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Notes on Differential Forms and Spacetime

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Foreword

In the early 1960s I attended a small seminar on Differential Forms. At that time I did not understand all that was said, but nevertheless, I was thrilled. Here was the dream way of understanding physical phenomena! In 1970 an article by Deschamps [1] became available. I also bought two excellent books on the subject, Flanders [2] and Misner, Thorne, Wheelon [3]. My daily work being something totally else, I read the article and those books whenever I got bored with the work. The following are simply my private notes, written for myself so that I did not need to start every time from scratch.

The remarkable fact is that the Maxwell equations, the Einstein Field Equations and Schrödinger's equation in quantum mechanics can be understood as features of the 4-dimensional Minkowski space; nothing else is needed. The Differential Forms are the best technique to describe this fact.

I have edited the notes so that they are almost readable, I hope. I'm here making them available to everybody. Remember that this is not a textbook nor a report. I'm convinced, though, that this approach would be the best way to teach the basics of electromagnetics and gravity. If there is anybody there who feels like writing a textbook along these lines, please let me know. I'll be the first one to buy your book! (You might mention in your Foreword that you got the spark from my Notes.)

Differential forms will be denoted by bold letters, vectors are italic bold. The components of the vectors may be differential forms.

The basic formula

All of the following happens in a 4-D Minkowski space (ct, x, y, z) with the signature (− + + +). Such a space is called spacetime.

> Why should we believe that we live in a Minkowski space? Start from the definition of speed cdt = ds, square it and write it as $0 = -(cdt)^2 + dx^2 + dy^2 + dz^2$, and that's why!

The notation is mainly from [Flanders], the ideas are from [Misner, Thorne, Wheelon]. Note that multiplication between two forms is always exterior (although ∧ is not shown), so that e.g. dxdy = −dydx and dxdx = 0. $\alpha d\mathbb{V}$ means $\alpha \rfloor d\mathbb{V}$. A detailed explanation of the product with ⌋ is found in e.g. [Lindell]. It is the product between a (multi)vector and a differential form.

All coordinate systems are locally orthogonal.

The theory is based on the general formula

$$\alpha d\mathbb{V} = (d\alpha)\mathbb{V}$$

where $\mathbb{V}$ is a volume of n dimensions ($d\mathbb{V}$ is its boundary) and α is a differential (n-1)-form. (For n = 2 this formula is usually called the Stokes theorem, for n = 3 the Gauss theorem.)

Now postulate that the space $\mathbb{V}$ is closed (meaning that there is no boundary, $d\mathbb{V} = 0$, which makes $\alpha d\mathbb{V} = 0$), then clearly $d\alpha$ has to be zero everywhere. Because spacetime is a closed 3-dimensional surface in a 4-dimensional space, this means that the differential of any 3-form in spacetime is zero. And that's all that is needed for Maxwells' equations, the Einstein Field equations and Schrödinger's equation of quantum mechanics. No other assumptions are needed.

For the purpose of what follows I will call this fact the **principle A: in spacetime the differential of any 3-form = 0**.

> Why should we believe that we live in the 3-dimensional surface of a 4-dimensional sphere?
>
> Mainly, I think, because in many cases the density of something (ρ) in a volume $\mathbb{V}$ diminishes only as a consequence of there being a stream (vector $\boldsymbol{J}$) in time of that something out of that volume, through the surrounding surface $d\mathbb{V}$.
>
> The equation of that phenomenon is simply $\nabla\cdot\boldsymbol{J} = -\partial\rho/\partial t$.
>
> This equation can be written out as
>
> $$[\partial\rho/\partial t + \partial J_x/\partial x + \partial J_y/\partial y + \partial J_z/\partial z]dtdxdydz = 0$$
>
> dtdxdydz is added to show that what is in the brackets is a 4-form. Then it is clear that, using differential forms, the equation can also be written
>
> $$d[\rho dxdydz - J_x dtdydz - J_y dtdzdx - J_z dtdxdy] = 0$$
>
> (note that $-dxdtdydz = dtdxdydz$ etc.). Denote the 3-form within the brackets $\boldsymbol{\gamma}$. So the differential $d\gamma$ is zero. Then the 3-dimensional space is closed. The simplest closed 3-dimensional subspace in the 4-dimensional space is the surface of a 4-dimensional sphere. (There are others).

Multiply the last equation with c and insert $J_x = \rho v_x$. Then

$$\gamma = \rho(cdxdydz - v_x d(ct)dydz - v_y d(ct)dzdx - v_z d(ct)dxdy)/c$$

If ρ is taken to mean the electric charge density (Coulombs/m^3) then what follows will the electromagnetic theory. If ρ is taken to be the mass density m (kg/m^3) then we get the theory of gravity. In quantum mechanics ρ is a probability density. Note that even if the basics of all these cases are the same, all of them have different boundary conditions.

The other fact needed for all the theory that follows is de Rham's theorem:

if $d\boldsymbol{\gamma} = 0$ where $\boldsymbol{\gamma}$ = is a p-form
then (in a starlike space)
$\boldsymbol{\gamma} = d\boldsymbol{\psi}$, where $\boldsymbol{\psi}$ is a (p-1) form

This formula I will call **principle B**. (In the other direction this is trivial, because two successive differentiations give always 0: $d\boldsymbol{\gamma} = d(d\boldsymbol{\psi}) = 0$.)

Maxwell's equations

$\boldsymbol{\gamma}$ is a 3-form in spacetime. Then by principle A

$$d\boldsymbol{\gamma} = 0$$

so that by principle B

$$\boldsymbol{\gamma} = d\boldsymbol{\psi}$$

where $\boldsymbol{\psi}$ is a 2-form. This equation contains half of Maxwell's equations.

$*\boldsymbol{\psi}$ is also a 2-form ($*$ is the Hodge star operator: e.g. $*(d(ct)dx) = dydz$, so that $d(ct)dx*(d(ct)dx) = d(ct)dxdydz$; $*dydz = -d(ct)dx$ (note the minus-sign, which comes from the signature); if $\boldsymbol{\alpha}$ is a p-form ($p \leq 4$) then in spacetime $**\boldsymbol{\alpha} = (-1)^{p(4-p)+1}\boldsymbol{\alpha}$). The remaining half of Maxwell's equations result from the assumption that there exist a potential **A** (a 1-form) such that

$$*\boldsymbol{\psi} = d\mathbf{A}$$

from which then follows

$$d*\boldsymbol{\psi} = 0$$

The pair

$$d\boldsymbol{\psi} = \boldsymbol{\gamma}$$
$$d*\boldsymbol{\psi} = 0$$

are the Maxwell equations in their most compact form. Usually the textbooks derive this representation starting from the familiar 3-dimensional vector equations, whereby they loose sight of the fact that they are simply a consequence of the Minkowski spacetime being closed.

The electromagnetic theory is mainly concerned with the six components of the 2-forms $\boldsymbol{\psi}$ and $*\boldsymbol{\psi}$:

$$\boldsymbol{\psi} = D_x dydz + D_y dzdx + D_z dxdy + d(ct)(H_x dx + H_y dy + H_z dz)$$
$$*\boldsymbol{\psi} = *[D_x dydz + D_y dzdx + D_z dxdy] + *[d(ct)(H_x dx + H_y dy + H_z dz)]$$
$$= -d(ct)\varepsilon(E_x dx + E_y dy + E_z dz) + (B_x dydz + B_y dzdx + B_z dxdy)/\mu c$$

with $D_x = \varepsilon E_x$ and $\mu H_x = B_x/c$ etc. (c comes from using the engineering units: differentiation does not change units, so H has the dimension of As/m^2, instead of the conventional A/m. This is corrected by adding c.) Note that the vectors $\boldsymbol{E} = (E_x, E_y, E_z)$ etc. do not have a components in the (ct)-direction.

The local features of the vectors ***E, D, H*** and ***B*** can be calculated case by case from Maxwell's equations with given boundary conditions.

The electromagnetic theory gets special variety of the fact that there are two kinds of charges, $\rho = +q$ and $\rho = -q$. Charges of different signs attract each other, of the same sign expel. Otherwise e.g. shielding would not be possible. Another special feature is the electric current which flows in conductors.

In gravity there is only one kind of mass $\rho = m$, and two masses always attract each other. The mass can flow, in normal temperatures, only in free space or in pipes with a hole. The above formulation is valid and has been used e.g. by [Tajmar, M., de Matos, C.J.]. Vector $\boldsymbol{E}$ is denoted $\boldsymbol{g}$ and called the acceleration.

The theory of gravity is more often concerned with the motion of mass particles that move freely in spacetime in each others potential field. The distances can be very large, so the curvature of space becomes important. (From densities to particles use the Dirac delta-function.)

It should be possible to apply Maxwell's equations also in quantum mechanics.

In all cases the remote effects must be calculated from the potential field, which can be found using the wave equation.

The wave equation

The wave equation for the potential **A** (a 1-form) is easily derived from Maxwell's equations and $*\boldsymbol{\psi} = d\mathbf{A}$. Apply $(d{*}d{*} + {*}d{*}d)$ to **A** to get

$$(d{*}d{*} + {*}d{*}d)\mathbf{A} = -*\boldsymbol{\gamma}$$

(noting that $**\boldsymbol{\psi} = -\boldsymbol{\psi}$). The first term $d{*}d{*}\mathbf{A}$ is identically zero because $*\mathbf{A}$ is a 3-form in a closed space (principle A), but it is valuable because it generates terms that modify certain terms in $*d{*}d\mathbf{A}$. This would otherwise be pretty complicated.

The wave equations for $\boldsymbol{\psi}$ and $*\boldsymbol{\psi}$ are equally easily written.

When $(d{*}d{*} + {*}d{*}d)$ is worked out in spacetime it (naturally) turns out to be the d'Alembertian operator $\Box = \frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$.

Writing $\quad \mathbf{A} = = A_{ct}d(ct) + A_x dx + A_y dy + A_z dz$

it is seen that $\quad \Box\mathbf{A} = (\Box A_{ct})d(ct) + (\Box A_x)dx + (\Box A_y)dy + (\Box A_z)dz$

so that the wave equation shall be applied separately in turn to each component A_{ct}, A_x, A_y, A_z.

From $\Box = \frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$ it is clear that e.g. $[A_x(x-ct), A_y = 0, A_z = 0]$ is a solution in empty space ($\boldsymbol{\gamma} = 0$). It is recognized that c is the velocity with which the phenomenon propagates in the xyz-space. It is the local speed of light. It is the same for all phenomena (electromagnetism, gravity, quantum probability) in spacetime.

The wave equation is in daily use for the electromagnetic waves. It is also applied for gravity waves, see [Tajmar, M., de Matos, C.J.]. There is an important difference between the electromagnetic and the gravity waves: the wave impedance of empty space, E/H, is in the electromagnetic case about 377 ohms, whereas in the case of gravity waves it is practically zero (of the order 10^{-17}). This would be a major problem should anybody plan to use the gravity waves for telecommunication.

In quantum mechanics the wave equation leads to the Schrödinger equation.

Curved space

The curvature of the space can be taken into account simply in the following way (from [Flanders]). Let the 4-velocity vector be $\boldsymbol{U} = (c\mathbf{e}_o + v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z) = \mathbf{u}\boldsymbol{e}$ and differentiate for both $\mathbf{u}$ and $\boldsymbol{e}$ to get

$$d\boldsymbol{U} = (d\mathbf{u} - \mathbf{u}\boldsymbol{\Omega})\boldsymbol{e}$$

where $d\boldsymbol{e} = \boldsymbol{\Omega}\boldsymbol{e}$. The condition that there be no acceleration in curved spacetime can now be written as

$$d\mathbf{u} - \mathbf{u}\boldsymbol{\Omega} = 0$$

which allows the path of the particle to be calculated once $\boldsymbol{\Omega}$ is known. This equation is usually derived using variational principles.

The Einstein Field Equations

The Field Equations can be seen as a consequence of the Minkowski space being closed, when the space is allowed to be curved. Principles A and B lead to rules concerning the curvature coefficients of the space.

To take the curvature of the space into account write

$$d\boldsymbol{P} = \boldsymbol{\sigma}\boldsymbol{e}$$
$$d\boldsymbol{e} = \boldsymbol{\Omega}\boldsymbol{e}$$

where $\boldsymbol{\sigma} = (\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ is a row vector of 1-forms and $\boldsymbol{\Omega}$ is a 4x4 matrix of 1-forms. $\boldsymbol{e}$ is the column of unit vectors $\boldsymbol{e} = (e_0, e_1, e_2, e_3)^T$. $d\boldsymbol{P}$ is a small step away from the point P. $d\boldsymbol{e}$ gives the curvature induced change of the coordinate axes at the new point.

2 examples:

Example 1.
For the simplest case (no curvature) $\boldsymbol{\sigma}_0 = d(ct)$, $\boldsymbol{\sigma}_1 = dx$, $\boldsymbol{\sigma}_2 = dy$, $\boldsymbol{\sigma}_3 = dz$, $\boldsymbol{\Omega} = 0$, and

$$ds^2 = -\,d(ct)^2 + dx^2 + dy^2 + dz^2$$

Example 2.
For the Schwarzschild metric we have

$$\boldsymbol{\sigma}_0 = (1-\delta)d(ct)$$
$$\boldsymbol{\sigma}_1 = 1/(1-\delta)dr$$
$$\boldsymbol{\sigma}_2 = r d\vartheta$$
$$\boldsymbol{\sigma}_3 = r\sin\vartheta\, d\varphi$$

$$ds^2 = -(1-\delta)^2 d(ct)^2 + [1/(1-\delta)^2]dr^2 + r^2[d\vartheta^2 + \sin^2\vartheta\, d\varphi^2]$$

and
$$\boldsymbol{\Omega} = \begin{vmatrix} 0 & -\delta'\sigma_0 & 0 & 0 \\ \delta'\sigma_0 & 0 & \frac{1-\delta}{r}\sigma_2 & \frac{1-\delta}{r}\sigma_3 \\ 0 & -\frac{1-\delta}{r}\sigma_2 & 0 & \frac{\cot\vartheta}{r}\sigma_3 \\ 0 & -\frac{1-\delta}{r}\sigma_3 & -\frac{\cot\vartheta}{r}\sigma_3 & 0 \end{vmatrix}$$

where $\delta = GM/rc^2$; $\delta' = \partial\delta/\partial r = -GM/r^2c^2$
G = gravitational constant; M = mass

d***P*** and d***e*** are small steps, not differentials. They can be differentiated:

$$d^2\boldsymbol{P} = (d\boldsymbol{\sigma} - \boldsymbol{\sigma\Omega})\boldsymbol{e} = \boldsymbol{\tau}\,\boldsymbol{e}$$
$$d^2\boldsymbol{e} = (d\boldsymbol{\Omega} - \boldsymbol{\Omega}^2)\boldsymbol{e} = \boldsymbol{\theta}\,\boldsymbol{e}$$

$\boldsymbol{\tau}$ is the row of torsion coefficients and $\boldsymbol{\theta}$ is the matrix of the curvature coefficients. For a geodesic path $\boldsymbol{\tau} = 0$.

$*\boldsymbol{\sigma}$ is a 3-form. We can make more 3-forms by multiplying the 2-forms $\boldsymbol{\theta}$ or $*\boldsymbol{\theta}$ with $\boldsymbol{\sigma}$. $\boldsymbol{\sigma\theta}$ is identically zero, $\boldsymbol{\sigma}*\boldsymbol{\theta} \neq 0$. Consider the 3-form ($*\boldsymbol{\sigma}$ - $\kappa\boldsymbol{\sigma}*\boldsymbol{\theta}$). ($\kappa$ is a constant.). By principle A

$$d(*\boldsymbol{\sigma} - \kappa\boldsymbol{\sigma}*\boldsymbol{\theta}) = 0$$

which by principle B leads to

$$(*\boldsymbol{\sigma} - \kappa\boldsymbol{\sigma}*\boldsymbol{\theta}) = d\boldsymbol{\beta}$$

where $\boldsymbol{\beta}$ is a 2-form. If we postulate $d\boldsymbol{\beta} = 0$ then this equation gives Einstein's Field Equations

$$*\boldsymbol{\sigma} = \kappa\,\boldsymbol{\sigma}*\boldsymbol{\theta}$$

This innocent looking equation actually contains 4x4 differential equations and 20 unknown non-vanishing coefficients. Four coefficients (functions of time and location) remain to be solved from other equations case by case.

The meaning of the Field Equations is that $\boldsymbol{\sigma}$ can be substituted by $*\kappa(\boldsymbol{\sigma}*\boldsymbol{\theta})$, if the curvature coefficients are determined in such a way that the Field Equations are satisfied. (The 2-form $\boldsymbol{\beta}$ can be other than zero. $d\boldsymbol{\beta}$ makes then the famous case of the cosmology factor.)

It is quite an achievement to develop a new coordinate system that fulfills the equation $*\boldsymbol{\sigma} = \kappa\boldsymbol{\sigma}*\boldsymbol{\theta}$.

Momentum and Forces

When a particle with mass M and charge Q is moving with the speed $\mathbf{u} = (c, v_x, v_y, v_z)$ in an environment with gravity potential $\mathbf{A}_g$ and an electromagnetic potential $\mathbf{A}_e$, the rate of change of the momentum $\mathbf{p} = M\mathbf{u}$ of the particle will be proportional to the change of field potentials along the path (for the present purpose we assume that v << c, so that $\sqrt{(1-(v/c)^2)} \approx 1$, and that the curvature of space can be neglected):

$$\partial(M\mathbf{u})/\partial t \;\; = \mathbf{F}_g + \mathbf{F}_e = -\mu_g cM(\mathbf{u}|d\mathbf{A}_g) - \mu cQ(\mathbf{u}|d\mathbf{A}_e)$$

To get a picture of these terms, take the electromagnetic case:

$$\mathbf{F}_e = -\mu cQ(\mathbf{u}|d\mathbf{A}_e)$$

$$= -\mu cQ(c\mathbf{e}_o + v_x\mathbf{e}_x + v_y\mathbf{e}_y + v_z\mathbf{e}_z) \mid [-d(ct)\varepsilon(E_x dx + E_y dy + E_z dz) + (B_x dydz + B_y dzdx + B_z dxdy)/\mu c]$$

$$= -(Q/c)(v_xE_x + v_yE_y + v_zE_z)d(ct) +$$

$$+ Q(E_x dx + E_y dy + E_z dz) +$$

$$+ Q(v_zB_y - v_yB_z)dx + (v_xB_z - v_zB_x)dy + (v_yB_x - v_xB_y)dz$$

where use has been made of $\varepsilon\mu c^2 = 1$. (Note that $\mathbf{e}_o|d(ct) = 1$, $\mathbf{e}_o|dx = 0$, $\mathbf{e}_x|dx = 1$ etc.)

We get the corresponding vector components from $\mathbf{F}$ as the product $F_o = \mathbf{e}_o|\mathbf{F}$ etc.:

$$\boldsymbol{F} = -(Q/c)\boldsymbol{v}\cdot\boldsymbol{E}\,\mathbf{e}_o + Q(\boldsymbol{E} + \boldsymbol{v}x\boldsymbol{B})$$

Remember that $\boldsymbol{v}$, $\mathbf{E}$ and $\boldsymbol{B}$ are vectors in xyz-space, but $\boldsymbol{F}$ has all four components. The component in the direction of time causes the energy loss (heat) occurring in conductors: $(Q/c)\boldsymbol{v}\cdot\boldsymbol{E} = \boldsymbol{J}\cdot\boldsymbol{E}/c$.

Concluding remark

The above should be enough to show that there can be no doubt about the blessings of the differential forms in spacetime.

In fact, there is not very much to choose from: vector analysis cannot be used in 4-dimensions (there is no exterior product) and the tensors give no physical insight (they are just tables of coefficients).

So who is going to write the textbook?

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