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Mathematics

On the asymptotic eigenvalue distribution of a pseudo-differential operator

(uncertainty principle/canonical transformations/packings of unit cubes)

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ABSTRACT A description of the number $N(K)$ of eigenvalues less than K for a pseudo-differential operator with positive symbol is given in terms of the number of unit cubes canonically imbedded in the subset of phase space where the symbol is less than CK. This gives back in particular the order of magnitude of $N(K)$ for elliptic symbols.

This paper is devoted to a description of the growth of the number $N(K)$ of eigenvalues less than K of a pseudo-differential operator with positive symbol. Very precise information on $N(K)$ has previously been obtained by various authors under more restrictive conditions, notably by Hörmander (1) for elliptic operators and by Menikoff and Sjöstrand (2) for certain classes of subelliptic operators with loss of one derivative.

We first state our main results for symbols which have already been localized in phase space. Thus, let $a(x,\xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ be a positive function satisfying the inequalities

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta} M^{2-|\beta|} \quad \text{for} \quad (x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n \qquad [1]$$

$$a(x,\xi) \geq CM^2 \quad \text{when} \quad \max\{|x|,|\xi|/M\} \geq 1 \qquad [2]$$

let $a(x,D)$ be the corresponding pseudo-differential operator, and denote by $S(a,K)$ the set

$$S(a,K) = \{(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n; \; a(x,\xi) < K\}.$$

It was shown in ref. 3 that the first eigenvalue of $a(x,D)$ is bounded below (up to multiplicative constants) by the first K for which $S(a,K)$ contains the image of the unit cube in phase space by a canonical transformation with suitable bounds; this then raises the question of whether $N(K)$ can be compared to the number in $S(a,K)$ of disjoint images of the unit cube by canonical imbeddings. The following theorem provides an answer to this question.

THEOREM 1. *There exists an algorithm, "a canonical packing", associating to each* $K(C_\epsilon M^\epsilon \leq K \leq M^2)$ *a set* $Q^n(a,K)$ *of disjoint images of the unit cube by canonical transformations, which are all contained in* $S(a,C'_\epsilon K)$ *and have the following property:*

If we define

$L(K) =$ *number of elements in* $Q^n(a,K)$

$N(K) =$ *number of eigenvalues* $<K$ *of the quadratic form* $Re\langle a(x,D)u,u\rangle$

then

(A) $N(c_\epsilon K) \leq C''_\epsilon \dot{L}(K)$
(B) $N(C''_\epsilon K) \geq c_\epsilon \dot{L}(K)$.

Here $C_\epsilon, C'_\epsilon, C''_\epsilon, c_\epsilon$ *are constants depending only on* ϵ *and the dimension* n.

To state global results, consider a compact manifold X equipped with a positive smooth measure μ. If $a(x,\xi)$ is a positive classical second-order symbol defined on $T^*(X)$, then the quadratic form $\text{Re}\langle a(x,D)u,u\rangle_{L^2(d\mu)}$ is specified modulo errors $O(\|u\|^2)$. To ensure that $\text{Re}\langle a(x,D)u,u\rangle$ has discrete eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots$ tending to infinity we assume the subelliptic estimate

$$\text{Re}\langle a(x,D)u,u\rangle + C\|u\|^2 \geq c_\epsilon \|u\|^2_{(\epsilon)}$$

where $\|u\|^2_{(\epsilon)}$ denotes a Sobolev norm. Note that ref. 5 allows us to check whether the above estimate holds for a given positive $a(x,\xi)$ and that the order of magnitude of λ_k is independent of the choice of μ.

A slight variant of the "canonical packing" of *Theorem 1* produces a family $\mathbf{Q}(a,K)$ of pairwise disjoint canonical images of the unit cube, all contained in $\{(x,\xi) \in T^*(X); a(x,\xi) < K\}$. Set $L(K)$ equal to the number of cubes in $\mathbf{Q}(a,K)$, and set $N(K)$ equal to the number of eigenvalues $< K$. Then

THEOREM 2. *Under the above hypotheses there exist constants* K_0, C, C', c, c' *for which* $N(cK) \leq C\, L(K)$, $N(C'K) \geq c'\, L(K)$, *whenever* $K \geq K_0$.

It is not difficult to deduce *Theorem 2* from *Theorem 1*, so we confine our further discussion to the local result.

We now describe the algorithm mentioned above and sketch a proof of *Theorem 1*. The arguments involved rely heavily on the $S^{M,m}_{\Phi,\phi}$ symbolic calculus of Beals and Fefferman (3), the sharp Gårding inequality with gain of two derivatives of ref. 4, and the microlocalization procedures of ref. 5.

The construction of $\mathbf{Q}^n(a,K)$ is by induction on the dimension n. If $n = 0$, a is a real number, phase space consists of a single point z_0, and we set

$$Q^0(a,K) = \begin{cases} \phi & \text{if} \quad a \leq K \\ z_0 & \text{if} \quad a > K. \end{cases}$$

Assuming the algorithm for $\mathbf{Q}^{n-1}(a,K)$ is known, $\mathbf{Q}^n$ is obtained as follows.

Start with $B = \{(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n; \; |x|,|\xi|/M \leq 1\}$. Cut B dyadically, retaining those B_js for which one of the following occurs:

(*a*) $a \geq cM_j^2$ on B_j^{****}
(*b*) $M_j^2 \leq cK$ but a is false
(*c*) a and b are false, and

$$\max_{\substack{|\alpha|+|\beta|=2 \\ (x,\xi)\in B_j^{****}}} |D_x^\alpha D_\xi^\beta a(x,\xi)| \, \delta_j^{|\alpha|}(M\delta_j)^{|\beta|} \geq CM_j^2.$$

Here we have denoted by $\delta_j \times M\delta_j$ the sides of B_j, by B_j^* the dilate of B_j by a large constant, and written M_j for $M\delta_j^2$.

Let J_a, J_b, J_c be the sets of indices j corresponding to the three different types of boxes B_j above. Given a box B, it is convenient to introduce $S^m(B)$ as the space of all C^∞ functions $p(x,\xi)$ satisfying

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leq C_{\alpha\beta}(\text{diam}_x B)^{m-|\alpha|}(\text{diam}_\xi B)^{m-|\beta|}.$$

When B is the form $1 \times M$ we shall also write $S^m(M)$ for $S^m(B)$.

For each B_j we now define a collection $\mathbf{Q}_j$ of canonical images of the unit cube as follows:

For $j \in J_a$ take $\mathbf{Q}_j = \emptyset$.

For $j \in J_b$ take $\mathbf{Q}_j$ to be the set of all pieces obtained by cutting B_j into equal blocks of sizes $M_j^{-1/2} \times M_j^{1/2}$.

For $j \in J_c$ we first find a canonical transformation $\Phi_j:\hat{B}_j^{***} \to B_j^{***}$ with $\hat{B}^j$ having sizes $1 \times M_j$ and $a \circ \Phi_j = \xi_1^2 + p_j(x,\xi')$ on $\hat{B}_j^{**}$ ($\xi' = (\xi_2, \ldots, \xi_n)$). Choose symbols $\theta_j(x',\xi') \in S^0(M_j)$ with $0 \le \theta_j \le 1$, $\theta_j = 1$ if $|x'|,|\xi'|/M \le 100$, $\theta_j = 0$ if $|x'| + |\xi'|/M \ge 300$, and define

$$p_{j\ell}(x',\xi') = K^{1/2}\theta_j(x',\xi') \int_{|x_1-\ell K^{-1/2}| \le K^{-1/2}/2} p_j(x,\xi')dx_1 + M_j^2[1 - \theta_j(x',\xi')]$$

for each integer ℓ satisfying $|\ell| \le 2K^{1/2}$. We now let $\hat{\mathbf{Q}}_j$ be the collection of regions of the form

$$\{(x_1,\xi_1);\ \ |x_1 - \ell K^{-1/2}| < K^{-1/2}/2,\ \ |\xi_1| < 2K^{1/2}\} \times Q$$

where $|\ell| \le 2K^{1/2}$ and $Q \in \mathbf{Q}^{n-1}(p_{j\ell},K)$ and set

$$\mathbf{Q}_j = \{\Phi_j(Q);\ \ Q \in \hat{\mathbf{Q}}_j\}.$$

For each j, $\mathbf{Q}_j$ is then a set of pairwise disjoint canonical images of the unit cube contained in $S(a,CK) \cap B_j^*$. Next let $\#(B_j) =$ number of elements in $\mathbf{Q}_j$ and define a finite sequence $B_{j_1}, \ldots, B_{j_L}$ by picking B_{j_1} to maximize $\#(B_{j_1})$, and $B_{j_{s+1}}$ to satisfy $B_{j_{s+1}}^* \cap B_{j_r}^* = \emptyset$ for $r \le s$, with $\#(B_{j_{s+1}})$ as large as possible. The sequence $\{B_{j_s}\}$ eventually stops since the set of all B_js is finite. The $B_{j_s}^*$s are then pairwise disjoint and $\Sigma_s \#(B_{j_s}) \sim \Sigma \#(B_j)$; to see this associate to each B_j the first B_{j_s} with $B_{j_s}^* \cap B_j^* \neq \emptyset$. We then have $\#(B_{j_s}) \ge \#(B_j)$, while each B_{j_s} is paired to a bounded number of B_j since $\mathrm{Vol}(B_{j_s}) \sim \mathrm{Vol}(B_j)$ and $B_j \subset B_{j_s}^{***}$. Finally, we set

$$Q^n(a,K) = \bigcup_{s=1}^{L} \mathbf{Q}_{j_s}$$

which is then a collection of pairwise disjoint canonical images of the unit cube all contained in $S(a,CK)$. Observe that the number of elements in $\mathbf{Q}^n(a,K)$ is $\sim \Sigma_j \#(B_j)$. The construction is complete.

Proof of *Part A* of *Theorem 1*. To prove A, it will suffice to construct a subspace $\mathbf{H}$ of codimension $\le C_\epsilon L(K)$ in $L^2(\mathbf{R}^n)$ such that

$$\mathrm{Re}\langle a(x,D)u,u\rangle \ge c_\epsilon K\|u\|^2 \quad \text{for all} \quad u \in \mathbf{H}. \qquad [3]$$

Let $\tau \in S^0(M)$, $\sigma_j,\psi_j \in S^0(B_j)$ be symbols satisfying $\tau = 0$ in B^*, $\tau = 1$ outside of B^{**}, $1 = \tau + \Sigma\, \sigma_j^2$, supp $\sigma_j \subset B_j^*$, supp $\psi_j \subset \subset$ (supp $\sigma_j)^*$, $\psi_j = 1$ on supp σ_j.

For B_j satisfying b, denote by (x_j,ξ_j) the center of B_j and let $\{v_j^s\}$ be the collection of eigenfunctions with eigenvalues $\le AK$ of the Hermite operator

$$H_j(x,\xi) = (M\delta_j)^2|x - x_j|^2 + \delta_j^2|\xi - \xi_j|^2.$$

Here A is a large constant to be determined later.

For B_j satisfying c and ℓ integer, $|\ell| \le 2K^{1/2}$, let $\{w_{j\ell}^s\}$ be the collection of eigenfunctions with eigenvalues $\le CK$ of the quadratic form $\mathrm{Re}\langle p_{j\ell}(x',D')v,v\rangle$, and let U_j be Fourier integral operators such that

$$\mathrm{Re}\langle q(x,D)u,u\rangle = \mathrm{Re}\langle (q \circ \Phi_j)(y,D)U_ju,U_ju\rangle + 0(\|u\|^2)$$

for $q \in S^2(B_j)$ with support included in (supp $\sigma_j)^*$. The existence of U_j is guaranteed by the sharp Egorov principle, and the symbol of U_j may be assumed to be supported in

$$\{(x,\xi;\, y,\eta) \in \text{graph } \Phi_j;\ \ (x,\xi) \in (\text{supp } \sigma_j)^{**},\ \ |y| \le 2\}.$$

The space $\mathbf{H}$ is then the space of all $u \in L^2(\mathbf{R}^n)$ which satisfy the two sets of conditions:

$$(C_j^s)\ \langle \sigma_j(x,D)u,v_j^s\rangle = 0 \quad \text{for} \quad j \in J_b$$

$$(C_{j\ell}^s)\ \iint_{|y_1-\ell K^{-1/2}| \le K^{-1/2}/2} (U_j\sigma_j(x,D)u)(y)w_{j\ell}^s(y')dy'dy_1 = 0 \ (\text{for}\ \ j \in J_c\ \ \text{and}\ \ \ell \in \mathbf{Z},\ \ |\ell| \le 2K^{1/2}).$$

Since the number of $\{v_j^s\}$ in case b is $\sim \#(B_j)$ and the number of $\{w_{j\ell}^s\}$ in case c is $\le C\ \#\{\mathbf{Q}^{n-1}(p_{j\ell},CK)\}$ by the inductive hypothesis it follows that the codimension of $\mathbf{H}$ is $\le CL(K)$.

The proof of [3] will be easy once the following estimates have been established.

$$\mathrm{Re}\langle a(x,D)\sigma_j(x,D)u,\sigma_j(x,D)u\rangle \ge cK\|\sigma_j(x,D)u\|^2 - C_\epsilon M^{-N}\|u\|^2, \quad [4]$$

$$\text{for} \quad j \in J_a \cup J_c, \quad u \in \mathbf{H};$$

$$c\|u\|^2 \le \left\{\sum_{j\in J_a \cup J_c} \|\sigma_j(x,D)u\|^2 + \|\tau(x,D)u\|^2\right\},\ u\ \in\ \mathbf{H}. \ [5]$$

In fact [4], [5], the ellipticity of $a(x,\xi)$ outside of B, and the sharp Gårding inequality applied to the boxes B_js of case b together imply:

$$\mathrm{Re}\langle a(x,D)u,u\rangle \ge cK\left\{\sum_{j\in J_a\cup J_c} \|\sigma_j(x,D)u\|^2 + \|\tau(x,D)u\|^2\right\} - c\sum_{j\in J_b}\|\sigma_j(x,D)u\|^2 - C\|u\|^2 \ge c'K\|u\|^2,\ \ u \in \mathbf{H},$$

which is the desired inequality.

To prove [5] observe that if $j \in J_b$ and $u \in \mathbf{H}$ then

$$\sum_{j\in J_b} \mathrm{Re}\langle H_j(x,D)\sigma_j(x,D)u,\sigma_j(x,D)u\rangle \ge AK\sum_{j\in J_b}\|\sigma(x,D)u\|^2 \qquad [6]$$

in view of the conditions (C_j^s); since the left-hand side of [6] is $\mathrm{Re}\langle \Sigma_{j\in J_b}(H_j\sigma_j^2)(x,D)u,u\rangle + 0(\|u\|^2)$ and $(\Sigma_{j\in J_b}H_j\sigma_j^2)(x,\xi) \le CK$, this implies

$$\sum_{j\in J_b}\|\sigma_j(x,D)u\|^2 \le \|u\|^2(C/A),$$

which for A large yields [5] as a simple consequence. As for [4], it is trivial for $j \in J_a$ since $a(x,\xi)$ is then elliptic, and we need only consider the case $j \in J_c$. Thus, let

$$q_j(y,\eta') = \theta_j(y',\eta')p_j(y,\eta') + M_j^2[1 - \theta_j(y',\eta')]$$

$$v_j = U_j\sigma_j(x,D)u$$

$$v_{j\ell}(y') = K^{1/2}\int_{|y_1-\ell K^{-1/2}|<K^{-1/2}/2} v_j(y)dy_1,\quad |\ell| \le 2K^{1/2}$$

$$v_{jr}^\dagger(y) = v_j(y_1 + rK^{-1/2}/L,y'),\quad |r| \le L,\quad L \gg 1,$$

and $\gamma_{M_j}(\eta_1)$ be a C^∞ function equal to η_1^2 for $|\eta_1| \le M_j$, to M_j^2 for $|\eta_1| \ge 2M_j$ with good bounds; we then have

$$\mathrm{Re}\langle a(x,D)\sigma_j(x,D)u,\sigma_j(x,D)u\rangle \ge c\langle \gamma_{M_j}(D_1)v_j,v_j\rangle + c\,\mathrm{Re}\langle q_j(y,D')v_j,v_j\rangle - C\|\sigma_j(x,D)u\|^2$$

$$\ge c\sum_{\substack{|r|\le L\\ |l|\le 2K^{1/2}}} \int_{|y_1-\ell K^{-1/2}|<K^{-1/2}/2} \{K\|v_{jr}^\dagger(y_1,\cdot) - v_{j\ell}\|^2_{L^2(\mathbf{R}^{n-1})} + \mathrm{Re}\langle q_j(y_1 + rK^{-1/2}/L,y',D')v_{jr}^\dagger,y_{jr}^\dagger\rangle\}\,dy_1 - C\|\sigma_j(x,D)u\|^2. \quad [7]$$

The spectral decomposition theorem of [5] now shows that up to multiplicative constants the integrand for each j,ℓ is bounded below by

$$\inf_{w\in L^2(\mathbf{R}^{n-1})} \{K\|w - v_{j\ell}\|^2 + \mathrm{Re}\langle\{\sum_{|r|\le L} q_j(y_1 + rK^{-1/2}/L,y',D')\}w,w\rangle\} \quad [8]$$

which is in turn greater than

$$c \inf_{w\in L^2(\mathbf{R}^{n-1})} \{K\|w - v_{j\ell}\|^2 + \mathrm{Re}\langle p_{j\ell}(y',D')w,w\rangle\}, \quad c > 0 \quad [9]$$

since $q_j(y,\eta')$ is non-negative and $p_{j\ell}(y',\eta')$ is just the average of q over the interval $|y_1 - \ell K^{-1/2}| < K^{-1/2}/2$ for each (y',η'). However, conditions $(C^s_{j\ell})$ simply say that $\langle v_{j\ell},w^s_{j\ell}\rangle = 0$ and thus

$$\inf_{w\in L^2(\mathbf{R}^{n-1})} \{K\|w - v_{j\ell}\|^2 + \mathrm{Re}\langle p_{j\ell}(y',D')w,w\rangle\} \ge cK\|v_{j\ell}\|^2. \quad [10]$$

In view of (7), [8], [9], and [10], we can now write

$$\mathrm{Re}\langle a(x,D)\sigma_j(x,D)u,\sigma_j(x,D)u\rangle \ge cK \sum_{|\ell|\le 2K^{1/2}} K^{-1/2}\|v_{j\ell}\|^2 - C\|\sigma_j(x,D)u\|^2,$$

which, together with the estimate

$$\mathrm{Re}\langle a(x,D)\sigma_j(x,D)u,\sigma_j(x,D)u\rangle \ge cK \sum_{|\ell|\le 2K^{1/2}} \int_{|y_1-\ell K^{-1/2}|<K^{-1/2}/2} \|v_j(y_1,\cdot) - v_{j\ell}\|^2_{L^2(\mathbf{R}^{n-1})}\, dy_1 - C\|\sigma_j(x,D)u\|^2.$$

and the fact that σ_j is supported in $|y_1| \le 2$, yield

$$\mathrm{Re}\langle a(x,D)\sigma_j(x,D)u,\sigma_j(x,D)u\rangle \ge cK\|v_j\|^2 - C\|\sigma_j(x,D)u\|^2.$$

The desired estimate follows by applying the symbolic calculus and the theorem of Egorov. The proof of A is complete.

Proof of *Part B* of *Theorem 1*. Recall that $L(K) = \Sigma_{j_s\in J_b\cup J_c} \#(B_{j_s})$. For $j_s \in J_c$ let W_s be Fourier integral operators such that

$$\mathrm{Re}\langle q(y,D)v,v\rangle = \mathrm{Re}\langle (q \circ \Phi_{j_s}^{-1})(x,D)W_sv,W_sv\rangle + 0(\|v\|^2),$$

for $q \in S^2(M_{j_s})$ supported in a dilate of $|y| + |\eta|/M_{j_s} \le 1$. Let $\phi \in C_0^\infty(\mathbf{R})$ be a fixed function supported in $|t| \le 1/2$ which does not vanish identically and define

$$\tilde\phi_\ell(y_1) = \phi(-\ell + K^{1/2}y_1) \quad |\ell| \le 2K^{1/2}$$

$\mathbf{H}_s = \oplus$ eigenspaces with eigenvalues $\le K/A$ of $H_{j_s}(x,D)$ $(j_s \in J_b)$

$$\mathbf{H}_s = \left\{ \sum_{|\ell|\le 2K^{1/2}} \tilde\phi_\ell(y_1)\psi_\ell(y');\ \psi_\ell \in \oplus \text{ eigenspaces with eigenvalue} \le AK \text{ of } \mathrm{Re}\langle P_{j_s\ell}(y',D')v,v\rangle \right\} (j_s \in J_c)$$

$\phi_s \in S^0(M_{j_s})$, $\quad 0 \le \phi_s \le 1$,

$$\phi_s = 0 \text{ outside } |x| + |\xi|/M_{j_s} \ge 100$$
$$\phi_s = 1 \text{ in } |x| + |\xi|/M_{j_s} \le 50.$$
$$\hat\phi_s = \phi_s \circ \Phi_{j_s}{}^{-1}$$

The main properties of $\mathbf{H}_s$ are the following:

$$\dim \mathbf{H}_s \ge c\ \#(B_{j_s}), \quad j_s \in J_b \cup J_c, \quad [11]$$

$$\|\hat\phi_s(x,D)W_sv\|^2 \ge c\|v\|^2 \quad \text{if} \quad j_s \in J_c, \quad v \in \mathbf{H}_s \quad [12]$$

$$\mathrm{Re}\langle(\hat\phi_s^2a)(x,D)W_sv,W_sv\rangle \le CK\|v\|^2 \quad \text{if} \quad j_s \in J_c, \quad v \in \mathbf{H}_s \quad [13]$$

$$\|\hat\phi_s(x,D)v_s\|^2 \ge \frac{1}{2}\|v_s\|^2 \quad \text{if} \quad j_s \in J_b, \quad v \in \mathbf{H}_s. \quad [14]$$

[11] is a consequence of the inductive hypothesis [Part B of *Theorem 1* in $(n - 1)$ variables]. To derive [12] and [13] we observe that if $j_s \in J_c$, $v \in \mathbf{H}_s$ then $\|D_1v\|^2 \le CK\|v\|^2$ and

$$\mathrm{Re}\langle q_{j_s}(y,D')v,v\rangle \le CK\|v\|^2,$$

since $q_{j_s}(y,\eta') \le CP_{j_s\ell}(y',\eta')$ for y_1 satisfying $|y_1 - \ell K^{-1/2}| < K^{-1/2}/2$. Choose a function $\beta(y_1)$ with $0 \le \beta \le 1$, $\beta = 0$ if $|y_1| \le 3$, $\beta = 1$ if $|y_1| \ge 5$; then it is easily seen that

$$\mathrm{Re}\langle(D_1^2 + q_{j_s}(y,D') + M_{j_s}^2\beta(y_1))v,v\rangle \le CK\|v\|^2 \quad [15]$$

from which it follows that

$$BK\{\|v\|^2 - \|\hat\phi_s(x,D)W_sv\|^2\} + 0(\|v\|^2) \le CK\|v\|^2 \quad [16]$$

where B is a large constant which may be assumed to be $\le M_{j_s}^2/K$ (we would be in case b otherwise). In fact

$$BK(1 - \hat\phi_s^2 \circ \Phi_{j_s}) \le \eta_1^2 + M_{j_s}^2\beta(y_1) + M_{j_s}^2(1 - \theta_{j_s})$$

since the right side is elliptic on the support of the left side, and [16] follows from [15] by applications of the sharp Gårding inequality and the symbolic calculus. It is now easy to derive [12] from [16].

As for [13], note that

$$(\hat\phi_s^2 \circ \Phi_{j_s})(p \circ \Phi_{j_s}) \le \eta_1^2 + q_{j_s},$$

and thus in view of [15] and the sharp Gårding inequality

$$\mathrm{Re}\langle((\hat\phi_s^2 \circ \Phi_{j_s})(p \circ \Phi_{j_s})(y,D)v,v\rangle \le CK\|v\|^2.$$

The sharp Egorov principle yields [13].

Finally, if $j_s \in J_b$ and $v_s \in \mathbf{H}_s$ then $K(1 - \phi_s^2) \le CH_{j_s}(x,\xi)$ and

$$\mathrm{Re}\langle H_{j_s}(x,D)v,v\rangle \le K\|v\|^2/A$$

and arguing as before shows that

$$K\{\|v\|^2 - \|\phi_s(x,D)v\|^2\} + 0(\|v\|^2) \le K\|v\|^2/A,$$

which implies [14].

Now, define an operator $L\colon \oplus\, \mathbf{H}_s \to L^2(\mathbf{R}^n)$ by letting

$$Lw = \sum_{j_s\in J_b} \hat\phi_s(x,D)v_s + \sum_{j_s\in J_c} \hat\phi_s(x,D)W_sv_s^\dagger$$

for $w = [(v_s)_{j_s\in J_b}, (v_s^\dagger)_{j_s\in J_c}]$. We shall show that

$$\|Lw\|^2 \ge c\|w\|^2 \quad w \in \oplus\mathbf{H}_s, \quad [17]$$

$$\mathrm{Re}\langle a(x,D)Lw,Lw\rangle \le CK\|w\|^2. \quad [18]$$

This will complete the proof of B since [17] shows that Image (L) has dimension $\Sigma_{j_s\in J_a\cup J_b} \dim \mathbf{H}_s \ge cL(K)$, while [17] and [18] together yield

$$\mathrm{Re}\langle a(x,D)u,u\rangle \le CK\|u\|^2 \quad \text{for} \quad u \in \text{Image}\,(L).$$

Now symbolic calculus and the fact that the $\hat\phi_s(x,\xi)$ have pairwise disjoint supports imply

$$\|Lw\|^2 = \sum_{j_s\in J_b} \|\phi_s(x,D)v_s\|^2 + \sum_{j_s\in J_c} \|\phi_s(x,D)W_sv_s^\dagger\|^2 + 0(M^{-N}\|v_s\|^2 + M^{-N}\|v_s^\dagger\|^2)$$
$$\ge c\left(\sum_{j_s\in J_b} \|v_s\|^2 + \sum_{j_s\in J_c} \|v_s^\dagger\|^2\right) + M^{-N}\,0(\|w\|^2)$$

in view of [12] and [14]. Similarly

$$\mathrm{Re}\langle a(x,D)Lw,Lw\rangle \leq \sum_{j_s \epsilon J_b} \mathrm{Re}\langle(\phi_s^2 a)(x,D)v_s,v_s\rangle$$

$$+ \sum_{j_s \epsilon J_c} \mathrm{Re}\langle(\phi_s^2 a)(x,D)\, W_s v_s^\dagger, W_s V_s^\dagger\rangle + 0(M^{-N}\|w\|^2)$$

$$\leq CK \sum_{j_s \epsilon J_b} \|v_s\|^2 + CK \sum_{j_s \epsilon J_c} \|v_s^\dagger\|^2 + 0(M^{-N}\|w\|^2).$$

Here we have used [13] and the fact that $\phi_s^2 a \leq CK$ when $j_s \epsilon J_b$. [18] follows. q.e.d.

When $a(x,\xi)$ is an elliptic symbol of second order on a compact manifold X of dimension n, it follows easily from *Theorem 2* that $N(K) \sim \mathrm{Vol}(X)K^{n/2}$; this of course is the order of magnitude given by the more precise formula of ref. 1, where sharp estimates for the error terms are also derived. It would also be interesting to relate our results to those of ref. 2 for subelliptic operators with loss of one derivative; this may require more careful considerations from symplectic geometry.

Finally, we observe that $N(K)$ is here evaluated in terms of the number of unit cubes in a specific canonical packing of $S(a,K) = \{(x,\xi);a(x,\xi) < K\}$. We may ask whether other canonical packings would lead to the same result and, in particular, whether the given packing contains essentially the maximum number of cubes that can be disjointly imbedded in $S(a,K)$ by canonical transformations. Closely related to this question is the one of bounds for the kth eigenvalue λ_k; it would be of interest to determine when the upper and lower bounds for λ_k given by *Theorem 2* are comparable. We expect this usually to be the case.

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