. More precisely, a C^1 function defined in a domain $\Omega \subset \mathbb{H}$ satisfying $\frac{\partial f}{\partial x_0}(z) \neq 0$ for all $z \in \Omega$ is conformal in the sense of Gauss, if and only if there are three $C^0(\Omega)$ functions Ψ_1, Ψ_2, Ψ_3 satisfying

$$< \Psi_i(z), \Psi_j(z) >= \delta_{ij} \qquad (1 \leq i,j \leq 3) \qquad \forall z \in \Omega$$

such that

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_0} \Psi_k(z) \qquad k = 1,2,3 \tag{2}$$

or in other words if and only if the limit

$$\lim_{\Delta z^{[\Psi]} \to 0} (\Delta f)(\Delta z^{[\Psi]})^{-1} \tag{3}$$

exists, where $\Delta z^{[\Psi]} = \Delta x_0 + \sum_{i=1}^{3} \Delta x_i \Psi_i(z_0)$ with $\Delta x_k = x_k - x_k^*$.

Relation (3) provides an analytic characterization of Möbius transformations in terms of a formal condition of differentiability putting the differential of a function in relation with a variable differential form of first order.

One observes further that the use of arbitrary orthonormal frames, also called structural sets, which have often been used in works of V. Kravchenko, M. Shapiro and N. Vasilevski (cf. e.g. [17] and [25]) is really essential here. However, it is crucial to underscore that one obtains for every single point a different structural set if and only if the conformal mapping is composed by inversions.

We further proceed to study the relation between the structural sets appearing in the system of differential equations (2) and its associated solution

Moreover, we observe that the system (2) can be rewritten in the form of the following system of non-linear partial differential equations

$$< \frac{\partial f}{\partial x_i}(\frac{\partial f}{\partial x_0})^{-1}, \frac{\partial f}{\partial x_j}(\frac{\partial f}{\partial x_0})^{-1} >= \delta_{ij}. \tag{4}$$

It is remarkable from the point of view of the theory of partial differential equations that the general solution of (4) can be represented according to Liouville's theorem in the form

$$f(z) = (az+b)(cz+d)^{-1}$$

with the global parameters $a, b, c, d \in \mathbb{H}$. The system (4) provides an example of a system of non-linear differential equations which is completely characterized by a finite number of parameters.

2 Preliminaries

2.1 Basic notions

$\mathbf{H}$ denotes the Hamiltonian skew field. An arbitrary element $z \in \mathbb{H}$ can be written in the form

$$z = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3, \tag{5}$$

where $e_0 := 1$ and e_1, e_2, e_3 are the canonical quaternionic units satisfying

$$e_1 e_2 = e_3, \quad e_2 e_3 = e_1, \quad e_3 e_1 = e_2$$

and

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_i e_j = -e_j e_i \ \forall i \neq j, \quad i, j \in \{1, 2, 3\}.$$

One can identify the quaternionic skew field with the vector space $\mathbf{R}^4$ considering the canonical vector space isomorphism $\Theta : \mathbf{R}^4 \mapsto \mathbf{H}$ defined by

$$\tilde{z} := (x_0, x_1, x_2, x_3)^T \ \mapsto \ z = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3. \tag{6}$$

x_0 is called the real part of z and is denoted by $Re(z)$.

The complementary expression $x_1 e_1 + x_2 e_2 + x_3 e_3$ is called the pure quaternionic part of z and is denoted by $Pu(z)$ like in [26]. We further denote the set of quaternions satisfying $Re(z) = 0$ by $Pu(\mathbb{H})$.

To every $z \in \mathbf{H}$ represented as in (5), the conjugated quaternion $\overline{z}$ is defined by:

$$\overline{z} = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3.$$

A scalar product between two quaternions z and $w = y_0 + y_1 e_1 + y_2 e_2 + y_3 e_3$ can be introduced by

$$< z, w > := \frac{1}{2}(z\overline{w} + w\overline{z}). \quad (7)$$

This scalar product coincides with the Euclidean scalar product in $\mathbb{R}^4$, considering the associated vectors $\tilde{z} = (x_0, x_1, x_2, x_3)^T$ and $\tilde{w} = (y_0, y_1, y_2, y_3)^T$. It further induces a norm on $\mathbb{H}$ which coincides with the Euclidean norm in $\mathbb{R}^4$:

$$|z| := \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}.$$

2.2 Quaternionic differential forms

We proceed to introduce quaternionic differential forms. For a detailed description of the theory of quaternionic differential forms and their properties we refer to [26], [19] and [12].
Let $\Omega \subset \mathbb{H}$ be an open set and

$$f : \Omega \to \mathbf{H} \quad f(z) = \sum_{i=0}^{3} e_i f_i(z)$$

be a quaternion valued function, where f_i denote its real-valued components. Furthermore, let f be real differentiable in the usual sense. Its differential at the point $z_0 \in \Omega$ is then an $\mathbf{R}$ linear map $df(z_0) : \Omega \to \mathbb{H}$. If one identifies the tangential space in every $z_0 \in \Omega$ with $\mathbf{H}$ itself, then one can regard this differential as a quaternionic 1-form:

$$df = \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} dx_i, \quad (8)$$

where the forms dx_i are the canonical real 1-forms.

Conversely, in view of [26], one can consider every quaternion valued 1-form

$$\omega = \sum_{i=0}^{3} \alpha_i dx_i \quad (\alpha_i \in \mathbb{H})$$

as an $\mathbf{R}$-linear map $\omega : \mathbb{H} \to \mathbb{H}$ being uniquely defined by:

$$\omega(\sum_{i=0}^{3} e_i x_i) = \sum_{i=0}^{3} \alpha_i x_i.$$

In view of this definition, the differential of the identity function is

$$dz = \sum_{i=0}^{3} e_i dx_i. \quad (9)$$

This differential is said to be the canonical quaternionic 1-form.
For two arbitrary quaternionic 1-forms

$$\omega := \sum_{i=0}^{3} \alpha_i dx_i \text{ and } \theta := \sum_{i=0}^{3} \beta_i dx_i \quad (\alpha_i, \beta_i \in \mathbb{H})$$

the wedge product is defined by

$$\omega \wedge \theta := \sum_{i,j=0}^{3} \alpha_i \beta_j dx_i \wedge dx_j \tag{10}$$

which provides a quaternionic 2-form. This definition allows to introduce the quaternionic surface 3-form $d\sigma$:

$$d\sigma(z) = \sum_{i=0}^{3} (-1)^i e_i \, \hat{dx_i} \quad \text{with} \quad \hat{dx_i} := \bigwedge_{j=0, i\neq j}^{3} dx_j$$

which is the crucial ingredient in the description of quaternionic-analytic functions by differential forms in [26] and [12]. We finally introduce the exterior derivative of a quaternionic differential form ω of arbitrary degree by

$$d\omega := \sum_{i=0}^{3} e_i d\omega_i, \tag{11}$$

where $\omega_0, \omega_1, \omega_2$ and ω_3 denote the real-valued components of ω.

2.3 Quaternionic differentiability and analyticity

The hypercomplex differential form calculus provides several approaches to generalize differentiability and complex-analyticity to hypercomplex analysis. In this paper we restrict ourselves to quaternionic analysis.

The most straightforward way to generalize complex differentiability to quaternions is to start from a usual differential quotient. This approach was firstly discussed by G. Scheffers in 1893 (cf. [24]) and later on by A.S. Melijhzon in 1948 (cf. [20]).

Definition 1. *Let $\Omega \subseteq \mathbb{H}$ be an open set, and let $z_0 \in \mathbb{H}$. $f : \Omega \to \mathbb{H}$ is called left M-differentiable (left quaternionic differentiable) at z_0, if*

$$\lim_{z \to z_0} [f(z) - f(z_0)](z - z_0)^{-1} = \lim_{\Delta z \to 0} (\Delta f)(\Delta z)^{-1}$$

exists.

A function $f : \Omega \to \mathbb{H}$ is called right M-differentiable (right quaternionic differentiable) at z_0, if

$$\lim_{z \to z_0} (z - z_0)^{-1}[f(z) - f(z_0)]^{-1} = \lim_{\Delta z \to 0} (\Delta z)^{-1}(\Delta f)$$

exists.

f is called left (right) M-differentiable in Ω, if f is left (right) M-differentiable in every point $z \in \Omega$.

It is already due to G. Scheffers and A.S. Melijhzon that

Lemma 1. *Let $\Omega \subseteq \mathbb{H}$ be a domain and $f \in C^1(\Omega)$. Then f is left-M-differentiable in Ω if and only if $f(z) = az + b$, with $a, b \in \mathbf{H}$.*
f is right-M-differentiable in Ω if and only if $f(z) = z\alpha + \beta$ with $\alpha, \beta \in \mathbf{H}$.

An elegant proof of this statement has also been given in [26] by identifying $\mathbf{H}$ with $\mathbb{C}^2$.
We can express M-differentiability by a relation between two quaternionic differential forms of first order:

Theorem 1. *Suppose $\Omega \subset \mathbb{H}$ is a domain and $f \in C^1(\Omega)$. Then f is left M-differentiable if and only if*

$$df = \frac{\partial f}{\partial x_0} dz. \tag{12}$$

Further, f is right M-differentiable if and only if

$$df = dz \frac{\partial f}{\partial x_0}. \tag{13}$$

Proof: By a straightforward calculation we verify that a function $f(z) = az + b$ with $a \neq 0$ satisfies (12) and that further every function $g(z) = z\alpha + \beta$ with $\alpha \neq 0$ satisfies (13).
Conversely we observe that for a $C^1(\Omega)$ function satisfying (12) we obtain

$$\lim_{\Delta z \to 0} (\Delta f)(\Delta z)^{-1} = \frac{\partial f}{\partial x_0} \quad \forall z \in \Omega \tag{14}$$

Thus, f is left M-differentiable. Analogously one verifies that a C^1-function satisfying (13) is right M-differentiable. □

A more efficient approach to generalize complex analyticity to hypercomplex analysis is the Cauchy Riemann approach considering $C^1(\Omega)$ functions which are in the kernel of the generalized Cauchy-Riemann operator in $\mathbb{H}$ given by

$$D := \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i} \tag{15}$$

Definition 2. *(cf.* [7], [11])
Let $U \subset \mathbb{H}$ be an open set and let $f : U \to \mathbb{H}$ be a real differentiable function. Then f is called left monogenic (right monogenic) in U, if $Df = 0$ ($fD = 0$).

A. Sudbery showed in 1979 (cf. [26]) that one can also describe this function class by a modified notion of differentiability based on the consideration of relations of quaternionic differential forms of second and third order. In [26] A. Sudbery introduces the following notion:

Definition 3. *A function $f : \Omega \to \mathbb{H}$ is left regular [right regular] at $z_0 \in \Omega$, if it is real-differentiable at z_0 and if there exists an $f_l'(z_0)$[resp. $f_r'(z_0)$] $\in \mathbb{H}$ such that*

$$d(dz \wedge dz\ f) = d\sigma(z) f_l'(z_0) \quad [resp.\ d(f\ dz \wedge dz) = f_r'(z_0) d\sigma(z)]. \tag{16}$$

It can be shown (cf. [26]) that the set of left (right) regular functions coincides exactly with the set of the left (right) monogenic functions and that the associated left (right) derivative $f_l'(z)$ ($f_r'(z)$) equals $2\frac{\partial f}{\partial x_0}$.

We observe that the concept of monogenicity is not compatible with the notion of M-differentiability, since every non-constant linear affine function is not monogenic, but actually M-differentiable.

2.4 Möbius transformations

The function class of complex Möbius transformations plays a crucial role in geometric function theory. It is well-known that every Möbius transformation in the complex plane can be represented in the form

$$f(z) = \frac{az + b}{cz + d} \qquad a, b, c, d \in \mathbb{C} \quad ad - bc \neq 0.$$

Möbius transformations in higher dimensions were firstly treated by K. Th. Vahlen in 1904 in [27]. M. L. Sarasin, a student of R. Fueter, analyzed in her PhD thesis [23] (1930) geometric mapping questions of Möbius transformations in quaternions in a detailed way.
Further contributions on Möbius transformations in view of geometric questions have been presented for example by L. Ahlfors (cf. e.g. [1], [2]), by G. Zöll [28], by S. Kraußhar [16] and recently by R. M. Porter [22] and by many other authors.
We recall (cf. [28]) that in the quaternionic skew field a Möbius transformation can be represented by

$$f(z) = (az+b)(cz+d)^{-1}$$

where a, b, c, d are quaternions satisfying $|b - ac^{-1}d||c| \neq 0$ if $c \neq 0$ or $|ad| \neq 0$ if $c = 0$ or equivalently by

$$f(z) = (z\gamma + \delta)^{-1}(z\alpha + \beta),$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{H}$ such that $|\beta - \delta\gamma^{-1}\alpha||\gamma| \neq 0$ if $\gamma \neq 0$, or $|\alpha\delta| \neq 0$ if $\gamma = 0$.

Already R. Fueter discovered (cf. e.g. [11]) that Möbius transformations play a crucial role in quaternionic analysis since a monogenic function composed with a Möbius transformation gives up to a conformal weight again a monogenic function.
However, we observe that quaternionic Möbius transformations themselves are neither left nor right monogenic. One further verifies directly that the non-constant left and right M-differentiable functions are strictly included in the set of Möbius transformations.

3 Conformality in quaternions

Using quaternionic differential forms one can rewrite the classical definition of conformality in the sense of Gauss in $\mathbb{R}^4$ given e.g. in [6] and [5] equivalently in terms of quaternions:

Definition 4. *Let $\Omega \subset \mathbb{H}$ be a domain.*
A real differentiable function $f : \Omega \to \mathbb{H}$ is called conformal in the sense of Gauss, if there exists a positive real valued continuous function $\lambda : \mathbb{H} \to \mathbb{R}^{>0}$ $\quad z \mapsto \lambda(z)$ such that

$$|df|^2 = \lambda(z)|dz|^2, \tag{17}$$

In the sequel we simply use the expression conformality for the notion conformality in the sense of Gauss.

We recall that in the complex case the class of the conformal mappings consists exactly of the holomorphic functions satisfying $\frac{\partial f}{\partial z}(z) \neq 0$ and antiholomorphic functions satisfying $\frac{\partial f}{\partial \bar{z}} \neq 0$.
In spaces of dimension $n \geq 3$ the set of conformal mappings is restricted to the set of Möbius transformations as firstly shown by J. Liouville in 1850 for the three dimensional case. We state:

Theorem 2. *(Liouville's theorem)*
Let $\Omega \subset \mathbf{H}$ be a domain. A C^1 function $f : \Omega \to \mathbf{H}$ is a conformal mapping if and only if f is a Möbius transformation.

J. Liouville proved this theorem in 1850 (cf. [18]) under the condition of f being at least a C^3 homeomorphism. It turned out to be quite difficult to weaken this differentiability hypothesis. In 1958 P. Hartman managed to prove this assertion in [13] for C^1 homeomorphisms. According to T. Iwaniec and G. Martin [15] one may also drop the condition of f being an homeomorphism.

Since the set of conformal mappings coincides with the set of Möbius transformations, one has actually a closed description of them.
However, the question, if it is possible to characterize them by a certain kind of analytic concept, i.e. by a certain notion of differentiability treated by quaternions, remained open.

4 An analytic characterization of conformal mappings

We proceed to weaken the notion of M-differentiability in such a way that precisely the complete set of Möbius transformations and hence conformal mappings will be included.
To this end we deduce a characterization of conformal mappings by a system of quaternionic differential equations. We start with the following proposition:

Proposition 1. *Let $\Omega \subset \mathbb{H}$ be a domain. Then a $C^1(\Omega)$ function f is conformal in Ω if and only if for every $z \in \Omega$*

$$|\frac{\partial f}{\partial x_0}|^2 = |\frac{\partial f}{\partial x_1}|^2 = |\frac{\partial f}{\partial x_2}|^2 = |\frac{\partial f}{\partial x_3}|^2 = \lambda(z) > 0 \tag{18}$$

$$Re\{\frac{\partial f}{\partial x_i}\frac{\partial \overline{f}}{\partial x_k}\} = 0 \quad i < k \quad i,k = 0,1,2,3. \tag{19}$$

Proof: According to the definition, f is conformal if there is a positive real-valued continuous function $\lambda : \mathbb{H} \to \mathbb{R}^{>0}$ with

$$|df|^2 = \lambda(z)\sum_{k=0}^{3} dx_k^2. \tag{20}$$

We consider the expression:

$$\begin{aligned} |df|^2 &= (\sum_{i=0}^{3}\frac{\partial f}{\partial x_i}dx_i)(\sum_{i=0}^{3}\frac{\partial \overline{f}}{\partial x_i}dx_i) \\ &= \sum_{i=0}^{3}[\sum_{r=0}^{3}(\frac{\partial f_r}{\partial x_i})^2]dx_i^2 + 2\sum_{j<i}(\sum_{r=0}^{3}\frac{\partial f_r}{\partial x_j}\frac{\partial f_r}{\partial x_i})dx_j\, dx_i, \end{aligned}$$

and we observe that one can rewrite (20) in the following equivalent way:

$$\sum_{i=0}^{3}[\sum_{r=0}^{3}(\frac{\partial f_r}{\partial x_i})^2]dx_i^2 + 2\sum_{j<i}(\sum_{r=0}^{3}\frac{\partial f_r}{\partial x_j}\frac{\partial f_r}{\partial x_i})dx_j\, dx_i = \lambda(z)\sum_{k=0}^{3} dx_k^2. \tag{21}$$

By a comparison of coefficients, one can infer that f is conformal if and only if the following system of differential equations is satisfied:

$$\sum_{i=0}^{3}(\frac{\partial f_i}{\partial x_k})^2 = \lambda(z) \quad k = 0,1,2,3 \tag{22}$$

$$\sum_{i=0}^{3}\frac{\partial f_i}{\partial x_j}\frac{\partial f_i}{\partial x_k} = 0, \quad j < k = 0,1,2,3. \quad \square \tag{23}$$

We can rewrite the system of differential equations (18) and (19) in the form of orthogonal relations providing the following characterization:

Proposition 2. *Let $\Omega \subset \mathbb{H}$ be a domain. Then a real differentiable function $f : \Omega \to \mathbb{H}$ is conformal in Ω if and only if for every $z \in \Omega$*

$$< \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_k} > = \delta_{i,k}\,\lambda(z), \tag{24}$$

where $\delta_{i,k}$ denotes the Kronecker symbol.

Remarks and motivations: The non-constant left (right) M-differentiable functions form actually a very special subset of conformal mappings which is characterized by the following system of differential equations

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_0} e_k \quad \text{or} \quad \frac{\partial f}{\partial x_k} = e_k \frac{\partial f}{\partial x_0} \quad \text{respectively, where } k = 1,2,3 \tag{25}$$

following directly by (12) and (13). In order to extend the class of M-differentiable functions to obtain all Möbius transformations we will replace the canonical units e_1, e_2, e_3 in the system (25) by a general set of orthonormal pure quaternions $[\Psi] = (\Psi_1, \Psi_2, \Psi_3)$. These sets are often called structural sets. They have been used for example by V. Kravchenko, M. Shapiro and N. Vasilevski (cf. e.g. [25], [17]). They represented quaternionic variables and operators including the Cauchy-Riemann operator in a general basis $[\Psi]$ in order to obtain a more general quaternionic function theory. For what follows the use of general structural sets is crucial. But it turns out to be essential not to consider only fixed structural sets. It is crucial to endow in general every single point with a different structural set or in other words to consider moving frames. Substituting the canonical basis elements in (25) by elements of an orthonormal pure quaternionic continuously moving frame leads to the description of the complete set of Möbius transformations. The following theorem provides a more precise formulation and moreover a correction to [3]:

Theorem 3. *(Local characterization of quaternionic conformal mappings by a system of differential equations):*
Let $\Omega \subset \mathbb{H}$ be a domain. A continuously real differentiable function $f : \Omega \to \mathbb{H}$ with $\frac{\partial f}{\partial x_0} \neq 0 \; \forall z \in \Omega$ is conformal in $\Omega \subset \mathbb{H}$ if and only if there exist three C^0 functions $\Psi_k : \Omega \to \mathbf{H}$ satisfying

$$\begin{aligned} Re\{\Psi_i(z)\} &= 0 \quad i = 1,2,3, \; \forall z \in \Omega \\ < \Psi_i(z), \Psi_j(z) > &= \delta_{i,j} \quad i,j \in \{1,2,3\}, \forall z \in \Omega \end{aligned} \tag{26}$$

such that

$$\frac{\partial f}{\partial x_k} = \Psi_k(z) \frac{\partial f}{\partial x_0}. \tag{27}$$

Proof: Let f be a conformal mapping in Ω satisfying $\frac{\partial f}{\partial x_0} \neq 0$.
Then we define for every $k = 1,2,3$ at each point $z \in \Omega$:

$$\Psi_k(z) := \left(\frac{\partial f}{\partial x_k}\right)\left(\frac{\partial f}{\partial x_0}\right)^{-1}. \tag{28}$$

The functions $\Psi_k(z)$ are well defined elements of $C^0(\Omega)$, since $\frac{\partial f}{\partial x_0} \neq 0$. Now we show that the system $[\Psi(z)] := (\Psi_1(z), \Psi_2(z), \Psi_3(z))$ is an orthonormal system of pure quaternions at each point of Ω. We observe immediately that $|\Psi_k(z)| = 1$ for all $z \in \Omega$, since f satisfies (18).
In order to show that the function Ψ_k take only pure quaternionic values in Ω we consider

$$\begin{aligned} Re\{\Psi_k(z)\} &= Re\left\{\left(\frac{\partial f}{\partial x_k}\right)\left(\frac{\partial f}{\partial x_0}\right)^{-1}\right\} \\ &= \frac{1}{|\frac{\partial f}{\partial x_0}|^2} Re\left\{\frac{\partial f}{\partial x_k} \frac{\partial \overline{f}}{\partial x_0}\right\} = 0 \end{aligned}$$

which follows by (19), since f is conformal at every point of Ω.

In order to verify that the functions $\Psi_k(z)$ form an orthogonal system at every single point $z \in \Omega$ we consider

$$\begin{aligned} 2 < \Psi_i(z), \Psi_j(z) > &= \Psi_i(z)\overline{\Psi_j(z)} + \Psi_j(z)\overline{\Psi_i(z)} \\ &= \frac{\partial f}{\partial x_i}(\frac{\partial f}{\partial x_0})^{-1}\overline{(\frac{\partial f}{\partial x_0})^{-1}\frac{\partial f}{\partial x_j}} \\ &+ \frac{\partial f}{\partial x_j}(\frac{\partial f}{\partial x_0})^{-1}\overline{(\frac{\partial f}{\partial x_0})^{-1}\frac{\partial f}{\partial x_i}} \\ &= 2|\frac{\partial f}{\partial x_0}|^{-2} Re\{\frac{\partial f}{\partial x_i}\frac{\partial \overline{f}}{\partial x_j}\} = 0, \end{aligned}$$

since equation (19) is satisfied. Finally, by (28), the frame $\Psi_k(z)$ satisfies the system (27).

Conversely, suppose that we can associate with each point of Ω an orthonormal system of pure quaternions $[\Psi] = [\Psi(z)]$ such that for $k = 1,2,3$ the system (27) is satisfied. In order to show that f is conformal in Ω, we verify that f satisfies in Ω the system of differential equations (18) and (19). Since $|\Psi_k(z)| = 1$ at each $z \in \Omega$ the property (18) follows directly. In order to show (19), we consider the following two expressions involving (27):

$$\begin{aligned} 2 < \frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_k} > &= \frac{\partial f}{\partial x_0}\frac{\partial \overline{f}}{\partial x_k} + \frac{\partial f}{\partial x_k}\frac{\partial \overline{f}}{\partial x_0} \\ &= \frac{\partial f}{\partial x_0}\frac{\partial \overline{f}}{\partial x_0}\overline{\Psi_k(z)} + \Psi_k(z)\frac{\partial f}{\partial x_0}\frac{\partial \overline{f}}{\partial x_0} \\ &= 2|\frac{\partial f}{\partial x_0}|^2 Re\{\Psi_k(z)\} = 0, \end{aligned}$$

since the functions $\Psi_k(z)$ are pure quaternionic valued in Ω. Further,

$$\begin{aligned} 2 < \frac{\partial f}{\partial x_j}, \frac{\partial f}{\partial x_k} > &= \frac{\partial f}{\partial x_j}\frac{\partial \overline{f}}{\partial x_k} + \frac{\partial f}{\partial x_k}\frac{\partial \overline{f}}{\partial x_j} \quad (j \neq k, \ \ j,k \neq 0) \\ &= \Psi_j(z)\frac{\partial f}{\partial x_0}\frac{\partial \overline{f}}{\partial x_0}\overline{\Psi_k(z)} + \Psi_k(z)\frac{\partial f}{\partial x_0}\frac{\partial \overline{f}}{\partial x_0}\overline{\Psi_j(z)} \\ &= 2|\frac{\partial f}{\partial x_0}|^2 < \Psi_j(z), \Psi_k(z) > = 0, \end{aligned}$$

since $(\Psi_i(z))_{i=1,2,3}$ is an orthonormal system at each single point $z \in \Omega$. Applying Proposition 1 leads to the assertion. □

We observe that one can also characterize the set of quaternionic conformal mappings by a similar system of differential equations namely by writing the elements of the structural set on the right-hand side of the expression $\frac{\partial f}{\partial x_0}$. Considering

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_0}\Psi_k(z) \quad k = 1,2,3. \tag{29}$$

we can prove the same result by setting

$$\Psi_k(z) := (\frac{\partial f}{\partial x_0})^{-1}\frac{\partial f}{\partial x_k} \quad \text{for every} \quad k = 1,2,3. \tag{30}$$

Because of the non-commutativity with respect to multiplication in the quaternionic skew field the previous statement is not evident.

Since conformality in quaternions is characterized by the systems (27) and (29) we infer that the function classes

$$\begin{aligned} \mathcal{C}^{(r)} &:= \{f:\Omega\to\mathbf{H} | f\in C^1(\Omega), \frac{\partial f}{\partial x_0}\neq 0, \ \frac{\partial f}{\partial x_k} = \Psi_k(z)\frac{\partial f}{\partial x_0}, k=1,2,3\} \\ \mathcal{C}^{(l)} &:= \{f:\Omega\to\mathbf{H} | f\in C^1(\Omega), \frac{\partial f}{\partial x_0}\neq 0, \ \frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_0}\Psi_k(z), k=1,2,3\}, \end{aligned}$$

where $\Psi_k:\Omega\to\mathbb{H}$ are functions satisfying (26) of Theorem 3, are equivalent. We want to discuss Theorem 3 in view of the planar case. Let $D\subset\mathbb{C}$ be a domain. We observe that in the complex case there are only two $C^0(D)$ functions Ψ satisfying $Re(\Psi(z))=0$ and $|\Psi(z)|=1$ in D, namely $\Psi(z)\equiv e_1$ and $\Psi(z)\equiv -e_1$. Thus, a real differentiable complex valued function $f(x_0+e_1x_1)$ satisfying $\frac{\partial f}{\partial x_0}(x_0+e_1x_1)\neq 0$ in a domain D is conformal there if and only if

$$\frac{\partial f}{\partial x_1}(z) = e_1\frac{\partial f}{\partial x_0}(z) \quad\text{or}\quad \frac{\partial f}{\partial x_1}(z) = -e_1\frac{\partial f}{\partial x_0}(z) \quad \forall z\in D.$$

In the first case f is holomorphic in D and in the second case f is antiholomorphic there. Thus, conformality in the planar case is characterized by the analytic concept of holomorphy or antiholomorphy, respectively.

In the quaternionic case we have actually many more possibilities for the range of values of the functions Ψ_1,Ψ_2 and Ψ_3.
In order to characterize quaternionic conformal mappings by an analytic notion we reformulate Theorem 3 in terms using variable quaternionic differential forms. In view of Theorem 3 we can say that a non-constant $C^1(\Omega)$ function f is conformal at a point $z_0\in\mathbb{H}$ if and only if there exists a structural set $[\Psi(z_0)]=(\Psi_1(z_0),\Psi_2(z_0),\Psi_3(z_0))$ of pure quaternions such that

$$df = \frac{\partial f}{\partial x_0}dz^{[\Psi]} \quad\text{or}\quad df = dz^{[\Psi]}\frac{\partial f}{\partial x_0},$$

with the variable quaternionic 1-form

$$dz^{[\Psi]} := dx_0 + \Psi_1(z_0)dx_1 + \Psi_2(z_0)dx_2 + \Psi_3(z_0)dx_3.$$

This reformulation leads to the following definition:

Definition 5. *(C-differentiability)*
Let $\Omega\subset\mathbb{H}$ be an open set and let $z^*\in\Omega$ with $z^* = x_0^* + \sum_{i=1}^{3} e_i x_i^*$. Then f is called left C-differentiable at z^*, if and only if there exist three pure quaternions $\Psi_1(z^*),\Psi_2(z^*),\Psi_3(z^*)$ with the property $<\Psi_i(z^*),\Psi_j(z^*)>=\delta_{ij}$ such that

$$\lim_{\Delta z^{[\Psi]}\to 0}(\Delta f)(\Delta z^{[\Psi]})^{-1}$$

exists, where $\Delta z^{[\Psi]} = \Delta x_0 + \sum_{i=1}^{3}\Delta x_i\Psi_i(z^*)$ with $\Delta x_k = x_k - x_k^*$.
f is called left C-differentiable in Ω, if f is left C-differentiable at every point $z\in\Omega$.
$f:\Omega\to\mathbb{H}$ is called right C-differentiable at z^*, if and only if there exist three pure quaternions $\Psi_1(z^*),\Psi_2(z^*),\Psi_3(z^*)$ with $<\Psi_i(z^*),\Psi_j(z^*)>=\delta_{ij}$, such that

$$\lim_{\Delta z^{[\Psi]}\to 0}(\Delta z^{[\Psi]})^{-1}(\Delta f)$$

exists.
f is called right C-differentiable in Ω if f is right C-differentiable at every point $z\in\Omega$.

The limit

$$\lim_{\Delta z^{[\Psi]} \to 0} (\Delta f)(\Delta z^{[\Psi]})^{-1}$$

can be considered as a linearization of the function f at the point z_0 with respect to $[\Psi]$ and is equal to the expression $\frac{\partial f}{\partial x_0}$, which may be regarded as the left C-derivative of f at the point z_0. The formal notion of C-differentiability provides a further justification to consider the expression $\frac{\partial f}{\partial x_0}$ as derivative of a quaternionic function.

If we consider the special case $\Psi_k(z) \equiv e_k$, then we obtain precisely the set of M-differentiable functions.

Thus, C-differentiability is actually an extension of M-differentiability. The set of non-constant C-differentiable functions coincides precisely with the set of Möbius transformations.

5 Classification of the frames

Now we want to analyze the relation between the functions Ψ_k in the differential equation (27) (or resp. (29)) and its solution which must be a function of the type $f(z) = (az+b)(cz+d)^{-1}$ according to the version of Liouville's theorem proved in [15].

The case $f(z) = az + b$ or $f(z) = z\alpha + \beta$ has already been discussed.

So we consider now Möbius transformations of the form $f(z) = (az+b)d^{-1}$ or $f(z) = \delta^{-1}(z\alpha+\beta)$. In this case we can easily prove using the definition of $\Psi_k(z)$ in (28) (or in (30)) that the functions $\Psi_k(z)$ are constant functions in Ω, but $\Psi_k(z) \neq e_k$.

If we consider Möbius transformations being also composed by inversions, then we will observe that the functions $\Psi_k(z)$ are not constant in Ω.

We consider for example the standard inversion $f(z) = z^{-1}$ concentrating on the case where the $\Psi_k(z)$ are written on the right-hand side of the expression $\frac{\partial f}{\partial x_0}$, then we get

$$\Psi_k(z) = [\sum_{i=0}^{3} x_i^2 - 2x_0(x_0 - \sum_{i=1}^{3} e_i x_i)]^{-1}[-e_k(\sum_{i=0}^{3} x_i^2) - 2x_k(x_0 - \sum_{i=1}^{3} e_i x_i)]. \tag{31}$$

At every point of Ω we obtain a different structural set which is illustrated in the following examples. At the point $z_1 = e_2$ we get:

$$\Psi_1(z_1) = -e_1 \quad \Psi_2(z_2) = e_2 \quad \Psi_3(z) = -e_3,$$

but at the point $z_2 := 1 + e_1 + e_2$ we obtain:

$$\Psi_1(z_2) = \frac{1}{3}(e_1 + 2e_2 - 2e_3) \quad \Psi_2(z_2) = \frac{1}{3}(2e_1 + e_2 + 2e_3) \quad \Psi_3(z_2) = \frac{1}{3}(2e_1 - 2e_2 - e_3).$$

The following theorem reveals a relationship between the functions $\Psi_k(z)$ and the coefficients of the Möbius transformation being solution of (29) in the most general case:

Theorem 4. *Let $\Omega \subset \mathbb{H}$ be a domain and let $f : \Omega \to \mathbb{H}$ be a Möbius transformation written in the form:*

$$f(z) = (az + b)(cz + d)^{-1}.$$

Then the associated functions $\Psi_k : \Omega \to Pu(\mathbb{H})$ in the differential equation

$$\frac{\partial f}{\partial x_k} = \frac{\partial f}{\partial x_0}\Psi_k(z)$$

are represented by

$$\begin{aligned}\Psi_k(z) &= [2x_0 a\bar{c} + b\bar{c} + a\bar{d} - (az+b)(cz+d)^{-1}A_0(z)]^{-1} \\ & \quad [2x_k a\bar{c} - be_k\bar{c} + ae_k\bar{d} - (az+b)(cz+d)^{-1}A_k(z)] \end{aligned} \tag{32}$$

where

$$A_j(z) := \frac{\partial}{\partial x_j}\{|cz+d|^2\} \quad j = 0, 1, 2, 3.$$

Proof: Compute the partial derivatives of f with respect to the four components. Using the definition (30) of the Ψ_k we arrive at the result. □

Remarks:

1. By (32) we infer that $\Psi_k \equiv const$ if and only if $c = 0$.

2. We observe that the system (27) can be rewritten in the form of the following system of non-linear partial differential equations

$$< \frac{\partial f}{\partial x_i}(\frac{\partial f}{\partial x_0})^{-1}, \frac{\partial f}{\partial x_j}(\frac{\partial f}{\partial x_0})^{-1} >= \delta_{ij}. \tag{33}$$

It is remarkable that the general solution of (33) can be represented according to Liouville's theorem in the form $f(z) = (az+b)(cz+d)^{-1}$ where $a, b, c, d \in \mathbb{H}$ are global parameters. The system (33) provides an example of a system of non-linear differential equations which is uniquely characterized by a finite number of parameters.

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