The Riemann Hypothesis

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Abstract

Riemann proved (see [53], [37] and preferably Edwards' book [15]), that $\zeta(s)$ has an analytic continuation to the whole plane apart from a simple pole at $s = 1$. Moreover, he showed that $\zeta(s)$ satisfies the functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s).$$

Riemann uses the functional equation to deduce an approximate formula for $\pi(x) = \#\{p \leq x; p \text{ prime}\}$: that is

$$\pi(x) \sim Li(x) + \sum_{n=2}^{N} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}}), \quad N > \log x / \log 2.$$

In the paper (see [53]), he stated also that $\zeta(s)$ had infinitely many non trivial roots and that it seemed probable that they all have real part $1/2$.

1   Distribution of prime numbers

Among the positive integers there is a sub-class of special importance, that is, the class of primes. Some questions about primes are: a) How many prime numbers are there? and b) How are the prime numbers distributed?

The answer to the question of how many prime numbers are there is given by the theorem of Euclid (*Elements*, Book 9, Prop. 20): *There exist infinitely many prime numbers.* In the proof of Euclid, to prove the theorem, it will suffice to prove that if $\{p_1, p_2, \cdots, p_n\}$ is any finite set of primes, then we can find a prime number that is not in the set.

As usual $\mathbb{Z}$ will denote the ring of integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers. Also we shall denote by $\mathbb{N}$ the set of positive (natural) integers, not including 0.

In 1737, Euler proved the existence of an infinity of primes by a new method, which shows moreover that the series $\sum_{p_n} p_n^{-1}$ is divergent. Euler's work is based on the idea of using an identity in which the primes appear on one side but not on the other. Stated formally his identity is

$$\prod_p \{1 - p^{-s}\}^{-1} = \sum_{n=1}^{\infty} n^{-s}, \quad (s > 1) \tag{1}$$

where p ranges over all prime numbers. This formula results from expanding each of the factors on the left

$$(1 - x)^{-1} = 1 + x + x^2 + \ldots, \quad x = p^{-s}.$$

As their product is a sum of terms of the form $(p_1^{\alpha_1} \cdots p_r^{\alpha_r})^{-s}$, where $p_1 \neq p_2 \neq \cdots \neq p_r$, the formula (1) is deduced by using the fundamental theorem of arithmetic: *Each natural number $n > 1$ can be decomposed uniquely, up to order of the factors, as a product of prime numbers.*

From an old theorem of Nicole Oresme (1360), we have that: *The harmonic series $\sum_{n=1}^{\infty} 1/n = \zeta(1)$ is divergent.*

Function $\pi(x)$. We introduce a function $\pi(x)$ which has become universally accepted and means the number of primes not exceeding x. If $p_1 = 2, p_2 = 3, \cdots$, and p_n denote the nth prime number, then for each integer n we have $\pi(p_n) = n$. It follows from Euclid's theorem that $\pi(x) \to \infty$ as $x \to \infty$.

Legendre. Experimentally, Legendre conjectured in 1798 and again in 1808 the hypothesis that

$$\pi(x) \sim \frac{x}{\log x - A(x)}, \quad \lim_{x \to \infty} A(x) = 1,08366\ldots$$

Gauss. Gauss [20], in a letter to the astronomer Enke in 1849, stated that he had found in his early years (at age 17, in 1792), that the number $\pi(x)$ of primes up to x and the integral

$$\int_2^x \frac{dt}{\log t}$$

are asymptotically equal. However, Gauss does not mention Euler's formula and he gives no analytic basis for the approximation, which he presents only on the basis of extensive computations. So, Gauss conjectured that

$$\pi(x) \sim \frac{x}{\log x}, \quad x \to \infty$$

This conjecture, now a theorem, is called the "Prime Number Theorem".

Chebyshev. The first serious work on the function $\pi(x)$ is due to the Russian mathematician Chebyshev. On May 24, 1848, Chebyshev read at the Academy of St. Petersburg his

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first memoir on the distribution on prime numbers, later published in 1850. He proved, using elementary methods, that for every $\epsilon > 0$ there exists $x_0 > 0$ such that if $x > x_0$, then

$$(C' - \epsilon)\frac{x}{\log x} < \pi(x) < (C + \epsilon)\frac{x}{\log x}$$

where $C' = 2^{1/2}3^{1/3}5^{1/5}30^{-1/30} < 1$, $C = (6/5)C'$. Moreover, Chebyshev showed that if the limit

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x}$$

exists, it must be equal to 1. He deduced also, that Legendre's approximation of $\pi(x)$ cannot be true, unless $1,08366$ is replaced by 1.

Dirichlet. In 1837, Dirichlet proved his famous theorem of the existence of infinitely many primes in any arithmetic progression $n \equiv h \pmod{k}$, with h and k positive co-primes integers. The main novelty in his proof consisted in making use of characters modulo k, that is, homomorphisms from the group $(\mathbb{Z}/k\mathbb{Z})^*$ of invertible residues $\pmod{k}$ into the multiplicative group of complex numbers of modulus 1, (see Tenenbaum [59] cap II.8, for all these facts).

Riemann. Riemann considers the zeta-function defined by the generalized harmonic series $\zeta(s) = \sum n^{-s}$, where the letter s denotes a complex variable $s = \sigma + it$ $\sigma > 1$ its real part and t its imaginary part. If $\sigma > 1$ we can represent $\zeta(s)$ by an absolutely convergent infinite product, namely

$$\zeta(s) = \prod_p \{1 - p^{-s}\}^{-1}. \tag{2}$$

The Riemann hypothesis appears in a paper over the number of primes less than a given magnitude, published in 1859 by Riemann (see-preferably- Edwards' book [15], also [53], or [37]). It is the only work that Riemann published in number theory, but most of Riemann's ideas have been incorporated in later to the work of many mathematicians.

In 1859 Dirichlet died and Riemann was appointed to the chair of mathematics at Göttingen. A few days later Riemann was elected to the Berlin Academy of Sciences. He had been proposed by Kummer, Borchardt and Weierstrass. Riemann, newly elected to the Berlin Academy of Sciences had to report on their most recent research and he sent a report on *Über die Anzahl der Prinzahlen unter einer gegebener Gröse*. Riemann style is extremely difficult, his paper is extremely condensed and in particular [53], is probably a summary of very extensive researches which he never found the time to expound at grater length. The main purpose of the paper was to give estimates for the number of primes less than a given number. In this paper he stated that $\zeta(s)$ had infinitely many non trivial roots and that it seemed probable that they all have real part 1/2. Many of

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the results which Riemann obtained in this paper were later proved by Hadamard, de la Vallée-Poussin, Hardy and von Mangoldt.

2 Analytic continuation

The development of complex analysis was a central preoccupation of Riemann's and so it comes as no surprise that from the beginning Riemann considered the zeta-function as an analytic function. He first showed that $\zeta(s)$ had an analytic continuation to the complex plane as a meromorphic function which has only one singularity, a simple pole of residue 1 at $s = 1$.

Riemann derives his functional equation for $\zeta(s)$ from the representation of the gamma-function as an integral, for which Riemann still used the symbol Π

$$\Gamma(s) = \Pi(s-1) = \int_0^\infty e^{-u} u^{s-1} du. \tag{3}$$

The gamma-function defined by (3) has meromorphic continuation to all of $\mathbb{C}$, and is analytic except at $s = 0, -1, -2, \cdots, -n, \cdots$, where it has simple poles.

If n is a positive integer, we have, on writing nx for u

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty e^{-nx} x^{s-1} dx.$$

Hence

$$\Gamma(s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-nx} x^{s-1} dx = \int_0^\infty \frac{x^{s-1} dx}{e^x - 1}, \quad Re(s) > 1$$

if the inversion of the order of summation and integration can be justified. Next he considers the integral

$$I(s) = \int_C \frac{(-x)^{s-1}}{e^x - 1} dx$$

where the contour C starts at infinity on the positive real axis, encircles the origin once in the positive direction (counterclockwise), excluding the points $\pm 2i\pi n, n \in \mathbb{N}$ and returns up the positive real axis to $+\infty$. In the many-valued function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ the $\log(-x)$ is determined in such a way that it is real for negative values of x.

Thus Riemann deduced that for each $s \neq 1$

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s) \tag{4}$$

(see the proof in [59] pag. 139 or [60], Chap. II, §2.3). The equation (4) gives a relation between $\zeta(s)$ and $\zeta(1-s)$ which, by making use of properties of the gamma-function, can be formulated as the statement that $\Gamma(s/2)\pi^{-s/2}\zeta(s)$ remains unchanged when s is replaced by $1-s$. Indeed, on writing $\pi n^2 x$ for u in $\Gamma(s/2)$ we have

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \int_0^\infty \theta(x) x^{s/2-1} dx, \quad (\sigma > 1)$$

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where $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x}$, and using the equation

$$2\theta(x) + 1 = x^{-1/2}[2\theta(1/x) + 1]$$

for the theta-function of Jacobi, one obtains the relation

$$\Gamma(s/2)\pi^{-s/2}\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} \theta(x)(x^{(s/2)-1} + x^{-(1+s)/2})dx.$$

The last integral is convergent for all values of s, and so the formula holds, by analytic continuation, for all values of $s \neq 1$. Now the right-hand side is unchanged if s is replaced by $1 - s$. Hence we can write the functional equation (4) in the more symmetric form

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s). \tag{5}$$

(see [59], [11] or [15]). The symmetry of the functional equation and the poles at $s = 0$, and $s = 1$ of $\pi^{-s/2}\Gamma(s/2)\zeta(s)$, suggests the introduction of the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \tag{6}$$

that is an entire function that non vanishing in $Re(s) > 1$. Then from (5) and (6) we have that the function $\xi(s)$ verifies the functional equation

$$\xi(s) = \xi(1-s). \tag{7}$$

it also has no zeros in $Re(s) < 0$, apart from the trivial zeros at $s = -2, -4, -6, \ldots$. Thus all the zeros have their real parts between 0 and 1.

Riemann considered $\xi(s)$ with $s = 1/2 + it$ and stated that there is a product representation

$$\xi(t) = \xi(0)\prod_{\alpha}\left(1 - \frac{t^2}{\alpha^2}\right) \tag{8}$$

with zeros α of $\xi(t)$ that correspond to the critical zeros ρ of the zeta-function. Hadamard studied ([21]) entire functions and their representations as infinite products. One consequence is that the product formula (8) is valid.

Theorem. (Riemann) *The zeta function $\zeta(s)$ is analytic in the whole complex plane except for a simple pole at $s = 1$ with residue 1. It satisfies the functional equation (4).*

Riemann hypothesis. *The non trivial zeros of the function $\zeta(s)$ have real part equal to 1/2.* In 1900 Hilbert included the resolution of Riemann hypothesis as the 8th problem of his 23 problems for mathematicians of the twentieth century to work on. In the later years of the nineteenth century several mathematicians had developed Riemann's work to the point that Hadamard and C. de la Vallée Poussin independently of one another in 1896 proved the Prime Number Theorem.

3   Riemann's explicit formula

Riemann gave an explicit formula for $\pi(x)$ in terms of the complex zeros of $\zeta(s)$. In his derivation of the formula, he uses complex integrals and the Cauchy residue theorem. The starting point is the product representation (2) of $\zeta(s)$ in $Re(s) > 1$. Thus, the relation (2) implies that $\log \zeta(s) = -\sum \log(1 - p^{-s})$. Expanding $\log(1 - p^{-s})$ and summing we obtain

$$\log \zeta(s) = \sum_p \sum_n (1/n)p^{-ns}, \quad Re(s) > 1. \tag{9}$$

With the assistance of these methods (as Riemann says), the number of prime numbers that are smaller than x can now be determined. Let $F(x)$ be equal to this number when x is not exactly equal to a prime number; but let it be greater by $1/2$ when x is a prime number, so that, for any x at which there is a jump in the value in $f(x)$

$$F(x) = \frac{F(x+0) + F(x-0)}{2}$$

and states: If in the identity (9) one now replaces p^{-ns} by $s \int_{p^n}^{\infty} x^{-s-1}dx$, $(Re(s) > 1)$ for each $n \in \mathbb{N}$, then one obtains

$$\frac{\log \zeta(s)}{s} = \int_0^{\infty} f(x)x^{-s-1}dx, \quad Re(s) > 1 \tag{10}$$

if one denotes

$$F(x) + (1/2)F(x^{1/2}) + (1/3)F(x^{1/3}) + \dots$$

by $f(x)$. If we think of $\zeta(s)$ as given and $f(x)$ as required, then (10) represents an integral equation to be solved for $f(x)$.

He applies Fourier inversion to (10) to obtain that

$$f(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s)x^s \frac{ds}{s}, \quad (a > 1). \tag{11}$$

First, Riemann integrates by parts to obtain

$$f(x) = -\frac{1}{2\pi i} \cdot \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left(\frac{\log \zeta(s)}{s}\right) x^s ds, \quad (a > 1) \tag{12}$$

From the relations (6) and (8), the following formula for $\log \zeta(s)$ is obtained

$$\log \zeta(s) = \log \xi(0) + \sum_\alpha \log\left(1 + \frac{(s-1/2)^2}{\alpha^2}\right) - \log \Gamma(s/2 + 1) + (s/2)\log \pi - \log(s - 1). \tag{13}$$

and substituting (13) in (12) one obtains his result

$$f(x) = Li(x) - \sum_{Im\rho>0}[Li(x^\rho) + Li(x^{1-\rho})] + \int_x^{\infty} \frac{dt}{t(t^2 - 1)\log t} + \log \xi(0) \tag{14}$$

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which is the Riemann's formula except that, as noted by Edwards in [15], $\log \xi(0)$ equals $-\log 2$, and

$$Li(x) = \lim_{\epsilon \to 0} \left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t} \right) = \int_2^x \frac{dt}{\log t} + 1,04\cdots$$

To deduce the Rieman's formula for $\pi(x)$, that is, for the number of primes less than any given magnitude x, we observe that the functions $f(x)$ and $\pi(x)$ are related by the formula

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots + \frac{1}{n}\pi(x^{\frac{1}{n}}) + \cdots \tag{15}$$

But if $x^{1/n} < 2$ for any given x and n sufficiently large, then $\pi(x^{\frac{1}{n}}) = 0$. Thus the series in (15) is finite. Riemann inverts this relation (15) by means of the Möbius inversion formula (see §10.9 [15]) to obtain

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{\frac{1}{n}}). \tag{16}$$

From (16) and (14) one obtains the Riemann's formula for $\pi(x)$

$$\pi(x) \sim Li(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} Li(x^{\frac{1}{n}}). \tag{17}$$

Riemann made in their paper the following assertions related to zeros of $\zeta(s)$:

(1) There are infinitely many complex zeros of $\zeta(s)$ which lie in the critical strip. Zeros of the zeta-function in the critical strip are denoted $\rho = \beta + i\gamma$. Let $T > 0$, and let $N(T)$ denote the number of zeros of the function $\zeta(s)$ in the region $0 \le Re(s) \le 1$, $0 < Im(s) \le T$. That is

$$(T) = \#\{\rho = \beta + i\gamma \mid 0 \le \beta \le 1, 0 < \gamma \le T\}. \tag{18}$$

According to Riemann, the number of zeros with imaginary parts between 0 and T is about

$$\frac{T}{2\pi} \log(\frac{T}{2\pi}) - \frac{T}{2\pi}.$$

(2) Riemann stated also, that the number of zeros of the zeta-function between this limits, in the critical line $Re(s) = 1/2$, is "about" the same.

(3) If ρ is a complex zero of $\zeta(s)$, then the series $\sum |\rho|^{-2}$ converges and the series $\sum |\rho|^{-1}$ diverges.

(4) The entire function $\xi(s) = s(s-1)\pi^{-(1/2)s}\Gamma(s/2)\zeta(s)$ can be written as Weierstrass' product.

(5) All complex zeros of $\zeta(s)$ lie on the critical line $Re(s) = 1/2$.

(6) The relation

$$f(x) = Li(x) - \sum_\rho Li(x^\rho) + \int_x^\infty \frac{dt}{t(t^2-1)\log t} - \log 2, \ x > 1$$

where

$$Li(x^\rho) = Li(e^{\rho \log x}), \ Li(e^w) = \int_{-\infty+iv}^{u+iv} \frac{e^z}{z}dz, \ w = u + iv, \ \sum_\rho Li(x^\rho) = \lim_{T\to+\infty} \sum_{|\rho|\leq T} Li x^\rho$$

hods true.

A simpler variant of his formula is

$$\psi(x) = \sum_{n\leq x} \Lambda(n) = x - \sum_\rho \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2}),$$

valid for x not a prime power ($x \neq p^m$). Note that the Prime Number Theorem is equivalent to $\psi(x) \sim x$, $x \to \infty$ and that $|x^\rho| = x^\beta$; thus it was necessary to show that $\beta < 1$ in order to conclude of Gauss's conjecture, that is, the Prime Number Theorem.

Hypothesis (1), (3), (4) were proved by Hadamard [21]. The estimate of $N(T)$ and (6) were proved by Mangoldt [45].

4 Complex zeros of the zeta function

In order to prove the convergence of the product (8) Riemann needed to investigate the distribution of roots of $\xi(s)$, then he begins to observing that if $Re(s) > 1$, then by the Euler product, $\zeta(s) \neq 0$ (because a convergent infinite product can be zero only if one of its factors is zero). Moreover if $s = 1 + it$ we have the following result

Theorem (Hadamard-de la Vallée Poussin). *If $t \neq 0$, then $\zeta(1+it) \neq 0$.*

As consequence, the zeta function has no zeros in the closed half plane $Re(s) \geq 1$.

If $Re(s) < 0$, then $Re(1-s) > 1$, the right-side in the functional equation (5) is not zero, so the zeros must be exactly the poles of $\Gamma(s/2)$. Then, the zeros of $\zeta(s)$ are

1. Simple zeros at the points $s = -2, -4, -6, \cdots$, which are called the trivial zeros

2. Zeros in the critical strip consisting of the complex zeros $\rho = \beta + i\gamma$ with $0 \leq Re(\rho) \leq 1$. In 1914, Hardy proved that there are infinitely many roots ρ of $\zeta(s) = 0$ on the line $Re(s) = 1/2$. (see [24] or [11]).

Since $\zeta(\bar{s}) = \overline{\zeta(s)}$, the zeros of $\zeta(s)$ lie symmetrically with respect to the real axis, so, it suffices to consider the zeros in the upper half of the critical strip. It is common to list the complex zeros $\rho_n = \beta_n + i\gamma_n$, with $\gamma_n > 0$, in order of increasing imaginary parts as $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots$. Here zeros are repeated according to their multiplicity.

Van de Lune, te Riele and Winter [44] have determined that the first 1.500.000.001 complex zeros of $\zeta(s)$ are all simple, lie on the critical line, and have imaginary part with $0 < \gamma < 545.439.823, 215$.

(i) Let $T > 0$, and let $N(T)$ denote the number of zeros of the function $\zeta(s)$ in the region $0 \leq Re(s) \leq 1$, $0 < Im(s) \leq T$. That is

$$N(T) = \#\{\rho = \beta + i\gamma \mid 0 < \beta \leq 1, 0 < \gamma \leq T\}.$$

By the functional equation and the argument principle, one obtains

$$N(T) = \frac{T}{2\pi} \log(\frac{T}{2\pi e}) + \frac{7}{8} + S(T) + O(\frac{1}{T}) \tag{19}$$

where $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT) \ll \log T$ with the argument obtained by continuous variation from 2 to $2 + iT$, and thence to $1/2 + iT$, along straight lines. It was conjectured by Riemann and proved by von Mangoldt. This is the so called Riemann-von Mangoldt formula, (the sign $\ll$ of I.M. Vinogradov is taken, as usual, in the sense O).

If the Riemann hypothesis (henceforth **RH** for short) is true, then it is known that

$$S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT) \ll \frac{\log T}{\log \log T}. \tag{20}$$

(ii) Now, let $N_0(T)$ be the zero counting function

$$N_0(T) = \#\{\rho = 1/2 + i\gamma \mid 0 < \gamma \leq T\},$$

that it, $N_0(T)$ counts the number of zeros of $\zeta(s)$ in the critical line, up to height T. The **RH** is equivalent to $N(T) = N_0(T)$ for all T.

(iii) Let $N(\sigma, T)$ the function which counts the number of zeros of $\zeta(s)$ in the critical strip up to height T, and to the right of σ-line.

A method to approach Riemann's hypothesis is to estimate for any given σ, $1/2 \leq \sigma \leq 1$, the number $N(\sigma, T)$ of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$, with $\sigma \leq \beta$ and $0 < t \leq T$ (where T is sufficiently large):

$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \mid \beta \geq \sigma, 0 < \gamma \leq T\}$$

$1/2 \leq \sigma \leq 1$. In this case, the **RH** is equivalent to the property $N(1/2, T) = 0$ for all T.

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Estimates for $N(\sigma, T)$ may be written in the form

$$N(\sigma, T) \ll T^{a(\sigma)(1-\sigma)} \log^b T, \quad b \geq 0. \tag{21}$$

In view of formula (19) one has $a(\sigma)(1-\sigma) = 1$ and $b = 1$ in (21) for $0 \leq \sigma \leq 1/2$, while $a(\sigma)(1-\sigma) \leq 1$ for $\sigma > 1/2$ and $a(\sigma)(1-\sigma)$ is non increasing.

The hypothesis $a(\sigma) \leq 2$ in (21) is known as "the density hypothesis". A. Selberg [57] has proved that

$$N(\sigma, T) \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})} \log T \tag{22}$$

uniformly for $1/2 < \sigma \leq 1$. The **RH** is equivalent to $N(\sigma, T) = 0$ for $\sigma > 1/2$.

Gaps between zeros. The most detailed study of the zeros on the critical line involves estimates for the difference $\gamma_{n+1} - \gamma_n$ between the ordinates of consecutive zeros $s = \frac{1}{2} + i\gamma$, $\gamma \in \mathbb{R}$. For a long time, the result of Hardy and Littlewood (1918) that $\gamma_{n+1} - \gamma_n \ll \gamma_n^{1/4+\epsilon}$ was the best. Later it was superseded, with the use of finer methods by Moser, Balasubramanian, Karatzsuba, Ivic [31].

It is known that the average size of $\gamma_{n+1} - \gamma_n$ is $\sim 2\pi/\log \gamma_n$. Thus if

$$\lambda := \limsup_{n \to \infty} (\gamma_{n+1} - \gamma_n) \frac{\log \gamma_n}{2\pi}, \quad \mu := \liminf_{n \to \infty} (\gamma_{n+1} - \gamma_n) \frac{\log \gamma_n}{2\pi}$$

we have $\mu \leq 1 \leq \lambda$, and it is known (unconditionally) that $\mu < 1 < \lambda$. It is conjectured that $\mu = 0$ and $\lambda = \infty$, but assuming the generalized **RH** Conrey-Ghosh- Gonek [9] proved only that $\lambda > 2.68$. Also under the **RH** they had proved $\mu < 0.5172$.

If (20) holds, then from the estimate (19) one obtains

$$N(T + H) - N(T) > 0, \text{ for } H = C/\log \log T, C > 0, T \geq T_0.$$

Hence assuming the **RH** the following estimate holds

$$\gamma_{n+1} - \gamma_n \ll \frac{1}{\log \log \gamma_n.}$$

Instead of working with the complex zeros of $\zeta(s)$ on the critical line $Re(s) = 1/2$, it is convenient to introduce the function

$$Z(t) = \chi^{-1/2}(\frac{1}{2} + it)\zeta(\frac{1}{2} + it)$$

where $\chi(s)$ is given by $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\Gamma(1 - s)$. As $\chi(s)\chi(1 - s) = 1$ and $\overline{\Gamma(s)} = \Gamma(\bar{s})$, it follows that $| Z(t) | = | \zeta(\frac{1}{2} + it) |$, $Z(t)$ is even, and $\overline{Z(t)} = Z(t)$. Hence $Z(t)$ is real if t is real and the zeros of $Z(t)$ correspond to the zeros of $\zeta(s)$ on the critical line $Re(s) = 1/2$.

5 Consequences of the Riemann hypothesis

There are many equivalent forms for the Riemann hypothesis. The study and classification of these assertions sharpens our understanding of the problem. Writing $\Xi(t) = \xi(1/2 + it)$, and as consequence of the functional equation (7), we obtain $\Xi(t) = \Xi(-t)$ which is real for real t an even function of t. Then **RH** is the assertion to all zeros of $\Xi(t)$ are reals.

Riemann obtained that $\Xi(t)$ verifies the relation

$$\Xi(t) = 4 \int_1^\infty \frac{d[x^{3/2}\theta'(x)]}{dx} x^{-1/4} \cos\left(\frac{t}{2}\log x\right) dx \tag{23}$$

where $\theta(x) = \sum_{n=1}^\infty e^{-n^2\pi x}$. We can write (23) in the form

$$\Xi(t) = 2 \int_0^\infty \phi(u) \cos ut \, du$$

where $\phi(u) = 2\sum_{n=1}^\infty (2n^4\pi^2 e^{9t/2} - 3n^2\pi e^{5t/2}) e^{-\pi n^2 e^{2t}}$. A necessary and sufficiently condition to of zeros of $\Xi(t)$ are reals is

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \phi(\alpha)\phi(\beta) e^{i(\alpha+\beta)x} e^{(\alpha-\beta)y} (\alpha - \beta)^2 d\alpha d\beta \geq 0$$

for all x, y reals (see Chap X and XIV of [60]).

A. Speiser [58], proved that **RH** is equivalent to the non-vanishing of the derivate $\zeta'(s)$ in the left-half of the critical strip $0 < \sigma < 1/2$.

The order of zeta-function on the critical line. Hardy and Littlewood proved that $\zeta(1/2 + it) \ll t^{\frac{1}{4}+\epsilon}$ for every positive ϵ, and $t \geq t_0 > 0$ (since $\overline{\zeta(1/2 + it)} = \zeta(1/2 - it)$, t may be assumed to be positive. H. Weyl improved the bound to $t^{1/6+\epsilon}$, Bombieri-Iwaniec obtained 9/56, instead to 1/6, A. Watt improved the bound to 89/560 and M.N. Huxley ([29]) improved the bound to

$$\zeta(1/2 + it) \ll t^{\frac{89}{570}+\epsilon}.$$

Lindelöf conjectured that

$$\zeta(1/2 + it) \ll_\epsilon t^\epsilon, \quad t \geq t_0 > 0$$

for every positive ϵ. In 1912, Littlewood proved that the Lindelöf hypothesis is true if the **RH** is true. The converse theorem cannot be made.

For the function $S(T) = \frac{1}{\pi}\arg\zeta(\frac{1}{2} + iT)$ we have that the **LH** is equivalent to

$$\int_0^T S(t)dt = o(\log T), \quad (T \to \infty)$$

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while the **RH** implies much more

$$\int_0^T S(t)dt \ll \frac{\log T}{\log \log T}, \quad (T \to \infty).$$

The order of zeta-function on the 1-line. E.C. Titchmarsh, was proved ([60] theorems 8.5 and 8.8), that each one of the inequalities

$$| \zeta(1+it) | > A \log \log t, \quad 1/ | \zeta(1+it) | > A \log \log t$$

is satisfies for some arbitrary large values of t, if A is a suitable constant and considers the question of how large the constant can be in the two cases.

On **RH** we have

$$e^\gamma \leq \limsup_{t \to \infty} \frac{\zeta(1+it)}{\log \log t} \leq 2e^\gamma$$

$$\frac{6}{\pi^2} e^\gamma \leq \limsup_{t \to \infty} \frac{1/\zeta(1+it)}{\log \log t} \leq \frac{12}{\pi^2} e^\gamma$$

where γ is Euler's constant, (see Titchmarsh [60] 8.9).

Riemann hypothesis and distribution of primes. We know that the zeta function was introduced as an analytic tool for studying prime numbers and some of the most important applications of the zeta functions belong to prime number theory. The equivalence between the error term in the Prime Number Theorem and the real part of the zeros of $\zeta(s)$ is as follow

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(x^{\theta + \epsilon})$$

or

$$\psi(x) = \sum_{p^m \leq x} \log p = x + O(x^\theta \log^2 x)$$

is equivalent to $\zeta(s) \neq 0$ for $Re(s) > \theta$, $1/2 \leq \theta < 1$ (Th. 12.3 [31]). If we use the strongest zero-free region then we deduce the results

$$\pi(x) - \int_2^x \frac{dt}{\log t} \ll x^{1/2} \exp(-C(\log x)^{3/5}(\log \log x)^{-1/5})$$

$$\psi(x) - x \ll x^{1/2} \exp(-C(\log x)^{3/5}(\log \log x)^{-1/5})$$

but the Riemann hypothesis is equivalent to each of the following statements related to prime numbers

(1) $\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x} \log x)$. [35]

(2) $\psi(x) = \sum_{p^m \leq x} \log p = x + O(\sqrt{x} \log^2 x)$, $x > 0$.

In the problem related to the difference of two consecutive prime numbers, Pilts conjectured that for every $\epsilon > 0$, $p_{n+1} - p_n \ll p_n^\epsilon$ where p_n denote the nth prime number, but this

has never been proved. On the other hand, it is known that the relation $p_{n+1} - p_n \ll \log p_n$ is certainly not true.

(3) The **RH** is equivalent to $p_{n+1} - p_n \ll p_n^{1/2} \log p_n$.

The Möbius function. The Euler product formula for $\zeta(s)$ implies that

$$\frac{1}{\zeta(s)} = \prod_p \{1 - p^{-s}\} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + it, \sigma > 1.$$

The coefficient $\mu(n)$ is known as Möbius function.

It has the property $\sum_{d|n} \mu(d) = 1$, $(n = 1)$ and 0 if $n > 1$, where $d|n$ means that d is a divisor of n. Thus $\mu(n)$ verifies

$$\mu(1) = 1, \mu(n) = - \sum_{d|n, d<n} \mu(d), (n > 1).$$

Each one of the following properties is equivalent to the **RH** (see [60])

(4) $M(x) = \sum_{n \le x} \mu(n) \ll x^{1/2+\epsilon}$, for all $\epsilon > 0$,

(5) the series $\sum_{n=1}^{\infty} \mu(n)/n^{-s}$ is convergent, and its sum is $1/\zeta(s)$, for every s with $\sigma > 1/2$.

The function w(n). Let $\omega(n)$ be the number of prime factors of the positive integer n counted without multiplicity, and $q \in \mathbb{N}$. D. Wolke proved that the **RH** is true if and only if

$$\sum_{n \le x} \omega^q(n) - x \sum_{0 \le j \le \frac{1}{2} \log x} R_{jq}(\log \log x)(\log x)^{-j} \ll x^{1/2+\epsilon}$$

holds for every $\epsilon > 0$, where $R_{jq}(t)$ are certain polynomials such that $deg\ R_{0q} \le q$, and $deg\ R_{jq} \le q - 1$, $(j \ge 1)$ (see [63]).

Sum of divisors. Let $\sigma(n)$ denote the sum of divisors of n, that is $\sigma(n) = \sum_{d|n} d$. G. Robin (see[54]) proved that **RH** is true, if and only if

$$\sigma(n) < e^\gamma n \log \log n$$

for large n, where γ is the Euler's constant. J.C. Lagarias [39], proved that under the **RH**

$$\sigma(n) < H_n + e^{H_n} \log H_n, \quad H_n = \sum_{j=1}^{n} 1/j, (n \ge 2).$$

Bertrand's postulate. In 1852, Chebyshev proved Bertrand's postulate according to which each interval $(n, 2n]$, $n \ge 1$, contains at least one prime. O. Romare, Y. Saoutier

[55] under the **RH** proved that for all $x \geq 2$ in the interval $(x - \frac{8}{5}\sqrt{x}\log x, x]$ there exist a prime number.

Pythagorean triangles. The positive integers a, b, c are said to form a primitive Pythagorean triples (a, b, c) if $a \leq b$, a, b, c are coprimes and $a^2 + b^2 = c^2$. Let $P(x)$ denote the number of primitive Pythagorean triangles, whose area is less than x. In 1955, J. Lambek y L. Moser [Pacific J. Math. 5(1955), 73-83, MR 16, 796h] proved that $P(x) = c_1 x^{1/2} + O(x^{1/3})$. Muler, Nowak and Menzer ([50]), assuming the **RH** proved that $P(x) = c_1 x^{1/2} + c_2 x^{1/3} + R(x)$, with $R(x) \ll x^{127/560+\epsilon}$. The error term is connected with the Möbius function and then with the zeros of the zeta function. W. Zhai [70] under the **RH** deduced

$$R(x) \ll x^{127/616}(\log x)^{963/308}, \ 127/616 = 0.2061\ldots$$

A Divisor problem. Let a, b be positive coprime integers such that $1 \leq a < b$, and let $\Delta_{a,b}(x)$ be the error term for the asymptotic formula of

$$D^*(a, b; x) = \sum_{m^a n^b \leq x, \ (m,n)=1} 1, \ (1 \leq a < b, \ (a,b) = 1.$$

Lu y Zhai [43], proved under the **RH** that

$$\Delta_{a,b}(x) \ll x^{\frac{a+6b}{(2a+7b)(a+b)}+\epsilon}, \ b \leq \frac{3a}{2}; \quad \Delta_{a,b}(x) \ll x^{\frac{a+2b}{(2a+2b)(a+b)}+\epsilon}, \ b > \frac{3a}{2}.$$

The primitive circle problem. Let

$$P(x) = \sum_{m^2+n^2 \leq x, (m,n)=1} 1$$

and let $\Delta(x) = P(x) - (6/\pi)x$. The "primitive circle problem" is to obtain an upper bound for $\Delta(x)$. The best known unconditional result is

$$\Delta(x) \ll x^{1/2} \exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}).$$

Jie Wu ([68]), under the Riemann hypothesis proved that $\Delta(x) \ll x^{221/608+\epsilon}$.

Powerful numbers. Let $k \geq 2$ be a fixed integer and let N_k denote the set of all positive integers n with the property that if a prime $p \mid n$, then $p^k \mid n$. Thus the set of powerful numbers N_k contains 1 and the numbers whose canonical representation is $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $\alpha_i \geq k$ for all $i = 1, 2, \cdots, r$. We put $f_k(n) = 1$ if $n \in N_k$ and $f_k(n) = 0$ if $n \notin N_k$ and the Dirichlet series representation is

$$F_k(s) = \sum_{n=1}^{\infty} f_k(n)n^{-s} = \prod_p (1 + \frac{p^{-ks}}{1-p^{-s}}), \ Re(s) > 1/k,$$

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then $f_k(n)$ is the characteristic function of N_k, (see E. Kratzel [36] Chap. 7). The problem is to obtain an estimate for the number $N_k(x)$ of k-full integers not exceeding x:

$$N_k(x) = \sum_{n \leq x} f_k(n), \quad k \geq 2$$

the main term is deduced from the residues of $F_k(s)$ at the simples poles $s = 1/u$, $u = k, k+1, \cdots, 2k-1$ and we have the following relation

$$N_k(x) = \sum_{t=k}^{2k-1} c_{t,k} x^{1/k} + \Delta_k(x), \; c_{t,k} = \operatorname*{Res}_{s=1/t} F_k(s)/s.$$

Let $\gamma_k = \inf\{\rho_k \mid \Delta_k(x) \ll x^{\rho_k}\}$. A. Ivić showed that we would have under the assumption of the truth of Lindelöf hypothesis, $\gamma_k \leq 1/2k$, but in special cases, it is possible to prove much more. If $k = 2$, we have

$$F_2(s) = \frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}.$$

Let $N_2(x)$ be denote the number of square-full integers $n \leq x$, and $\Delta_2(x)$ the error term. The best unconditional O-estimate is

$$\Delta_2(x) = N_2(x) - \frac{\zeta(3/2)}{\zeta(3)} x^{\frac{1}{2}} - \frac{\zeta(2/3)}{\zeta(2)} x^{\frac{1}{3}} \ll x^{\frac{1}{6}} \exp(-c(\log x)^{\frac{3}{5}} (\log\log x)^{-\frac{1}{5}}),$$

essentially due to P. T. Bateman and E. Grosswald [3] see Theorem 14.4 and page 438 in A. Ivić's book [31].

Under the **RH**, J. Wu [67] proved that $\Delta_2(x) \ll x^{(12/85)+\epsilon}$, for any $\epsilon > 0$, which can be compared with the complementary estimate $\Delta(x) = \Omega(x^{1/10})$, due to R. Balasubramanian, K. Ramachandra and M. V. Subbarao (Acta Arith. 50 (1988), no. 2, 107-118). Bateman and Grosswald also noted that the statement $\Delta_2(x) \ll x^\alpha$, for any constant exponent $\alpha < 1/6$, is equivalent to some quasi-Riemann hypothesis, i.e. to the assertion that $\zeta(s) \neq 0$ for all s with $\sigma > \sigma_0$ and $\sigma_0 < 1$.

A similar situation arises in case of cube-full integers.

Farey series and the RH. A sequence (x_n) of real numbers is uniformly distributed (mod 1) if and only if for every Riemann-integrable function f on $[0, 1]$ one has

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f(\{x_n\}) = \int_0^1 f(x)dx$$

where $\{x\}$ is the fractional part of x, ([10], Th. 3 Cahp. VIII). Let $F_x = F_{[x]}$ denote the sequence of all irreducible fractions with denominators $\leq x$, arranged in increasing order of magnitude, that is

$F_x = \{r_k = a_k/b_k; 0 < a_k \le b_k \le x, (a_k, b_k) = 1, k = 1, 2, \cdots, \phi(x)\}$, for any $x \ge 1$, where $\phi(x) = \sum_{n \le x} \varphi(n)$, $(\varphi(n) = \#\{k \le n; (k, n) = 1\}$ is the Euler function. $F_{[x]}$ is called the Farey series of order x, (Note that the Farey series is not a series at all but a finite sequence).

Since the Farey fractions are uniformly distributed (mod 1) we have that

$$\lim_{N \to \infty} \frac{1}{\phi(N)} \sum_{\substack{q \le N}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} f(a/q) = \int_0^1 f(x)dx \qquad (24)$$

for every Riemann-integrable function f on $[0, 1]$. Relation (24) suggests the problem of estimation of the error term

$$E_f(N) = \sum_{\substack{q \le N}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} f(a/q) - \phi(N) \int_0^1 f(x)dx.$$

Franel discovered a connection between Farey series and the Riemann hypothesis. He proved ([17]) that the estimation

$$\sum_{k \le \phi(x)} \left(r_k - \frac{k}{\phi(x)} \right)^2 \ll_\epsilon x^{2(\beta-1)+\epsilon}$$

for every $\epsilon > 0$ is equivalent to $\zeta(s) \ne 0$ for $Re(s) > \beta$. Thus

RH is truth if and only if

$$\sum_{k \le \phi(x)} \left(r_k - \frac{k}{\phi(x)} \right)^2 \ll x^{-1+\epsilon}.$$

Another version (Landau *Vorlesungen ii,* 167-177), is that the **RH** is equivalent to the statement

$$\sum_{k \le \phi(x)} \mid r_k - \frac{k}{\phi(x)} \mid \ll x^{\frac{1}{2}+\epsilon},$$

for all $\epsilon > 0$ as $x \to \infty$.

M. Mikolás, (see [46]-[49]) proved that the **RH** is equivalent to the relation

$$\sum_{k=1}^{\phi(x)} f(r_k) - \phi(x) \int_0^1 f(u)du \ll x^{1/2+\epsilon}$$

for all $\epsilon > 0$, where $f(u)$ can be $\sin(\lambda u)$, $\cos(\lambda u)$, $(|\lambda| \ne \pi)$ or a polynomial of degree ≤ 3. Huxley [28] has generalized Franel's theorem to the case of Dirichlet L-functions and Fujii [18] gives another equivalence with the **RH** in terms of the Farey series.

Kanemitsu and Yoshimoto [33] showed that each one of the estimates are equivalent to the **RH**

$$\sum_{r_k \leq 1/3} \left(r_k - \frac{h(1/3)}{2\phi(x)} \right) \ll x^{1/2+\epsilon}, \quad \sum_{r_k \leq 1/4} \left(r_k - \frac{h(1/4)}{2\phi(x)} \right) \ll x^{1/2+\epsilon}$$

where $h(t) = \sum_{r_k \leq t} 1$. Moreover, Kanemitsu and Yoshimoto obtain conditions equivalents for a great class of functions (see also [34], [69]).

The Riesz sum. M. Riesz (see [60]) considers the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{(n-1)! \zeta(2n)}.$$

An application of the calculus of residues gives

$$f(x) = \frac{i}{2} \int_{a-i\infty}^{a+i\infty} \frac{x^s ds}{\Gamma(s)\zeta(2s)\sin(\pi s)} = \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(1-s)x^s ds}{\zeta(2s)},$$

where $1/2 < a < 1$. Taking $a > 1/2$ it follows that $f(x) \ll x^{1/2+\epsilon}$. On the **RH** we can move the line of integration to $a = 1/4 + \epsilon$ and obtain the estimation $f(x) \ll x^{1/4+\epsilon}$. Conversely, by Mellin's inversion formula

$$\frac{\Gamma(1-s)}{\zeta(2s)} = -\int_0^{\infty} f(x) x^{-s-1} dx$$

and if $f(x) \ll x^{1/4+\epsilon}$ holds, then the integral converges uniformly for $\sigma \geq \sigma_0 > 1/4$, the analytic function represented is regular for $\sigma > 1/4$ and the truth of the **RH** follows.

Hardy and Littlewood in 1918 [60] stated that **RH** holds if and only if

$$\sum_{n=1}^{\infty} \frac{(-x)^n}{n! \zeta(2n+1)} \ll x^{-1/4}, \ x \to \infty.$$

Nyman-Beurling criterion. The Riemann zeta function on may see as a Mellin transform in the critical strip by means of the following formula

$$\frac{\zeta(s)}{s} = -\int_0^{\infty} \rho(1/x) x^{s-1} dx, 0 < \sigma < 1$$

where $\rho(x)$ is the fractional part function $\rho(x) = x - [x]$.

B. Nyman and A. Beurling had the idea that it should be possible to translate the Riemann hypothesis into a property of $\rho(x)$. Beurling's linear space $N_{(0,1)}$ consists of all functions

$$x \to f(x) = \sum_{k=1}^{n} a_k \rho(\frac{\theta_k}{x}), \ 0 < \theta_k \leq 1$$

where the a_k are constants such that, $\sum_{k=1}^{n} a_k \theta_k = 0$. For $1 < p < \infty$, let B^p be the closure of B in $L^p(0,1)$. In his thesis, Nyman [51] proved the following theorem:

Theorem. *The Riemann hypothesis is true if and only if $N_{(0,1)}$ is dense in $L^2(0,1)$.*

Beurling gives in his paper [4] a generalization of Nyman's theorem

Theorem. *Let $1 < p < \infty$. The following properties are equivalent.*
(i) $\zeta(s)$ has no zeros in the half plane $Re(s) > 1/p$
(ii) $N_{(0,1)}$ is dense in $L^p(0,1)$
(iii) The characteristic function $\chi_{(0,1)}$ is in the closure of $N_{(0,1)}$ in $L^p(0,1)$.

L. Baez-Duarte [1], stated a new version of Nyman-Beurling criterion:

Theorem. *Let $M(y)$ be denote the summatory function of $\mu(n)$, and let*

$$G_n(x) = \int_1^n M(\theta)\rho(\frac{1}{\theta x})\frac{d\theta}{\theta}.$$

(1) If $\lambda(x) = \chi_{(0,1)}(x)\log x$, then $\| G_n - \lambda \|_p \to 0$, as $n \to \infty$ implies that $\zeta(s) \neq 0$ para $Re(s) > 1/p$. (2) If $\zeta(s) \neq 0$ for $Re(s) > 1/p$ then $\| G_n - \lambda \|_r \to 0$, as $n \to \infty$ whenever $r \in (1, p)$.

Riemann ξ-function and positivity criterion. The Riemann ξ-function (6), is an entire function of order one which is real-valued on the real axis and satisfies the relation $Re\left(\frac{\xi'(s)}{\xi(s)}\right) > 0$ when $Re(s) > 1$. The Riemann hypothesis is equivalent to the positivity condition

$$Re\left(\frac{\xi'(s)}{\xi(s)}\right) > 0, \text{ when } Re(s) > 1/2.$$

(Hinkkanen [27]). J.C. Lagarias [38] defined an arbitrary discrete set Ω in $\mathbb{C}$ which represents the set of zeros of an entire function $f_\Omega(z)$ counted with multiplicity. Lagarias call admissible to the set Ω if complex conjugate zeros ρ and $\overline{\rho}$ occur with the same multiplicity, and the zeros satisfy the convergence condition

$$\sum_{\rho \in \Omega} \frac{1+ \mid Re(\rho) \mid}{1+ \mid \rho \mid^2} < \infty.$$

Theorem. *([38]) Let Ω be an admissible zero of set in $\mathbb{C}$. Then the following conditions are equivalent*
(1) All elements $\rho \in \Omega$ have $Re(\rho) \leq \theta$
(2) The function $f'_\Omega(s)/f_\Omega(s)$ satisfies the positivity condition

$$Re(f'_\Omega(s)/f_\Omega(s)) > 0, \text{ for } Re(s) > \theta.$$

If Ω are the non trivial zeros of $\zeta(s)$, and

$$h_{\mathbb{Q}}(\sigma) = \inf\{Re\left(\frac{\xi'(\sigma + it)}{\xi(\sigma + it)}\right) : -\infty < t < \infty\},$$

then under the **RH** Lagarias proved that $h(\sigma) = \xi'(\sigma)/\xi(\sigma)$, for $\sigma > 1/2$

This result is improved by R. Garunkstis [19], who obtained that if $\zeta(s) \neq 0$ for $\sigma > a$, $1/2 \leq a < 1$ then $h(\sigma) = \xi'(\sigma)/\xi(\sigma)$ for $\sigma > a$.

Xian-Jin Li (see [40]) showed that a necessary and sufficient condition for the non trivial zeros of the Riemann zeta function to lie on the critical line is that $\lambda_n = ((n-1)!)^{-1}(d^n/ds^n)(s^{n-1} \log \xi(s)) |_{s=1}$ is non negative for every positive integer. Thus the **RH** holds if and only if $\lambda_n \geq 0$, for each $n = 1, 2, 3, \ldots$.

He also showed that an identical result applies to the Riemann hypothesis for the Dedekind zeta-function $\zeta_K(s)$ of a number field.

The number λ_n can be written in terms of the complex zeros of $\zeta(s)$ as

$$\lambda_n = \sum_\rho (1 - (1 - 1/\rho)^n)$$

where the sum over ρ is understood as $\sum_\rho = \lim_{T \to \infty} \sum_{|Im(\rho) \leq T}$.

Bombieri y Lagarias showed that Li's criterion follows as a consequence of a general set of inequalities for an arbitrary multiset of complex numbers ρ and therefore is not specific to zeta function (see [5]).

Weil-Bombieri and RH. In 1952, A. Weil [62] gives a generalization of the Riemann explicit formula and in the same paper Weil proved that from his formula one can define a quadratic functional whose positivity is equivalent to the Riemann hypothesis.

E. Bombieri [6], studies the Weil explicit formula. First of all, he provides a very clear proof of the formula which relates the values of a smooth function f summed over the primes to the sum of its Mellin transform $\tilde{f}$ summed over the complex zeros of the Riemann zeta-function.

Explicit Formula. Define $f^*(x) = f(1/x)/x$. If $f(x)$ is any smooth complex-valued function with compact support in $(0, \infty)$ and Mellin transform

$$\tilde{f}(x) = \int_0^\infty f(x) x^{s-1} dx$$

then

$$\sum_\rho \tilde{f}(\rho) = \int_0^\infty f(x) dx + \int_0^\infty f^*(x) dx - \sum_{n=1}^\infty \Lambda(n)(f(n) + f^*(n))$$

$$-(\log 4\pi + \gamma)f(1) - \int_1^\infty \left(f(x) + f^*(x) - \frac{2}{x}f(1) \right) \frac{x}{x^2 - 1} dx.$$

Bombieri next proves a strong version of Weil's criterion for the Riemann hypothesis. One takes a function $f(x) = g * \overline{g}^*$ in the Weil formula, where $*$ is the multiplicative convolution of a function g and its transpose conjugate $\overline{g}^*$.

Theorem (Bombieri). *The Riemann Hypothesis holds if and only if*

$$\sum_{\rho} \tilde{g}(\rho)\overline{\tilde{g}}(1-\rho) > 0$$

for every complex-valued $g(x) \in C_0^{\infty}((0, \infty))$, not identically 0.

6 Generalizations of the Riemann hypothesis

Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence of complex numbers, and $\sum_{n=1}^{\infty} a(n)n^{-s}$, $(s \in \mathbb{C})$ the associated Dirichlet series. When $a(n) \equiv 1$ we have the Riemann zeta function.

L-functions. Let k be a fixed positive integer. A Dirichlet character χ of conductor k is a completely multiplicative function on the integers: that is $\chi(mn) = \chi(m)\chi(n)$ for every pair of integers m, n, periodic with period k, $\quad \chi(n + k) = \chi(n)$, and such that $\chi(n) = 0$ if $(n, k) > 1$. The principal character verifies $\chi(n) = 1$ if $(n, k) = 1$ and $\chi(n) = 0$ if $(n, k) > 1$.

One can then define a *L*-function associated to $\chi(n)$ by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

and as $\chi(n)$ is bounded, we have that the series is absolutely convergent for $Re(s) > 1$. More than this is true, however: the series for $L(s, \chi)$, (χ non principal) converges for $Re(s) > 0$. One has also an expansion as an Euler product

$$L(s, \chi) = \prod_{p}(1 - \chi(p)p^{-s})^{-1}.$$

It was introduced by Dirichlet for solving the problem of existence of infinity prime numbers on arithmetic progressions.

Dirichlet's theorem. *There are infinitely many primes of the form $kt + \ell$ if and only if ℓ and k are coprime integers $(\ell, k) = 1$.* (For instance, see Ellison Mendes-France's book for a proof [16]).

The Dirichlet series simplest after $\zeta(s)$ is the Dirichlet L-function, $L(s, \chi_3)$, for the non trivial character of conductor 3:

$$L(s, \chi_3) = \sum_{r=0}^{\infty}\{(3r + 1)^{-s} - (3r + 2)^{-s}\}.$$

This can be written as an Euler product

$$L(s, \chi_3) = \prod_{p \equiv 1 \pmod 3} \{1 - p^{-s}\}^{-1} \prod_{p \equiv 2 \pmod 3} \{1 + p^{-s}\}^{-1}.$$

The L-function $L(s, \chi_3)$ satisfies the functional equation

$$\xi(s, \chi_3) = (\pi/3)^{-(s+1)/2} \Gamma((s+1)/2)) L(s, \chi_3) = \xi(1 - s, \chi_3).$$

It is expected to have all of its nontrivial zeros on the critical line $1/2$.

One defines a character of conductor 4 as $\chi_4(n) = 0$ if n is even; and $\chi_4(n) = (-1)^{(n-1)/2}$ if n is odd. Then

$$L(s, \chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s} = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots, \quad Re(s) > 1.$$

This can be written as an Euler product

$$L(s, \chi_4) = \prod_{p \equiv 1 \pmod 4} \{1 - p^{-s}\}^{-1} \prod_{p \equiv 3 \pmod 4} \{1 + p^{-s}\}^{-1}.$$

The L-function $L(s, \chi_4)$ also satisfies an analogue functional equation.

Theorem. *Let χ be a non principal character, primitive of conductor k, $(k > 1)$. The L-function $L(s, \chi)$ has an analytic continuation to the complex plane which is an entire function of s. It verifies a functional equation*

$$\xi(s, \chi) = E(\chi) \xi(1 - s, \bar{\chi})$$

where

$$\xi(s, \chi) = \left(\frac{\pi}{k}\right)^{-\frac{s+a}{2}} \Gamma(\frac{s+a}{2}) L(s, \chi)$$

and $a = 0$ if $\chi(-1) = 1$; $a = 1$ if $\chi(-1) = -1$. The bar means complex conjugation and

$$E(\chi) = k^{-1/2} \sum_{r=1}^{k-1} \chi(r) e^{\frac{2\pi i}{k}}.$$

For the proof, see for instance [11].

Since $L(s, \chi) \neq 0$ for $\sigma > 1$, $\xi(s, \chi) \neq 0$ and $\xi(s, \bar{\chi}) \neq 0$ for $\sigma > 1$. By the functional equation we obtain that $\xi(s, \chi) \neq 0$ for $\sigma < 0$ so that all the zeros of $\xi(s, \chi)$ must lie in the strip $0 \leq \sigma \leq 1$.

Corollary. *If $a = 0$, the function $L(s, \chi)$ has simple zeros at $s = 0, -2n, n \in \mathbb{N}$. If $a = 1$, then $L(s, \chi)$ has simple zeros at $s = -(2n - 1), n \in \mathbb{N}$. These are called the trivial zeros.*

As in the case of $\zeta(s)$, one can deduce the existence of an infinity of non-real zeros of $L(s, \chi)$ either by using to the theory of entire functions, or by an estimate of the Riemann-von Mangoldt type. One can prove that if $N(T)$ denotes the number of zeros $\{\rho = \beta + i\gamma, 0 < \beta < 1, 0 < \gamma \le T\}$ of $L(s, \chi)$ then

$$N(T) = \frac{T}{2\pi} \log T + AT + O(\log T)$$

where A is a constant which depends on k, the modulus of character χ.

The theorem of the existence of an infinity of zeros of the critical line, proved by Hardy for the $\zeta(s)$, has been extended to a general class of Dirichlet series, including the L-functions, by E. Hecke.

Generalized Riemann Hypothesis. The conjecture is that all the zeros of L-functions are situated on the critical line.

Let $\left(\frac{d}{n}\right)$ denote Kronecker's extension of the Legendre symbol for quadratic residues, for $n = 1, 2, \cdots$. It is known that if d is the discriminant of a quadratic number field (that is, d is square-free and $d \equiv 1 \pmod 4$; or $d = 4N$ where $N \equiv 2$ or $3 \pmod 4$, and square-free), and $\chi_d(n) = \left(\frac{d}{n}\right)$, then χ_d is a real primitive Dirichlet character modulus $|d|$. Let

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n) n^{-s}$$

be the associated Dirichlet L-function. It is an entire function (for $d \ne 1$) which satisfies the functional equation

$$\xi(s, \chi_d) = \left(\frac{\pi}{|d|}\right)^{-s/2} \Gamma\left(\frac{s + a_d}{2}\right) L(s, \chi_d) = \xi(1 - s, \chi_d)$$

where $a_d = 1$ if $d < 0$ and $a_d = 0$ if $d > 0$.

Zeta functions of number field. For a number field K of degree n, the zeta function of K is

$$\zeta_K(s) = \sum_{m=1}^{\infty} \frac{F_K(m)}{m^s},$$

where $F_K(m)$ is the number of ideals whose norm is precisely m. This sum converges for complex s with $Re(s) > 1$. Often $\zeta_K(s)$ is referred to as the Dedekind zeta function. When $K = \mathbb{Q}$, is just the Riemann zeta function: $\zeta_{\mathbb{Q}}(s) = \zeta(s)$.

The function $\zeta_K(s)$ has an analytic continuation to the entire complex s-plane except for a first order pole at $s = 1$. The residue of $\zeta_K(s)$ at $s = 1$ is

$$Res_{s=1}\zeta_K(s) = \frac{2^{r_1 + r_2} \pi^{r_2} hR}{w\sqrt{|D|}}$$

where r_1 is the number of real conjugate fields of K, $2r_2$ is the number of complex conjugate fields of K, n is the degree of K, h is the class-number of K, R is the regulator of K, w is the number of roots of unity in K, and D is the discriminant of K. There is also a functional equation relating values at $s = 1$ to values at $1 - s$,

$$\xi_K(s) = \xi_K(1 - s)$$

where

$$\xi_K(s) = \left(\frac{|D|}{2^{2r_2} \pi^n} \right)^{s/2} \Gamma^{r_1}(s/2) \Gamma^{r_2}(s) \zeta_K(s)$$

The function $\zeta_K(s)$ has no zeros in $Re(s) > 1$. It has trivial zeros with $Re(s) < 0$ of order r_2 at $-1, -3, -5, \ldots$; of order $r_1 + r_2$ at $-2, -4, -6, \ldots$; and one zero of order $r_1 + r_2 - 1$ at $s = 0$. All other zeros are in the critical strip. (See [65] chap. 6: *Galois Theory, Algebraic Number Theory, and Zeta Functions* from H.M. Stark).

Extended Riemann Hypothesis. The Extended Riemann Hypothesis is the assertion that the non trivial zeros of Dedekind zeta function of any algebraic number field lie on the critical line.

Some consequences.

(1) Given the **GRH**, the Siegel-Walfisz theorem (see Th-5 [59], cap II.8) can be improved to

$$\psi(x; h, k) = \sum_{\substack{n \leq x \\ n \equiv h \pmod{k}}} \Lambda(n) = \frac{x}{\varphi(k)} + O(\sqrt{x} \log^2 x), \ k \leq x.$$

(2) Also we have

$$\pi(x; h, k) = \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} 1 = \frac{x}{\varphi(k) \log x} + O(\sqrt{x} \log x).$$

(3) D. Wolke (see [64]), obtained the following equivalence for L-functions: Let $\chi_0, \chi_1, \chi_2, \ldots$ be the sequence of all Dirichlet characters (in which the principal character χ_0 occurs only once), ordered which increasing moduli, and let $\{\alpha_k\}_{k \in \mathbb{N}}$ a sequence of real numbers $\alpha_k \to 0$ for $k \to \infty$ which, in addition, satisfies the condition

$$\sum_{k \in \mathbb{N}} \alpha_k \log(k + 1) \mid \log \alpha_k \mid < \infty.$$

Define the multiplicative function $f : \mathbb{N} \to \mathbb{C}$ by

$$\sum_{n=1}^{\infty} f(n) n^{-s} = (\zeta(s))^{-1} \prod_{k=1}^{\infty} (L(s, \chi_k))^{\alpha_k}, \ Re(s) > 1$$

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then Wolke proved that the **GRH** in true iff

$$\sum_{n \leq x} f(n) \ll x^{1/2+\epsilon}, \quad for \ all \ \epsilon > 0.$$

(4) *The Goldbach conjecture.* The three prime Goldbach conjecture states that every odd integer ≥ 9 is a sum of three odd primes. The conjecture was proved under the **GRH** by Deshouillers-Effinger-te Riele-Zinoviev [14].

Quasi Riemann hypothesis. The term "Quasi Riemann hypothesis" related to a zeta-function is used to mean that this zeta-function has no zeros in a half-plane $\sigma > \sigma_0$ for some $\sigma_0 < 1$.

Modified Generalized Riemann Hypothesis. Say that an L-function satisfies the Modified Great Riemann Hypothesis, if its zeros are all on the critical line or the real axis.

The function $\omega(n) = \sum_{p|n} 1$ has "normal" order $\log \log n$ and a famous theorem of Turan bounds the variance of this distribution. Now let $f_a(n)$ be the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$, with a relatively prime to n. Saidak ([56]) obtained results of Turan type for the function $\omega(f_a(n))$, thus assuming a quasi-Riemann hypothesis for the Dedekind zeta functions of certain nonabelian number fields, he proves that for each squarefree integer $a \geq 2$,

$$\sum_{\substack{p \leq x (p,a)=1}} \Big(\omega(f_a(p)) - \log \log p \Big)^2 \ll \pi(x) \log \log x$$

$$\sum_{\substack{n \leq x (n,a)=1}} \Big(\omega(f_a(n)) - \frac{1}{2}(\log \log n)^2 \Big)^2 \ll x(\log \log x)^3.$$

The Ramanujan zeta function. Let

$$L(s,\tau) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}, \quad \sigma > \frac{13}{2} \tag{25}$$

be the L-function attached to the Ramanujan function $\tau(n)$. The function $\tau(n)$ is multiplicative (proved by Mordell, 1917). The order of magnitude for $\tau(n)$ is $|\tau(n)| \leq n^{11/2}d(n)$, which was conjectured by Ramanujan in 1916 and proved by P. Deligne 60 years later [13].

The Euler product corresponding to $L(s,\tau)$ is

$$L(s,\tau) = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}, \quad (\sigma > \frac{13}{2})$$

and the functional equation, becomes

$$(2\pi)^{-s}\Gamma(s)L(s,\tau) = (2\pi)^{s-12}\Gamma(12-s)L(12-s,\tau).$$

It has the trivial zeros $s = 0, -1, -2, -3, \cdots$ but no other zeros for $\sigma \le \frac{11}{2}$ and $\sigma \ge \frac{13}{2}$.

The critical strip $0 < \sigma < 1$ corresponding to the function $\zeta(s)$ is the strip $\frac{11}{2} < \sigma < \frac{13}{2}$. The analytic continuation of $L(s,\tau)$ has an infinity of zeros on the critical line $Re(s) = 6$ and the Riemann hypothesis for $L(s,\tau)$ asserts that all its complex zeros lie on that line.

The Davenport-Heilbronn zeta function. This function was introduced by H. Davenport and H. Heilbronn as

$$f(s) = 5^{-s}(\zeta(s,1/5) + \tan\theta\,\zeta(s,2/5) - \tan\theta\,\zeta(s,3/5) - \zeta(s,4/5)) \qquad (26)$$

where

$$\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s} \quad (0 < a \le 1), \ Re(s) > 1$$

is the Hurwitz zeta function defined for $Re(s) \le 1$ by analytic continuation. For $\theta = \arctan((\sqrt{10-2\sqrt{5}}-2)/(\sqrt{5}-1))$ it can be shown (see [12] or [60] cap. X), that $f(s)$ satisfies the functional equation

$$f(s) = 2 \cdot 5^{-s+\frac{1}{2}}(2\pi)^{s-1}\Gamma(1-s)\cos(\frac{\pi s}{2})f(1-s).$$

Moreover $f(s)$ has an infinity of zeros on the line $\sigma = 1/2$ and it also has an infinity of zeros in the half-plane $Re(s) > 1$, ($\zeta(s) \ne 0$ in $Re(s) > 1$). Thus the function $f(s)$ of Davenport and H. Heilbronn, satisfies a functional equation similar to the functional equation for $\zeta(s)$, but has no Euler product, and the analogue of the Riemann hypothesis is FALSE for $f(s)$.

7   Remarks.

The work of Riemann on the distribution of primes is thoroughly studied in Edwards' book [15], that I recommend strongly (a jewel), but a moderate knowledge of elementary number theory and complex analysis is required of the reader. I suggest for example to study the Tenenbaum's book (also the book of Ellison-Mendés). Another important text is the great classic book of E.C. Titchmarh with an extensive treatment of the theory of $\zeta(s)$ up to 1951 and its second edition revised by D.R. Heath-Brown (see Chapters XIII-XV for Riemann Hypothesis). A. Ivić [31] develops the theory of the Riemann zeta-function and some applications, and the results proved in this text are unconditional, that is, they do not depend on any hitherto unproved hypothesis.

There are so many results and conjectures having to do with the Riemann hypothesis that it is very difficult to mention all of them. The reader interested on Riemann hypothesis can consult the Mathematical Reviews with about 1600 references (up today).

References

[1] Baez-Duarte, L. *New versions of the Nyman-Beurling criterion for the Riemann Hypothesis.* Int. J. Math. Math. sci. 31 (2002), 7, 387-406.

[2] Balazard, M. *Completeness problems and the Riemann hypothesis; An annotated bibliography.* Collection: Number theory for the Milleniunm (2002), 21-48.

[3] Bateman, P.T.; Grosswald, E. *On a theorem of Erdos and Szekeres.* Illinois-J.-Math. 2 (1958), 88-98.

[4] Beurling, A. *A closure problem related to the Riemann zeta-function.* Proc. Nat. Acad. Sci. U.S.A. (1955), 41, 312-314.

[5] Bombieri, E. Lagarias, J.C. *Complements to Li's criterion for the Riemann hypothesis.* J. Number Theory 77, 274-287, (1999).

[6] Bombieri, E. *Remarks on Weil's quadratic functional in the theory of prime numbers, I.* Att. Accad Naz. Licei (9) Mat. Appl. 11 (2000) n-3, 183-233.

[7] Calderón, C. *La función zeta de Riemann.* Rev. Real Academia de Ciencias de Zaragoza. **57**, (2002), 67-87.

[8] Conrey,J.B. *The Riemann Hypothesis.* Not. Amer. Math. Soc. 50 (2003),no. 3, 341-353.

[9] Conrey,J.B.-Ghosh, A.-Gonek,S.M. *Large gaps between zeros on the zeta function.* Mathematika, 33 (1986) 2, 212-338 (1987).

[10] Chandrasekharan, K. *Introduction to analytic number theory.*Springer Verlag New York 1968.

[11] Chandrasekharan, K. *Arithmetical functions.* Springer Verlag New York 1969.

[12] Davenport, H.-Heilbrom, H. *On the zeros of certain Dirichlet series, I, II.*J. London Math. Soc. 11 (1936), 181-185, 307-312.

[13] Deligne, P. *La conjeture de Weil I.* I.H.E.S. Publ. Math. 43 (1974), 273-307.

[14] Deshouillers, J.M.-Effinger, G.-te-Riele, H.-Zinoviev, D. *A complete Vinogradov 3-primes theorem under the Riemann hypothesis.* Electron Res. Announc. Amer. Math. Soc. 3 (1997), 99-104.

[15] Edwards, H.M. *Riemann's zeta function* Acad. Press. New York. 1974.

[16] Ellison, W. J. and Mendès-France, M. *Les nombres premiers.* Hermann Paris (1975).

[17] Franel, J. *Les suites de Farey et le probléme des nombres premiers.* Göttinger Nachrichten (1924), 198-201.

[18] Fujii, A. *A remark on the Riemann hypothesis.* Comment. Math. Univ. St. Paul. 29 (1980), 195-201.

[19] Garunkstis, R. *On a positivity property of the Riemann ξ-function* Lithuanian Math. Journal 42 (2002) n-2, 140-145.

[20] Gauss, C.F. *Carta a Enke fechada el 24 de Diciembre de 1849,* "Werke" Vol.II, pp. 444-447. Königlichen Gesellschaft der Wissenchaften zu Göttingen.

[21] Hadamard, J. *Étude sur les propriétés des fonctions entiéres et en particulier d'une fonction considerée par Riemann.* J. Math. Pures Appl. (4) 9, 171-215 (1893)

[22] —— *Sur la distribution des zéros de la fonction ζ(s) et ses consécuences arithmétiques.* Bull. Soc. Math. France 24, 199-220 (1896).

[23] Hardy, G.H. *Comptes Rendus,* Acad. Sci. Paris, 158 (1914), 1012-1014.

[24] —— *Sur les zéros de la function ζ(s) de Riemann.* C.R. Acad. Sci. Paris 158, 1012-1014 (1914).

[25] Hardy, G.H. and Wright, E.M. *An Introduction to the theory of numbers.* Oxford. Clarendon Press. 1979.

[26] Hardy, G.H.-Littlewood, J.E. *Contributions to the theory of the Riemann zeta function and the distributions of primes.* Acta Math. 41, 19-196 (1918).

[27] Hinkkanen, A. *On functions of bounded type.* Complex Var. Th. Appl. Int. Jour. 34 (1997) no. 1-2, 119-139.

[28] Huxley, N.M. *The distribution of Farey points I.* Acta Arith. 18 (1971), 281-287.

[29] —— *Exponential sums and the Riemann zeta function IV.* Proc. London Math. Soc. (1993) (3) 66, 1-40

[30] Ingham, A. *The distribution of prime numbers.* Cambridge University Press, Cambridge, 1932.

[31] Ivić, A. *The Riemann zeta function.* Jhon Wiley, New York 1985.

[32] —— *On some results concerning the Riemann hypothesis.* Collection: Analytic number theory London Math. Soc. Lecture Notes Ser. 247, 139-167, (1997).

[33] Kanemitsu, S- Yoshimoto, M. *Farey series and the Riemann hypothesis.* Acta Arith. 75 (1996) n-4, 351-374.

[34] Kanemitsu,-Shigeru; Yoshimoto,-Masami, *Farey series and the Riemann hypothesis. III.* International Symposium on Number Theory (Madras, 1996). Ramanujan-J. 1 (1997), no. 4, 363-378.

[35] Koch, H. von. *Sur la distribution des nombres premiers.* Acta Math. 24 (1901), 159-182

[36] Kratzel, E. *Lattice points,* VEB Deutscher Verlag, Berlin 1988.

[37] Laugwitz, D. *Bernhard Riemann 1826-1866. Turning points in the conception of Mathematics.* Translated by Abe Shenitzer, Birkhäuser, Boston Basel Berlin, 1999.

[38] Lagarias, J.C. *On a positivity property of the Riemann ξ function.* Acta Arith. 89, (1999),3, 217-234.

[39] —— *An elementary problem equivalent to the RiemannHypothesis.* Amer. Math. Montly, 109 (2002) 6, 534-543.

[40] Li, Xian Jin. *The positivity of a sequence of numbers and the Riemann hypothesis.* J. Number Theory 65 (1997) n-2, 325-333.

[41] Littlewood, J.E. *Quelques consécuences de l'hypothèse que la fonction $\zeta(s)$ de Riemann n'a pas de zéros dans la demi-plan $Re(s) > 1/2$.* Comptes Rendus de l'Acad. des Sciences (Paris) 154 (1912), 263-266.

[42] Littlewood, J.E. *On the zeros of Riemann zeta-function.* Proc. Camb. Phil. Soc. 22 (1924), 295-318.

[43] Lu, G.S. , Zhai, W. *A special divisor problem.* Acta Math. Sinica 45 (2002) 5, 991-1004.

[44] van de Lune,J. - te Riele, H.J.J.-Winter, D.T. *On the zeros of the Riemann zeta function IV.* Math. Comp. 46 (1986) n-174, 667-681.

[45] Mangoldt, H. von *Zur Verteilung der Nullstellen der Riemannschen Funktion $\xi(t)$* Math. Annalen, 60 (1905), 1-19.

[46] Mikolas, M. *Sur l'hypothese de Riemann.* C.-R.-Acad.-Sci.-Paris 228, (1949). 633–636

[47] —— *Farey series and their connection with the prime number problem. I. (Errata: EA 13,1138)* Acta-Univ.-Szeged.-Sect.-Sci.-Math. 13, (1949). 93–117

[48] —— *Farey series and their connection with the prime number problem. II.* Acta-Sci.-Math.-Szeged 14, (1951). 5–21

[49] —— *An equivalence theorem concerning Farey series.* Mat.-Lapok 2, (1951). 46–53

[50] Muler, W. , Nowak, W.G. and Menzer, H. *On the number of primitive Pythagorean triangles.* Ann. Sci. Math. Quebec 12 (1988) 2, 263-273.

[51] Nyman, B. *On the one-dimensional translation group abd semi-group in certain functions spaces. Ph.D.* Thesis Upsala, 1950.

[52] Ribenboim, P. *The book of prime number records.* Springer-Verlag, 1989.

[53] Riemann, B. *Über die Anzahl der Prinzahlen unter einer gegebener Gröse.* Mo- nastsber. Akad. Berlin, 671-680, (1859).

[54] Robin, G. *Large values of the sum of divisors function and the Riemann hypothesis.* J. Math. Pures Appl. (9) 63 (1984) n-2, 187-213.

[55] Romare, O. and Saoutier, Y. *Short effective intervals containig primes.* J. Number Theory, 98 (2003), 1, 10-33

[56] Saidak, F. *The normal number of prime factors of $f_a(n)$.* J.-Ramanujan-Math.-Soc. 17 (2002), no. 1, 19-33.

[57] Selberg, A. *Contributions to the theory of the Riemann zeta function.* Arch. Math. og. Naturv. B, 48 (1946) n-5.

[58] Speiser, A. *Geometrisches zur Riemannschen Zetafuncktion.* M.A. 110 (1934), 514-521.

[59] Tenenbaum, G. *Introduction to analytic and probabilistic number theory.* Cambridge Univ. Press 46 (1995).

[60] Titchmarsh, E.C. *The theory of the Riemann Zeta-Function.* Clarendon Press. Oxford 1988. Revised by D.R. Heath Brown.

[61] Vallée-Poussin, Ch. de la *Recherches analytiques sur la theorie des nombres. La función $\zeta(s)$ de Riemann et les nombres premiers en general.* Annales de la Soc. Sci. de Bruxelles 20, (1896), 183-256.

[62] Weil,A. *Sur les "formules explicites" de la theorie des nombres premiers.* Comm.-Sem.-Math.-Univ.-Lund 1952, (1952). Tome Supplementaire, 252-265.

[63] Wolke, D. *On the number theoretic function w(n).* Acta Arith. 55 (1990), 4, 323-331.

[64] —— *On a certain class of multiplicative functions.* J. Number Theory 37 (1991),n-3, 279-287.

[65] Waldschmidt, M. ; Moussa, P. ; Luck, J.M. ; Itzykson, C. *From Number theory to Physics.* Springer-Verlag,1989.

[66] Walfisz, A. *Weylsche Exponentiansummen in der Neueren Zahlentheorie.* VEB Deutscher Verlag, Berlin 1963.

[67] Wu, J. *On the distribution of square-full integers.* Arch. Math. (2001) 233-240.

[68] —— *On the primitive circle problem.* Monatsh.-Math. 135 (2002), no. 1, 69-81.

[69] Yoshimoto, M. *Farey series and the Riemann hypothesis. IV.* Acta-Math.-Hungar. 87 (2000), no. 1-2, 109-119. Part II: Acta Math. Hungar. 78 (1998), no. 4, 287-304.

[70] Zhai, Wen-Guang, *On the number of primitive Pythagorean triangles.* Acta Arith. 105 (2002) 4, 387-403.