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NAVIER–STOKES PROBLEMS MODELED BY EVOLUTION HEMIVARIATIONAL INEQUALITIES

Stanislaw Migórski and Anna Ochal

Jagiellonian University
Faculty of Mathematics and Computer Science
Institute of Computer Science
ul. Nawojki 11, 30-072 Krakow, Poland

Dedicated to Professor Zdzislaw Denkowski on the occasion of his 65th birthday

Abstract. In this paper we study an inequality problem for the evolution Navier-Stokes type operators related to the model of motion of a viscous incompressible fluid in a bounded domain. The equations are nonlinear Navier-Stokes ones for the velocity and pressure with non-standard boundary conditions. We assume the nonslip boundary condition together with a Clarke subdifferential relation between the pressure and the normal components of the velocity. The existence of weak solutions to the model is proved by applying the regularized Galerkin method.

1. **Introduction.** In this paper we examine a class of hemivariational inequality problems for the evolution Navier–Stokes operators. The main feature of this class is a nonmonotone and possibly multivalued boundary condition which is expressed by the generalized Clarke subdifferential. The motivation for our study comes from the fluid flow control problems and the flow problems for semipermeable walls and membranes. More precisely, the problem under consideration describes a model in which we regulate the boundary orifices in a channel (or a tube) to reduce the pressure of the fluid on the boundary when the normal velocity reaches a prescribed value. The multivalued subdifferential boundary condition can be used to model a control problem when the pressure is regulated by a hydraulic control device.

Considering the nonmonotone character of the multivalued boundary condition, a convex analysis approach to the problem is not possible. We are naturally lead to a mathematical model involving the Clarke subdifferential of a locally Lipschitz superpotential. Such formulation is called a hemivariational inequality and it was introduced and studied in the early 1980's by Panagiotopoulos [18, 19]. The hemivariational inequalities are natural generalizations of variational inequality problems and their origin is in nonsmooth mechanics. For the description of origins of hemivariational inequalities and the mathematical theory, we refer to Panagiotopoulos [18], Naniewicz and Panagiotopoulos [17], Goeleven et al. [9] and the references

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therein, for recent results on parabolic and hyperbolic hemivariational inequalities cf. [12, 13, 14, 16].

The goal of our paper is to extend the main results of Konovalova [10] and Migórski and Ochal [15] to the case of evolution Navier–Stokes problem. We prove the existence of solutions for the equations modeling the motion of a viscous incompressible fluid. In [10] the dynamic variational inequality for the Navier–Stokes equation was considered while in [15] the existence of solutions to stationary hemivariational inequality has been obtained by a surjectivity argument. In the present paper we use a Galerkin method for a regularized problem. The regularization is applied for the nonsmooth superpotential and a solution to the hemivariational inequality is obtained as a limit of a sequence of solutions to a regularized problem.

The related problems for Navier–Stokes equations with boundary conditions involving the pressure have been considered in Conca et al. [7]. For other flow problems dealing with semipermeable media as well as for the flow through porous media, we refer to Panagiotopoulos [18], Naniewicz and Panagiotopoulos [17] (Chapter 5.5.3), Chebotarev [4, 5] and the references therein. We also refer to Selmani et al. [20] for results on variational inequalities which model the stationary flow of Bingham fluid with friction. An existence result for Bingham fluids in a laminar flow in a cylindrical pipe with nonmonotone boundary condition can be found in [11].

The paper is organized as follows. In Section 2 we recall some notation and definitions and in Section 3 we formulate the evolution hemivariational inequality for fluid flow problem. The main result on an evolution inclusion associated with the hemivariational inequality is given in Section 4.

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2. **Preliminaries.** In this section we introduce the notation and recall some definitions needed in the sequel.

Let Ω be a domain in $\mathbb{R}^d$, $d = 2, 3$ with regular boundary Γ. Let n denote the outward unit normal vector to Γ. Given $v \in H^{1/2}(\Gamma;\mathbb{R}^d)$ we denote by v_N and v_T the usual normal and the tangential components of v on the boundary Γ, i.e. $v_N = v \cdot n$, $v_T = v - v_N n$, where $v \cdot n = \sum_{i=1}^n v_i n_i$.

We introduce the spaces which are needed in the weak formulation of the problem under consideration. Let

$$W = \{w \in C^\infty(\Omega;\mathbb{R}^d) \,:\, \operatorname{div} w = 0 \text{ in } \Omega,\ w_T = 0 \text{ on } \Gamma\}$$

and let $\delta \in (\frac{1}{2}, 1)$. We denote by V, Z and H the closure of W in the norm of $H^1(\Omega;\mathbb{R}^d)$, $H^\delta(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively. Then

$$V \subset Z \subset H \simeq H^* \subset Z^* \subset V^*$$

with all embeddings being continuous and compact. Denoting by $i\colon V \to Z$ the embedding injection, by $\gamma\colon Z \to L^2(\Gamma;\mathbb{R}^d)$ and $\gamma_0\colon H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ the trace operators, for all $v \in V$ we have $\gamma_0 v = \gamma(iv)$. For simplicity we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for $v \in V$.

Given a finite time interval $(0,T)$, we define the spaces

$$\mathcal{V} = L^2(0,T;V),\ \mathcal{H} = L^2(0,T;H) \text{ and } \mathcal{W} = \{w \in \mathcal{V} : w' \in \mathcal{V}^*\},$$

where the time derivative involved in the definition is understood in the sense of vector valued distributions. We have the following continuous embeddings $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$. Equipped with the norm $\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}^*}$ the space $\mathcal{W}$ becomes a separable reflexive Banach space. It is well known (cf. e.g. [8]) that the space $\mathcal{W}$ is embedded continuously in $C(0,T;H)$ (the space of continuous functions on $[0,T]$ with values in H), i.e. every element of $\mathcal{W}$, after a possible modification on a set of measure zero, has a unique continuous representative in $C(0,T;H)$. Moreover, since V is embedded compactly in H, then so does $\mathcal{W}$ into $\mathcal{H}$ (cf. [8]). The inner products in Hilbert spaces H and $\mathcal{H}$ are denoted by $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, respectively.

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function $f\colon E \to \mathbb{R}$, where E is a Banach space (see Clarke [6]). The generalized directional derivative of f at $x \in E$ in the direction $v \in E$, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v) = \limsup_{y\to x,\ \lambda\downarrow 0} \frac{f(y+\lambda v) - f(y)}{\lambda}.$$

The generalized gradient of f at x, denoted by $\partial f(x)$, is a subset of a dual space E^* given by $\partial f(x) = \{\zeta \in E^* : f^0(x;v) \geq \langle \zeta, v\rangle_{E^*\times E} \text{ for all } v \in E\}$. The locally Lipschitz function f is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $f'(x;v)$ exists and satisfies $f^0(x;v) = f'(x;v)$ for all $v \in E$. It is well known that a locally Lipschitz convex function is regular (cf. Proposition 2.3.6 of [6]).

3. **Problem statement.** In this section we formulate the initial–boundary value problem for the evolution Navier-Stokes equation with a subdifferential boundary condition.

Let Ω be a bounded simply connected domain in $\mathbb{R}^d$, $d = 2$ or 3 with connected boundary Γ of class C^2. We consider the following system of evolution Navier-Stokes equations

$$u' - \nu\triangle u + (u\cdot\nabla)u + \nabla p = f, \qquad \nabla\cdot u = 0 \quad \text{in } Q, \tag{1}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2}$$

This system describes an incompressible viscous fluid flow in the domain Ω, where $u = \{u_i(x,t)\}_{i=1}^d$ is a velocity, $f = \{f_i(x,t)\}_{i=1}^d$ is an external forces vector field, $p = p(x,t)$ is the pressure, ν is a kinematic viscosity of the fluid ($\mathrm{Re} = \frac{1}{\nu}$ is the Reynolds number), $t \in (0,T)$ represents time, u_0 is the initial velocity and $Q = \Omega \times (0,T)$. The nonlinear term $(u\cdot\nabla)u$ in (1) (often called the convective term) is a symbolic notation for the vector $\{\sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}\}_{i=1}^d$. The divergence free condition $\operatorname{div} u = \nabla\cdot u = 0$ is the equation for law of mass conservation and it states that the motion is incompressible.

Following the papers of Konovalova [10] and Chebotarev [4, 5] in order to give the variational formulation of (1)-(2), it is convienient to rewrite the problem in the standard Lamb form. Applying the well-known formulas of vector analysis, we have an equivalent form of this problem:

$$u' - \nu \operatorname{rot}\operatorname{rot} u + \operatorname{rot} u \times u + \nabla\widetilde{p} = f \quad \text{in } Q, \tag{3}$$

$$\operatorname{div} u = 0 \quad \text{in } Q, \quad u(0) = u_0 \quad \text{in } \Omega, \tag{4}$$

where $\widetilde{p} = p + \frac{1}{2}|u|^2$ denotes the total head of the fluid (called also a total pressure or a Bernoulli pressure). In what follows we assume that on the boundary Γ the tangential components of the velocity vector are prescribed and without loss of generality we put them equal to zero (the nonslip condition):

$$u_T = u - u_N\, n = 0 \quad \text{on } \Gamma \times (0,T). \tag{5}$$

Furthermore, we suppose the following subdifferential boundary condition

$$\widetilde{p}(x,t) \in \partial j(x,t,u_N(x,t)) \quad \text{on } \Gamma \times (0,T), \tag{6}$$

where $j\colon \Gamma \times (0,T) \times \mathbb{R} \to \mathbb{R}$ is a given superpotential function which is locally Lipschitz in the third variable and ∂j stands for the Clarke subdifferential of $j(x,t,\cdot)$.

We introduce the operators which are needed in the weak formulation of the problem (3)-(6). Let us define the operators $A\colon V \to V^*$ and $B[\,\cdot\,]\colon V \to V^*$ by

$$\langle Au, v\rangle = \nu \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v\, dx, \tag{7}$$

$$\langle B(u,v), w\rangle = \int_\Omega (\operatorname{rot} u \times v)\cdot w\, dx, \qquad B[v] = B(v,v) \tag{8}$$

for $u, v, w \in V$. It is known (cf. [15]) that in the case the domain Ω is simply connected, the bilinear form $(\!(u,v)\!)_V = \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v\, dx$ generates a norm in V, $\|u\|_V = (\!(u,u)\!)_V^{1/2}$ which is equivalent to the $H^1(\Omega;\mathbb{R}^d)$-norm. Hence, it is clear that the operator A is coercive with a constant $\alpha > 0$.

Assuming sufficient regularity of the functions involved in the problem, multiplying (3) by $v \in V$ and applying the Green formula, we obtain

$$\langle u'(t) + Au(t) + B[u(t)], v\rangle + \int_\Gamma \widetilde{p}\, v_N\, d\sigma(x) = \langle F(t), v\rangle \quad \text{a.e. } t \in (0,T), \tag{9}$$

where $\langle F(t), v\rangle = \int_\Omega f(t)\cdot v\, dx$. From the relation (6) and the definition of the Clarke subdifferential we have

$$\int_\Gamma \widetilde{p}(x,t)\, v_N(x)\, d\sigma(x) \le \int_\Gamma j^0(x,t,u_N(x,t); v_N(x))\, d\sigma(x), \tag{10}$$

where $j^0(x,t,\xi;\eta)$ denotes the generalized directional derivative of $j(x,t,\cdot)$ at the point $\xi \in \mathbb{R}$ in the direction $\eta \in \mathbb{R}$.

From (9) and (10) we deduce the following weak formulation of the problem which is a hemivariational inequality: find $u \in \mathcal{W}$ such that

$$\begin{cases} \langle u'(t) + Au(t) + B[u(t)], v\rangle + \displaystyle\int_\Gamma j^0(x,t,u_N(x,t); v_N(x))\, d\sigma(x) \\ \qquad\qquad \ge \langle F(t), v\rangle \quad \text{for all } v \in V, \ \text{a.e. } t \in (0,T) \\ u(0) = u_0. \end{cases} \tag{11}$$

We observe that since $\mathcal{W} \subset C(0,T;H)$ continuously the initial condition $u(0)$ makes sense in H.

We have proved that the hemivariational inequality (11) is derived from (3)-(6). The following shows that in some sense the converse statement also holds.

Remark 1. If $u \in \mathcal{W}$ is a solution to the hemivariational inequality (11) and u is sufficiently smooth, then there exists a distribution $\widetilde{p}$ such that the conditions (3) and (6) hold. Indeed, since $u \in \mathcal{W}$ from the definition of $\mathcal{V}$ we have div $u = 0$ in Q and $u_T = 0$ on $\Gamma \times (0,T)$. Let us now take $v = \pm w$, where $w \in V \cap C_0^\infty(\Omega;\mathbb{R}^d)$ in (11). Since w is arbitrary and $j^0(x,t,u_N(x,t);0) = 0$, we obtain $\langle u'(t) + Au(t) + B[u(t)], w\rangle = \langle F(t), w\rangle$ for a.e. $t \in (0,T)$. From Proposition 1.1 in Chapter I of Temam [21] it follows that $u'(t) + Au(t) + B[u(t)] + \nabla\widetilde{p}(t) = F(t)$ for a.e. $t \in (0,T)$ which implies (3). Next let $v \in V$. Multiplying the last equation by v and integrating by parts over Ω we get

$$\langle u'(t) + Au(t) + B[u(t)], v\rangle + \int_\Gamma \widetilde{p}(x,t)\, v_N(x)\, d\sigma(x) = \langle F(t), v\rangle \quad \text{a.e. } t \in (0,T).$$

Comparing this equality with (11), we have

$$\int_\Gamma \big(j^0(x,t,u_N(x,t);v_N(x)) - \widetilde{p}(x,t)\, v_N(x)\big)\, d\sigma(x) \ge 0$$

for every $v \in V$ and a.e. $t \in (0,T)$. Arguing as in Proposition 3.3.1 of Panagiotopoulos [18], we deduce $j^0(x,t,u_N(x,t);v_N(x)) \ge \widetilde{p}(x,t)\, v_N(x)$ on $\Gamma \times (0,T)$. This shows that the subdifferential condition (6) holds.

Remark 2. The condition (6) arises in the problem of motion of a fluid through a channel or a tube. The fluid pumped into Ω can leave the tube at the boundary orifices while a device can change the sizes of the latter. In this problem we regulate the normal velocity of the fluid on the boundary to reduce the total pressure on Γ. For instance, we consider the boundary condition (6) with

$$\partial j(x,t,s) = h(x,t) \times \begin{cases} 0 & \text{if } s < 0 \\ \dfrac{c}{a} s & \text{if } 0 < s < a \\ [b,c] & \text{if } s = a \\ \dfrac{b}{a^\sigma} s^\sigma & \text{if } s > a, \end{cases}$$

where $h \in L^\infty(\Gamma \times (0,T))$, $a > 0$, $0 \le b < c$ and $0 \le \sigma \le 1$. The condition $u_N > 0$ is interpreted as the outflow of the fluid through the boundary. If $u_N \in (0,a)$ the orifices on the boundary allow the fluid to infiltrate outside the tube. When the velocity increases so does the total pressure, say, linearly from the value 0 to the value c. If u_N reaches the value a, a mechanism opens the orifices wider and allows the fluid to pass through Γ. Therefore the pressure drops to a value b and we may assume that $\widetilde{p} = c_1(u_N)^\sigma + c_2$ for $u_N > a$ with suitable constants c_1 and c_2. Moreover, in (6) we allow j to depend on the variable $(x,t) \in \Gamma \times (0,T)$ which means that the subdifferential boundary condition can be of different character on different parts of Γ at different time instances. For other examples, see Example 18 in [15].

4. **An existence result.** The goal of this section is to give a result on the existence of weak solutions to the problem (11). We associate with the hemivariational inequality (11) the following problem

$$\begin{cases} u'(t) + Au(t) + B[u(t)] + \partial j(x,t,u_N(x,t)) \ni f(t) \quad \text{a.e. } t \in (0,T) \\ u(0) = u_0. \end{cases} \tag{12}$$

Definition 1. A function $u \in \mathcal{W}$ is said to be a solution to (12) if there exists $\xi \in L^2(0,T;L^2(\Gamma))$ such that

$$\langle u'(t) + Au(t) + B[u(t)], v\rangle + (\xi(t), v_N)_{L^2(\Gamma)} = \langle f(t), v\rangle$$

for all $v \in V$ and a.e. $t \in (0,T)$, $\xi(x,t) \in \partial j(x,t,u_N(x,t))$ a.e. on $\Gamma \times (0,T)$ and $u(0) = u_0$.

Remark 3. From the definition of the Clarke subdifferential, it follows that every solution to (12) is also a solution to (11).

In what follows we restrict the analysis to superpotential j which is independent of (x,t) and which subdifferential is obtained by "filling in the gaps" procedure. Let $\beta \in L^\infty_{loc}(\mathbb{R})$. For every $\varepsilon > 0$ and $t \in \mathbb{R}$, we define

$$\underline{\beta}_\varepsilon(t) = \operatorname*{ess\,inf}_{|s-t|\le\varepsilon} \beta(s), \qquad \overline{\beta}_\varepsilon(t) = \operatorname*{ess\,sup}_{|s-t|\le\varepsilon} \beta(s).$$

For t fixed, $\underline{\beta}_\varepsilon$ is an increasing function of ε and $\overline{\beta}_\varepsilon$ is decreasing in ε. Let

$$\underline{\beta}(t) = \lim_{\varepsilon\to 0^+} \underline{\beta}_\varepsilon(t), \qquad \overline{\beta}(t) = \lim_{\varepsilon\to 0^+} \overline{\beta}_\varepsilon(t)$$

and let $\widehat{\beta}\colon \mathbb{R} \to 2^{\mathbb{R}}$ be a multifunction defined by

$$\widehat{\beta}(t) = [\underline{\beta}(t), \overline{\beta}(t)] \ \text{ for all } \ t \in \mathbb{R},$$

i.e. $\widehat{\beta}(t)$ is represented by the interval with the initial and end points given by $\underline{\beta}(t)$ and $\overline{\beta}(t)$, respectively. Roughly speaking $\widehat{\beta}(t)$ results from β by filling in the gaps at points where β is discontinuous. From Chang [3] we know that a locally Lipschitz function $j\colon \mathbb{R} \to \mathbb{R}$ can be determinated up to an additive constant by the relation

$$j(t) = \int_0^t \beta(s)\, ds \ \text{ for all } t \in \mathbb{R}$$

such that $\partial j(t) \subset \widehat{\beta}(t)$ for all $t \in \mathbb{R}$. If moreover, the limits $\beta(t \pm 0)$ exist for every $t \in \mathbb{R}$, then $\partial j(t) = \widehat{\beta}(t)$ for $t \in \mathbb{R}$.

Remark 4. The above construction can be also repeated for a function $\beta = \beta(x,t,r)$, $\beta\colon \Gamma \times (0,T) \times \mathbb{R} \to \mathbb{R}$ which is locally essentially bounded in $r \in \mathbb{R}$, satisfies a measurability hypothesis with respect to $(x,t) \in \Gamma \times (0,T)$, and $|\beta(x,t,r)| \le c_0(1 + |r|^\sigma)$ for a.e. $(x,t) \in \Gamma \times (0,T)$, for all $t \in \mathbb{R}$ with $c_0 > 0$ and $0 \le \sigma \le 1$, cf. e.g. Section 1.2.3 of [9] and [12].

We admit the following hypothesis.

$\underline{H(\beta)}$: $\beta \in L^\infty_{loc}(\mathbb{R})$ is such that the left and right limits $\beta(t \pm 0)$ exist for every $t \in \mathbb{R}$ and it verifies the growth condition $|\beta(t)| \le c_0(1 + |t|^\sigma)$ for all $t \in \mathbb{R}$ with $c_0 > 0$ and $0 \le \sigma \le 1$.

Theorem 1. *Let the operators A and B be given by (7) and (8), respectively and let the function β satisfy $H(\beta)$, $f \in \mathcal{V}^*$ and $u_0 \in H$. If $H(\beta)$ holds with $\sigma = 1$, we suppose additionally that $\alpha > 2\sqrt{2}c_0\|\gamma\|^2$. Then the problem (12) admits a solution.*

Proof. The existence of solution to (12) will be proved by applying the Galerkin method to a regularized problem. First we introduce the regularization β_n of β. We choose a mollifier $\varrho \in C_0^\infty((-1,1))$, $\varrho \ge 0$ and $\int_{\mathbb{R}} \varrho(s)\, ds = 1$ and define $\varrho_n\colon \mathbb{R} \to \mathbb{R}$

by $\varrho_n(s) = n\varrho(ns)$ for $s \in \mathbb{R}$ and $n \in \mathbb{N}$. We consider $\beta_n : \mathbb{R} \to \mathbb{R}$ defined by the convolution

$$\beta_n(t) = \int_{\mathbb{R}} \varrho_n(s)\, \beta(t-s)\, ds \quad \text{for } t \in \mathbb{R}.$$

It is easy to show that β_n is continuous for all $n \in \mathbb{N}$ and it satisfies the same growth condition as β for all $n \in \mathbb{N}$.

Let $\{\varphi_1, \varphi_2, \ldots\}$ be a basis in V, i.e. $\{\varphi_i\}$ forms an at most countable sequence of elements of V, finitely many $\varphi_1, \ldots, \varphi_n$ are linearly independent and $V = \overline{\bigcup_n V_n}$ with $V_n = \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$. Since V is separable the existence of such a basis is guaranted. Moreover, the family $\{V_n\}$ of finite dimensional subspaces of V satisfies

$$\forall\, v \in V \quad \exists\, \{v_n\},\ v_n \in V_n \text{ such that } v_n \to v \text{ in } V, \text{ as } n \to \infty.$$

Let $\{u_{0n}\}$ be such that $u_{0n} \in V_n$ for $n \in \mathbb{N}$ and

$$u_{0n} \to u_0 \quad \text{in } H, \text{ as } n \to \infty.$$

We consider the following regularized Galerkin system of finite dimensional differential equations associated with (12):

$$\begin{cases} \text{find } u_n \in L^2(0,T;V_n) \text{ such that } \ u_n' \in L^2(0,T;V_n) \ \text{ and} \\ \langle u_n'(t) + Au_n(t) + B[u_n(t)], v_n\rangle + (\beta_n(u_{nN}(t)), v_{nN})_{L^2(\Gamma)} = \langle f(t), v_n\rangle \\ \hfill \text{for a.e. } t \in (0,T) \text{ and all } v_n \in V_n \\ u_n(0) = u_{0n}. \end{cases} \tag{13}$$

Substitution of $u_n(t) = \sum_{k=1}^n c_{kn}(t)\varphi_k$ gives an initial value problem for a system of first order ordinary differential equations for $c_{kn}(\cdot)$, $k = 1, \ldots, n$. Its solvability on some interval $[0, t_n)$ follows from the Caratheodory theorem. Then, this solution can be extended on the closed interval $[0,T]$ by using the a priori estimates below (cf. also [11]).

Next, we obtain estimates on the sequence $\{u_n\}$ of solutions to (13). Choosing $u_n(t)$ as a test function in (13), using the coercivity of A, properties of B (cf. Section 4 of [15]) and the Young inequality, we have

$$\frac{1}{2}\frac{d}{dt}|u_n(t)|_H^2 + \alpha\|u_n(t)\|_V^2 + (\beta_n(u_{nN}(t)), u_{nN}(t))_{L^2(\Gamma)} \le \frac{\alpha}{2}\|u_n(t)\|_V^2 + \frac{2}{\alpha}\|f(t)\|_{V^*}^2$$

for a.e. $t \in (0,T)$. Integrating over $(0,t)$, we get

$$\begin{aligned} &\frac{1}{2}|u_n(t)|_H^2 - \frac{1}{2}|u_{0n}|_H^2 + \frac{\alpha}{2}\int_0^t \|u_n(s)\|_V^2\, ds + \int_0^t (\beta_n(u_{nN}(s)), u_{nN}(s))_{L^2(\Gamma)}\, ds \\ &\le \frac{2}{\alpha}\|f\|_{\mathcal{V}^*}^2 \text{ for all } t \in [0,T]. \end{aligned} \tag{14}$$

From the estimate $|\beta_n(s)| \le c_0(1+|s|^\sigma)$ for $s \in \mathbb{R}$ and the continuity of the trace operator, we have

$$\begin{aligned}\|\beta_n(u_{nN}(t))\|^2_{L^2(\Gamma)} &= \int_\Gamma |\beta_n(u_{nN}(x,t))|^2\, d\Gamma(x)\\ &\le 2c_0^2 \int_\Gamma (1+|u_{nN}(x,t)|^{2\sigma})\, d\Gamma(x)\\ &\le 2c_0^2 \int_\Gamma (1+|u_n(x,t)|^{2\sigma}_{\mathbb{R}^d})\, d\Gamma(x)\\ &\le 2c_0^2 m(\Gamma) + 2c_0^2 m(\Gamma)^{1-\sigma}\|u_n(t)\|^{2\sigma}_{L^2(\Gamma;\mathbb{R}^d)}\\ &\le 2c_0^2 m(\Gamma) + 2c_0^2 m(\Gamma)^{1-\sigma}\|\gamma\|^{2\sigma}\,\|u_n(t)\|^{2\sigma}_V,\end{aligned}$$

where $\|\gamma\| = \|\gamma\|_{\mathcal{L}(V,L^2(\Gamma;\mathbb{R}^d))}$. Hence, we have

$$\begin{aligned}\|\beta_n(u_{nN})\|^2_{L^2(0,t;L^2(\Gamma))} &= \int_0^t \|\beta_n(u_{nN}(s))\|^2_{L^2(\Gamma)}\, ds\\ &\le 2c_0^2\, t\, m(\Gamma) + 2c_0^2\|\gamma\|^{2\sigma}\, m(\Gamma)^{1-\sigma}\, t^{1-\sigma}\|u_n\|^{2\sigma}_{L^2(0,t;V)}\end{aligned}$$

for $t \in [0,T]$ and consequently

$$\|\beta_n(u_{nN})\|_{L^2(0,t;L^2(\Gamma))} \le c_1 + c_2\|u_n\|^\sigma_{L^2(0,t;V)} \quad \text{for } t\in[0,T] \tag{15}$$

with $c_1 = c_0\sqrt{2\,t\,m(\Gamma)}$ and $c_2 = c_0\|\gamma\|^\sigma\sqrt{2(t\,m(\Gamma))^{1-\sigma}}$. We also calculate

$$\begin{aligned}\left|\int_0^t (\beta_n(u_{nN}(s)), u_{nN}(s))_{L^2(\Gamma)}\, ds\right| &\le \int_0^t \|\beta_n(u_{nN}(s))\|_{L^2(\Gamma)}\|u_{nN}(s)\|_{L^2(\Gamma)}\, ds\\ &\le \|\beta_n(u_{nN})\|_{L^2(0,t;L^2(\Gamma))}\,\|u_{nN}\|_{L^2(0,t;L^2(\Gamma))}\\ &\le \Big(c_1 + c_2\|u_n\|^\sigma_{L^2(0,t;V)}\Big)\,\|\gamma\|\,\|u_n\|_{L^2(0,t;V)}.\end{aligned}$$

Inserting the latter to (14), we have

$$\begin{aligned}&\frac{1}{2}|u_n(t)|^2_H + \frac{\alpha}{2}\|u_n\|^2_{L^2(0,t;V)}\, ds \le\\ &\frac{1}{2}|u_{0n}|^2_H + \frac{2}{\alpha}\|f\|^2_{\mathcal{V}^*} + c_1\|\gamma\|\,\|u_n\|_{L^2(0,t;V)} + c_2\|\gamma\|\,\|u_n\|^{\sigma+1}_{L^2(0,t;V)}\end{aligned}$$

for all t $\in [0,T]$. If $0\le\sigma<1$, then $\{u_n\}$ remains in a bounded subset of $L^2(0,T;V)$. If $\sigma=1$, then $\{u_n\}$ is also bounded in $L^2(0,T;V)$ provided $\alpha > 2\sqrt{2}c_0\|\gamma\|^2$. Furthermore, we deduce that $\{u_n\}$ is bounded in $L^\infty(0,T;H)$, so passing to a subsequence, if necessary, we have

$$u_n \to u \ \text{ weakly in } \mathcal{V} \text{ and weakly}-* \text{ in } L^\infty(0,T;H),$$

where $u \in \mathcal{V}\cap L^\infty(0,T;H)$.

The basic problem is now to get the weak convergence of the nonlinear term $B[u_n]$. For the case $d=2$, we obtain from Temam [21] that

$$\|B[u_n]\|_{\mathcal{V}^*} \le c_4\|u_n\|_{L^\infty(0,T;H)}\,\|u_n\|_{\mathcal{V}} \ \text{ with } c_4>0.$$

Hence, exploiting (13), (15) and the boundedness of A, we obtain that $\{u_n'\}$ is bounded in $\mathcal{V}^*$. Thus, by passing to a next subsequence, if necessary, it follows

$$u_n' \to u' \ \text{ weakly in } \mathcal{W} \text{ with } u\in\mathcal{W}.$$

Using the facts that $\mathcal{W} \subset C(0,T;H)$ continuously, $\mathcal{W} \subset \mathcal{H}$ compactly and $\mathcal{W} \subset L^2(0,T;L^2(\Gamma;\mathbb{R}^d))$ compactly, we have $u \in C(0,T;H)$ and

$$u_n \to u \ \text{ in } \ \mathcal{H}, \quad \gamma u_n \to \gamma u \ \text{ in } \ L^2(0,T;L^2(\Gamma;\mathbb{R}^d)).$$

Since $u_n \to u$ weakly in $\mathcal{V}$ and in $\mathcal{H}$, analogously as in Ahmed [1], we have $B[u_n] \to B[u]$ weakly in $\mathcal{V}^*$. We remark that if $d = 3$ the convergence of $B[u_n] \to B[u]$ weakly in $\mathcal{V}^*$ is more difficult to obtain. In this case we proceed as in the proof of Theorem 1 in Ahmed [1] employing a compactness embedding theorem.

From (15) we may suppose

$$\beta_n(u_{nN}) \to \xi \ \text{ weakly in } \ L^2(0,T;L^2(\Gamma)) \ \text{ with } \ \xi \in L^2(0,T;L^2(\Gamma)).$$

Since the mapping $\mathcal{W} \ni w \mapsto w(0) \in H$ is linear and continuous, we have $u_n(0) \to u(0)$ weakly in H, which together with $u_{0n} \to u_0$ in H entails $u(0) = u_0$.

Let $\psi \in C_0^\infty(0,T)$ and $v \in V$. Then, there exists $\{v_n\}$, $v_n \in V_n$ such that $v_n \to v$ in V, as $n \to \infty$. Denoting $\Psi_n(x,t) = \psi(t)v_n(x)$ and $\Psi(x,t) = \psi(t)v(x)$, we have $\Psi_n \to \Psi$ in $\mathcal{W}$. From (13), it follows

$$\int_0^T \langle u_n'(t) + Au_n(t) + B[u_n(t)], \Psi_n(t)\rangle\, dt + \int_0^T (\beta_n(u_{nN}(t)), \Psi_{nN}(t))_{L^2(\Gamma)}\, dt$$
$$= \int_0^T \langle f(t), \Psi_n(t)\rangle\, dt.$$

Using the above convergences, letting $n \to \infty$, we obtain

$$\int_0^T \langle u'(t) + Au(t) + B[u(t)], v\rangle\, \psi(t)\, dt + \int_0^T (\xi(t), v_N)_{L^2(\Gamma)}\, \psi(t)\, dt$$
$$= \int_0^T \langle f(t), v\rangle\, \psi(t)\, dt.$$

Since ψ is arbitrary, we deduce that

$$\langle u'(t) + Au(t) + B[u(t)], v\rangle + (\xi(t), v_N)_{L^2(\Gamma)} = \langle f(t), v\rangle$$

for a.e. $t \in (0,T)$ and for all $v \in V$.

It remains to prove that $\xi(x,t) \in \widehat{\beta}(u_N(t))$ on $\Gamma \times (0,T)$. We apply the convergence theorem of Aubin and Cellina [2] to the multifunction ∂j. First, we observe that $\partial j \colon \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous. Next, since $\gamma u_n \to \gamma u$ in $L^2(0,T;L^2(\Gamma;\mathbb{R}^d))$, we obtain $u_{nN} \to u_N$ in $L^2(0,T;L^2(\Gamma))$ and consequently

$$u_{nN}(x,t) \to u_N(x,t) \ \text{ a.e. } \ (x,t) \in \Gamma \times (0,T).$$

By the definition of $\widehat{\beta}$, we deduce that for a.e. $(x,t) \in \Gamma \times (0,T)$ and for every neighborhood $\mathcal{N}$ of zero in $\mathbb{R}^2$, there exists $n_0 = n_0(x,t,\mathcal{N}) \in \mathbb{N}$ such that

$$(u_{nN}(x,t), \beta_n(u_{nN}(x,t))) \in Gr\, \partial j + \mathcal{N} \ \text{ for all } \ n \geq n_0.$$

From the convergences

$$u_{nN}(x,t) \to u_N(x,t) \ \text{ for a.e. } \ (x,t) \in \Gamma \times (0,T),$$
$$\beta_n(u_{nN}) \to \xi \ \text{ weakly in } \ L^2(0,T;L^2(\Gamma)),$$

we have

$$\xi(x,t) \in \overline{\operatorname{conv}}\, \partial j(u_N(x,t)) = \partial j(u_N(x,t)) \ \text{ for a.e. } \ (x,t) \in \Gamma \times (0,T),$$

which completes the proof. $\square$

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E-mail address: `migorski@softlab.ii.uj.edu.pl`
E-mail address: `ochal@softlab.ii.uj.edu.pl`