A Simple One-dimensional Model for the Three-dimensional Vorticity Equation

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Abstract

A simple qualitative one-dimensional model for the 3-D vorticity equation of incompressible fluid flow is developed. This simple model is solved exactly; despite its simplicity, this equation retains several of the most important structural features in the vorticity equations and its solutions exhibit some of the phenomena observed in numerical computations for breakdown for the 3-D Euler equations.

1. Introduction

In regions far away from boundaries, the physical mechanism of vortex stretching is an important factor responsible for the complexity of incompressible fluid flow. In two space dimensions, where vortex stretching does not occur, the conservation of vorticity leads to the global existence of smooth solutions for the incompressible Euler equations. In three space dimensions, where vortex stretching is a prominent effect, it is an outstanding unsolved problem of mathematical fluid dynamics to determine whether solutions of the Euler equations develop singularities in finite time. This problem is important from the physical point of view because the existence of such singularities signifies the onset of turbulence in high Reynolds number flows and the structure of this conceivable singularity has direct bearing on the inertial cascade in such turbulent flows (see [7]). The possible breakdown of solutions and the structure of singularitites has been studied recently through a wide range of ingenious numerical methods by many authors (see [2], [3], [4], [8], [10]). Several recent theorems support the link between vortex stretching, breakdown for the Euler equations, and the onset of

Communications on Pure and Applied Mathematics, Vol. XXXVIII 715–724 (1985)

CCC 0010-3640/85/060715-10$04.00

turbulence. In [1], the authors proved that the only way in which smooth solutions of the Euler equations can become singular is that the vorticity become infinite in finite time in a precise fashion; i.e., vortex stretching is the controlling mechanism for breakdown. Another recent theorem (see [5]) establishes that on any closed interval of time, where solutions of the Euler equations remain smooth, the Navier-Stokes equations have a unique smooth regular solution for sufficiently high Reynolds numbers.

In this paper, a simple qualitative one-dimensional mathematical model for the 3-D vorticity equation is developed. Despite its simplicity, this equation retains several of the most important structural features in the vorticity equation, and its solutions exhibit some of the phenomena observed in numerical computations for breakdown of the 3-D Euler equations; a detailed discussion is given at the end of this paper. One great advantage of the simple model which we present is that it can be integrated explicitly. Since the numerical computation of solutions forming singularities involves many subtle issues, this simplified model provides a class of elementary unambiguous test problems for the numerical methods used in studying the breakdown for the 3-D Euler equations. This work is in progress and will be described elsewhere.

2. Heuristic Derivation of the Model Vorticity Equation

The Euler equations for the velocity $v = {}^t(v_1, v_2, v_3)$ and scalar pressure p are given by

(1)
$$\begin{aligned} \frac{Dv}{Dt} &= -\nabla p, \qquad x \in \mathbb{R}^3,\ t > 0, \\ \operatorname{div} v &= 0, \\ v(x,0) &= v_0(x), \end{aligned}$$

where D/Dt is the convective derivative, $D/Dt = \partial/\partial t + \Sigma_{i=1}^3 v_j \partial/\partial x_j$. With $\omega = \nabla \times v$, the vorticity, the Euler equations can be written in the equivalent form

(2.A)
$$\begin{aligned} \frac{D\omega}{Dt} &= \omega \cdot \nabla v, \\ \omega(x,0) &= \omega_0(x) = \nabla \times v_0, \end{aligned}$$

where the velocity v is determined by the vorticity ω from the equations

$$\operatorname{div} v = 0, \qquad \operatorname{curl} v = \omega,$$

resulting in the familiar Biot-Savart formula,

(2.B)
$$v(x,t) = -\frac{1}{4\pi} \int \frac{(x-y)}{|x-y|^3} \times \omega(y,t)\, dy.$$

The vector $\omega = \text{curl } v$ belongs to the nullspace of the antisymmetric part of the matrix ∇v. Therefore, the term $\omega \cdot \nabla v$ can be replaced by $D\omega$, where the deformation matrix D is the symmetric part of ∇v:

$$D = \tfrac{1}{2}(\nabla v + {}^T\nabla v) = (D_{ij}).$$

Formula (2.B) can be differentiated to express D as a strongly singular integral operator acting on ω:

$$D(\omega) = (D_{ij}(\omega)) = \sum_{l=1}^{3} \text{P.V.}\int \mathscr{D}_{ij}^{l}(x-y)\,\omega_l(y)\,dy; \tag{2.C}$$

here $\mathscr{D}_{ij}^{l}(\lambda\vec{x}) = \lambda^{-3}\mathscr{D}_{ij}^{l}(\vec{x})$ for $\vec{x} \neq 0$ and the mean of $\mathscr{D}_{ij}^{l}$ over the unit sphere vanishes. The explicit formulae for the kernels $\mathscr{D}_{ij}^{l}$ are not needed in the developments below. Through the formulae in (2.B) and (2.C) we obtain an integro-differential equation for the vorticity above which is equivalent to the Euler equations in (1),

$$\frac{D\omega}{Dt} = D(\omega)\omega, \qquad x \in \mathbb{R}^3,\ t > 0, \tag{3}$$
$$\omega(x,0) = \omega_0(x).$$

For completeness, we remark here that with v defined by (2.B) every smooth solution of (2.A) automatically satisfies $(D/Dt)\,\text{div}\,\omega = 0$. Since $\text{div}\,\omega_0 = 0$, the fact that $\text{div}\,\omega = 0$ automatically guarantees that the velocity v from (2.B) satisfies both $\text{div}\,v = 0$ and $\nabla \times v = \omega$.

In two space dimensions, $D(\omega)\omega \equiv 0$, and vorticity is conserved, i.e., $D\omega/Dt = 0$. In three dimensions, the matrix $D(\omega)$ is a symmetric matrix with $\text{tr}\,D = 0$ and vortex-stretching occurs when ω roughly aligns with an eigenvector of $D(\omega)$ corresponding to a positive eigenvalue. Thus, essential differences in fluid behavior in two and three space dimensions are manifested through the appearance of the term $D(\omega)\omega$ on the right-hand side of (3).

The reformulation of the Euler equations in (3) and the above comments motivate the qualitative model which we present next. The matrix-valued function D depends linearly on the function ω; the operator relating ω to $D\omega$ is a linear singular integral operator that commutes with translation, i.e., it is given by the convolution of ω with a kernel homogeneous of degree -3 and with mean value on the unit sphere equal to zero. In the one space dimension, there is only one such operator, the Hilbert transform,

$$H(\omega) = \frac{1}{\pi}\text{P.V.}\int \frac{\omega(y)}{(x-y)}\,dy. \tag{4}$$

The quadratic term $H(\omega)\omega$ is a scalar one-dimensional analogue of the vortex stretching term $D(\omega)\omega$. We replace the convective derivative D/Dt by $\partial/\partial t$ in order to have a one-dimensional incompressible flow and arrive at the *model vorticity equation*,

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega, \qquad \omega(x,0) = \omega_0(x). \tag{5}$$

For the Euler equations, the velocity is determined from the vorticity by convolution with a mildly singular kernel, homogeneous of degree $1 - N$, and the analogue of the velocity for the model is defined within a constant by such a convolution, i.e.,

$$v = \int_{-\infty}^{x} \omega(y,t)\,dy. \tag{6}$$

Since the Hilbert transform is a skew-symmetric operator,

$$\int_{-\infty}^{\infty} H(\omega)\omega\,dy = (H\omega, \omega) = 0.$$

Integrating (5) with respect to y on $\mathbb{R}$ shows then that all smooth solutions of (5) that decay sufficiently rapidly as $|y| \to \infty$ satisfy for all t

$$\frac{d}{dt}\int_{-\infty}^{\infty} \omega(y,t)\,dy = 0. \tag{7}$$

Thus, if $\omega_0(x)$ is the derivative of a function vanishing for $|x| \to \infty$, smooth solutions of (5) also retain this property for $t > 0$. Many studies for the Euler equations concentrate on periodic fluid flow; there is an obvious analogue of the Hilbert transform in (4) on the circle and the periodic model vorticity equation can be defined as in (5). In this case, the mean of ω per period is conserved and v is defined unambiguously by

$$v = \int_{x_0}^{x} \omega(y,t)\,dy$$

provided that the initial data $\omega_0(x)$ satisfies

$$\int_{x_0}^{x_0+p} \omega_0(y)\,dy = 0$$

with p the period.

3. Integration of the Model Vorticity Equation and Explicit Breakdown of Solutions

The nonlinear equation in (5) is well posed in many standard function spaces, for example, $H^1(\mathbb{R})$, the Sobolev space of functions which are square integrable with square integrable first derivative. The local existence and uniqueness follows from the fact that $H^1(\mathbb{R})$ is a Banach algebra of continuous functions and the Hilbert transform maps $H^1(\mathbb{R})$ continuously into itself so that standard existence and uniqueness results for Lipschitz nonlinear ordinary differential equations in Banach space apply. We have the following explicit solution formula for the model equation in (5):

THEOREM. *Suppose $\omega_0(x)$ is a smooth function decaying sufficiently rapidly as $|x| \to \infty$ ($\omega_0 \in H^1(\mathbb{R})$ suffices). Then the solution to the model vorticity equation in* (5) *is given explicitly by*

$$\omega(x,t) = \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \tag{8}$$

Remark 1. The equation in (5) can also be defined on any smooth closed curve Γ, using the Hilbert transform H for that curve Γ. The explicit formulae and method of proof given below are exactly the same with this modification—in particular, when Γ is a circle. However, the constraint $\int_\Gamma \omega\, dz = 0$ is no longer automatically preserved by solutions of (5) except when Γ is the circle or the real line.

Remark 2. The proof also provides an explicit expression for $H(\omega)$,

$$(H\omega)(x,t) = \frac{2H\omega_0(x)(2 - tH\omega_0(x)) - 2t\omega_0^2(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \tag{9}$$

The formula in (8) immediately yields the following:

COROLLARY 1. (Breakdown of smooth solutions for the model vorticity equation). *The smooth solution to the differential equation in* (5) *blows up in finite time if and only if the set Z defined by*

$$Z = \{x | \omega_0(x) = 0 \quad \text{and} \quad H\omega_0(x) > 0\} \tag{10}$$

is not empty. In this case, $\omega(x,t)$ becomes infinite as $t \uparrow T$, where the blow-up time is given explicitly by $T = 2/M$ with $M = \sup\{H\omega_0)_+(x) | \omega_0(x) = 0\}$.

Proof of Theorem 1: To display the generality of the proof, we shall use the following identities for the Hilbert transform on the line which are also valid for any closed curve Γ (see [9]):

$$H(Hf) = +f, \tag{11.A}$$

$$H(fg) = fHg + gHf + H(Hf \cdot Hg). \tag{11.B}$$

From these identities it follows that

$$H(fHf) = \tfrac{1}{2}\left((Hf)^2 - f^2\right). \tag{12}$$

By applying H to the model vorticity equation and using (12), we obtain an equation satisfied by $(H\omega)(x,t)$:

$$\frac{\partial}{\partial t} H\omega = \tfrac{1}{2}\left((H\omega)^2 - \omega^2\right). \tag{13}$$

We introduce the quantity

$$z(x,t) = H\omega(x,t) + i\omega(x,t) \tag{14}$$

and by combining (5) and (13), we see that $z(x,t)$ satisfies the local equation

$$\frac{\partial z}{\partial t}(x,t) = \tfrac{1}{2}z^2(x,t) \tag{15}$$

with the explicit solution

$$z(x,t) = \frac{z_0(x)}{1 - \frac{1}{2}tz_0(x)}. \tag{16}$$

Formulas (8) and (9) are the real and imaginary parts of (16).

Formula (16) defines $z(x,t)$ as an analytic function in $\mathcal{I}m\, x < 0$. It is well known that the Hilbert transform on the line can be interpreted in terms of complex-valued functions z on the real axis which are boundary values of analytic functions in the lower half-plane that tend to zero sufficiently fast at infinity. The Hilbert transform relates the imaginary part of such a function to its real part on the real axis. Thus a function of the form

$$z = H\omega + i\omega$$

is always the boundary value of a function analytic in the lower half-plane. The identity in (11.A) is then merely the observation that if z is analytic in the lower half-plane, so is iz. The identity (11.B) is the observation that if z and w are analytic in the lower half-plane, so is their product $z \cdot w$. Equation (5) is then the imaginary part of (15); by analyticity (15) holds.

Next, we give an instructive explicit example in the 2π-periodic case.

EXAMPLE. We choose $\omega_0(x) = \cos(x)$ so that $H(\omega_0(x)) = \sin(x)$ and compute that

$$\omega(x, 2t) = \frac{\cos(x)}{1 + t^2 - 2t\sin(x)} \tag{17}$$

and

$$v(x, 2t) = \int_0^x \omega(x', 2t)\, dx' = (2t)^{-1}\log(1 + t^2 - 2t\sin(x)).$$

In this specific example, the breakdown time is $T = 2$ and, as $t \nearrow T$, $\omega(x,t)$ develops a non-integrable local singularity like $1/x$ near $x = 0$. There are two interesting facets to this breakdown process. First,

$$\int_{-\pi}^{\pi} |\omega(x,t)|^p\, dx \nearrow \infty \quad \text{as} \quad t \nearrow T \tag{18.A}$$

for any fixed p with $1 \leqq p < +\infty$. Also, there are finite constants M_p such that

$$\int_{-\pi}^{\pi} |v(x,t)|^p\, dx \leqq M_p \quad \text{as} \quad t \nearrow T \tag{18.B}$$

for any p with $1 \leqq p < \infty$. In particular, the kinetic energy of v remains uniformly bounded as t approaches the breakdown time T. The behavior in the above example is typical for solutions of the model vorticity equation as the following corollary of the theorem indicates:

COROLLARY 2. *Given the initial data $\omega_0(x)$ for the model vorticity equation, assume that the points x_0 with $\omega_0(x_0) = 0$ and defining the breakdown time T are simple zeroes of $\omega_0(x)$. Then as $t \nearrow T$ with T the breakdown time, $\omega(x, t)$ and $v(x, t)$ have the same properties as given in* (18.A) *and* (18.B).

We omit the proof since it is a straightforward but tedious calculation using the explicit solution formulae. It is easy to show that if the initial data ω_0 satisfies the assumption of Corollary 2, then, for $T < t < T + \tau$ with T the breakdown time and τ small enough, the analytic function

$$1 - \tfrac{1}{2}tz_0(x)$$

has a zero in the lower half-plane $\mathcal{I}m\, x < 0$. Thus for such t the function $z(x,t)$ defined by (16) has a pole in the lower half-plane, and therefore its imaginary part given by the formula (8) is *not* related to its real part by the Hilbert transform. In particular, for such t, formula (9) does *not* hold and therefore $\omega(x,t)$ as given in (8) does *not* continue the solution of (5) for $T < t < T + \tau$.

4. Qualitative Comparison of Solutions for the Model Vorticity Equation and the 3-D Euler Equations

First, we recall that $H(\omega)$ in the model vorticity equation has the analogous role as the deformation matrix in the 3-D Euler equations. With this identification, the qualitative fact that blow-up for solutions of the model vorticity equation occurs only at points where $H\omega$ has positive sign is reminiscent of the fact that vorticity for solutions of the 3-D Euler equations increases when it roughly aligns with eigenvectors of the deformation matrix with positive eigenvalues. Below, in discussing properties regarding the conjectured breakdown of solutions for the 3-D Euler equations, we refer to information which can be extracted from various numerical experiments (see [2], [3], [4]).

Smooth solutions of the 3-D Euler equations have the following well-known elementary properties:

SCALE INVARIANCE: If $v(x,t)$ satisfies (1), then for constants λ, α,

$$v_{\lambda,\alpha}(x,t) = \lambda v(\lambda^{\alpha}x, \lambda^{1+\alpha}t) \quad \text{also satisfies} \quad (1).$$

CONSERVATION OF ENERGY: For solutions of (1),

$$\int |v(x,t)|^2\, dx = \int |v_0(x)|^2\, dx \quad \text{for all} \quad t.$$

The reader can easily verify that the function $v(x,t)$ defined in (6) through solutions of the model vorticity equation has the same scale invariant properties for $x \in \mathbb{R}^1$ as for solutions of the 3-D Euler equations with $x \in \mathbb{R}^3$. We mention this explicitly here because some of the numerical methods for studying the blow-up of solutions for the 3-D Euler equations exploit this scale invariance in the numerical algorithm (see [3]). The functions $v(x,t)$ associated with solutions of the model vorticity equation do not satisfy conservation of L^2 norm. However, the computations reported in [3], [4] suggest that for the 3-D Euler equations, as t approaches the conjectured breakdown time T, the vorticity satisfies

$$\int |\omega|^p(x,t)\, dx \nearrow \infty \quad \text{as} \quad t \nearrow T \quad \text{for any} \quad 1 \leqq p < +\infty,$$

while the kinetic energy of $v(x,t)$ remains constant as $t \nearrow T$. Corollary 2 establishes that typical solutions of the model vorticity equation behave in an analogous fashion as the breakdown time is approached. Chorin also reports in

[3], [4], the results of two different numerical procedures which indicate that

$$\text{(19)}\quad \begin{array}{l}\text{the set of breakdown points is a set of Lebesgue measure zero in } R^3\\ \text{with Hausdorff dimension } \sim 2.5.\end{array}$$

Such a relation was first suggested by Benoit Mandelbrot. For typical solutions of the model vorticity equation, according to Corollary 1, the set of conceivable breakdown points is also a set of Lebesgue measure zero contained in the zero set of $\omega_0(x)$ and typically consists of a finite number of points. In the inviscid calculations for the 3-D Euler equations in [2], it is reported that the deformation matrix $D(\omega)$ becomes large on open sets where ω vanishes. In the model vorticity equation, it is easy to construct explicit examples in which $H(\omega)$ becomes arbitrarily large on an open set where ω vanishes—in fact, an earlier non-constructive breakdown argument for special initial data by Constantin [6] directly exploits this property.

We end this section by remarking that the properties of solutions of the model vorticity equation described in Corollary 2 and also referred to below (16) are *never* satisfied for solutions which blow up for the local scalar quadratic equation,

$$\frac{\partial \omega}{\partial t} = \omega^2,$$

$$\omega(x,0) = \omega_0(x),$$

as the reader can easily verify. Of course, this quadratic equation arises from the characteristic form of the equation for $\omega = u_{\bar{x}}$, where u satisfies the local quadratic equation, $u_t - uu_{\bar{x}} = 0$.

Acknowledgment. The authors thank Sergiu Klainerman for his perceptive comments and active interest during the course of this work.

The work of the first author was performed while he was a visiting member at the Courant Institute. The work of the second author was partially supported by Dept. of Energy under contract DE-AC02-76ER03077, that of the third was partially supported by National Science Foundation Grant DMS84-0223.

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Received May, 1985.