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PURELY INFINITESIMAL GEOMETRY
(EXCERPT)

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1. INTRODUCTION: CONCERNING THE RELATION
BETWEEN GEOMETRY AND PHYSICS

The real world, into which we have been placed by virtue of our consciousness, *is not there* simply and all at once, but *is happening*; it passes, annihilated and newly born at each instant, a continuous one-dimensional succession of states in *time*. The arena of this temporal happening is a three-dimensional Euclidean *space*. Its properties are investigated by *geometry*, the task of *physics* by comparison is to conceptually comprehend the real that exists in space and to fathom the laws persisting in its fleeting appearances. Therefore, physics is a science which has geometry as its foundation; the concepts however, through which it represents reality—matter, electricity, force, energy, electromagnetic field, gravitational field, etc.—belong to an entirely different sphere than the geometrical.

This old view concerning the relation between the form and the content of reality, between geometry and physics, has been overturned by Einstein’s theory of relativity.¹ The *special theory of relativity* led to the insight that space and time are fused into an indissoluble whole which shall here be called the *world*; the world, according to this theory, is a four-dimensional Euclidean manifold—Euclidean with the modification that the underlying quadratic form of the world metric is not positive definite but is of inertial index 1. The *general theory of relativity*, in accordance with the spirit of modern physics of local action [*Nahewirkungsphysik*], admits that as valid only in the infinitely small, hence for the world metric it makes use of the more general concept of a metric [*Maßbestimmung*] based on a quadratic *differential* form, developed by Riemann in his habilitation lecture. | But what is new in principle in this is the insight that the metric is not a property of the world in itself, rather, spacetime as the form of appearances is a completely formless four-dimensional continuum in

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1 I refer to the presentation in my book *Raum, Zeit, Materie*, Springer 1918 (in the sequel cited as RZM), and the literature cited there.

the sense of analysis situs. The metric, however, expresses something real that exists in the world, which produces physical effects on matter by means of centrifugal and gravitational forces, and whose state is in turn determined according to natural laws by the distribution and composition of matter. By removing from Riemannian geometry, which claims to be a purely “local geometry,” [*Nahe-Geometrie*] an inconsequence still currently adhering to it, ejecting one last element of non-local geometry [*ferngeometrisches Element*] which it had carried along from its Euclidean past, I arrived at a world metric from which not only arises gravitation, but also the electromagnetic effects, and therefore, as one may assume with good reason, accounts for all physical processes.² According to this theory, *everything real that exists in the world is a manifestation of the world metric*; the physical concepts are none other than the geometric ones. The only difference that exists between geometry and physics is that geometry fathoms in general what lies in the nature of the metric concepts,³ whereas physics has to determine the law by which the real world is distinguished among all the four-dimensional metric spaces possible according to geometry and pursue its consequences.⁴

In this note, I want to develop that *purely infinitesimal geometry* which, according to my conviction, contains the physical world as a special case. The construction of the local geometry proceeds adequately in three steps. On the first step stands the *continuum* in the sense of analysis situs, without any metric—physically speaking, *the empty world*; on the second the *affinely connected continuum*—I so call a manifold in which the concept of infinitesimal parallel displacement of vectors is meaningful; in physics, the affine connection appears as *the gravitational field*—; finally on the third, the *metric continuum*—physically: *the “aether,”* whose states are manifested in the phenomena of matter and electricity.

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2. SITUS-MANIFOLD (EMPTY WORLD)

As a consequence of the difficulty in grasping the intuitive character of the continuous connection by means of a purely logical construction, a completely satisfactory analysis of the concept of an *n-dimensional manifold* is not possible today.⁵ The following is sufficient for us: An *n-dimensional manifold* refers to *n* coordinates $x_1 x_2 \dots x_n$, of which each possesses at each point of the manifold a particular numerical value: different sets of values of the coordinates correspond to different

2 A first communication about this appeared under the title “Gravitation und Elektrizität” in *Sitzungsber. d. K. Preuß. Akad. d. Wissenschaften* 1918, p. 465.

3 Naturally, traditional geometry leaves the path of this, its principal task, and immediately takes on the less specific one by not making space itself anymore the object of its investigation, but the structures possible in space, special classes and their properties they are endowed with on the basis of the space-metric.

4 I am bold enough to believe that the totality of physical phenomena can be derived from a single universal world law of greatest mathematical simplicity.

5 See also H. Weyl, *Das Kontinuum* (Leipzig 1918), specifically pp. 77 ff.

points; if $\bar{x}_1 \bar{x}_2 \dots \bar{x}_n$ is a second system of coordinates then there exist between the x - and the \bar{x} -coordinates of the same arbitrary point regular relations

$$x_i = f_i(\bar{x}_1 \bar{x}_2 \dots \bar{x}_n) \quad (i = 1, 2, \dots, n),$$

where f_i denote purely logically-arithmetically constructible functions; of these we presuppose not only that they are continuous, but also that they possess continuous derivatives

$$\alpha_{ik} = \frac{\partial f_i}{\partial \bar{x}_k},$$

whose determinant does not vanish. The last condition is necessary and sufficient for the affine geometry to be valid in the infinitely small, namely that there exist invertible linear relationships between the coordinate differentials in the two systems:

$$dx_i = \sum_k \alpha_{ik} d\bar{x}_k. \tag{1}$$

We assume the existence and continuity of higher order differentials where required during the course of the investigation. In any case, the concept of the continuous and continuously differentiable point-function, if necessary also the 2, 3, ... times continuously differentiable, has therefore an invariant meaning independent of the coordinate system. The coordinates themselves are such functions. An n -dimensional manifold for which we regard no properties other than those lying within the concept of an n -dimensional manifold, we call—in physical terminology—an (n -dimensional) *empty world*.¹

The relative coordinates dx_i of a point $P' = (x_i + dx_i)$ infinitely close to the point $P = (x_i)$ are the components of a *line element* in P , or an *infinitesimal displacement* $\overrightarrow{PP'}$ of P . In going to a different coordinate system the formulae (1) apply for these components, the α_{ik} denoting the corresponding derivatives at the point P . More generally, on the basis of a definite coordinate system in the neighborhood of P , any n numbers ξ^i ($i = 1, 2, \dots, n$) given in a definite order, characterize at the point P a *vector* (or a *displacement*) at P . The components ξ^i respectively $\bar{\xi}^i$ of the same vector in any two coordinate systems, the “unbarred” one and the “barred” one, are related by the same linear transformation equations (1):

$$\xi^i = \sum_k \alpha_{ik} \bar{\xi}^k.$$

Vectors at P can be added and multiplied by numbers; thus they form a “linear” or “affine” totality [*Gesamtheit*]. With each coordinate system are associated n “unit vectors” e_i at P , namely those vectors which in the coordinate system in question have the components

$$\begin{array}{c|l} e_1 & 1, 0, 0, \dots, 0 \\ e_2 & 0, 1, 0, \dots, 0 \\ \dots & \dots \dots \dots \dots \dots \\ e_n & 0, 0, 0, \dots, 1 \end{array}$$

Any two (linearly independent) line elements at P with the components dx_i and δx_i respectively span a (two-dimensional) area element at P with the components

$$dx_i \delta x_k - dx_k \delta x_i = \Delta x_{ik},$$

each three (independent) line elements $dx_i, \delta x_i, \delta x_i$ at P , a (three-dimensional) volume element with the components

$$\begin{vmatrix} dx_i & dx_k & dx_l \\ \delta x_i & \delta x_k & \delta x_l \\ \delta x_i & \delta x_k & \delta x_l \end{vmatrix} = \Delta x_{ikl};$$

etc. A linear form depending on an arbitrary line- or area- or volume- or ... element at P is called a *linear tensor* of order 1, 2, 3... respectively. By using a particular coordinate system, the coefficients a of this linear form

$$\sum_i a_i dx_i, \text{ resp. } \frac{1}{2!} \sum_{ik} a_{ik} \Delta x_{ik}, \frac{1}{3!} \sum_{ikl} a_{ikl} \Delta x_{ikl}, \dots$$

[388] It can be uniquely normalized through the alternation requirement; e.g., for the case just written down this implies that the triple of indices (ikl) , which arise through an even permutation of itself corresponds to the same coefficient a_{ikl} , whereas under odd permutations the coefficient changes into its negative, that is

$$a_{ikl} = a_{kli} = a_{lik} = -a_{kil} = -a_{lki} = -a_{ilk}.$$

The coefficients normalized in this manner are called the *components* of the tensor in question. From a scalar field f one obtains through differentiation a linear tensor field of order 1 with the components

$$f_i = \frac{\partial f}{\partial x_i};$$

from a linear tensor field f_i of order 1, one of 2nd order:

$$f_{ik} = \frac{\partial f_i}{\partial x_k} - \frac{\partial f_k}{\partial x_i},$$

from one of order 2, a linear tensor field of order 3:

$$f_{ikl} = \frac{\partial f_{kl}}{\partial x_i} + \frac{\partial f_{li}}{\partial x_k} + \frac{\partial f_{ik}}{\partial x_l};$$

etc. These operations are independent of the coordinate system used.⁶

A linear tensor of the 1st order at P we will call a *force* acting there. Assuming a definite coordinate system, such a force is thus characterized by n numbers ξ_i , which transform contragrediently to the components of the displacement under a change to another coordinate system:

$$\xi_i = \sum_k \alpha_{ki} \xi_k.$$

If η^i are the components of an arbitrary displacement at P , then

$$\sum_i \xi_i \eta^i$$

is an invariant. By a *tensor* at P , one generally understands a linear form of one or more arbitrary displacements and forces at P . For example, if we are dealing with a linear form of three arbitrary displacements ξ, η, ζ and two arbitrary forces ρ, σ :

$$\sum a_{ikl}^{pq} \xi^i \eta^k \zeta^l \rho_p \sigma_q,$$

then we speak of a tensor of order 5, with the components a being covariant with respect to the indices ikl and contravariant with respect to the indices pq . A displacement is itself a contravariant 1st order tensor, the force a covariant one. The fundamental operations of tensor algebra are:⁷ [389]

1. Addition of tensors and multiplication by a number;
2. Multiplication of tensors;
3. Contraction.

Accordingly, tensor algebra can already be constructed in the empty world—it does not presuppose any metric [*Maßbestimmung*]⁷—of tensor analysis, however, only that of “linear” tensors.

A “*motion*” in our manifold is given, if to each value s of a real parameter is assigned a point in a continuous manner; by using the coordinate system x_i , the motion is expressed by the formulae $x_i = x_i(s)$, in which the x_i on the right are to be understood as function symbols. If we presuppose continuous differentiability, then we obtain, independently of the coordinate system, for each point $P = (s)$ of the motion a vector at P with the components:

6 RZM, §13.

7 RZM, §6.

$$u^i = \frac{dx_i}{ds},$$

the *velocity*. Two motions, arising from one another through continuous monotonic transformation of the parameter s describe the same *curve*.

3. AFFINELY CONNECTED MANIFOLD (WORLD WITH GRAVITATIONAL FIELD)

3.1 The Concept of the Affine Connection

If P' is infinitely close to the fixed point P , then P' is *affinely connected* with P , if for each vector at P it is determined into which vector at P' it will transform under *parallel displacement* from P to P' . The parallel displacement of all vectors at P from there to P' must evidently satisfy the following requirement.

A. *The transfer of the totality of vectors from P to the infinitely close point P' by means of parallel displacement produces an affine transformation of the vectors at P to the vectors at P' .*

If we use a coordinate system in which P has the coordinates x_i , P' the coordinates $x_i + dx_i$, an arbitrary vector at P the components ξ^i , and the vector at P' , that results from it through parallel displacement to P' , the components $\xi^i + d\xi^i$, then $d\xi^i$ must therefore depend linearly on the ξ^i : |

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$$d\xi^i = -\sum_r d\gamma^i_r \xi^r.$$

$d\gamma^i_r$ are infinitesimal quantities which depend only on the point P and the displacement $\overrightarrow{PP'}$ with the components dx_i , but not on the vector ξ subject to parallel displacement. From now on, we consider affinely connected manifolds; in such a manifold, each point P is affinely connected to all its infinitely close points. A second requirement is still to be imposed on the concept of parallel displacement, that of *commutativity*.

B. *If P_1, P_2 are two points infinitely close to P and if the infinitesimal vector $\overrightarrow{PP_1}$ becomes $\overrightarrow{P_2P_{21}}$ under parallel displacement from P to P_2 , and $\overrightarrow{PP_2}$ becomes $\overrightarrow{P_1P_{12}}$ under parallel displacement to P_1 , then the points P_{12} and P_{21} coincide. (An infinitely small parallelogram results.)*

If we denote the components of $\overrightarrow{PP_1}$ by dx_i , and those of $\overrightarrow{PP_2}$ by δx_i , then the requirement in question obviously implies that

$$d\delta x_i = -\sum_r d\gamma^i_r \cdot \delta x_r \tag{2}$$

is a symmetric function of the two line elements d and δ . Consequently, $d\gamma^i_r$ must be a linear form of the differentials dx_i ,

$$d\gamma^i_r = -\sum_s \Gamma^i_{rs} dx_s,$$

and the coefficients Γ , the “*components of the affine connection*,” which depend only on the location of P , must satisfy the symmetry condition

$$\Gamma^i_{sr} = \Gamma^i_{rs}.$$

Because of the way in which the infinitesimal quantities are dealt with in the formulation of the requirement **B**, it could be objected that the latter lacks a precise meaning. Therefore, we want to determine explicitly through a rigorous proof that the symmetry of (2) is a condition independent of the coordinate system. For this purpose, we make use of a (twice differentiable) scalar field f . From the formula for the total differential

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i$$

we infer, that if ξ^i are the components of an arbitrary vector at P , |

$$df = \sum_i \frac{\partial f}{\partial x_i} \xi^i \tag{391}$$

is an invariant independent of the coordinate system. We form its variation under a second infinitesimal displacement δ , in which the vector ξ shall be displaced parallel to itself from P to P_2 , and obtain

$$\delta df = \sum_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} \xi^i \delta x_k - \sum_{ir} \frac{\partial f}{\partial x_i} \cdot d\gamma^i_r \xi^r.$$

If we replace in this expression ξ^i again by dx_i and subtract from this equation the one obtained by interchanging d and δ , then the invariant

$$\Delta f = (\delta d - d\delta)f = \sum_i \left\{ \frac{\partial f}{\partial x_i} \sum_r (d\gamma^i_r \delta x_r - \delta\gamma^i_r dx_r) \right\}.$$

results. The relations

$$\sum_r (d\gamma^i_r \delta x_r - \delta\gamma^i_r dx_r) = 0$$

contain the necessary and sufficient condition that for any scalar field f the equation $\Delta f = 0$ is satisfied.

In physical terms, an affinely connected continuum is to be described as a world in which a *gravitational field* exists. The quantities Γ^i_{rs} are the components of the gravitational field. The formulae, according to which these components transform in changing from one coordinate system to another, we need not state here. Under linear transformations the Γ^i_{rs} behave with respect to r and s like the covariant components of a tensor and with respect to i like the contravariant components, but lose this character under non-linear transformations. However, the changes $\delta\Gamma^i_{rs}$, which are experienced by the quantities Γ , if one arbitrarily varies the affine connection of the manifold, form the components of a generally-invariant tensor of the given character.

What is to be understood by *parallel displacement of a force at P* from there to the infinitely close point P' results from the requirement that the invariant product of this force and an arbitrary vector at P is preserved under parallel displacement. If ξ_i are the components of the force, η^i those of the displacement, then⁸

$$d(\xi_i \eta^i) = (d\xi_i \cdot \eta^i) + \xi_i d\eta^i = (d\xi_i - d\gamma^r_i \xi_r) \eta^i = 0$$

yields the formula

$$d\xi_i = \sum_r d\gamma^r_i \xi_r.$$

[392] | At each point P , one can introduce a coordinate system x_i of a kind—I call it *geodesic at P*—such that in it, the components of the affine connection Γ^i_{rs} vanish at the point P . If x_i are initially arbitrary coordinates that vanish at P , and Γ^i_{rs} designate the components of the affine connection at the point P in this coordinate system, then one obtains a geodesic coordinate system \bar{x}_i via the transformation

$$x_i = \bar{x}_i - \frac{1}{2} \sum_{rs} \Gamma^i_{rs} \bar{x}_r \bar{x}_s. \tag{3}$$

Namely, if we consider the \bar{x}_i as independent variables and their differentials $d\bar{x}_i$ as constants, then one has in the sense of Cauchy at $P(\bar{x}_i = 0)$:

$$dx_i = d\bar{x}_i, \quad d^2x_i = -\Gamma^i_{rs} d\bar{x}_r d\bar{x}_s,$$

therefore,

$$d^2x_i + \Gamma^i_{rs} d\bar{x}_r d\bar{x}_s = 0.$$

Because of their invariant nature, the last equations in the coordinate system \bar{x}_i become:

$$d^2\bar{x}_i + \bar{\Gamma}^i_{rs} d\bar{x}_r d\bar{x}_s = 0.$$

8 In the following we will use Einstein's convention that summation is always to be carried out over indices which occur twice in a formula without our finding it necessary to always place a summation sign in front of it.

For arbitrary constant $d\bar{x}_i$, these are, however, satisfied only if all the $\bar{\Gamma}^i_{rs}$ vanish. Therefore, through an appropriate choice of the coordinate system, the gravitational field can always be made to vanish at a single point. Through the requirement of “geodesy” at P the coordinates in the neighborhood of P are determined up to linear transformation excluding terms of third order; i.e., if x_i, \bar{x}_i are two coordinate systems geodesic at P , and if the x_i as well as the \bar{x}_i vanish at P , then by neglecting terms in \bar{x}_i of order 3 and higher, linear transformation equations $x_i = \sum_k \alpha_{ik} \bar{x}_k$ with constant coefficients α_{ik} apply.

3.2 Tensor Analysis, Straight Line

Only in an affinely connected space can tensor analysis be fully established. If for example f^k_i are the components of a 2nd order tensor field, covariant in i and contravariant in k , then with the aid of an arbitrary displacement ξ and a force η at the point P , we form the invariant

$$f^k_i \xi^i \eta_k$$

and its change under an infinitely small displacement d of the point P , in which ξ and η are displaced parallel with respect to themselves. We have

$$d(f^k_i \xi^i \eta_k) = \frac{\partial f^k_i}{\partial x_l} \xi^i \eta_k dx_l - f^k_r \eta_k d\gamma^r_i \xi^i + f^r_i \xi^i d\gamma^k_r \eta_k,$$

and therefore

[393]

$$f^k_{il} = \frac{\partial f^k_i}{\partial x_l} - \Gamma^r_{il} f^k_r + \Gamma^k_{rl} f^r_i$$

are the components of 3rd order tensor field, covariant in il and contravariant in k , which arises from the given 2nd order tensor field in a coordinate independent manner.

In the affinely connected space, the concept of *straight or geodesic line* gains a definite meaning. The straight line arises as the trajectory of the initial point of the vector which is displaced in its own direction keeping it parallel to itself; it can therefore be described as that curve the direction of which remains unchanged. If u^i are the components of that vector, then during the course of the motion the equations

$$\begin{aligned} du^i + \Gamma^i_{\alpha\beta} u^\alpha dx_\beta &= 0, \\ dx_1 : dx_2 : \dots : dx_n &= u^1 : u^2 : \dots : u^n \end{aligned}$$

should always hold. The parameter s used in describing the curve can thus be normalized in such a way that

$$\frac{dx_i}{ds} = u^i$$

identically along s , and the differential equations of the straight line are then

$$w^i \equiv \frac{d^2 x_i}{ds^2} + \Gamma_{\alpha\beta}^i \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0.$$

For each arbitrary motion $x_i = x_i(s)$, the left hand sides of these equations are the components of a vector invariantly linked to the motion at the point s , the acceleration. Actually, if ξ_i is an arbitrary force at that point, which during the transition to the point $s + ds$ is displaced parallel to itself, then

$$\frac{d(u^i \xi_i)}{ds} = w^i \xi_i.$$

A motion, whose acceleration vanishes identically is called a *translation*. A straight line—this is another way of grasping our above explanation—is to be understood as the trajectory of a translation.

3.3 Curvature

If P and Q are two points connected by a curve, and a vector is given at the first point, then one can displace this vector parallel to itself along the curve from P to Q . The resulting *vector transfer* is however in general *not integrable*; i.e. the vector [394] which one ends up with at Q depends on the path along which the transport takes place. Only in the special case of integrability does it make sense to speak of the *same* vector at two different points P and Q ; these are understood to be vectors which arise from one another under parallel transport. In this case, the manifold is called *Euclidean*. In such a manifold, special “linear” coordinate systems can be introduced which are distinguished by the fact that equal vectors at different points have equal components. Any two such linear coordinate systems are related by linear transformation equations. In a linear coordinate system the components of the gravitational field vanish identically.

On the infinitely small parallelogram constructed above (§3, I., B.), we attach at the point P an arbitrary vector with components ξ^i and in the first case displace it parallel to itself to P_1 , and from there to P_{12} , and in the second case first to P_2 , and from there to P_{21} . Since P_{12} and P_{21} coincide, we can form the difference of these two vectors at this point and through this obviously obtain there a vector with the components

$$\Delta \xi^i = \delta d \xi^i - d \delta \xi^i.$$

From

$$d \xi^i = -d \gamma_k^i \xi^k = -\Gamma_{kl}^i dx_l \xi^k,$$

it follows that

$$\delta d\xi^i = -\frac{\partial \Gamma^i_{kl}}{\partial x_m} dx_l \delta x_m \xi^k - \Gamma^i_{kl} \delta dx_l \cdot \xi^k + d\gamma^i_r \delta \gamma^r_k \xi^k,$$

and because of the symmetry of δdx_l :

$$\Delta \xi^i = \left\{ \left(\frac{\partial \Gamma^i_{km}}{\partial x_l} - \frac{\partial \Gamma^i_{kl}}{\partial x_m} \right) dx_l \delta x_m + (d\gamma^i_r \delta \gamma^r_k - d\gamma^r_k \delta \gamma^i_r) \right\} \xi^k.$$

Therefore, we obtain

$$\Delta \xi^i = \Delta R^i_k \xi^k,$$

where the ΔR^i_k are linear forms of the two displacements d and δ , or rather of the area element spanned by them, independent of the vector ξ and with the components

$$\Delta x_{lm} = dx_l \delta x_m - dx_m \delta x_l,$$

$$\Delta R^i_k = R^i_{klm} dx_l \delta x_m = \frac{1}{2} R^i_{klm} \Delta x_{lm} \quad (R^i_{kml} = -R^i_{klm}), \tag{4}$$

$$R^i_{klm} = \left(\frac{\partial \Gamma^i_{km}}{\partial x_l} - \frac{\partial \Gamma^i_{kl}}{\partial x_m} \right) + (\Gamma^i_{lr} \Gamma^r_{km} - \Gamma^i_{mr} \Gamma^r_{kl}). \tag{5}$$

If η_i are the components of an arbitrary force at P , then $\eta_i \Delta \xi^i$ is an invariant; consequently, R^i_{klm} are the components of a 4th order tensor at P , covariant in klm and contravariant in i , the *curvature*. That the curvature vanishes identically is the necessary and sufficient condition for the manifold to be Euclidean. In addition to the condition of “skew” symmetry given beside (4), the curvature components satisfy the condition of “cyclic” symmetry: [395]

$$R^i_{klm} + R^i_{lmk} + R^i_{mkl} = 0.$$

By its nature, the curvature at a point P is a linear map or transformation $\Delta \mathbf{P}$, which assigns to each vector ξ there another vector $\Delta \xi$; this transformation itself depends linearly on an element of area at P :

$$\Delta \mathbf{P} = \mathbf{P}_{ik} dx_i \delta x_k = \frac{1}{2} \mathbf{P}_{ik} \Delta x_{ik} \quad (\mathbf{P}_{ki} = -\mathbf{P}_{ik}).$$

Accordingly, the curvature is best described as a “linear transformation-tensor of 2nd order.”

In order to counter objections to the proof of the invariance of the curvature tensor, which could be raised against the above considerations involving infinitesimals, one uses a force field f_i , and forms the change $d(f_i \xi^i)$ of the invariant product $f_i \xi^i$ in such a way that under the infinitely small displacement d the vector ξ is displaced parallel to itself. Replacing in the expression obtained the infinitesimal displacement

dx with an arbitrary vector ρ at P , one obtains an invariant bilinear form of two arbitrary vectors ξ and ρ at P . From this one forms the change which corresponds to a second infinitely small displacement δ , by parallelly taking along the vectors ξ and ρ , and replacing thereafter the second displacement by a vector σ at P . One obtains the form

$$\delta d(f_i \xi^i) = \delta d f_i \cdot \xi^i + d f_i \delta \xi^i + \delta f_i d \xi^i + f_i \delta d \xi^i.$$

Through the interchange of d and δ and subsequent subtraction, this yields, because of the symmetry of $\delta d f_i$, the invariant

$$\Delta(f_i \xi^i) = f_i \Delta \xi^i,$$

and thus the desired proof has been completed.

4. METRIC MANIFOLD (THE AETHER)

4.1 The Concept of The Metric Manifold

[396] A manifold carries at the point P a metric, if the line elements at P can be compared with respect to their lengths. For this purpose, we assume the validity of the Pythagorean-Euclidean laws in the infinitely small. Hence, to any two vectors ξ, η at P shall correspond a number $\xi \cdot \eta$, the *scalar product*, which is a symmetric bilinear form with respect to the two vectors. This bilinear form is certainly not absolute, but is only determined up to an arbitrary non-zero factor of proportionality. Hence, it is actually not the form $\xi \cdot \eta$, that is given but only the equation $\xi \cdot \eta = 0$; two vectors which satisfy this equation are called *perpendicular* to one another. We presuppose that this equation is non-degenerate, i.e. that the only vector at P , to which all vectors at P can be perpendicular is the 0 vector. We do not however presuppose that the associated quadratic form $\xi \cdot \xi$ is positive definite. If it has the index of inertia q , and if $n - q = p$, then we say in brief, the manifold at the point considered is $(p + q)$ -dimensional. As a result of the arbitrary factor of proportionality, the two numbers p, q are only determined up to their order. We now assume that our manifold carries a metric [*Maßbestimmung*] at each point P . For the purpose of analytic representation, we consider (1) a definite coordinate system, and (2) the factor of proportionality appearing in the scalar product and which can be arbitrarily chosen at each point as fixed; with this, a “*frame of reference*”⁹ for the analytic representation is obtained. If the vector ξ at the point P with the coordinates x_i has the components ξ^i , and η the components η^i , then one has

$$(\xi \cdot \eta) = \sum_{ik} g_{ik} \xi^i \eta^k \quad (g_{ki} = g_{ik}),$$

9 I thus differentiate between “coordinate system” and “frame of reference.”

where the coefficients g_{ik} are functions of the x_i . The g_{ik} should not only be continuous, but also be twice continuously differentiable. Since they are continuous and their determinant g by assumption does not vanish anywhere, the quadratic form $(\xi \cdot \xi)$ has the same index of inertia q at all points; therefore, we can describe the manifold in its entirety as $(p + q)$ -dimensional. If we retain the coordinate system, but make a different choice for the undetermined factor of proportionality, then instead of the g_{ik} we obtain for the coefficients of the scalar product the quantities

$$g'_{ik} = \lambda \cdot g_{ik},$$

where λ is a nowhere vanishing continuous (and twice continuously differentiable) function of position.

According to the previous assumption, the manifold is only equipped with an *angle-measurement*; the geometry which is solely based on this, would be described as “*conformal geometry*”; it has, as is well known, in the realm of two-dimensional manifolds (“*Riemannian surfaces*”) experienced extensive development, because of its importance for complex function theory. If we make no further assumptions, then the individual points of the manifold remain completely isolated from one another with respect to metrical properties. The manifold becomes endowed with a metric connection from point to point, only when a *principle exists for the transfer of the unit of length from a point P to an infinitely close one*. Instead, Riemann made the much farther reaching assumption, that line elements can be compared not only at the same location, but that they can be compared as to their lengths at two finitely distant locations. *But the possibility of such a “non-local geometric” comparison definitely cannot be admitted in a purely infinitesimal geometry*. Riemann’s assumption has also entered the Einsteinian world geometry of gravitation. Here, this inconsequence shall be removed.

[397]

Let P be a fixed point and P_* an infinitely close point obtained from P through the displacement with the components dx_i . We assume a definite frame of reference. In relation to the unit of length thus defined at P (as well as at all other points in the space), the square of the length of an arbitrary vector ξ at P is given by

$$\sum_{ik} g_{ik} \xi^i \xi^k.$$

Now, if we transfer the unit of length chosen at P to P_* , which we presuppose as possible, the square of the length of an arbitrary vector ξ_* at P_* is given by

$$(1 + d\varphi) \sum_{ik} (g_{ik} + dg_{ik}) \xi_*^i \xi_*^k,$$

where $1 + d\varphi$ is a factor of proportionality deviating infinitesimally from 1; $d\varphi$ must be a homogeneous function of degree 1 of the differentials dx_i . Namely, if we transplant the unit of length chosen at P from point to point along a curve leading from P to a finitely distant point Q , then on the basis of the unit of length so

obtained at Q we obtain for the square of the length of an arbitrary vector at Q the expression $g_{ik} \xi^i \xi^k$, multiplied by the factor of proportionality which results from the product of the infinitely many individual factors of the form $1 + d\varphi$, which arise each time that we move from one point on the curve to the next.

$$\prod (1 + d\varphi) = \prod e^{d\varphi} = e^{\sum d\varphi} = e^{\int_P^Q d\varphi}.$$

[398] | In order that the integral appearing in the exponent makes sense, $d\varphi$ must be a function of the differentials of the kind asserted.

If one replaces g_{ik} by $g'_{ik} = \lambda g_{ik}$, then in place of $d\varphi$ a different quantity $d\varphi'$ will appear. If λ denotes the value of this factor at the point P , one must have

$$(1 + d\varphi')(g'_{ik} + dg_{ik}) = \lambda(1 + d\varphi)(g_{ik} + dg_{ik}),$$

and this yields

$$d\varphi' = d\varphi - \frac{d\lambda}{\lambda}. \quad (6)$$

Of the initially possible assumptions about $d\varphi$, that it is a linear differential form, or the root of a quadratic one, or the cubic root of a cubic one etc., only the first, as we can now see from (6), has an invariant meaning. We have thus arrived at the following result.

The metric of a manifold is based on a quadratic and on a linear differential form

$$ds^2 = g_{ik} dx_i dx_k \quad \text{and} \quad d\varphi = \varphi_i dx_i. \quad (7)$$

However, conversely these forms are not absolutely determined by the metric, but each pair of forms ds'^2 and $d\varphi'$, which arise from (7) according to the equations

$$ds'^2 = \lambda \cdot ds^2, \quad d\varphi' = d\varphi - \frac{d\lambda}{\lambda} \quad (8)$$

is equivalent to the first pair in the sense that both express the same metric. In this λ is an arbitrary, nowhere vanishing continuous (more precisely: twice continuously differentiable) function of position. Into all quantities or relations which represent metric relations analytically, the functions g_{ik} , φ_i must thus enter in such a way that invariance holds (1) with respect to an arbitrary coordinate transformation ("coordinate-invariant"), and (2) with respect to the replacement of (7) by (8) ("measure-invariance").

$$\frac{d\lambda}{\lambda} = d \lg \lambda$$

is a total differential. Hence, whereas in the quadratic form ds^2 , a factor of proportionality remains arbitrary at each location, the indeterminacy of $d\varphi$ consists of an additive total differential.

A metric manifold we describe physically as a world filled with *aether*. The particular metric existing in the manifold represents a particular state of the world filling aether. This state is thus to be described relative to a frame of reference through the specification (arithmetic construction) of the functions g_{ik}, φ_i . †

From (6) it follows that the linear tensor of 2nd order with the components

[399]

$$F_{ik} = \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i}$$

is uniquely determined by the metric of the manifold; I call it the *metric vortex*. It is the same, I believe, as what in physics one calls the *electromagnetic field*. It satisfies the “first system of Maxwell’s equation”

$$\frac{\partial F_{kl}}{\partial x_i} + \frac{\partial F_{li}}{\partial x_k} + \frac{\partial F_{ik}}{\partial x_l} = 0.$$

Its vanishing is the necessary and sufficient condition for the transfer of length to be integrable, i.e., for those conditions which Riemann placed at the foundations of metric geometry to prevail. We understand from this how Einstein through his world geometry, which mathematically follows Riemann, could only account for gravitation but not for the electromagnetic phenomena.

4.2 Affine Connection of a Metric Manifold

In a metric space, in place of the requirement A imposed on the concept of parallel displacement in §3, I., we have the more specific one

*A**: that the parallel displacement of all vectors at a point P to an infinitely close point P' , must not only be an affine but also a congruent transfer of the totality of these vectors.

Using the previous notation, this requirement yields the equation

$$(1 + d\varphi)(g_{ik} + dg_{ik})(\xi^i + d\xi^i)(\xi^k + d\xi^k) = g_{ik}\xi^i\xi^k. \tag{9}$$

For all quantities a^i , which carry an upper index (i), we define the “lowering” of the index through the equations

$$a_i = \sum_k g_{ik}a^k.$$

(and the reverse process of raising an index through the inverse equations). Using this symbolism, for (9) we can write

$$(g_{ik}\xi^i\xi^k)d\varphi + \xi^i\xi^k dg_{ik} + 2\xi_i d\xi^i = 0.$$

The last term is

$$= -2\xi_i \xi^k d\gamma^i_k = -2\xi^i \xi^k d\gamma_{ik} = -\xi^i \xi^k (d\gamma_{ik} + d\gamma_{ki});$$

[400] | and therefore

$$d\gamma_{ik} + d\gamma_{ki} = dg_{ik} + g_{ik}d\varphi. \quad (10)$$

This equation can certainly be satisfied only if $d\varphi$ is a linear differential form; an assumption to which we were already driven above as the only reasonable one. From (10) or

$$\Gamma_{i,kr} + \Gamma_{k,ir} = \frac{\partial g_{ik}}{\partial x_r} + g_{ik}\varphi_r \quad (10^*)$$

follows, as a consequence of the symmetry property $\Gamma_{r,ik} = \Gamma_{r,ki}$:

$$\Gamma_{r,ik} = \frac{1}{2} \left(\frac{\partial g_{ir}}{\partial x_k} + \frac{\partial g_{kr}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_r} \right) + \frac{1}{2} (g_{ir}\varphi_k + g_{kr}\varphi_i - g_{ik}\varphi_r); \quad (\Gamma_{r,ik} = g_{rs}\Gamma^s_{ik}). \quad (11)$$

It turns out that on a metric manifold the concept of the infinitesimal parallel displacement of a vector is uniquely determined through the requirements put forward.¹⁰ I consider this as the *fundamental fact of infinitesimal geometry*, that with the metric also the affine connection of a manifold is given, that *the principle of transfer of length inherently carries with it that of transfer of direction*, or expressed physically, *that the state of the aether determines the gravitational field*.

If the quadratic form $g_{ik}dx_i dx_k$ is indefinite, then among the geodesic lines, the null lines are distinguished as those along which the form vanishes. They depend only on the ratios of the g_{ik} , but not at all on the φ_i , they are thus structures of conformal geometry.¹¹

We had imposed certain axiomatic requirements on the concept of parallel transport and shown that they can be satisfied on a metric manifold in one and only one way. However, it is also possible to define that concept explicitly in a simple manner. If P is a point in our metric manifold, then we call a frame of reference *geodesic* in P , if upon its use the φ_i vanish at P and the g_{ik} assume stationary values:

$$\varphi_i = 0, \quad \frac{\partial g_{ik}}{\partial x_r} = 0.$$

[401] | D. For each point P there exist geodesic frames of reference. If ξ is a given vector at P , and P' is an infinitely close point to P , then we understand by the vector which arises from x through parallel transport to P' that vector at P' , which has the same components as ξ in the geodesic coordinate system belonging to P . This definition is independent of the choice of the geodesic frame of reference.

10 See also Hessenberg, "Vektorielle Begründung der Differentialgeometrie," *Math. Ann.* vol. 78 (1917), p. 187-217, especially p. 208.

11 With this comment, I would like to correct a mistake on page 183 of my book *Raum, Zeit, Materie*.

It is not difficult to demonstrate the assertion contained in this explanation independently of the train of thought followed here through direct calculation, and to show by the same means that the process of parallel transport so defined is, in an arbitrary coordinate system, described by the equation

$$d\xi^r = -\Gamma^r_{ik}\xi^i dx_k \quad (12)$$

with the coefficients Γ to be taken from (11).¹² But here, where the invariant meaning of equation (12) is already established, we conclude more simply as follows. According to (11), the Γ^r_{ik} vanish in a geodesic frame of reference and the equations (12) reduce to $d\xi^r = 0$. Hence, the concept of parallel transfer that we derived from the axiomatic requirements agrees with the one defined in D. Only the existence of a geodesic frame of reference is left to be shown. For this purpose, we choose a coordinate system x_i , geodesic at P , having the point P as its origin ($x_i = 0$). If the unit of length at P and in its vicinity is for the time being chosen arbitrarily, and if furthermore the φ_i denote the value of these quantities at P , then one only needs to complete the transition from (7) to (8) with

$$\lambda = e^{\sum_i \varphi_i x_i},$$

in order to obtain that, besides the Γ^i_{rs} , the φ_i also vanish at P . From this then follows—see (10*)—the geodesic nature of the frame of reference so obtained. The coordinates of a frame of reference geodesic at P are in the immediate vicinity of P determined up to terms of 3rd order, leaving aside linear transformation, and the unit of length up to terms of 2nd order, leaving aside the addition of a constant factor.

12 In this one could follow the approach I have taken in RZM, §14.