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Commutators of Calderón-Zygmund operators related to admissible functions on spaces of homogeneous type and applications to Schrödinger operators

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Abstract Let $\mathcal{X}$ be an RD-space. In this paper, the authors establish the boundedness of the commutator $T_b f = bTf - T(bf)$ on L^p, $p \in (1, \infty)$, where T is a Calderón-Zygmund operator related to the admissible function ρ and $b \in BMO_\theta(\mathcal{X}) \supseteq BMO(\mathcal{X})$. Moreover, they prove that T_b is bounded from the Hardy space $H^1_\rho(\mathcal{X})$ into the weak Lebesgue space $L^1_{\text{weak}}(\mathcal{X})$. This can be used to deal with the Schrödinger operators and Schrödinger type operators on the Euclidean space $\mathbb{R}^n$ and the sub-Laplace Schrödinger operators on the stratified Lie group $\mathbb{G}$.

Keywords commutator, spaces of homogeneous type, stratified Lie groups, admissible function, Hardy space, reverse Hölder inequality, Riesz transform, Schrödinger operators

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1 Introduction

Let $(\mathcal{X}, d, \mu)$ be an RD-space with a regular Borel measure μ such that all balls defined by the quasi-metric d have finite and positive measure and are open sets. For any $x \in \mathcal{X}$ and $r > 0$, set the ball $B(x,r) = \{y \in \mathcal{X} : d(x,y) < r\}$. In what follows, for any $x, y \in \mathcal{X}$ and $r \in (0,\infty)$, set $V_r(x) = \mu(B(x,r))$ and $V(x,y) = \mu(B(x, d(x,y)))$. Also, let T be a bounded operator on $L^p(\mathcal{X})$ for some $p \in (1,\infty)$. A measurable function $K(x,y)$ is called the kernel of T provided that

$$T(f)(x) = \int_{\mathcal{X}} K(x,y) f(y) d\mu(y) \tag{1.1}$$

holds for each continuous function f with compact support, and for almost all x not in the support of f. In this paper, we consider the commutator

$$T_b(f)(x) = T(bf)(x) - b(x)Tf(x), \quad x \in \mathcal{X}, \tag{1.2}$$

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where $b \in BMO_\infty(\rho)$ (see (1.9)).

It is well known that when T is a Calderón-Zygmund operator, Coifman et al. [10] proved that $[b, T]$ is a bounded operator on L^p for $1 < p < \infty$ if and only if $b \in BMO(\mathbb{R}^n)$. See [20, 32, 37] for the research development of the commutator T_b on the Euclidean space $\mathbb{R}^n$ and [3, 11, 35] on the spaces of homogeneous type.

In recent years, the harmonic analysis problems of differential operators (for example, Schrödinger operators and elliptic operators, and so on) have received many people's attention. On the one hand, some scholars pay more attention to the investigation of the Schrödinger operators; see [2, 13–15, 25, 28–30, 36] and their references. Moreover, Yang et al. extended some important problems related to the Schrödinger operators to the more abstract setting (cf. [39–41]). On the other hand, some scholars concentrated on the research of other differential operators; see [7, 12, 18, 19, 21–24, 27, 38, 42] and their references.

Motivated by [2, 15, 26, 39], in this paper we investigate the L^p estimates and the endpoint estimates for T_b on the space of the homogeneous type $\mathcal{X}$ when the kernel $K(x, y)$ satisfies some conditions related to the admissible function. Our main results can be used to study the Schrödinger operators and Schrödinger type operators on $\mathbb{R}^n$ and to study the sub-Laplace Schrödinger operators on the stratified Lie group $\mathbb{G}$, and then to derive some new results including Lemma 4.1 and Corollary 4.2.

The notion of admissible functions on the spaces of homogeneous type was first introduced by Yang and Zhou in [41]. A positive function ρ on $\mathcal{X}$ is called admissible if there exist positive constants C_3 and k_0 such that for all $x, y \in \mathcal{X}$,

$$\rho(y) \leqslant C_3[\rho(x)]^{\frac{1}{1+k_0}}[\rho(x) + d(x, y)]^{\frac{k_0}{1+k_0}}. \tag{1.3}$$

A nontrivial class of admissible function is the well-known reverse Hölder class $\mathcal{B}_q(\mathcal{X}, d, \mu)$. Recall that a nonnegative function potential U is said to belong to $\mathcal{B}_q(\mathcal{X}, d, \mu)$ with $q \in (1, \infty]$ if there exists a positive constant C such that for all balls B,

$$\left(\frac{1}{\mu(B)}\int_B U(y)^q\, d\mu(y)\right)^{\frac{1}{q}} \leqslant C\left(\frac{1}{\mu(B)}\int_B U(y)\, d\mu(y)\right) \tag{1.4}$$

with usual modification when $q = \infty$. Following [36] and [39], for all $x \in \mathcal{X}$, set

$$\rho(x) \doteq \sup_{r>0}\left\{r : \frac{r^2}{V_r(x)}\int_{B(x,r)} U(y)\, dy \leqslant 1\right\}. \tag{1.5}$$

It follows from Proposition 2.1 in [39] that if the measure $U(z)d\mu(z)$ has the doubling property, then ρ as in (1.5) is an admissible function, where $q \in (\max\{1, \frac{n}{2}\}, \infty]$ with n appearing in (2.3) and $U \in \mathcal{B}_q(\mathcal{X}, d, \mu)$.

Let T be an operator defined as in (1.1) with the kernel $K(x, y)$. In this paper, we always assume that T is a Calderón-Zygmund operator related to the admissible function ρ, that is, T and its kernel $K(x, y)$ satisfy the following conditions:

(a) T is a bounded operator on $L^2(\mathcal{X})$;

(b) For every l there exists a positive constant C_l such that

$$|K(x, y)| \leqslant \frac{C_l}{(1 + \frac{d(x,y)}{\rho(x)})^l V(x, y)}; \tag{1.6}$$

(c) For every l there exists a positive constant C_l such that

$$|K(x, z) - K(y, z)| \leqslant \frac{C_l}{(1 + \frac{d(x,z)}{\rho(x)})^l}\frac{d(x, y)^\delta}{V(x, z)d(x, z)^\delta}, \tag{1.7}$$

or

$$|K(z, x) - K(z, y)| \leqslant \frac{C_l}{(1 + \frac{d(x,z)}{\rho(x)})^l}\frac{d(x, y)^\delta}{V(x, z)d(x, z)^\delta}, \tag{1.8}$$

whenever $d(x, y) < \frac{1}{2}d(x, z)$, $\delta \in (0, 1]$.

Remark 1.1. It follows from [8] that the above operator T is bounded on $L^p(\mathcal{X})$ for $1 < p < \infty$ and is of weak type $(1,1)$.

Following [2], we define the class $BMO_\theta(\rho)$ of locally integrable function b such that

$$\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |b(y) - b_B| d\mu(y) \leqslant C\left(1 + \frac{r}{\rho(x)}\right)^\theta, \tag{1.9}$$

for all $x \in \mathcal{X}$ and $r > 0$, where $\theta > 0$ and $b_B = \frac{1}{\mu(B)}\int_B b(y)d\mu(y)$. A norm for $b \in BMO_\theta(\rho)$, denoted by $[b]_\theta$ is given by the infimum of the constants satisfying (1.9), after identifying functions that differ upon a constant. If $\theta = 0$ in (1.9), then $BMO_\theta(\rho)$ is exactly the John-Nirenberg space $BMO(\mathcal{X})$. Denote $BMO_\infty(\rho) = \bigcup_{\theta>0} BMO_\theta(\rho)$. It is easy to see that $BMO \subset BMO_\theta(\rho) \subset BMO_{\theta'}(\rho)$ for $0 < \theta \leqslant \theta'$. Hence $BMO(\mathcal{X}) \subset BMO_\infty(\rho)$. When $\mathcal{X} = \mathbb{R}^n$ and ρ is defined as (1.5), Bongioanni et al. [2] gave some examples to clarify that the space $BMO(\mathbb{R}^n)$ is a subspace of $BMO_\infty(\rho)$. Moreover, it follows from [39] that $BMO_\rho(\mathcal{X})$ is the dual space of $H^1_\rho(\mathcal{X})$. Then by duality we have $BMO_\rho(\mathcal{X}) \subseteq BMO(\mathcal{X})$. Therefore, $BMO_\rho(\mathcal{X})$ is also a subspace of $BMO_\infty(\rho)$.

Now, we are in a position to state our first result.

Theorem 1.2. *Let ρ be an admissible function and $b \in BMO_\infty(\rho)$. Assume that T is an operator satisfying the above conditions* (a), (b) *and* (c). *Then, for $1 < p < \infty$,*

$$\|T_b f\|_{L^p(\mathcal{X})} \leqslant C[b]_\theta \|f\|_{L^p(\mathcal{X})} \tag{1.10}$$

for all $f \in L^p(\mathcal{X})$, where $\theta > 0$.

To obtain the endpoint estimate for T_b, we need to introduce the Hardy space $H^1_\rho(\mathcal{X})$ defined by the grand maximal function associated to ρ (see [39] or Subsection 2.2).

Our second result can be stated as follows.

Theorem 1.3. *Let ρ be an admissible function and $b \in BMO_\theta(\rho)$ with $\theta < \frac{\delta}{k_0+1}$, where k_0 appears in* (2.6) *and δ appears in* (1.7) *and* (1.8). *Assume that T is an operator satisfying the above conditions* (a), (b) *and* (c). *Then, for any $\lambda > 0$,*

$$\mu(\{x \in \mathcal{X} : |T_b f(x)| > \lambda\}) \leqslant \frac{C[b]_\theta}{\lambda}\|f\|_{H^1_\rho(\mathcal{X})}, \quad \forall f \in H^1_\rho(\mathcal{X}). \tag{1.11}$$

Namely, the commutator T_b is bounded from $H^1_\rho(\mathcal{X})$ into $L^1_{\text{weak}}(\mathcal{X})$.

It is worth mentioning that Theorem 1 in [2] is a special case of Theorem 1.2 in this paper and Theorem 4.1 in [26] is also a special case of Theorem 1.3 in this paper when $\mathcal{X} = \mathbb{R}^n$ and $U \in \mathcal{B}_q(\mathbb{R}^n, |\cdot|, dx)$ with $q \geqslant n$. Similar to the case in [2], we can enlarge the class of functions b with respect to the classical case because the kernels of T have stronger decay and some continuity.

Compared with the proofs in [2] and [26], the proofs of our main results in this paper become more complicated on the space of the homogeneous type than on the Euclidean space. In particular, our results can be applied to handle the Schrödinger type operator on $\mathbb{R}^n$ and the sub-Laplace Schrödinger operators on the stratified Lie group $\mathbb{G}$, while they were not investigated in [2] and [26]. Moreover, our main results can also be applied to handle divergence form elliptic operators plus a positive potential satisfying the reverse Hölder inequality when their matrix coefficients and potential satisfy a stronger smoothness condition. Here, we omit the details for this problem.

This paper is organized as follows. In Section 2, we recall some basic facts for the spaces of the homogeneous type, the admissible function $\rho(x)$ and the Hardy space H^1_ρ. Moreover, we give some lemmas related to BMO spaces $BMO_\theta(\rho)$. In Section 3, we prove Theorems 1.2 and 1.3. Section 4 gives some applications of our main results in this paper.

Throughout this paper, the letter C stands for a constant and is not necessarily the same at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leqslant \frac{B_1}{B_2} \leqslant C$. Moreover, for the ball $B = B(x,r)$, we denotes the ball MB by $MB = B(x, Mr)$, where M is a positive constant.

2 Preliminary lemmas and propositions

In this section, we first recall the spaces of the homogeneous type in the sense of Coifman and Weiss [8,9] and RD-spaces in [17].

Given a set $\mathcal{X}$, a function $d:\mathcal{X}\times\mathcal{X}\to R_0^+$ is called a quasi-metric on $\mathcal{X}$ if the following conditions are satisfied:

(i) For every x and y in $\mathcal{X}$, $d(x,y)\geqslant 0$ and $d(x,y)=0$ if and only if $x=y$;

(ii) For every x and y in $\mathcal{X}$, $d(x,y)=d(y,x)$;

(iii) There exists a constant $K\geqslant 1$ such that

$$d(x,y)\leqslant K(d(x,z)+d(z,y)) \tag{2.1}$$

for every x,y and z in $\mathcal{X}$. We shall say that two quasi-metrics d and d' on $\mathcal{X}$ are equivalent if there exist two positive constants c_1 and c_2 such that $c_1d'(x,y)\leqslant d(x,y)\leqslant c_2d'(x,y)$ for all $x,y\in\mathcal{X}$. In particular, equivalent quasi-metrics induce the same topology on $\mathcal{X}$.

Let μ be a regular Borel measure on the σ-algebra of the subsets of $\mathcal{X}$ which contains the balls $B(x,r)=\{y:d(x,y)<r\}$. The triple $(\mathcal{X},d,\mu)$ is called a space of the homogeneous type if there exists a positive constant C_1 such that for all $x\in\mathcal{X}$ and $r>0$,

$$\mu(B(x,2r))\leqslant C_1\mu(B(x,r))<\infty. \tag{2.2}$$

Moreover, the triple $(\mathcal{X},d,\mu)$ is called an RD-space if there exist constants $0<\kappa\leqslant n$ and $C_2\geqslant 1$ such that for all $x\in\mathcal{X}$, $0<r<\frac{\mathrm{diam}(\mathcal{X})}{2}$ and $1\leqslant\lambda<\frac{\mathrm{diam}(\mathcal{X})}{2r}$,

$$(C_2)^{-1}\lambda^{\kappa}\mu(B(x,r))\leqslant\mu(B(x,\lambda r))\leqslant C_2\lambda^n\mu(B(x,r)), \tag{2.3}$$

where $\mathrm{diam}(\mathcal{X})=\sup_{x,y\in\mathcal{X}}d(x,y)$ and the parameter n is a measure of the dimension of the space.

The following proposition is due to Macías and Segovia [33] (see also Theorem 2.3 in [1]).

Proposition 2.1. *Let $(\mathcal{X},d,\mu)$ be a space of the homogeneous type. Then there exists a quasi-metric δ on $\mathcal{X}$ which is equivalent to d such that, for $x\in\mathcal{X}$, $0<r\leqslant 6K^3R$ and $y\in B_\delta(x,R)=\{y\in\mathcal{X}:\delta(x,y)<R\}$, we have*

$$\mu(B_\delta(x,R)\cap B_\delta(y,r))\geqslant C\mu(B_\delta(y,r)), \tag{2.4}$$

where $C>0$ depends only on the constants of the space. Moreover,

$$\delta(x,y)\leqslant d(x,y)\leqslant 3K^2\delta(x,y), \tag{2.5}$$

for every x and y in $\mathcal{X}$. The balls $B_\delta(x,R)$ endowed with the restrictions of the quasi-metric δ and the measure μ become bounded spaces of the homogeneous type with constants K' and C_1, satisfying (2.1) *and* (2.2) *respectively, independent of $R>0$ and $x\in\mathcal{X}$.*

Following the above proposition, we can always assume that the balls B in $\mathcal{X}$ endowed with the restrictions of the quasi-metric d and the measure μ become bounded spaces of the homogeneous type and the balls B always satisfy (2.4) throughout the paper. In addition, we also assume that $\mathcal{X}$ is an RD-space and $\mu(\mathcal{X})=\infty$.

In particular, we should point out that the results which we cite in [39] are valid even if d is a quasi-metric instead of metric, see Section 2 in [39].

2.1 Properties of admissible functions

In this subsection, we recall some properties of admissible functions proved in Subsection 2.1 in [39].

Lemma 2.2. *Let ρ be an admissible function. Then*

(i) *for any $\widetilde{C}>0$, there exists a positive constant C, depending on $\widetilde{C}>0$, such that if $d(x,y)\leqslant\widetilde{C}\rho(x)$, then $C^{-1}\rho(y)\leqslant\rho(x)\leqslant C\rho(y)$;*

(ii) *there exists a positive constant C such that for all $x, y \in \mathcal{X}$,*

$$C^{-1}[\rho(x) + d(x,y)] \leqslant \rho(y) + d(x,y) \leqslant C[\rho(x) + d(x,y)];$$

(iii) *there exists a positive constant C_4 such that for all $x, y \in \mathcal{X}$,*

$$\rho(y) \geqslant C_4[\rho(x)]^{1+k_0}[\rho(x) + d(x,y)]^{-k_0}.$$

Using (ii) and (iii) of Lemma 2.2, we immediately obtain the following corollary.

Corollary 2.3. *There exists $k_0 > 0$ such that, for any x and y in $\mathcal{X}$,*

$$C^{-1}\rho(x)\left(1 + \frac{d(x,y)}{\rho(x)}\right)^{-k_0} \leqslant \rho(y) \leqslant C\rho(x)\left(1 + \frac{d(x,y)}{\rho(x)}\right)^{\frac{k_0}{k_0+1}}. \tag{2.6}$$

A ball $B(x, \rho(x))$ is called critical. Due to Lemma 2.3 in [39], we have the following covering lemma on $\mathcal{X}$.

Proposition 2.4. *There exists a sequence of points $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{X}$, such that the family of critical balls $Q_k = B(x_k, \rho(x_k)), k \geqslant 1$, satisfies*

(i) $\bigcup_k Q_k = \mathcal{X}$;

(ii) *there exists $N = N(\rho)$ such that for every $k \in \mathbb{N}$,* $\mathrm{card}\{j : 4Q_j \cap 4Q_k \neq \emptyset\} \leqslant N$.

2.2 Hardy space $H^1_\rho(\mathcal{X})$

The Hardy space $H^1_\rho(\mathcal{X})$, which will be used to obtain the endpoint estimates of the commutators T_b, was introduced by Yang and Zhou [39].

For this purpose, we first recall the spaces of test functions $\mathcal{G}(x, r, \beta, \gamma)$ which play an important role in the theory of functions on a space of the homogeneous type (cf. [16, 17, 39]).

Definition 2.5. Let $x \in \mathcal{X}, r > 0, \beta \in (0, 1]$ and $\gamma > 0$. A function f on $\mathcal{X}$ is said to belong to the space of test functions, $\mathcal{G}(x, r, \beta, \gamma)$, if there exists a positive constant C_f such that

(i)

$$|f(y)| \leqslant C_f \frac{1}{V_r(x) + V(x,y)}\left[\frac{r}{r + d(x,y)}\right]^{\gamma} \quad \text{for all} \quad y \in \mathcal{X};$$

(ii)

$$|f(y) - f(y')| \leqslant C_f \left[\frac{d(y,y')}{r + d(x,y)}\right]^{\beta} \frac{1}{V_r(x) + V(x,y)}\left[\frac{r}{r + d(x,y)}\right]^{\gamma}$$

for all $y, y' \in \mathcal{X}$ satisfying the fact that $d(y, y') \leqslant \frac{[r+d(x,y)]}{2}$. Moreover, for any $f \in \mathcal{G}(x, r, \beta, \gamma)$, its norm is defined by

$$\|f\|_{\mathcal{G}(x,r,\beta,\gamma)} \equiv \inf\{C_f : \text{(i) and (ii) hold}\}.$$

Note that $\mathcal{G}(x, r, \beta, \gamma)$ is a Banach space. Let $\varepsilon \in (0, 1]$ and $\beta, \gamma \in (0, \varepsilon]$. Define the space $\mathcal{G}_0^\varepsilon(x, r, \beta, \gamma)$ to be the completion of the set $\mathcal{G}(x, r, \varepsilon, \varepsilon)$ in $\mathcal{G}(x, r, \beta, \gamma)$. For $f \in \mathcal{G}_0^\varepsilon(x, r, \beta, \gamma)$, define $\|f\|_{\mathcal{G}_0^\varepsilon(x,r,\beta,\gamma)} = \|f\|_{\mathcal{G}(x,r,\beta,\gamma)}$. Let $(\mathcal{G}_0^\varepsilon(x, r, \beta, \gamma))'$ be the set of all continuous linear functionals on $\mathcal{G}_0^\varepsilon(x, r, \beta, \gamma)$. Throughout this section, we fix $x_1 \in \mathcal{X}$ and write $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_1, 1, \beta, \gamma)$ and $(\mathcal{G}_0^\varepsilon(\beta, \gamma))' = (\mathcal{G}_0^\varepsilon(x_1, 1, \beta, \gamma))'$.

Definition 2.6. Let $\varepsilon_1 \in (0, 1], \varepsilon_2 > 0$, $\varepsilon \in (0, \min\{\varepsilon_1, \varepsilon_2\})$ and ρ be an admissible function. For any $\beta, \gamma \in (0, \varepsilon)$, $f \in (\mathcal{G}(\beta, \gamma))'$ and $x \in \mathcal{X}$, define the grand maximal function $G_\rho^{(\varepsilon,\beta,\gamma)}(f)$ associated to ρ by

$$G_\rho^{(\varepsilon,\beta,\gamma)}(f)(x) \equiv \sup\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{G}_0^\varepsilon(\beta, \gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \leqslant 1 \text{ for some } r \in (0, \rho(x))\}.$$

Definition 2.7. Let $\varepsilon \in (0, 1], \beta, \gamma \in (0, \varepsilon)$ and ρ be an admissible function. The Hardy space $H^1_\rho(\mathcal{X})$ associated to ρ is defined by

$$H^1_\rho(\mathcal{X}) \equiv \{f \in (\mathcal{G}(\beta, \gamma))' : \ \|f\|_{H^1_\rho(\mathcal{X})} \equiv \|G_\rho^{(\varepsilon,\beta,\gamma)}(f)\|_{L^1(\mathcal{X})} < \infty\}.$$

Definition 2.8. Let $1 < q \leqslant \infty$. A measurable function a is called a $(1,q)_\rho$-atom associated to the ball $B(x,r)$ if $r < \rho(x)$ and the following conditions hold:

(i) $\operatorname{supp} a \subset B(x,r)$ for some $x \in \mathcal{X}$ and $r > 0$,

(ii) $\|a\|_{L^q(\mathcal{X})} \leqslant \mu(B(x,r))^{\frac{1}{q}-1}$,

(iii) when $r < \frac{\rho(x)}{4}$, $\int_{\mathcal{X}} a(x)d\mu(x) = 0$.

Definition 2.9. Let $\varepsilon \in (0,1], \beta,\gamma \in (0,\varepsilon)$ and $q \in (1,\infty]$. The space $H^{1,q}_\rho(\mathcal{X})$ is defined to be the set of all $f = \sum_{j\in\mathbb{N}} \lambda_j a_j$ in $(\mathcal{G}^\varepsilon_0(\beta,\gamma))'$, where $\{a_j\}_{j\in\mathbb{N}}$ are $(1,q)_\rho$-atoms and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ such that $\sum_{j\in\mathbb{N}} |\lambda_j| < \infty$. For any $f \in H^{1,q}_\rho(\mathcal{X})$, define $\|f\|_{H^{1,q}_\rho(\mathcal{X})} \equiv \inf\{\sum_{j\in\mathbb{N}} |\lambda_j|\}$, where the infimum is taken over all the above decompositions of f.

Furthermore, Yang and Zhou [39] gave the atomic decomposition characterization of $H^1_\rho(\mathcal{X})$ which plays an important role in the proof of our second result.

Proposition 2.10. *Let ρ be an admissible function and $q \in (1,\infty]$. Then $H^1_\rho(\mathcal{X}) = H^{1,q}_\rho(\mathcal{X})$ with equivalent norms.*

2.3 Some lemmas related to BMO spaces $BMO_\theta(\rho)$

Similar to the proofs of Proposition 3 and Lemma 2 in [2], we have the following proposition and lemmas.

Proposition 2.11. *Let $\theta > 0$ and $1 \leqslant s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\left(\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |b(y) - b_B|^s d\mu(y)\right)^{\frac{1}{s}} \leqslant C[b]_\theta \left(1 + \frac{r}{\rho(x)}\right)^{\theta'}, \tag{2.7}$$

for all $B = B(x,r)$, with $x \in \mathcal{X}$ and $r > 0$, where $\theta' = (1+k_0)\theta$ and k_0 is the constant appearing in (iii) *in Lemma* 2.2.

Proof. From the John-Nirenberg inequality on a space of the homogeneous type (see [3] or [5]), given a ball B_0 and $g \in BMO(B_0)$ we obtain that, for every $1 \leqslant s < \infty$,

$$\left(\frac{1}{\mu(B)}\int_B |g - g_B|^s d\mu(y)\right)^{\frac{1}{s}} \leqslant C\|g\|_{BMO(B_0)}, \tag{2.8}$$

for every ball $B \subseteq B_0$, where the constant C is independent of the ball B_0. Therefore, to prove (2.7) we only need to show the claim: if $R \geqslant 1$ and Q is a critical ball, then we have $b \in BMO(RQ)$ and

$$\|b\|_{BMO(RQ)} \leqslant C[b]_\theta(1+R)^{(k_0+1)\theta}. \tag{2.9}$$

In fact, if (2.9) holds, by using (2.8) we conclude that for any ball $B \subseteq RQ$,

$$\left(\frac{1}{\mu(B)}\int_B |b - b_B|^s d\mu(y)\right)^{\frac{1}{s}} \leqslant C[b]_\theta(1+R)^{(k_0+1)\theta}. \tag{2.10}$$

Let $B = B(x,r)$ and $Q = B(x,\rho(x))$, with $x \in \mathcal{X}$ and $r > 0$. If $r \leqslant \rho(x)$, we choose $R = 1$ and apply (2.10) to get (2.7). In the case $r > \rho(x)$, $B = \frac{r}{\rho(x)}Q$. Then we apply (2.10) with $R = \frac{r}{\rho(x)}$ which yields (2.7).

It remains to prove the claim. Let $B = B(z,r) \subset RQ$, with $z \in \mathcal{X}$ and $r > 0$. Following (2.6), we have

$$\rho(x)(1+R^{-k_0}) \leqslant \rho(x)\left(1 + \frac{d(z,x)}{\rho(x)}\right) \leqslant C\rho(z).$$

Then, since $r < R\rho(x)$,

$$\frac{r}{\rho(z)} \leqslant C\frac{r}{\rho(x)}(1+R)^{k_0} \leqslant C(1+R)^{k_0+1}.$$

From the fact that $b \in BMO_\theta(\rho)$ it follows that

$$\frac{1}{\mu(B)}\int_B |b - b_B| d\mu(y) \leqslant C[b]_\theta(1+R)^{(k_0+1)\theta}.$$ □

Lemma 2.12. *Let $b \in BMO_\theta(\rho)$, $B = B(x_0, r)$ and $s \geqslant 1$. Then*

$$\left(\frac{1}{\mu(2^kB)}\int_{2^kB}|b(y)-b_B|^s d\mu(y)\right)^{\frac{1}{s}} \leqslant C[b]_\theta k\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\theta'}, \tag{2.11}$$

for all $k \in \mathbb{N}$, with θ' as in (2.7).

Proof. Due to Proposition 2.11, we have

$$\begin{aligned}
&\left(\frac{1}{\mu(2^kB)}\int_{2^kB}|b(y)-b_B|^s d\mu(y)\right)^{\frac{1}{s}} \\
&\quad\leqslant \left(\frac{1}{\mu(2^kB)}\int_{2^kB}|b(y)-b_{2^kB}|^s d\mu(y)\right)^{\frac{1}{s}} + \sum_{j=1}^{k}|b_{2^jB}-b_{2^{j-1}B}| \\
&\quad\leqslant C[b]_\theta \sum_{j=1}^{k}\left(1+\frac{2^jr}{\rho(x_0)}\right)^{\theta'} \leqslant C[b]_\theta k\left(1+\frac{2^kr}{\rho(x_0)}\right)^{\theta'}.
\end{aligned}$$

□

We borrow the idea from [2] and define the following maximal function on the space of the homogeneous type $\mathcal{X}$. Given $\alpha > 0$, we define the following maximal functions for $g \in L^1_{\rm loc}(\mathcal{X})$ and $x \in \mathcal{X}$,

$$M_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{\mu(B)}\int_B |g|,$$
$$M^\sharp_{\rho,\alpha}g(x) = \sup_{x\in B\in\mathcal{B}_{\rho,\alpha}} \frac{1}{\mu(B)}\int_B |g-g_B|,$$

where $\mathcal{B}_{\rho,\alpha} = \{B(y,r) : y \in \mathcal{X}, r \leqslant \alpha\rho(y)\}$.

Also, given a ball $Q \subset \mathcal{X}$, for $g \in L^1_{\rm loc}(Q)$ and $x \in Q$, we define

$$M_Q g(x) = \sup_{x\in B\in\mathcal{F}(Q)} \frac{1}{\mu(B\cap Q)}\int_{B\bigcap Q}|g|, \tag{2.12}$$

and

$$M^\sharp_Q g(x) = \sup_{x\in B\in\mathcal{F}(Q)} \frac{1}{\mu(B\cap Q)}\int_{B\bigcap Q}|g-g_B|, \tag{2.13}$$

where $\mathcal{F}(Q) = \{B(y,r) : y \in Q, r > 0\}$.

Lemma 2.13. *For $1 < p < \infty$, there exist β and γ such that if $\{Q_k\}_{k=1}^\infty$ is a sequence of balls as in Proposition* 2.4*, then*

$$\int_{\mathcal{X}}|M_{\rho,\beta}(g)|^p \leqslant C\left(\int_{\mathcal{X}}|M^\sharp_{\rho,\gamma}(g)|^p + \sum_k |Q_k|\left(\frac{1}{\mu(Q_k)}\int_{2Q_k}|g|\right)^p\right), \tag{2.14}$$

for all $g \in L^1_{\rm loc}(\mathcal{X})$.

Proof. Let $Q = B(x_0, \rho(x_0))$ be a critical ball and $x, y \in Q$. Then by (2.6) we have

$$\rho(y) \leqslant C_0\rho(x), \tag{2.15}$$

where the constant C_0 depends on the constants C and k_0 in (2.6).

Hence, for any $x \in Q$,

$$M_{\rho,\beta}g(x) \leqslant M_{2KQ}(g\chi_{2KQ})(x), \tag{2.16}$$

with $\beta = \frac{1}{2C_0^2}$ and K is the constant appearing in (2.1).

In fact, for any $x \in Q$ and $x \in B(y,r)$ with $r \leqslant \beta\rho(y)$, we have

$$d(y,x_0) \leqslant K(d(y,x)+d(x,x_0)) \leqslant K(\beta\rho(y)+\rho(x_0)) \leqslant K(\beta C_0\rho(x)+\rho(x_0)) \leqslant 2K\rho(x_0).$$

Therefore, $B(y,r) \subset \mathcal{F}(2KQ) = \{B(z,r) : z \in 2KQ, r > 0\}$. Hence, (2.16) holds true.

Also, for $x \in 2Q$,

$$M^\sharp_{2KQ}(g\chi_{2KQ})(x) \leqslant CM^\sharp_{\rho,12K^4}(g)(x). \tag{2.17}$$

In fact, given a ball $B = B(y,r) \subset \mathcal{F}(2KQ)$, when $r > 12K^4\rho(x_0)$, it is easy to see that

$$\mu(2KQ) = \mu(B \cap 2KQ).$$

In other words, $B \cap 2KQ$ has measure comparable to $2KQ$ which belongs to $\mathcal{B}_{\rho,12K^4}$. In addition, when $0 < r \leqslant 12K^4\rho(x_0)$, by using Proposition 2.1, we have

$$\mu(B) \geqslant \mu(B \cap 2KQ) \geqslant C\mu(B),$$

where the constant C depends only on the constants of the space $\mathcal{X}$. Hence, $\mu(B \cap 2KQ)$ is comparable with $\mu(B)$. Clearly, $B \in \mathcal{B}_{\rho,12K^4}$. All in all, (2.17) holds.

By the decomposition of $\mathcal{X}$ in Proposition 2.4, Proposition 3.4 in [35], (2.16) and (2.17), and the fact that the balls $2Q_k$ are also spaces of the homogeneous type, we obtain

$$\begin{aligned}\int_{\mathcal{X}} |M_{\rho,\beta}(g)|^p d\mu(y) &\leqslant \sum_k \int_{Q_k} |M_{\rho,\beta}(g)|^p d\mu(y)\\ &\leqslant \sum_k \int_{Q_k} |M_{2KQ_k}(g\chi_{2KQ_k})|^p d\mu(y)\\ &\leqslant C\sum_k \int_{2Q_k} |M^\sharp_{2KQ_k}(g\chi_{2KQ_k})|^p d\mu(y) + C\sum_k \mu(2Q_k)\bigg(\frac{1}{\mu(2Q_k)}\int_{2Q_k}|g|\bigg)^p\\ &\leqslant C\sum_k \int_{2Q_k} |M^\sharp_{\rho,12K^4}(g)|^p d\mu(y) + C\sum_k \mu(2Q_k)\bigg(\frac{1}{\mu(2Q_k)}\int_{2Q_k}|g|\bigg)^p\\ &\leqslant C_{\kappa,n}\int_{\mathcal{X}} |M^\sharp_{\rho,12K^4}(g)|^p d\mu(y) + C_{\kappa,n}\sum_k \mu(Q_k)\bigg(\frac{1}{\mu(Q_k)}\int_{2Q_k}|g|\bigg)^p,\end{aligned}$$

where we have used the finite overlapping property given by Proposition 2.4 in the last inequality and the constant $C_{\kappa,n}$ depending only on the κ, n in (2.3). □

3 Proofs of the main results

Firstly, in order to prove Theorem 1.2, we need the following lemmas. As usual, we denote by M the Hardy-Littlewood maximal function and for $f \in L^1_{\text{loc}}(\mathcal{X})$ we denote by M_s the s-maximal function which is defined as

$$M_s f(x) = \sup_{r>0}\bigg(\frac{1}{\mu(B(x,r))}\int_{B(x,r)} |f(y)|^s d\mu(y)\bigg)^{\frac{1}{s}}.$$

Lemma 3.1. *Let $b \in BMO_\theta(\rho)$. Assume that T is an operator satisfying the above conditions* (a), (b) *and* (c) *in Section* 1. *Then there exists a constant C such that*

$$\frac{1}{\mu(Q)}\int_Q |T_b f(y)| d\mu(y) \leqslant C[b]_\theta \inf_{y\in Q} M_s f(y),$$

for all $f \in L^s_{\text{loc}}(\mathcal{X})$ for $s > 1$ and every ball $Q = B(x_0, \rho(x_0))$.

Proof. Let $f \in L^s(\mathcal{X})$ and $Q = B(x_0, \rho(x_0))$. Writing $T_b f$ as

$$T_b f = (b - b_Q)Tf - T(f(b - b_Q)). \tag{3.1}$$

Via Hölder's inequality and Lemma 2.12, we get

$$\frac{1}{\mu(Q)}\int_Q |(b-b_Q)Tf(y)| d\mu(y) \leqslant \bigg(\frac{1}{\mu(Q)}\int_Q |(b-b_Q)|^{s'} d\mu(y)\bigg)^{\frac{1}{s'}}\bigg(\frac{1}{\mu(Q)}\int_Q |Tf(y)|^s d\mu(y)\bigg)^{\frac{1}{s}}$$

$$\leqslant C[b]_\theta \left(\frac{1}{\mu(Q)} \int_Q |Tf(y)|^s d\mu(y) \right)^{\frac{1}{s}}.$$

If we write $f = f_1 + f_2$ with $f_1 = f\chi_{2KQ}$, then

$$\begin{aligned} \left(\frac{1}{\mu(Q)} \int_Q |Tf_1(y)|^s d\mu(y) \right)^{\frac{1}{s}} &\leqslant C \left(\frac{1}{\mu(Q)} \int_{2Q} |f(y)|^s d\mu(y) \right)^{\frac{1}{s}} \\ &\leqslant C \inf_{y \in Q} M_s f(y). \end{aligned}$$

For $x \in Q$, note that $\rho(x) \sim \rho(x_0)$ follows from (i) of Lemma 2.2. And it is easy to see that $d(x,z) \sim d(x_0,z)$ when $d(x_0,z) \geqslant 2\rho(x_0)$. Since $d(x,x_0) \leqslant \rho(x_0)$, there exist constants K_1 and K_2 such that

$$\mu(B(x_0, d(x_0,z))) \leqslant \mu(B(x, K_1 d(x,z))) \leqslant K_1^n C_2 \mu(B(x, d(x,z))),$$

and

$$\mu(B(x, d(x,z))) \leqslant \mu(B(x_0, K_2 d(x_0,z))) \leqslant K_2^n C_2 \mu(B(x_0, d(x_0,z))).$$

Hence,

$$V(x_0,z) = \mu(B(x_0, d(x_0,z))) \sim \mu(B(x, d(x,z))) = V(x,z).$$

By using the estimate (1.6) and Hölder's inequality, we have

$$\begin{aligned} |Tf_2(x)| &= \left| \int_{d(x_0,z) > 2K\rho(x_0)} K(x,z) f(z) d\mu(z) \right| \\ &\leqslant C \int_{d(x_0,z) > 2K\rho(x_0)} \frac{|f(z)|}{(1 + \frac{d(x,z)}{\rho(x)})^l V(x,z)} d\mu(z) \\ &\leqslant C \sum_{k \geqslant 1} \frac{2^{-lk}}{\mu(B(x_0, 2^k K\rho(x_0)))} \int_{2^k K\rho(x_0) \leqslant d(x_0,z) < 2^{k+1} K\rho(x_0)} |f(z)| d\mu(z) \\ &\leqslant C \sum_{k \geqslant 1} \frac{2^{-lk}}{\mu(B(x_0, 2^k K\rho(x_0)))} \int_{d(x_0,z) < 2^{k+1} K\rho(x_0)} |f(z)| d\mu(z) \\ &\leqslant C \sum_{k \geqslant 1} 2^{-lk} \left(\frac{1}{\mu(B(x_0, 2^k K\rho(x_0)))} \int_{d(x_0,z) < 2^{k+1} K\rho(x_0)} |f(z)|^s d\mu(z) \right)^{\frac{1}{s}} \\ &\leqslant C \inf_{y \in Q} M_s f(y). \end{aligned}$$

To deal with the second term of (3.1), we split again $f = f_1 + f_2$ with $f_1 = f\chi_{2KQ}$.

By using Hölder's inequality and boundedness of T on $L^p(\mathcal{X})$, where $p < s$,

$$\begin{aligned} \frac{1}{\mu(Q)} \int_Q |T((b - b_Q) f_1)(y)| d\mu(y) &\leqslant \left(\frac{1}{\mu(Q)} \int_Q |T((b - b_Q) f_1)(y)|^p d\mu(y) \right)^{\frac{1}{p}} \\ &\leqslant \left(\frac{1}{\mu(Q)} \int_{2Q} |((b - b_Q) f)(y)|^p d\mu(y) \right)^{\frac{1}{p}} \\ &\leqslant \left(\frac{1}{\mu(Q)} \int_Q |(b - b_Q)|^{p\tilde{s}} d\mu(y) \right)^{\frac{1}{p\tilde{s}}} \left(\frac{1}{\mu(Q)} \int_{2Q} |f(y)|^s d\mu(y) \right)^{\frac{1}{s}} \\ &\leqslant C[b]_\theta \inf_{y \in Q} M_s f(y), \end{aligned}$$

where $\frac{1}{\tilde{s}} + \frac{p}{s} = 1$, $p\tilde{s} > 1$ and we have used Proposition 2.11 in the last inequality.

For the remaining term, via the estimate (1.6), Hölder's inequality, and Lemma 2.12, we have

$$|T[f_2(b - b_Q)](x)| = \left| \int_{d(x_0,z) > 2\rho(x_0)} K(x,z)[f_2(b - b_Q)](z) d\mu(z) \right|$$

$$
\begin{aligned}
&\leqslant C\int_{d(x_0,z)>2\rho(x_0)}\frac{|[f_2(b-b_Q)](z)|}{(1+\frac{d(x,z)}{\rho(x)})^l V(x,z)}d\mu(z)\\
&\leqslant C\sum_{k\geqslant 1}\frac{2^{-lk}}{\mu(B(x_0,2^k\rho(x_0)))}\int_{2^k\rho(x_0)\leqslant d(x_0,z)<2^{k+1}\rho(x_0)}|[f_2(b-b_Q)](z)|d\mu(z)\\
&\leqslant C\sum_{k\geqslant 1}\frac{2^{-lk}}{\mu(B(x_0,2^k\rho(x_0)))}\int_{d(x_0,z)<2^{k+1}\rho(x_0)}|[f_2(b-b_Q)](z)|d\mu(z)\\
&\leqslant C\sum_{k\geqslant 1}2^{-lk}\bigg(\frac{1}{\mu(B(x_0,2^k\rho(x_0)))}\int_{d(x_0,z)<2^{k+1}\rho(x_0)}|f(z)|^s d\mu(z)\bigg)^{\frac{1}{s}}\\
&\quad\times\bigg(\frac{1}{\mu(B(x_0,2^k\rho(x_0)))}\int_{d(x_0,z)<2^{k+1}\rho(x_0)}|b-b_Q|^{s'} d\mu(z)\bigg)^{\frac{1}{s'}}\\
&\leqslant C\sum_{k\geqslant 1}2^{-lk+\theta' k}k[b]_\theta\inf_{y\in Q}M_s f(y)\\
&\leqslant C\inf_{y\in Q}M_s f(y),
\end{aligned}
$$

where $\frac{1}{s}+\frac{1}{s'}=1$ and l is large enough. Therefore, this completes the proof. □

Remark 3.2. Similarly, we can conclude that the above lemma also holds if the critical ball Q is replaced by $2Q$.

Lemma 3.3. *Let $b\in BMO_\theta(\rho)$. Assume that the kernel $K(x,y)$ of T satisfies the estimate* (1.7). *Then there exists a constant C such that*

$$
\int_{(2B)^c}|K(x,z)-K(y,z)||b(z)-b_B||f(z)|d\mu(z)\leqslant C[b]_\theta\inf_{y\in B}M_s f(y),
$$

for all $f\in L^s_{\rm loc}(\mathcal{X})$ for $s>1$ and $x,y\in B=B(x_0,r)$, with $r<\gamma\rho(x_0)$, where $\gamma\geqslant 1$.

Proof. Denote $Q=B(x_0,\gamma\rho(x_0))$. Note that $\rho(x)\sim\rho(x_0)$ and $d(x,z)\sim d(x_0,z)$. Similarly,

$$
V(x_0,z)=\mu(B(x_0,d(x_0,z)))\sim\mu(B(x,d(x,z)))=V(x,z).
$$

By using (1.7), we have

$$
\begin{aligned}
&\int_{(2B)^c}|K(x,z)-K(y,z)||b(z)-b_B||f(z)|d\mu(z)\\
&\quad\leqslant Cr^\delta\int_{Q\setminus 2B}\frac{|f(z)||b(z)-b_B|}{V(x_0,z)d(x_0,z)^\delta}d\mu(z)+Cr^\delta\rho(x_0)^l\int_{Q^c}\frac{|f(z)||b(z)-b_B|}{V(x_0,z)d(x_0,z)^{\delta+l}}d\mu(z)\\
&\quad=I_1+I_2.
\end{aligned}
$$

For I_1, by Hölder's inequality and Lemma 2.12, we have

$$
\begin{aligned}
I_1&\leqslant\sum_{j=2}^{j_0}\frac{2^{-j\delta}}{\mu(B(x_0,2^jr))}\int_{2^jB}|f(z)||b(z)-b_B|d\mu(z)\\
&\leqslant C\sum_{j=2}^{j_0}2^{-j\delta}j[b]_\theta\bigg(1+\frac{2^jr}{\rho(x_0)}\bigg)^{\theta'}\inf_{y\in B}M_s f(y)\\
&\leqslant C\sum_{j=2}^{\infty}2^{-j\delta}j[b]_\theta\inf_{y\in B}M_s f(y)\\
&\leqslant C[b]_\theta\inf_{y\in B}M_s f(y),
\end{aligned}
$$

where j_0 is the least integer such that $2^{j_0}\geqslant\frac{\gamma\rho(x_0)}{r}$.

To deal with I_2, we use Lemma 2.12 and choose $l > \theta'$ to derive

$$\begin{aligned} I_2 &\leqslant \frac{C\rho(x_0)^l}{r^l} \sum_{j=j_0-1}^{\infty} \frac{2^{-j(\delta+l)}}{\mu(B(x_0,2^jr))} \int_{2^jB} |f(z)||b(z)-b_B|d\mu(z) \\ &\leqslant \frac{C}{2}\frac{\rho(x_0)^l}{r^l} \sum_{j=j_0}^{\infty} 2^{-j(\delta+l)} j[b]_\theta \bigg(1+\frac{2^jr}{\rho(x_0)}\bigg)^{\theta'} \inf_{y\in B} M_sf(y) \\ &\leqslant C\sum_{j=j_0}^{\infty} j2^{-j\delta}\bigg(\frac{\rho(x_0)}{2^jr}\bigg)^{l-\theta'} [b]_\theta \inf_{y\in B} M_sf(y) \\ &\leqslant C[b]_\theta \inf_{y\in B} M_sf(y), \end{aligned}$$

where we have used the fact that $\frac{\rho(x_0)}{2^jr} \leqslant \frac{1}{\gamma}$ when $j \geqslant j_0$. □

Proof of Theorem 1.2. We start with a function $f \in L^p(\mathcal{X})$ for $1 < p < \infty$. Let $1 < s < p$. By Lemmas 2.13 and 3.1 and Remark 3.2, we have

$$\begin{aligned} \|T_bf\|_{L^p}^p &\leqslant \int_{\mathcal{X}} |M_{\rho,\beta}(T_bf)(x)|^p d\mu(x) \\ &\leqslant C\int_{\mathcal{X}} |M^\sharp_{\rho,\gamma}(T_bf)(x)|^p d\mu(x) + C\sum_k |Q_k|\bigg(\frac{1}{\mu(Q_k)}\int_{2Q_k} |T_bf(x)|d\mu(x)\bigg)^p \\ &\leqslant C\int_{\mathcal{X}} |M^\sharp_{\rho,\gamma}(T_bf)(x)|^p d\mu(x) + C[b]_\theta^p \sum_k \int_{2Q_k} |M_sf(x)|^p d\mu(x) \\ &\leqslant C\int_{\mathcal{X}} |M^\sharp_{\rho,\gamma}(T_bf)(x)|^p d\mu(x) + C[b]_\theta^p \|f\|_{L^p}^p, \end{aligned}$$

where we have used the finite overlapping property given by Proposition 2.4 and the boundedness of M_s in $L^p(\mathcal{X})$ for $s < p$.

Next, we consider the term $\int_{\mathcal{X}} |M^\sharp_{\rho,\gamma}(T_bf)(x)|^p d\mu(x)$. Our goal is to find a pointwise estimate of $M^\sharp_{\rho,\gamma}(T_bf)(x)$. Let $x \in \mathcal{X}$ and $B = B(x_0, r)$, with $r < \gamma\rho(x_0)$ such that $x \in B$. If $f = f_1 + f_2$, with $f_1 = f\chi_{2KB}$, then we write

$$T_bf = (b-b_B)Tf - T(f_1(b-b_B)) - T(f_2(b-b_B)). \tag{3.2}$$

Therefore, we need to control the mean oscillation on B of each term that we call J_1, J_2 and J_3. By Hölder's inequality and Proposition 2.11, we obtain

$$\begin{aligned} J_1 &\leqslant \frac{2}{\mu(B)}\int_B |(b-b_B)Tf(x)|d\mu(x) \\ &\leqslant C\bigg(\frac{2}{\mu(B)}\int_B |b-b_B|^{s'} d\mu(x)\bigg)^{\frac{1}{s'}} \bigg(\frac{1}{\mu(B)}\int_B |Tf(x)|^s d\mu(x)\bigg)^{\frac{1}{s}} \\ &\leqslant C[b]_\theta M_s(Tf)(x) \end{aligned}$$

since $\frac{r}{\rho(x_0)} < \gamma$.

To estimate J_2, let $1 < \tilde{s} < s$. Then,

$$\begin{aligned} J_2 &\leqslant \frac{2}{\mu(B)}\int_B |T[(b-b_B)f_1](x)|d\mu(x) \\ &\leqslant C\bigg(\frac{1}{\mu(B)}\int_B |T[(b-b_B)f_1](x)|^{\tilde{s}} d\mu(x)\bigg)^{\frac{1}{\tilde{s}}} \\ &\leqslant C\bigg(\frac{1}{\mu(B)}\int_B |(b-b_B)f_1(x)|^{\tilde{s}} d\mu(x)\bigg)^{\frac{1}{\tilde{s}}} \end{aligned}$$

$$\leqslant C\bigg(\frac{1}{\mu(B)}\int_B|b-b_B|^v d\mu(x)\bigg)^{\frac{1}{v}}\bigg(\frac{1}{\mu(B)}\int_B|f(x)|^s d\mu(x)\bigg)^{\frac{1}{s}}$$
$$\leqslant C[b]_\theta M_s(f)(x),$$

where $v=\frac{s\tilde{s}}{s-\tilde{s}}$.

For J_3, by Lemma 3.3 we obtain

$$\begin{aligned}J_3&\leqslant C\frac{1}{\mu(B)^2}\int_B\int_B|T(f_2(b-b_B))(u)-T(f_2(b-b_B))(y)d\mu(u)d\mu(y)\\&\leqslant C\frac{1}{\mu(B)^2}\int_B\int_B\int_{(2B)^c}|K(u,z)-K(y,z)||b(z)-b_B||f(z)|d\mu(z)d\mu(u)d\mu(y)\\&\leqslant C[b]_\theta M_s f(x).\end{aligned}$$

Therefore,

$$|M^\sharp_{\rho,\gamma}(T_b f)(x)|\leqslant C[b]_\theta(M_sTf(x)+M_sf(x)),$$

which gives the desired result. □

Proof of Theorem 1.3. For $f\in H^1_\rho(\mathcal{X})$, we can write $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, where each a_j is a $(1,q)_\rho$-atom and $\sum_{j=-\infty}^{\infty}|\lambda_j|\leqslant 2\|f\|_{H^1_\rho}$. Suppose that supp $a_j\subseteq B_j=B(x_j,r_j)$ with $r_j<\rho(x_j)$. Write

$$\begin{aligned}T_bf(x)&=\sum_{j=-\infty}^{\infty}\lambda_j(b(x)-b_{B_j})Ta_j(x)\chi_{8B_j}(x)+\sum_{j:r_j\geqslant\frac{\rho(x_j)}{4}}\lambda_j(b(x)-b_{B_j})Ta_j(x)\chi_{(8B_j)^c}(x)\\&\quad+\sum_{j:r_j<\frac{\rho(x_j)}{4}}\lambda_j(b(x)-b_{B_j})Ta_j(x)\chi_{(8B_j)^c}(x)-T\bigg(\sum_{j=-\infty}^{\infty}\lambda_j(b-b_{B_j})a_j\bigg)(x)\\&=A_1(x)+A_2(x)+A_3(x)+A_4(x).\end{aligned}$$

Using Hölder's inequality, (L^q,L^q)-boundedness of T and Proposition 2.11,

$$\begin{aligned}\|(b(x)-b_B)Ta_j(x)\chi_{8B_j}(x)\|_{L^1(\mathcal{X})}&\leqslant\bigg(\int_{8B_j}|b(x)-b_B|^{q'}d\mu(x)\bigg)^{\frac{1}{q'}}\|Ta_j\|_{L^q}\\&\leqslant\bigg(\int_{8B_j}|b(x)-b_B|^{q'}d\mu(x)\bigg)^{\frac{1}{q'}}\|a_j\|_{L^q}\\&\leqslant\bigg(\frac{1}{\mu(B_j)}\int_{8B_j}|b(x)-b_B|^{q'}d\mu(x)\bigg)^{\frac{1}{q'}}\\&\leqslant C[b]_\theta,\end{aligned}$$

since $r_j<\rho(x_j)$.

When considering the term $A_2(x)$, we note that $\rho(x_j)>r_j\geqslant\frac{\rho(x_j)}{4}$. Then

$$\begin{aligned}&\|(b(x)-b_{B_j})Ta_j(x)\chi_{(8B_j)^c}(x)\|_{L^1(\mathcal{X})}\\&\quad\leqslant C\int_{B_j}|a_j(y)|d\mu(y)\bigg\{\int_{d(x,x_j)\geqslant 8r_j}|K(x,y)||b(x)-b_{B_j}|d\mu(x)\bigg\}.\end{aligned}$$

Note that $d(x,x_j)\sim d(x,y)$ and

$$\bigg(1+\frac{d(x,y)}{\rho(x)}\bigg)\geqslant C\bigg(1+\frac{d(x,x_j)}{\rho(x)}\bigg)\geqslant C\bigg(1+\frac{d(x,x_j)}{\rho(x_j)}\bigg)^{\frac{1}{k_0+1}}.$$

Moreover, $V(x,y)\sim V(x,x_j)$. Then by Lemma 2.12 and the estimate (1.6),

$$\int_{d(x,x_j)\geqslant 8r_j}|K(x,y)||b(x)-b_{B_j}|d\mu(x)$$

$$
\begin{aligned}
&\leqslant C\sum_{k=1}^{\infty}\int_{2^{k+3}r_j\leqslant d(x,x_j)<2^{k+4}r_j}\frac{C_l}{\left(1+d(x,x_j)\,\rho(x_j)^{-1}\right)^{\frac{l}{l_0+1}}}\frac{1}{V(x,x_j)}|b(x)-b_{B_j}|d\mu(x)\\
&\leqslant C\sum_{k=1}^{\infty}2^{-\frac{(k+1)l}{l_0+1}}\frac{1}{V(x,2^{k+3}r_j)}\int_{d(x,x_j)<2^{k+4}r_j}|b-b_{B_j}|d\mu(x)\\
&\leqslant C\sum_{k=1}^{\infty}2^{-\frac{(k+1)l}{l_0+1}}\frac{C_2 2^n}{V(x,2^{k+4}r_j)}\int_{d(x,x_j)<2^{k+4}r_j}|b-b_{B_j}|d\mu(x)\\
&\leqslant C\sum_{k=1}^{\infty}2^{-\frac{(k+1)l}{l_0+1}}[b]_\theta k\left(1+\frac{2^{k+4}r_j}{\rho(x_j)}\right)^{(k_0+1)\theta}\\
&\leqslant C[b]_\theta,
\end{aligned}
$$

where l is large enough. Therefore,

$$\|(b(x)-b_{B_j})Ta_j(x)\chi_{(8B_j)^c}(x)\|_{L^1(\mathcal{X})}\leqslant C[b]_\theta.$$

For A_3, it follows from the vanishing condition of a_j and (1.8) that

$$
\begin{aligned}
&\|(b(x)-b_{B_j})Ta_j(x)\chi_{(8B_j)^c}(x)\|_{L^1(\mathcal{X})}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|d\mu(y)\left\{\int_{d(x,x_j)\geqslant 8r_j}|K(x,y)-K(x,x_j)||b(x)-b_{B_j}|d\mu(x)\right\}\\
&\quad\leqslant C\int_{B_j}|a_j(y)|d\mu(y)\sum_{k=1}^{\infty}\int_{2^{k+3}r_j\leqslant d(x,x_j)<2^{k+4}r_j}\frac{C_l}{(1+d(x,x_j)\,\rho(x_j)^{-1})^{\frac{l}{l_0+1}}}\\
&\qquad\times\frac{d(x_j,y)^\delta|b(x)-b_{B_j}|}{V(x,x_j)d(x,x_j)^\delta}d\mu(x)\\
&\quad\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}\frac{1}{V(x,2^{k+3}r_j)}\int_{d(x,x_j)<2^{k+4}r_j}|b-b_{B_j}|d\mu(x)\\
&\quad\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}\frac{C_2 2^n}{V(x,2^{k+4}r_j)}\int_{d(x,x_j)<2^{k+4}r_j}|b-b_{B_j}|d\mu(x)\\
&\quad\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta}[b]_\theta k\left(1+\frac{2^{k+4}r_j}{\rho(x_j)}\right)^{(k_0+1)\theta}\\
&\quad\leqslant C\sum_{k=1}^{\infty}2^{-(k+3)\delta+(k+2)(k_0+1)\theta}[b]_\theta k\\
&\quad\leqslant C[b]_\theta,
\end{aligned}
$$

where we have used the fact $\delta>(k_0+1)\theta$.

Thus, we obtain

$$\left|\left\{x\in\mathcal{X}:|A_i(x)|>\frac{\lambda}{4}\right\}\right|\leqslant\frac{C}{\lambda}\|A_i(x)\|_{L^1}\leqslant\frac{C[b]_\theta}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j|,\quad i=1,2,3.$$

Note that

$$
\begin{aligned}
\|(b-b_{B_j})a_j\|_{L^1}&\leqslant\left(\int_{B_j}|b(x)-b_B|^{q'}d\mu(x)\right)^{\frac{1}{q'}}\|a_j\|_{L^q}\\
&\leqslant\left(\frac{1}{\mu(B_j)}\int_{B_j}|b(x)-b_B|^{q'}d\mu(x)\right)^{\frac{1}{q'}}\\
&\leqslant C[b]_\theta\left(1+\frac{r_j}{\rho(x_j)}\right)^{\theta'}
\end{aligned}
$$

$$\leqslant C[b]_\theta,$$

where $r_j < \rho(x_j)$

By the weak $(1,1)$-boundedness of T, we get

$$\left|\left\{x \in \mathbb{R}^n : |A_4(x)| > \frac{\lambda}{4}\right\}\right| \leqslant \frac{C}{\lambda}\left\|\sum_{j=-\infty}^{\infty} \lambda_j (b - b(x_j)a_j\right\|_{L^1} \leqslant \frac{C[b]_\theta}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j|.$$

Therefore,

$$\left|\left\{x \in \mathbb{R}^n : |[b,T]f(x)| > \frac{\lambda}{4}\right\}\right| \leqslant C\sum_{i=1}^{4}\left|\left\{x \in \mathbb{R}^n : |A_i(x)| > \frac{\lambda}{4}\right\}\right| \leqslant \frac{C[b]_\theta}{\lambda}\sum_{j=-\infty}^{\infty}|\lambda_j| \leqslant \frac{C[b]_\theta}{\lambda}\|f\|_{H^1_\rho}.$$

This completes the proof of Theorem 1.3. □

4 Some applications

In this section, we present several applications of Theorems 1.2 and 1.3.

4.1 Schrödinger operators and Schrödinger type operators on $\mathbb{R}^n$

Let $n \geqslant 3$ and $\mathbb{R}^n$ be the n-dimensional Euclidean space endowed with the Euclidean norm $|\cdot|$ and the Lebesgue measure dx. The metric d induced by the Euclidean norm $|\cdot|$ is given by $d(x,y) = |x-y|$ for any $x, y \in \mathbb{R}^n$. Clearly, $(\mathbb{R}^n, |\cdot|, dx)$ is an RD-space. Denote the Laplace operator $\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^n$ by Δ. It is easy to check that the balls B in $\mathbb{R}^n$ endowed with the restrictions of the metric d and the Lebesgue measure dx become bounded spaces of the homogeneous type. Hence, $\mathbb{R}^n$ satisfies the assumption on the RD-space $\mathcal{X}$.

Let $q > \frac{n}{2}$ and $U \in \mathcal{B}_q(\mathbb{R}^n, |\cdot|, dx)$, where $\mathcal{B}_q(\mathbb{R}^n, |\cdot|, dx)$ is the reverse Hölder class as in Section 1. And let $\mathcal{L}_1 = -\Delta + U$ be the Schrödinger operator and $\mathcal{L}_2 = (-\Delta)^2 + U^2$ be the Schrödinger type operator. At this time, the Hardy space $H^{1,\infty}_\rho(\mathcal{X})$ is exactly the space $H^1_{\mathcal{L}_1}(\mathbb{R}^n)$ established by Dziubański and Zienkiewicz in [13] and $H^1_{\mathcal{L}_1}(\mathbb{R}^n) = H^1_{\mathcal{L}_2}(\mathbb{R}^n)$ (see Theorem 1.1 in [6]), where ρ defined in (1.5) is an admissible function.

When $\mathcal{X} = \mathbb{R}^n$, we will give three typical examples of Calderón-Zygmund operators T related to the admissible functions though we can give many other examples of T.

Case 1. Let $T = \mathcal{L}_1^{i\gamma}$, $\gamma \in \mathbb{R}$. Following Theorem 0.4 in [36], we know that $\mathcal{L}_1^{i\gamma}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Also, the kernel $K(x,y)$ of $\mathcal{L}_1^{i\gamma}$ satisfies (1.6) of Condition (b) in Section 1 by using (4.3) in [36]. We conclude from Theorem 2.7, (4.2) and the proof of Theorem 0.4 in [36] that the kernel $K(x,y)$ of $\mathcal{L}_1^{i\gamma}$ satisfies (1.7) and (1.8) of Condition (c) in Section 1. Therefore, the operator $\mathcal{L}_1^{i\gamma}$ satisfies the assumptions of Theorems 1.2 and 1.3.

Case 2. Assume $q \geqslant n$. Let $T = \nabla\mathcal{L}_1^{-\frac{1}{2}}$. Following Theorem 0.8 in [36], we know that $\nabla\mathcal{L}_1^{-\frac{1}{2}}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Also, the kernel $K(x,y)$ of $\nabla\mathcal{L}_1^{-\frac{1}{2}}$ satisfies (1.6) of Condition (b) in Section 1 by using (6.5) in [36]. Finally, we conclude from Theorem 2.7, Remark 4.9, (5.3) and the proof of Theorem 0.8 in [36] that the kernel $K(x,y)$ of $\nabla\mathcal{L}_1^{-\frac{1}{2}}$ satisfies (1.7) and (1.8) of Condition (c) in Section 1. Therefore, the operator $\nabla\mathcal{L}_1^{-\frac{1}{2}}$ satisfies the assumptions of Theorems 1.2 and 1.3.

Case 3. Assume $U \in \mathcal{B}_{2n}(\mathbb{R}^n, |\cdot|, dx)$ or $U \in \mathcal{B}_{\frac{n}{2}}(\mathbb{R}^n, |\cdot|, dx)$ and there exists a constant C such that $U(x) \leqslant C\rho(x)^{-2}$. Let $T = \nabla^2\mathcal{L}_2^{-\frac{1}{2}}$. Following Theorem 3 in [30], we conclude that $\nabla^2\mathcal{L}_2^{-\frac{1}{2}}$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Also, the kernels $K(x,y)$ of $\nabla^2\mathcal{L}_2^{-\frac{1}{2}}$ satisfy (1.6) of Condition (b) and (1.7) and (1.8) of Condition (c) in Section 1 by using Theorems 5 and 6 and Equality (9) in [30]. Therefore, the operator $\nabla^2\mathcal{L}_2^{-\frac{1}{2}}$ satisfies the assumptions of Theorems 1.2 and 1.3.

4.2 Sub-Laplace Schrödinger operators on stratified Lie groups

Let $\mathbb{G}$ be a stratified Lie group and $\mathfrak{g}$ be its Lie algebra. Namely, it is nilpotent, connected and simply connected, and its Lie algebra $\mathfrak{g}$ admits a vector space decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$ such that $[V_1, V_k] = V_{k+1}$ for $1 \leqslant k < m$ and $[V_1, V_m] = 0$. Let $X = \{X_1, \ldots, X_{d_1}\}$ be left invariant fields on $\mathbb{G}$ satisfying the Hörmander condition. Namely, X, together with their commutators of order $\leqslant m$, generates the tangent space of $\mathbb{G}$ at each point of $\mathbb{G}$. And assume that $\mathbb{G}$ is a Lie group with underlying manifold $\mathbb{R}^n$ for some positive integer n. $\mathbb{G}$ inherits dilations from $\mathfrak{g}$: if $g \in \mathbb{G}$ and $r > 0$, we write

$$rx = (r^{d_1} x_{_1}, \ldots, r^{d_n} x_{_n}), \tag{4.1}$$

where $1 \leqslant d_1 \leqslant \cdots \leqslant d_n$. The map $x \to rx$ is an automorphism of $\mathbb{G}$.

Denote by 0 the unit of $\mathbb{G}$ and let $\circ$ be the group law of $\mathbb{G}$. The left (or right) Haar measure on $\mathbb{G}$ is simply $dx = dx_{_1} \cdots dx_{_n}$, which is the Lebesgue measure on $\mathfrak{g}$. For any measurable set $E \subseteq \mathbb{G}$, denote by $|E|$ the measure of E. Let d_c be the Carnot-Carathéodory (control) metric on $\mathbb{G}$ associated to X. The ball of radius δ_0 centered at x is written by

$$B(x, \delta_0) = \{y \in \mathbb{G} : d_c(x, y) < \delta_0\}.$$

It follows from Section 5.4 in [39] that $(\mathbb{G}, d_c, \mu)$ is an RD-space.

We fix a homogeneous norm function $|\cdot|$ on $\mathbb{G}$ which is smooth away from 0. Thus, $|rx| = r|x|$ for all $x \in \mathbb{G}$, $r > 0, |x^{-1}| = |x|$ for all $x \in \mathbb{G}$, and $|x| > 0$ if $x \neq 0$. The homogeneous norm induces a quasi-metric d which is defined by $d(x, y) := |x^{-1}y|$. The Carnot-Caratheodory metric d_c is equivalent to the quasi-metric d. In fact, from the results of Nagel et al. in [34], we have that there exists a constant $a = a(\mathbb{G}) > 1$ such that for any $x, y \in \mathbb{G}$,

$$a^{-1} d_c(x, y) \leqslant d(x, y) \leqslant a d_c(x, y). \tag{4.2}$$

An important feature of both metrics d and d_c is that these distances and thus the associated metric balls are left-invariant. Hence, $|B(x, \delta_0)| \sim \delta_0^Q$ for any $\delta_0 > 0$.

It follows from Lemma 4.2 in [4] that the balls B in $\mathbb{G}$ endowed with the restrictions of the metric d and the Lebesgue measure dx become bounded spaces of the homogeneous type. Hence, $\mathbb{G}$ satisfies the assumption on the RD-space $\mathcal{X}$ in this paper.

In this section, we always assume $q \geqslant \frac{Q}{2}$ and $U \in \mathcal{B}_q(\mathbb{G}, d_c, \mu)$, where $\mathcal{B}_q(\mathbb{G}, d_c, \mu)$ is the reverse Holder class as in Section 1 and the number $Q = \sum_{j=1}^m j(\dim V_j)$ is called the homogeneous dimension of $\mathbb{G}$. At this time ρ defined in (1.5) is an admissible function.

The sub-Laplacian is given by $\Delta_{\mathbb{G}} = -\sum_{j=1}^{d_1} X_j^2$. The gradient operator ∇_G is denoted by $\nabla_G = (X_1, \ldots, X_{d_1})$. Note that $\Delta_{\mathbb{G}} = \nabla_G \cdot \nabla_G$. Let $\mathcal{L}_1 = \Delta_{\mathbb{G}} + U$ be the Schrödinger operator. At this time, the Hardy space $H_\rho^{1,\infty}(\mathcal{X})$ is exactly the space $H^1_{\mathcal{L}_1}(\mathbb{G})$ established by Lin et al. in [28]. Next, we will give one typical example of Calderón-Zygmund operator T related to the admissible functions when $\mathcal{X} = \mathbb{G}$.

Case 1. Let $q \geqslant Q$. Let $\Gamma(x, y, \lambda)$ be the fundamental solution of $\Delta_{\mathbb{G}} + U + \lambda$ with $\lambda \in [0, \infty)$.

Let $T = \nabla_G \mathcal{L}_1^{-\frac{1}{2}}$, where

$$\nabla_G \mathcal{L}_1^{-\frac{1}{2}} f(x) = \int_{\mathbb{G}} K(x, y) f(y) dy, \tag{4.3}$$

and

$$K(x, y) = \frac{1}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \nabla_{\mathbb{G},x} \Gamma(x, y, \lambda) d\lambda. \tag{4.4}$$

Next, we only need to show that $T = \nabla_G \mathcal{L}_1^{-\frac{1}{2}}$ satisfies Conditions (a), (b) and (c).

We conclude from Theorem C in [25] that$\nabla_G \mathcal{L}_1^{-\frac{1}{2}}$is bounded on $L^2(\mathbb{G})$, that is, it satisfies Condition (a).

Using (5.2) in [25] and (4.4), we conclude that the kernel $K(x, y)$ of $\nabla_G \mathcal{L}_1^{-\frac{1}{2}}$ satisfies (1.6), that is,

$$|K(x, y)| \leqslant \frac{C}{(1 + \frac{d_c(x,y)}{\rho(x)})^l d_c(x, y)^Q}.$$

We give the following lemma before we prove that the kernel $K(x,y)$ of $\nabla_G \mathcal{L}_1^{-\frac{1}{2}}$ satisfies Condition (c) in Section 1.

Lemma 4.1. *If $U \in \mathcal{B}_q(\mathbb{G}, d_c, \mu)$ for some $q \geqslant Q$ and $\Delta_{\mathbb{G}} u + (U+\lambda)u = 0$ in $B_0(x_0, 2R)$, then*

$$\left(\int_{B(x_0,R)} |\nabla_{\mathbb{G}}^2 u(x)|^q\right)^{\frac{1}{q}} \leqslant CR^{\frac{Q}{q}-2}(1+R\rho(x_0)^{-1}))^{l_0} \sup_{B(x_0,2R)} |u(x)|.$$

Proof. Lemma 3.2 in [25] implies the existence of the following cut-off function.

Let $\phi \in C_c^\infty(B(x_0,2R))$ such that $\phi \equiv 1$ on $B(x_0, \frac{R}{C^2})$, $0 < \phi \leqslant 1$, $|\nabla_{\mathbb{G}}\phi| \leqslant CR^{-1}$ and $|\nabla_{\mathbb{G}}^2 \phi| \leqslant CR^{-2}$, where $C \geqslant 1$ is a constant in Lemma 3.2 in [25]. Then

$$\begin{aligned} u(x)\phi(x) &= \int_{\mathbb{G}} \Gamma_0(x,y,\lambda)(\Delta_{\mathbb{G}}+\lambda)(u\phi)(y)\,dy \\ &= \int_{\mathbb{G}} \Gamma_0(x,y,\lambda)(U(y)u(y)\phi(y) + 2\nabla_{\mathbb{G}} u(y)\cdot\nabla_{\mathbb{G}}\phi(y) + u(y)\Delta_{\mathbb{G}}\phi(y))\,dy \\ &= \int_{\mathbb{G}} \Gamma_0(x,y,\lambda)(U(y)u(y)\phi(y) + u(y)\Delta_{\mathbb{G}}\phi(y))\,dy \\ &\quad + 2\int_{\mathbb{G}} u(y)\nabla_{\mathbb{G},y}\Gamma_0(x,y,\lambda)\cdot\nabla_{\mathbb{G}}\phi(y)dy. \end{aligned} \tag{4.5}$$

By using Theorem 4.1 in [25], we immediately obtain that $\Delta_{\mathbb{G}}(\Delta_{\mathbb{G}}+\lambda)^{-1}$ is bounded on $L^p(\mathbb{G})$ for all p, $1<p<\infty$ and $\nabla_{\mathbb{G}}^2(\Delta_{\mathbb{G}})^{-1}$ is a Calderón–Zygmund operator. Therefore, $\nabla_{\mathbb{G}}^2(-\Delta_{\mathbb{G}}+\lambda)^{-1}$ is bounded on $L^p(\mathbb{G})$ for all p, $1<p<\infty$. Using (4.5) and (3.9) and (3.10) in [25] we have, for $x \in B(x_0,R)$,

$$|\nabla_{\mathbb{G}}^2 u(x)| \leqslant |\nabla_{\mathbb{G}}^2\big(\Delta_{\mathbb{G}}+\lambda\big)^{-1}(Uu\phi)(x)| + \frac{C}{R^{Q+2}}\int_{B(x_0,2R)} |u(y)|\,dy.$$

Therefore

$$\begin{aligned} \left(\int_{B(x_0,R)} |\nabla_{\mathbb{G}}^2 u(x)|^q\,dx\right)^{\frac{1}{q}} &\leqslant C\sup_{B(x_0,2R)}|u(x)|\left(\left(\int_{B(x_0,2R)} U(x)^q\,dx\right)^{\frac{1}{q}} + R^{\frac{Q}{q}-2}\right) \\ &\leqslant CR^{\frac{Q}{q}-2}\sup_{B(x_0,2R)}|u(x)|\left(\frac{1}{R^{Q-2}}\int_{B(x_0,2R)} U(x)\,dx + 1\right) \\ &\leqslant CR^{\frac{Q}{q}-2}(1+R\rho(x_0)^{-1})^{l_0}\sup_{B(x_0,2R)}|u(x)|, \end{aligned}$$

where we have used Lemma 2.8 in [25]. □

In a similar manner to prove Lemma 4.1 via the fractional integral theorem on the stratified Lie group, we have the following corollary.

Corollary 4.2. *If $U \in \mathcal{B}_q(\mathbb{G}, d_c, \mu)$ for some $Q > q \geqslant \frac{Q}{2}$ and $\Delta_{\mathbb{G}} u + (U+\lambda)u = 0$ in $B_0(x_0,2R)$, then*

$$\left(\int_{B(x_0,R)} |\nabla_{\mathbb{G}} u(x)|^t\right)^{\frac{1}{t}} \leqslant CR^{\frac{Q}{q}-2}(1+R\rho(x_0)^{-1}))^{l_0}\sup_{B(x_0,2R)}|u(x)|,$$

where $\frac{1}{t} = \frac{1}{q} - \frac{1}{Q}$.

Now we are in a position to give the proof the kernel $K(x,y)$ of $\nabla_G\mathcal{L}_1^{-\frac{1}{2}}$ satisfying Condition (c).

We fix $x_0, y_0 \in \mathbb{G}$ and $\xi \in \mathbb{G}$. Let $R = \frac{d_c(x_0,y_0)}{4}$ and $u(x) = \Gamma(x,y_0,\lambda)$. Assume that $d_c(0,\xi) < \frac{R}{2}$. Then

$$|K(x_0\circ\xi, y_0) - K(x_0,y_0)| \leqslant \frac{1}{\pi}\int_0^\infty |\lambda|^{-\frac{1}{2}}|\nabla_{\mathbb{G},x}\Gamma(x_0\circ\xi,y_0,\lambda) - \nabla_{\mathbb{G},x}\Gamma(x_0,y_0,\lambda)|d\lambda.$$

By using Theorem 1.1 in [31] and Lemma 4.1, we have

$$|\nabla_{\mathbb{G},x}\Gamma(x_0\circ\xi,y_0,\lambda) - \nabla_{\mathbb{G},x}\Gamma(x_0,y_0,\lambda)| \leqslant d_c(0,\xi)^{1-\frac{Q}{q}}\left(\int_{B(x_0,2R)} |\nabla_{\mathbb{G}}^2\Gamma(x,y_0,\lambda)|^q dx\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&\leqslant C d_c(0,\xi)^{1-\frac{Q}{q}}\bigg(\int_{B(x_0,2R)}|\nabla^2_{\mathbb{G}}u(x)|^q\,dx\bigg)^{\frac{1}{q}}\\
&\leqslant C\bigg(\frac{d_c(0,\xi)}{R}\bigg)^{1-\frac{Q}{q}}\frac{1}{R}(1+R\,\rho(x_0)^{-1})^{l_0}\sup_{B(x_0,2R)}|\Gamma(x,y_0,\lambda)|\\
&\leqslant C\bigg(\frac{d_c(0,\xi)}{R}\bigg)^{1-\frac{Q}{q}}\frac{1}{R^{Q-1}}(1+R\,\rho(x_0)^{-1})^{-l+l_0}(1+\lambda^{\frac{1}{2}}R)^{-l}\\
&\leqslant \frac{C}{(1+\lambda^{\frac{1}{2}}R)^{l}(1+R\,\rho(x_0)^{-1})^{l-l_0}}\frac{Cd_c(0,\xi)^{\delta}}{d_c(x_0,y_0)^{Q+\delta}},
\end{aligned}
$$

where $\delta=1-\frac{Q}{q}>0$ and we choose l large enough. Then

$$
|K(x_0\circ\xi,y_0)-K(x_0,y_0)|\leqslant\frac{C}{(1+d_c(x_0,y_0)\,\rho(x_0)^{-1})^{l}}\frac{Cd_c(0,\xi)^{\delta}}{d_c(x_0,y_0)^{Q+\delta}}.
$$

(1.7) is proved for $d_c(0,\xi)<\frac{d_c(x_0,y_0)}{2}$, and similarly, we prove that (1.8) is valid for $d_c(0,\xi)<\frac{d_c(x_0,y_0)}{2}$. The proof is complete.

Therefore, the operator $\nabla_{\mathbb{G}}\mathcal{L}_1^{-\frac{1}{2}}$ satisfies the assumptions of Theorems 1.2 and 1.3. The main results in this paper are valid for $\nabla_{\mathbb{G}}\mathcal{L}_1^{-\frac{1}{2}}$.

Case 2. Let $q\geqslant\frac{Q}{2}$ and $T=\mathcal{L}_1^{i\gamma}$, $\gamma\in\mathbb{R}$. Because $\mathcal{L}_1$ is a self-adjoint and positive operator on $L^2(\mathbb{G})$, then it has a spectral resolution

$$
\mathcal{L}_1=\int_0^{\infty}\lambda dE_{\mathcal{L}_1}\lambda,
$$

where $E_{\mathcal{L}_1}\lambda$ are the spectral projection. For any $\gamma\in\mathbb{R}$, then we have

$$
\mathcal{L}_1^{i\gamma}=\int_0^{\infty}\lambda^{i\gamma}dE_{\mathcal{L}_1}\lambda.
$$

By the spectral theory we immediately conclude that $\mathcal{L}_1^{i\gamma}$ is bounded on $L^2(\mathbb{G})$. Namely, it satisfies Condition (a) in Section 1. Moreover, we can define $\mathcal{L}_1^{i\gamma}$ in another form as follows,

$$
\mathcal{L}_1^{i\gamma}f(x)=\int_{\mathbb{G}}K(x,y)f(y)dy, \tag{4.6}
$$

and

$$
K(x,y)=\frac{1}{\pi}\int_0^{\infty}\lambda^{i\gamma}\Gamma(x,y,\lambda)d\lambda. \tag{4.7}
$$

It follows from (1.10) in [25] that the kernel $K(x,y)$ of $\mathcal{L}_1^{i\gamma}$ satisfies Condition (b) in Section 1.

Finally, we show that the kernel $K(x,y)$ of $\mathcal{L}_1^{i\gamma}$ satisfies Condition (c) in Section 1. We fix $x_0,y_0\in\mathbb{G}$ and $\xi\in\mathbb{G}$. Let $R=\frac{d_c(x_0,y_0)}{4}$ and $u(x)=\Gamma(x,y_0,\lambda)$. Assume that $d_c(0,\xi)<\frac{R}{2}$. Then

$$
|K(x_0\circ\xi,y_0)-K(x_0,y_0)|\leqslant\frac{1}{\pi}\int_0^{\infty}|\lambda^{i\gamma}||\Gamma(x_0\circ\xi,y_0,\lambda)-\Gamma(x_0,y_0,\lambda)|d\lambda.
$$

By using Theorem 1.1 in [31] and Corollary 4.2, we have

$$
\begin{aligned}
|\Gamma(x_0\circ\xi,y_0,\lambda)-\Gamma(x_0,y_0,\lambda)|&\leqslant d_c(0,\xi)^{1-\frac{Q}{t}}\bigg(\int_{B(x_0,2R)}|\nabla_{\mathbb{G}}\Gamma(x,y_0,\lambda)|^t dx\bigg)^{\frac{1}{t}}\\
&\leqslant Cd_c(0,\xi)^{1-\frac{Q}{t}}\bigg(\int_{B(x_0,2R)}|\nabla_{\mathbb{G}}u(x)|^t\,dx\bigg)^{\frac{1}{t}}\\
&\leqslant C\bigg(\frac{d_c(0,\xi)}{R}\bigg)^{2-\frac{Q}{q}}\frac{1}{R}(1+R\,\rho(x_0)^{-1})^{l_0}\sup_{B(x_0,2R)}|\Gamma(x,y_0,\lambda)|
\end{aligned}
$$

$$\leqslant C\left(\frac{d_c(0,\xi)}{R}\right)^{2-\frac{Q}{q}}\frac{1}{R^{Q-2}}(1+R\,\rho(x_0)^{-1})^{-l+l_0}(1+\lambda^{\frac{1}{2}}R)^{-l}$$
$$\leqslant \frac{C}{(1+\lambda^{\frac{1}{2}}R)^l(1+R\,\rho(x_0)^{-1})^{l-l_0}}\frac{Cd_c(0,\xi)^\delta}{d_c(x_0,y_0)^{Q+\delta}},$$

where $\delta = 2-\frac{Q}{q} > 0$. Then

$$|K(x_0\circ\xi,y_0)-K(x_0,y_0)| \leqslant \frac{C}{(1+d_c(x_0,y_0)\rho(x_0)^{-1})^l}\frac{Cd_c(0,\xi)^\delta}{d_c(x_0,y_0)^{Q+\delta}}.$$

(1.7) is proved for

$$d_c(0,\xi) < \frac{d_c(x_0,y_0)}{2},$$

and similarly, we can prove that (1.8) is valid for

$$d_c(0,\xi) < \frac{d_c(x_0,y_0)}{2}.$$

The proof is complete.

Therefore, the operator $\mathcal{L}_1^{i\gamma}$ satisfies the assumptions of Theorems 1.2 and 1.3. The main results in this paper are valid for $\mathcal{L}_1^{i\gamma}$.

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References

1 Bernardis A L, Pradolini G, Lorente M, et al. Composition of fractional Orlicz maximal operators and A_1-weights on spaces of homogeneous type. Acta Math Sinica Engl Ser, 2010, 8: 1509–1518

2 Bongioanni B, Harboure E, Salinas O. Commutators of Riesz transforms related to Schrödinger operators. J Fourier Anal Appl, 2011, 17: 115–134

3 Bramanti M, Cerutti M. Commutators of singular integrals on homogeneous spaces. Boll Un Mat ltd, 1996, 10: 843–883

4 Bramanti M, Brandolini L. Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDEs. Rev Mat Iberoamericana, 2005, 21: 511–556

5 Burger N. Espace des fonctions à variation moyenne bornée sur un espace de nature homogène. C R Acad Sci Paris Sr, 1978, 286: 139–142

6 Cao J, Liu Y, Yang D C. Hardy spaces $H^1_{\mathcal{L}}(\mathbb{R}^n)$ associated to Schrödinger type operators. Houston J Math, 2010, 36: 1067–1095

7 Cao J, Yang D C. Hardy spaces $H^p_L(\mathbb{R}^n)$ associated with operators satisfying k-Davies-Gaffney estimates. Sci China Math, 2012, 55: 1403–1440

8 Coifman R R, Weiss G. Analyse Harmonique Non-Commutative sur Certain Espaces Homogènes. Lecture Notes in Math 242. Berlin: Springer-Verlag, 1971

9 Coifman R R, Weiss G. Extensions of Hardy spaces and their use in analysis. Bull Amer Math Soc, 1977, 83: 569–645

10 Coifman R R, Rochberg R, Weiss G. Factorization theorem for Hardy spaces in several variables. Ann Math, 1976, 103: 611–635

11 Duong X T, Yan L X. Commutators of BMO functions and singular integral operators with non-smooth kernels. Bull Austral Math Soc, 2003, 67: 187–200

12 Duong X T, Yan L X. Duality of Hardy and BMO spaces associated with operators with heat kernel. J Amer Math Soc, 2005, 18: 943–973

13 Dziubański J, Zienkiewicz J. Hardy space H^1 associated to Schrödinger operator with potential satisfying reverse Hölder inequality. Rev Mat Iberoamericana, 1999, 15: 279–296

14 Dziubański J, Garrigos G, Martinez T, et al. BMO spaces related to Schrödinger operators with potentials satisfying a reverse Hölder inequality. Math Z, 2005, 249: 329–356

15 Guo Z H, Li P T, Peng L Z. L^p boundedness of commutators of Riesz transform associated to Schrödinger operator. J Math Anal Appl, 2008, 341: 421–432

16 Han Y, Müller D, Yang D C. Littlewood-Paley characterizations for Hardy spaces on spaces of homogeneous type. Math Nachr, 2006, 279: 1505–1537

17 Han Y, Müller D, Yang D C. A theory of Besov and Triebel-Lizorkin spaces on metric measure space modeled on Carnot-Carathéodory spaces. Abstr Appl Anal, 2008, Art. ID 893409, 250 pp

18 Hofmann S, Lu G, Mitrea D, et al. Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. Mem Amer Math Soc, 2011, 1007: 1–78

19 Hofmann S, Mayboroda S. Hardy and BMO spaces associated to divergence form elliptic operators. Math Ann, 2009, 344: 37–116

20 Janson S. Mean oscillation and commutators of singular integrals operators. Ark Math, 1978, 16: 263–270

21 Jiang R, Yang D C. Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates. Commun Contemp Math, 2011, 13: 331–373

22 Jiang R, Yang D C. New Orlicz-Hardy spaces associated with divergence form elliptic operators. J Funct Anal, 2010, 258: 1167–1224

23 Jiang R, Yang D C. Generalized vanishing mean oscillation spaces associated with divergence form elliptic operators. Integral Equations Operator Theory, 2010, 67: 123–149

24 Jiang R, Yang D C, Zhou Y. Orlicz-Hardy spaces associated with operators. Sci China Ser A, 2009, 52: 1042–1080

25 Li H. Estimations L_p des opéateurs de Schröinger sur les groupes nilpotents. J Funct Anal, 1999, 161: 152–218

26 Li P T, Peng L Z. Enpoint estimates for commutators of Riesz transforms associated with Schrödinger operators. Bull Aust Math Soc, 2010, 82: 367–389

27 Liang Y, Yang D C, Yang S. Applications of Orlicz-Hardy spaces associated with operators satisfying Poisson estimates. Sci China Math, 2011, 54: 2395–2426

28 Lin C C, Liu H P, Liu Y. Hardy spaces associated with Schrödinger operators on the Heisenberg group. ArXiv:1106.4960

29 Lin C C, Liu H P. $BMO_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators. Adv Math, 2011, 228: 1631–1688

30 Liu Y, Dong J F. Some estimates of higher order Riesz transform related to Schrödinger type operator. Potential Anal, 2010, 32: 41–55

31 Lu G Z. Embedding theorem into Lipschitz and BMO spaces and applications to quasilinear subelliptic differential equations. Publ Mat, 1996, 40: 301–329

32 Lu S Z, Wu Q, Yang D C. Boundedness of commutators on Hardy type space. Sci China Ser A, 2002, 45: 984-997

33 Macías R, Segovia C. A well-behaved quasi-distance on spaces of homogeneous type. Trab Mat Inst Argentina Mat, 1981, 32: 1–18

34 Nagel A, Stein E M, Wainger S. Balls and metrics defined by vector fields I. Basic properties. Acta Math, 1985, 155: 103–147

35 Pradolini G, Salinas O. Commutators of singular integrals on spaces of homogeneous type. Czechoslov Math J, 2007, 57: 75–93

36 Shen Z W. L^p-estimates for Schrödinger operators with certain potentials. Ann Inst Fourier Grenoble, 1995, 45: 513–546

37 Sun Y Z, Su W Y. Hardy type estimates for commutators of singular integrals. Sci China Ser A, 2005, 48: 563-575

38 Yan L X. Classes of Hardy spaces associated with operators, duality theorem and applications. Trans Amer Math Soc, 2008, 360: 4383–4408

39 Yang D C, Zhou Y. Localized Hardy spaces H^1 related to admissible functions on RD-spaces and applications to Schrödinger operators. Trans Amer Math Soc, 2011, 363: 1197–1239

40 Yang D C, Yang D Y, Zhou Y. Localized BMO and BLO spaces on RD-spaces and applications to Schrödinger operators. Commun Pure Appl Anal, 2010, 9: 779–812

41 Yang D C, Yang D Y, Zhou Y. Localized Morrey-Campanato spaces on metric measure spaces and applications to Schrödinger operators. Nagoya Math J, 2010, 198: 77–119

42 Yang D C, Yang S B. Local Hardy spaces of Musielak-Orlicz type and their applications. Sci China Math, 2012, 55: 1677–1720