ARTICLE IN PRESS YJNTH:3811

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Transcendence of the log gamma function and some discrete periods

Sanoli Gun [1,2], M. Ram Murty [*,3], Purusottam Rath [1,4]

Department of Mathematics and Statistics, Queen's University, Jeffrey Hall, 99 University Avenue, Kingston, ON, Canada K7L 3N6

ARTICLE INFO

Article history:
Received 7 October 2008
Available online xxxx
Communicated by Michael A. Bennett

MSC:
11J81
11J86
11J91

Keywords:
Gamma function
Log gamma function
Schanuel's conjecture
Periods

ABSTRACT

We study transcendental values of the logarithm of the gamma function. For instance, we show that for any rational number x with $0 < x < 1$, the number $\log \Gamma(x) + \log \Gamma(1-x)$ is transcendental with at most one possible exception. Assuming Schanuel's conjecture, this possible exception can be ruled out. Further, we derive a variety of results on the Γ-function as well as the transcendence of certain series of the form $\sum_{n=1}^{\infty} P(n)/Q(n)$, where $P(x)$ and $Q(x)$ are polynomials with algebraic coefficients.

1. Introduction

The study of the nature of the values of the gamma function $\Gamma(z)$ at rational arguments has been in the focus from the times of Euler. But apart from a very few special cases, the (possible) transcendence of the gamma values at rational arguments is merely conjectural and even their irrationality is yet to be established. The result of Schneider who in 1941 [14] proved that the beta function

* Corresponding author.
E-mail addresses: sanoli@imsc.res.in (S. Gun), murty@mast.queensu.ca (M.R. Murty), rath@cmi.ac.in (P. Rath).

1 Coleman Research Fellow.

2 Present address: Institute for Mathematical Sciences, Chennai, 600 113, India.

3 Research partially supported by the Natural Sciences and Engineering Research Council (NSERC).

4 Present address: Chennai Mathematical Institute, Siruseri, 603 103, India.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.01.008

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

is transcendental when $a,b,a+b\in\mathbb{Q}\setminus\mathbb{Z}$ suggests a heuristic argument that these values are possibly transcendental. By the above result, we see that the numbers

$$B(1/4,1/2)=\frac{\Gamma(1/4)^2}{\sqrt{2\pi}}\quad\text{and}\quad B(1/3,1/2)=\frac{\sqrt{3}\Gamma(1/3)^3}{2^{4/3}\pi}$$

are transcendental. But this does not prove the transcendence of $\Gamma(1/3)$ and $\Gamma(1/4)$. The transcendence of $\Gamma(1/4)$ and $\Gamma(1/3)$ has been proved in 1976 by Chudnovsky [3] who proved the stronger assertion that the two numbers $\Gamma(1/4)$ and π are algebraically independent and so are the two numbers $\Gamma(1/3)$ and π. Later in 1996, Nesterenko [11] (see also [12, p. 6]) extended these results by showing the following:

Theorem *(Nesterenko). For any imaginary quadratic field with discriminant $-D$ and character ϵ, the numbers*

$$\pi,\quad e^{\pi\sqrt{D}},\quad \prod_{a=1}^{D-1}\Gamma(a/D)^{\epsilon(a)}$$

are algebraically independent. Consequently, the numbers $\Gamma(1/4),\pi$ and e^{π} are algebraically independent and so are the numbers $\Gamma(1/3),\pi$ and $e^{\pi\sqrt{3}}$.

Using the standard identities satisfied by the gamma function (to be given later), the transcendence of $\Gamma(1/6)$ can be deduced. Recently, Grinspan [4] showed that at least two of the three numbers $\Gamma(1/5)$, $\Gamma(2/5)$ and π are algebraically independent. Apart from these very few special cases, the algebraic nature of the gamma function at rational arguments remains enigmatic.

One of our goals in the present work is to explore the nature of the logarithm of the gamma function at rational arguments. Here, we prove

Theorem 3.1. *For any rational number $x\in(0,1)$, the number*

$$\log\Gamma(x)+\log\Gamma(1-x)$$

is transcendental with at most one possible exception.

The possible fugitive exception in the above theorem can be removed if we assume that the following conjecture due to Schanuel (Lang 1966 [8]) is true.

Schanuel's Conjecture. *Suppose $\alpha_1,\dots,\alpha_n$ are complex numbers which are linearly independent over $\mathbb{Q}$. Then the transcendence degree of the field*

$$\mathbb{Q}\big(\alpha_1,\dots,\alpha_n,e^{\alpha_1},\dots,e^{\alpha_n}\big)$$

over $\mathbb{Q}$ is at least n.

This conjecture is believed to include all known transcendence results as well as all reasonable transcendence conjectures on the values of the exponential function. In relation to transcendence of gamma values, we have

Theorem 3.4. *Schanuel's conjecture implies that for any $x\in\mathbb{Q}$, at least one of the following statements is true*:

(1) *Both $\Gamma(x)$ and $\Gamma(1-x)$ are transcendental.*
(2) *Both $\log\Gamma(x)$ and $\log\Gamma(1-x)$ are transcendental.*

In Section 3, we derive various consequences of Schanuel's conjecture. For instance, we have

Theorem 2.2. *Assume Schanuel's conjecture is true. If α is a Baker period then $1/\alpha$ is not a Baker period. In particular, $1/\pi$ is not a Baker period.*

Following [10], a Baker period is defined to be an element of the $\overline{\mathbb{Q}}$ vector space spanned by logarithms of non-zero algebraic numbers. Further, we have

Theorem 2.4. *Assume Schanuel's conjecture is true. If $\alpha_1, \dots, \alpha_n$ are non-zero algebraic numbers such that $\log \alpha_1, \dots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then $\log \alpha_1, \dots, \log \alpha_n, \log \pi$ are algebraically independent. In particular, $\log \pi$ is not a Baker period.*

Finally, in Section 4, we study the algebraic nature of series of the form $\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}$, where $P(x)$ and $Q(x)$ are polynomials with algebraic coefficients. Series of similar type where the roots of the denominator $Q(x)$ are primarily rational have been considered by several authors [1,2,10].

2. Schanuel's conjecture and consequences

Let $\mathcal{L}$ denote the logarithms of non-zero algebraic numbers, that is

$$\mathcal{L} := \left\{ \log \alpha \;\middle|\; \alpha \in \overline{\mathbb{Q}} \setminus \{0\} \right\}.$$

It is a linear space over $\mathbb{Q}$ and contains $i\pi$. The classical theorem of Hermite and Lindemann is the assertion that

$$\overline{\mathbb{Q}} \cap \mathcal{L} = \{0\}.$$

Gelfond and Schneider, independently, in 1934 proved that $\mathcal{L}$ is not a $\overline{\mathbb{Q}}$-linear space (i.e. $\overline{\mathbb{Q}} \cdot \mathcal{L} \nsubseteq \mathcal{L}$). More precisely, they proved

Theorem *(Gelfond–Schneider). If λ_1 and λ_2 are $\mathbb{Q}$-linearly independent elements of $\mathcal{L}$, then they are $\overline{\mathbb{Q}}$-linearly independent.*

Later, Baker in 1966 generalised the above to arbitrary number of logarithms of algebraic numbers. More generally, he proved the following:

Theorem *(Baker). If $\lambda_1, \lambda_2, \dots, \lambda_n$ are $\mathbb{Q}$-linearly independent elements of $\mathcal{L}$, then $1, \lambda_1, \lambda_2, \dots, \lambda_n$ are linearly independent over $\overline{\mathbb{Q}}$.*

An immediate consequence of the above theorem is that any non-zero element in the $\overline{\mathbb{Q}}$-vector space

$$\{\alpha_1\lambda_1 + \cdots + \alpha_n\lambda_n \mid n \in \mathbb{N},\ \alpha_i \in \overline{\mathbb{Q}},\ \lambda_i \in \mathcal{L}\}$$

is necessarily transcendental. As mentioned before, an element of this vector space will be called a Baker period.

On the other hand, the question of algebraic independence of transcendental numbers or even more specifically those of numbers connected with the exponential function is rather delicate. One of the very few general results is the following classical result due to Lindemann and Weierstrass [15].

Theorem *(Lindemann–Weierstrass). If $\beta_1, \dots, \beta_n$ are algebraic numbers which are linearly independent over $\mathbb{Q}$, then the numbers $e^{\beta_1}, \dots, e^{\beta_n}$ are algebraically independent.*

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

A more recent development is the striking result due to Nesterenko that π, e^{π} and $\Gamma(1/4)$ are algebraically independent.

The most far reaching conjecture in this set up is Schanuel's conjecture (Lang 1966 [8]) which is mentioned in the introduction. We deduce some important consequences of this conjecture. We start with noting the following special case of Schanuel's conjecture.

Weaker Schanuel's Conjecture. *Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers such that $\log\alpha_1, \dots, \log\alpha_n$ are linearly independent over $\mathbb{Q}$. Then these numbers are algebraically independent.*

We begin by proving the following consequence of the weaker Schanuel's conjecture; this will be of importance for us, especially for the results in the last section.

Lemma 2.1. *Assume the weaker Schanuel's conjecture. Let $\alpha_1, \dots, \alpha_n$ be non-zero algebraic numbers. Then for any polynomial $f(x_1, \dots, x_n)$ with algebraic coefficients such that $f(0, \dots, 0) = 0$, $f(\log\alpha_1, \dots, \log\alpha_n)$ is either zero or transcendental.*

Proof. We use induction on n. For $n = 1$, the lemma is true by Hermite–Lindemann's theorem. Suppose $f(x_1, \dots, x_n) \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$, $n \geqslant 2$ such that

$$f(\log\alpha_1, \dots, \log\alpha_n) = A, \quad A \text{ algebraic}. \tag{1}$$

By the weaker Schanuel's conjecture,

$$\log\alpha_1, \dots, \log\alpha_n$$

are linearly dependent over $\mathbb{Q}$. Then there exists integers $c_1, \dots, c_n$ such that

$$c_1 \log\alpha_1 + \cdots + c_n \log\alpha_n = 0.$$

Suppose $c_1 \neq 0$. Then $\log\alpha_1 = \frac{1}{c_1}(c_2 \log\alpha_2 + \cdots + c_n \log\alpha_n)$. Replacing this value of $\log\alpha_1$ in (1), we have

$$g(\log\alpha_2, \dots, \log\alpha_n) = A,$$

where $g(x_1, \dots, x_{n-1})$ is a polynomial with algebraic coefficients in $n-1$ variables. Then by induction hypothesis $A = 0$. This completes the proof of the lemma. $\square$

As a consequence, we have

Theorem 2.2. *Assume Schanuel's conjecture is true. If α is a Baker period then $1/\alpha$ is not a Baker period. In particular, $1/\pi$ is not a Baker period.*

Proof. Since α is a Baker period, we can write

$$\alpha = \beta_1 \log\delta_1 + \cdots + \beta_n \log\delta_n,$$

where $\beta_i, \delta_i \in \overline{\mathbb{Q}} \setminus \{0\}$. If $1/\alpha$ is also a Baker period, then

$$\frac{1}{\alpha} = \gamma_1 \log\alpha_1 + \cdots + \gamma_k \log\alpha_k,$$

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

where $\gamma_i, \alpha_i \in \overline{\mathbb{Q}} \setminus \{0\}$. This implies that

$$1 = f(\log\delta_1, \dots, \log\delta_n, \log\alpha_1, \dots, \log\alpha_k), \tag{2}$$

where f is a polynomial in $\overline{\mathbb{Q}}[x_1, \dots, x_{n+k}]$ with $f(0, \dots, 0) = 0$. Then by Lemma 2.1, the right-hand side of (2) is either zero or transcendental. In either case it is a contradiction and the result follows. □

We note that Kontsevich and Zagier [7] have introduced the notion of periods. A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of algebraic functions with algebraic coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients. Clearly all algebraic numbers are periods. An example of a transcendental period is π as it is expressible as

$$\pi = \iint\limits_{x^2+y^2 \leqslant 1} dx\,dy = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx.$$

Also, logarithms of algebraic numbers are periods and hence by Baker's theorem, we have an infinite (but countable) class of transcendental numbers which are periods. Moreover, the periods form a ring and the Baker periods form a subgroup of this ring. In view of the above results, it is tempting to wonder if the group of units of this ring contains only the obvious units, namely the non-zero algebraic numbers.

Now we proceed to derive some other interesting consequences of Schanuel's conjecture:

Proposition 2.3. *Assume that Schanuel's conjecture is true. Then we have the following*:

(1) *If $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, then $\log\alpha$ and $\log\log\alpha$ are algebraically independent. More generally, the numbers $\log_1\alpha, \log_2\alpha, \dots, \log_d\alpha$ are algebraically independent for any $d \in \mathbb{N}$, $d \geqslant 2$ (except in the case when $\log_i\alpha = 1$ for some i). Here $\log_1\alpha = \log\alpha$ and $\log_i\alpha = \log\log_{i-1}\alpha$.*

(2) *If $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$, then $\pi, e^{\alpha_1}, \dots, e^{\alpha_n}$ are algebraically independent. In particular, $e + \pi$, e/π and πe are transcendental.*

(3) *If $\alpha_1, \dots, \alpha_n$ are algebraic numbers such that $i, \alpha_1, \dots, \alpha_n$ are linearly independent over $\mathbb{Q}$, then $\pi, e^{\alpha_1\pi}, \dots, e^{\alpha_n\pi}$ are algebraically independent.* (The case $\alpha = 1$ is Nesterenko's theorem.)

(4) *For $\alpha, \beta \in \mathbb{Q}$ with $\alpha \neq \beta$ we have $\pi^\alpha, e^{\pi^\alpha}$ and e^{π^β} are algebraically independent.*

(5) *e, π and $\log\pi$ are algebraically independent. In particular, π^e is transcendental.*

Proof. (1) Note that for $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, $\log\alpha$ and $\log\log\alpha$ are linearly independent over $\mathbb{Q}$. As otherwise $\alpha^n = (\log\alpha)^m$ for $n, m \in \mathbb{Z}$, a contradiction. By applying Schanuel's conjecture, we see that the numbers $\log\alpha, \log\log\alpha$ are algebraically independent. The general case follows by induction.

(2) We apply Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $\alpha_1, \dots, \alpha_n$ and $i\pi$ to get the result.

(3) Apply Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $i\pi, \alpha_1\pi, \dots, \alpha_n\pi$ to conclude the result.

(4) Apply Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $i\pi, \pi^\alpha$ and π^β to get the result.

(5) By Nesterenko's theorem, we know that π and $\log\pi$ are linearly independent over $\mathbb{Q}$. We apply Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $1, i\pi$ and $\log\pi$ to conclude that e, π and $\log\pi$ are algebraically independent. Now apply Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $1, \log\pi, i\pi + e\log\pi, e\log\pi$. □

Further, we have

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

Theorem 2.4. *Assume Schanuel's conjecture is true. If* $\alpha_1, \dots, \alpha_n$ *are non-zero algebraic numbers such that* $\log\alpha_1, \dots, \log\alpha_n$ *are linearly independent over* $\mathbb{Q}$*, then* $\log\alpha_1, \dots, \log\alpha_n, \log\pi$ *are algebraically independent. In particular,* $\log\pi$ *is not a Baker period.*

Proof. Since $\log\alpha_1, \dots, \log\alpha_n$ are linearly independent over $\mathbb{Q}$, by Schanuel's conjecture the numbers $\log\alpha_1, \dots, \log\alpha_n$ are algebraically independent.

First suppose that $\pi, \log\alpha_1, \dots, \log\alpha_n$ are linearly dependent over $\overline{\mathbb{Q}}$, i.e.

$$\pi = \beta_1 \log\alpha_1 + \dots + \beta_n \log\alpha_n,$$

where $\beta_i \in \overline{\mathbb{Q}}$ and not all of them are zero. Without loss of generality, assume that $\beta_1 \neq 0$. Then $\pi, \log\alpha_2, \dots, \log\alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$. As otherwise

$$\pi = \delta_2 \log\alpha_2 + \dots + \delta_n \log\alpha_n, \quad \delta_i \in \overline{\mathbb{Q}}, \ \delta_i \neq 0 \text{ for some } i,$$

will force that $\log\alpha_1, \dots, \log\alpha_n$ are algebraically dependent, a contradiction. Now applying Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $i\pi, \log\alpha_2, \dots, \log\alpha_n, \log\pi$ we see that $\log\alpha_1, \dots, \log\alpha_n, \log\pi$ are algebraically independent.

Next suppose that π and $\log\alpha_1, \dots, \log\alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$. Then we apply Schanuel's conjecture to the $\mathbb{Q}$-linearly independent numbers $i\pi, \log\alpha_1, \dots, \log\alpha_n, \log\pi$ to get the required result. □

It is worthwhile to mention our motivation for studying the nature of $\log\pi$. The logarithms of the gamma function as well as $\log\pi$ are of central importance in studying the non-vanishing as well as algebraic nature of various special values of a general class of L-functions. An understanding of the nature of $\log\pi$ and $\log\Gamma(x)$ is central to such investigations. We refer to [5] for further elaborations.

3. Transcendence of the log gamma function

We begin by recalling some of the fundamental properties of the gamma function. The reciprocal of the gamma function is an entire function and hence has a product expansion given by

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{\frac{-z}{n}}.$$

Here γ is the elusive Euler's constant. Then, we have the following standard relations:

$$\Gamma(z+1) = z\Gamma(z) \quad \text{(Translation)},$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \text{(Reflection)},$$

$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-na} \Gamma(na) \quad \text{(Multiplication)}.$$

An interesting conjecture due to Rohrlich is the following:

Conjecture *(Rohrlich). Any multiplicative dependence relation of the form*

$$\pi^{n/2} \prod_{a \in \mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}, \quad n, m_a \in \mathbb{Z},$$

is a consequence of the above relations.

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

We shall also require the following properties of the digamma function $\psi(z)$, the logarithmic derivative of the gamma function. For $z \neq 0, -1, \ldots,$ where $\psi(z)$ has simple poles with residue -1, we have

$$\psi(1+z) = \psi(z) + \frac{1}{z}, \tag{3}$$

$$\psi(1-z) = \psi(z) + \pi \cot \pi z \quad \text{for } z \notin \mathbb{Z}, \quad \text{and} \tag{4}$$

$$-\psi(z) - \gamma = \frac{1}{z} + \sum_{n=1}^{\infty} \left\{ \frac{1}{z+n} - \frac{1}{n} \right\}. \tag{5}$$

Here, we consider the logarithm of the gamma function at rational arguments. Even though the gamma function is conjectured to take transcendental values at all rational non-integral arguments, the possibility that the logarithm of gamma function at rationals is algebraic is something which cannot be ruled out at the outset. In this connection, we have

Theorem 3.1. *For any rational number $x \in (0,1)$, the number*

$$\log \Gamma(x) + \log \Gamma(1-x)$$

is transcendental with at most one possible exception.

Proof. Using the reflection property of the gamma function, we have

$$\log \Gamma(x) + \log \Gamma(1-x) = \log \pi + \log \sin \pi x.$$

If x_1 and x_2 are distinct rational numbers with

$$\log \Gamma(x_i) + \log \Gamma(1-x_i) \in \overline{\mathbb{Q}}, \quad i = 1, 2,$$

then their difference $\log \sin \pi x_1 - \log \sin \pi x_2$ is an algebraic number. But this is a non-zero Baker period and hence transcendental. □

As an immediate corollary, we have

Corollary 3.2. *Except for at most one exceptional rational number $x \in (0,1)$, one of the numbers $\log \Gamma(x)$, $\log \Gamma(1-x)$ is transcendental.*

If we assume Schanuel's conjecture, the existence of the fictitious rational alluded above can be ruled out. More precisely

Proposition 3.3. *Schanuel's conjecture implies that*

$$\log \Gamma(x) + \log \Gamma(1-x)$$

is transcendental for every rational $0 < x < 1$.

Proof. As noticed in the previous section, Schanuel's conjecture implies that for any non-zero algebraic number α, the two numbers e^{α} and π are algebraically independent. Suppose $\alpha = \log \Gamma(x) + \log \Gamma(1-x)$ is algebraic. Then since $e^{\alpha} = \frac{\pi}{\sin(\pi x)}$, it contradicts the algebraic independence of e^{α} and π. □

We also have

Theorem 3.4. *Schanuel's conjecture implies that for any $x \in \mathbb{Q}$, at least one of the following statement is true:*

(1) *Both $\Gamma(x)$ and $\Gamma(1-x)$ are transcendental.*
(2) *Both $\log\Gamma(x)$ and $\log\Gamma(1-x)$ are transcendental.*

Proof. If (1) is true, there is nothing to prove. Without loss of generality, suppose that $\Gamma(x)$ is algebraic for some $x \in \mathbb{Q}$. Then $\log\Gamma(x)$ is a Baker period. Since

$$\log\Gamma(1-x) = -\log\Gamma(x) + \log\pi - \log\sin x\pi,$$

therefore by Theorem 2.4, it follows that $\log\Gamma(1-x)$ is transcendental. □

The Hurwitz zeta function is defined by

$$\zeta(s,x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad x \in \mathbb{R},\ x > 0,$$

for $\Re(s) > 1$. The series $\zeta(s,1)$ is the Riemann zeta function. Hurwitz [6] proved that this function extends meromorphically to the entire complex plane with a simple pole at $s = 1$ with residue 1. The continuation of $\zeta(s,x)$ can be enlarged to include all complex values x in the cut complex plane $\mathbb{C} \setminus (\infty, 0]$. In 1894, Lerch [9] established the following formula linking $\Gamma(x)$ and $\zeta(s,x)$:

$$\zeta'(0,x) = \log\Gamma(x) - \frac{1}{2}\log 2\pi.$$

Here the differentiation is with respect to the variable s. Consequently, we have

Theorem 3.5. *Assume Schanuel's conjecture is true. Then at least one of $\Gamma(x)$, $\zeta'(0,x)$, where $x \in \mathbb{Q}$, $0 < x \neq 1$ is transcendental.*

Proof. Note that it is sufficient to prove the result for $0 < x < 1$. If $\Gamma(x)$ is transcendental for all $0 < x < 1$, there is nothing to prove. Suppose for some $0 < x < 1$, $\Gamma(x)$ is algebraic. Then $\log\Gamma(x)$ is a Baker period. By Lerch, we have

$$\zeta'(0,x) = \log\Gamma(x) - \frac{1}{2}\log 2\pi.$$

Thus by Theorem 2.4, it follows that $\zeta'(0,x)$ is transcendental. □

Proposition 3.6. *Let $x \in \mathbb{Q}$ with $0 < x < 1$. Then $\zeta'(0,x) + \zeta'(0,1-x)$ is transcendental except for $x = 1/6$ or $x = 5/6$ where it takes the value zero.*

Proof. Since

$$\zeta'(0,x) + \zeta'(0,1-x) = -\log\sin\pi x - \log 2$$

is a Baker period, it is either zero or transcendental by Baker's theorem. This is zero only when $\sin\pi x = \frac{1}{2}$ i.e. when $x = 1/6$ or $x = 5/6$. □

Finally, in the other direction,

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

Proposition 3.7. *For any algebraic number x other than* 0 *and* 1, $\zeta'(0,x)-\zeta'(0,1+x)$ *is transcendental.*

Proof. Again by Lerch's identity and since $\Gamma(1+x)=x\Gamma(x)$, it is clear that

$$\zeta'(0,x)-\zeta'(0,1+x)=-\log x.$$

Thus by Baker's theorem, it is transcendental. □

4. Transcendence of series of rational function

In this section, we investigate the algebraic nature of some series of the form

$$\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)},\qquad \sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)}$$

where $P(x)$ and $Q(x)$ are polynomials with algebraic coefficients. Our aim is to consider such series with polynomials having arbitrary algebraic roots. We use the Lindemann–Weierstrass theorem and the theorem of Nesterenko to isolate the transcendence nature of many such sums. In some more general set up, these theorems are no longer strong enough and it is the conjecture of Schanuel which is of relevance.

Sums of these type can be regarded as discrete versions of the periods of Kontsevich and Zagier. Denoting the ring of periods by $\mathcal{P}$, we have the following chain of inclusions

$$\overline{\mathbb{Q}}\hookrightarrow\overline{\mathbb{Q}}(\pi)\hookrightarrow\overline{\mathbb{Q}}(\pi)\big(\zeta(3),\zeta(5),\ldots\big)\hookrightarrow\mathcal{P}.$$

Conjecturally, the transcendence degree in the second inclusion above is infinite and hence it is unlikely that we can conclude about the transcendental nature of all such series in total generality, even under an assertion as strong as Schanuel's. First, we have

Theorem 4.1.

(1) *Let α be a non-zero rational number and d be any natural number. Then $\sum_{n=1}^{\infty}\frac{1}{n^2+d\alpha^2}$ is transcendental.*
(2) *Let α be a non-integral rational number and $k>1$ be a natural number. Then $\sum_{n=-\infty}^{\infty}\frac{1}{(n+\alpha)^k}$ is transcendental.*

Proof. (1) Using the properties of the digamma function, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^2+d\alpha^2}&=\frac{1}{2i\sqrt{d}\alpha}\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n-i\sqrt{d}\alpha}-\frac{1}{n}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{n+i\sqrt{d}\alpha}-\frac{1}{n}\right)\right\}\\&=\frac{1}{2i\sqrt{d}\alpha}\left\{\left(-\psi(-i\sqrt{d}\alpha)-\gamma+\frac{1}{i\sqrt{d}\alpha}\right)+\left(\psi(i\sqrt{d}\alpha)+\gamma+\frac{1}{i\sqrt{d}\alpha}\right)\right\}\\&=\frac{1}{2i\sqrt{d}\alpha}\big(\psi(i\sqrt{d}\alpha)-\psi(-i\sqrt{d}\alpha)\big)-\frac{1}{d\alpha^2}\\&=\frac{1}{2i\sqrt{d}\alpha}\left(\frac{i}{\sqrt{d}\alpha}-\pi\cot(\pi i\sqrt{d}\alpha)\right)-\frac{1}{d\alpha^2}\\&=-\frac{1}{2d\alpha^2}-\frac{\pi}{2\sqrt{d}\alpha}\left(\frac{1+e^{2\pi\sqrt{d}\alpha}}{1-e^{2\pi\sqrt{d}\alpha}}\right).\end{aligned}$$

Since by Nesterenko's theorem, $\pi, e^{\pi\sqrt{d}}$ are algebraically independent, the above sum is transcendental.

(2) We know

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^k} = \frac{1}{\alpha} + \frac{(-1)^k}{(k-1)!} D^{k-1}(\pi \cot \pi z)|_{z=\alpha},$$

where $D = \frac{d}{dz}$. It is a consequence of a result of Okada [13] that $D^{k-1}(\pi \cot \pi z)|_{z=\alpha}$ is non-zero. But then it is π^k times a non-zero integer linear combination of algebraic numbers of the form $\csc \pi\alpha, \cot \pi\alpha$. Thus we have the result. □

Further, we have

Theorem 4.2.

(1) *Let $P(x)$ and $Q(x)$ be polynomials with algebraic coefficients. Suppose that* $\deg P \leqslant \deg Q - 2$ *and that Q has simple non-integral zeros $\alpha_1^2, \ldots, \alpha_r^2$ such that $\alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Q}$. Then the sum $\sum_{n=1}^{\infty} \frac{P(\pi n)}{Q(\pi^2 n^2)}$ is transcendental.*

(2) *Let $P(x)$ and $Q(x)$ be polynomials with algebraic coefficients. Suppose that* $\deg P \leqslant \deg Q - 2$ *and that Q has simple non-integral zeros $\alpha_1, \ldots, \alpha_r$. If $\alpha_1, \ldots, \alpha_r$ are linearly independent over $\mathbb{Q}$, the sum $\sum_{n=-\infty}^{\infty} \frac{P(\pi n)}{Q(\pi n)}$ is transcendental. Thus at least one of the two sums $\sum_{n=1}^{\infty} \frac{P(\pi n)}{Q(\pi n)}$, $\sum_{n=1}^{\infty} \frac{P(-\pi n)}{Q(-\pi n)}$ is transcendental.*

Proof. (1) Using partial fractions, we can write $\frac{P(n)}{Q(n)} = \sum_{j=1}^{r} \frac{c_j}{n-\alpha_j^2}$, where $c_j = P(\alpha_j^2)/Q'(\alpha_j^2)$. Then arguing as before, we have

$$\sum_{n=1}^{\infty} \frac{P(\pi n)}{Q(\pi^2 n^2)} = \sum_{j=1}^{r} c_j \sum_{n=1}^{\infty} \frac{1}{(\pi n)^2 - \alpha_j^2} = \sum_{j=1}^{r} \frac{c_j}{2\alpha_j^2} - \frac{i}{2} \sum_{j=1}^{r} \frac{c_j}{\alpha_j} \left(\frac{e^{2i\alpha_j}+1}{e^{2i\alpha_j}-1} \right).$$

By Lindemann–Weierstrass theorem, the second sum in the right-hand side is transcendental.

(2) As before, $\frac{P(n)}{Q(n)} = \sum_{j=1}^{r} \frac{c_j}{n-\alpha_j}$, where $c_j = P(\alpha_j)/Q'(\alpha_j)$. The restriction on the degree of P shows that $\sum_{j=1}^{r} c_j = 0$. Then we have

$$\sum_{n=-\infty}^{\infty} \frac{P(n)}{Q(n)} = \sum_{j=1}^{r} \frac{c_j}{\alpha_j} + \sum_{j=1}^{r} c_j \{ \psi(\alpha_j) - \psi(-\alpha_j) \},$$

and hence

$$\sum_{n=-\infty}^{\infty} \frac{P(\pi n)}{Q(\pi n)} = -i \sum_{j=1}^{r} c_j \left(\frac{e^{2i\alpha_j}+1}{e^{2i\alpha_j}-1} \right).$$

By Lindemann–Weierstrass theorem, the above sum is transcendental. □

Finally, we consider series with arbitrary algebraic coefficients and investigate from the standpoint of Schanuel's conjecture. First, we have

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008

Theorem 4.3. *Assume Schanuel's conjecture is true. Then we have the following:*

(1) *Let α be an algebraic number which is not a rational multiple of i. Then $\sum_{n=1}^{\infty}\frac{1}{n^2+\alpha^2}$ is transcendental. Further, $\sum_{n=1}^{\infty}\frac{1}{n^2+\pi^a}$ is transcendental for any rational number a.*

(2) *Let $P(x)$ and $Q(x)$ be polynomials with algebraic coefficients. Suppose that $\deg P \leqslant \deg Q - 2$ and that Q has simple non-integral zeros $\alpha_1^2,\dots,\alpha_r^2$ such that $1,\alpha_1,\dots,\alpha_r$ are linearly independent over $\mathbb{Q}$. Then the sum $\sum_{n=1}^{\infty}\frac{P(n)}{Q(n^2)}$ is transcendental.*

Proof. (1) As before,

$$\sum_{n=1}^{\infty}\frac{1}{n^2+\alpha^2} = -\frac{1}{2\alpha^2} - \frac{\pi}{2\alpha}\left(\frac{1+e^{2\pi\alpha}}{1-e^{2\pi\alpha}}\right).$$

By Proposition 2.3, $\frac{\pi}{2\alpha}(\frac{1+e^{2\pi\alpha}}{1-e^{2\pi\alpha}})$ is transcendental. Also since

$$\sum_{n=1}^{\infty}\frac{1}{n^2+\pi^a} = -\frac{1}{2\pi^a} - \frac{\pi}{2\pi^{a/2}}\left(\frac{1+e^{2\pi^{a/2+1}}}{1-e^{2\pi^{a/2+1}}}\right),$$

by Proposition 2.3, we get the desired result.

(2) Writing $\frac{P(n)}{Q(n)} = \sum_{j=1}^{r}\frac{c_j}{n-\alpha_j^2}$, where $c_j = P(\alpha_j^2)/Q'(\alpha_j^2)$, as before we have

$$\sum_{n=1}^{\infty}\frac{P(n)}{Q(n^2)} = \sum_{j=1}^{r} c_j \sum_{n=1}^{\infty}\frac{1}{n^2-\alpha_j^2} = \sum_{j=1}^{r}\frac{c_j}{2\alpha_j^2} - \frac{i\pi}{2}\sum_{j=1}^{r}\frac{c_j}{\alpha_j}\left(\frac{e^{2i\pi\alpha_j}+1}{e^{2i\pi\alpha_j}-1}\right).$$

By Proposition 2.3, the second sum above is necessarily transcendental. □

Finally considering sums over all integers, we have

Theorem 4.4. *Let $P(x)$ and $Q(x)$ be polynomials with algebraic coefficients. Suppose that $\deg P \leqslant \deg Q - 2$ and that Q has non-integral zeros $\alpha_1,\dots,\alpha_r$ with multiplicities $m_1,\dots,m_r$ respectively. If $1,\alpha_1,\dots,\alpha_r$ are linearly independent over $\mathbb{Q}$, the sum $\sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)}$ is transcendental under Schanuel's conjecture. In particular, at least one of the two sums $\sum_{n=1}^{\infty}\frac{P(n)}{Q(n)}$, $\sum_{n=1}^{\infty}\frac{P(-n)}{Q(-n)}$ is transcendental.*

Proof. Using partial fractions, we can write

$$\frac{P(n)}{Q(n)} = \sum_{j=1}^{r}\sum_{l=1}^{m_j}\frac{c_{j,l}}{(n-\alpha_j)^l},$$

where

$$c_{j,l} = D^{m_j-r}\left[\frac{P(x)}{Q(x)}(x-\alpha_j)^{m_j}\right]_{x=\alpha_j}, \quad D = \frac{d}{dx}.$$

Thus, we have

$$\sum_{n=-\infty}^{\infty}\frac{P(n)}{Q(n)} = \sum_{n=-\infty}^{\infty}\left\{\sum_{j=1}^{r}\sum_{l=1}^{m_j}\frac{c_{j,l}}{(n-\alpha_j)^l}\right\}$$

$$= \sum_{j=1}^{r}\sum_{l=1}^{m_j}\left\{\frac{c_{j,l}}{\alpha_j} + \frac{(-1)^l c_{j,l}}{(l-1)!} D^{l-1}(\pi \cot \pi x)|_{x=-\alpha_j}\right\}$$

$$= \sum_{j=1}^{r}\sum_{l=1}^{m_j}\frac{c_{j,l}}{\alpha_j} + \frac{(-1)^l c_{j,l}}{(l-1)!} \sum_{j=1}^{r}\sum_{l=1}^{m_j} D^{l-1}(\pi \cot \pi x)|_{x=-\alpha_j}.$$

Since $\sum_{j=1}^{r}\sum_{l=1}^{m_j} D^{l-1}(\pi \cot \pi x)|_{x=-\alpha_j}$ is an algebraic linear combination of rational functions involving $\pi, e^{i\pi\alpha_j}, e^{-i\pi\alpha_j}$, $1 \leqslant j \leqslant r$. Since $1, \alpha_1, \dots, \alpha_r$ are linear independent over $\mathbb{Q}$, using Proposition 2.3 we get the result. □

It is worth mentioning that in the above cases, we are not able to deal with the case when the polynomial $Q(x)$ has integer roots which seems to suggest that the transcendence of the Riemann zeta function at odd positive integers is beyond the realm of Schanuel's conjecture. Finally, while the theorem of Baker or even the conjecture of Schanuel helps in establishing the transcendence of such series, we do not seem to have any such general theory which is tailor-made to establish a weaker assertion, namely the irrationality of such series.

References

[1] S.D. Adhikari, N. Saradha, T.N. Shorey, R. Tijdeman, Transcendental infinite sums, Indag Math. 12 (1) (2001) 1–14.
[2] P. Bundschuh, Zwei Bemerkungen über transzendente Zahlen, Monatsh. Math. 88 (4) (1979) 293–304.
[3] G.V. Chudnovsky, Algebraic independence of constants connected with the exponential and elliptic functions, Dokl. Akad. Nauk Ukrain. SSR Ser. A 8 (1976) 698–701.
[4] P. Grinspan, Measures of simultaneous approximation for quasi-periods of abelian varieties, J. Number Theory 94 (1) (2002) 136–176.
[5] S. Gun, R. Murty, P. Rath, Transcendental nature of special values of L-functions, Canad. J. Math., in press.
[6] A. Hurwitz, Einige Eigenschaften der Dirichlet'schen Funktionen $F(s) = \sum(D/n)n^{-s}$, die bei der Bestimmung der Klassenzahlen Binärer quadratischer Formen auftreten, Z. Math. Phys. 27 (1882) 86–101.
[7] M. Kontsevich, D. Zagier, Periods, in: Mathematics Unlimited-2001 and Beyond, Springer-Verlag, 2001, pp. 771–808.
[8] Serge Lang, Introduction to Transcendental Numbers, Addison–Wesley Publishing Co., Reading, MA, 1966.
[9] M. Lerch, Dalsi studie v oboru Malmstenovskych rad, Rozpravy Ceske Akad. 18 (3) (1894), 63 pp.
[10] M. Ram Murty, N. Saradha, Transcendental values of the digamma function, J. Number Theory 125 (2) (2007) 298–318.
[11] Y.V. Nesterenko, Modular functions and transcendence, Mat. Sb. 187 (9) (1996) 65–96.
[12] Y.V. Nesterenko, P. Philippon (Eds.), Introduction to Algebraic Independence Theory, Lecture Notes in Math., 1752.
[13] T. Okada, On an extension of a theorem of S. Chowla, Acta Arith. 38 (4) (1980/1981) 341–345.
[14] T. Schneider, Zur Theorie der Abelschen Funktionen und Integrale, J. Reine Angew. Math. 183 (1941) 110–128.
[15] Michel Waldschmidt, Diophantine Approximation on Linear Algebraic Groups, Grundlehren Math. Wiss. (Fundamental Principles of Mathematical Sciences), vol. 326, Springer-Verlag, Berlin, 2000.

Please cite this article in press as: S. Gun et al., Transcendence of the log gamma function and some discrete periods, J. Number Theory (2009), doi:10.1016/j.jnt.2009.01.008