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On Finite Element Methods for Elliptic Equations on Domains with Corners*

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Abstract — Zusammenfassung

On Finite Element Methods for Elliptic Equations on Domains with Corners. A finite element method for approximating elliptic equations on domains with corners is proposed. The method makes use of the singular functions of the problem in the trial space and the kernel functions of the adjoint problem in the test space. This leads to good approximates of the coefficients of the singular functions. In the numerical computations, the method is compared with the well known Singular Function Method.

Key words: Polygonal domains, singular expansion.

AMS Subject Classification: 65N30.

Finite-Elemente-Verfahren zur numerischen Lösung elliptischer Differentialgleichungen auf Gebieten mit Ecken. Es wird eine Finite Elemente Methode zur Approximation elliptischer Differentialgleichungen auf Eckengebieten vorgeschlagen. Das Verfahren benutzt die Singulärfunktionen des Problems im Raum der Ansatzfunktionen und die Kernfunktionen des adjungierten Operators im Testraum. Dadurch erhält man gute Näherungen der Koeffizienten der Singulärfunktionen. In einem numerischen Beispiel wird das Verfahren mit der bekannten Methode der Singulärfunktionen verglichen.

1. Introduction

As a model problem we consider the Laplace equation

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \text{on} \quad \partial \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain with a corner σ with interior angle ω. For simplicity it is assumed that $\partial \Omega$ is smooth outside σ and coincides with a cone near the corner.

Denoting by (r, ϕ) the polar coordinates with respect to σ, the solution u of (1.1) admits the expansion

$$u = \sum_{i=1}^{n} k_i s_i + w, \tag{1.2}$$

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where the singular functions s_i are given by

$$s_i = \tau(r)\, r^{i\pi/\omega} \sin i\pi\phi/\omega$$

with a sufficiently smooth cut-off function τ with the properties

$$\tau(r) = \begin{cases} 1 & \text{for } 0 \le r \le a \\ 0 & \text{for } b \le r \end{cases}, \quad 0 < a < b.$$

The coefficients k_i in (1.2) are called the stress intensity factors and are of main interest in engineering applications. It will be shown in the next section that they depend on the right hand side f in a nontrivial way. The function w in (1.2) is smooth.

By the bad regularity of the first singular function in (1.2), the convergence rate of the usual Finite Element Method is considerably reduced in many cases, but several methods are known for constructing higher order approximations of the solution. Here we only mention the Singular Function Method (SFM) which consists in augmenting the spline space by the singular functions s_i, see Fix-Gulati-Wakoff [4] and Strang-Fix [10]. It is well known that the approximate stress intensity factors of the SFM do not converge in many cases (see Destuynder-Djaoua [2] and Schatz [9]). This leads to bad convergence of the approximate solution near the corner.

The outline of the paper is as follows: In Section 2 the basic results in the field of elliptic equations on domains with corners are reviewed and a representation for the stress intensity factors is given. The latter is initiated by an idea of Maz'ja-Plamenevskij [7]. In Section 3 the usual Finite Element Method on domains with corners is analyzed. It is shown that the method is not quasioptimal in the L^2-norms. Furthermore, some results on L^∞-convergence are reviewed. In Section 4 an iterative procedure for improving the FEM is analyzed. Using this approach, a new FEM for domains with corners is proposed in Section 5. Since this method is characterized by the use of the dual singular functions in the test space we call it the Dual Singular Function Method (DSFM). Some numerical computations for the SFM and the DSFM are reported in Section 6.

2. Singular Representation

The aim of this section is the study of the regularity problem in weighted Sobolev spaces. These are defined for $\alpha \in \mathbb{R}$ by

$$L^2_\alpha := \{u \in L^2_{\mathrm{loc}}(\Omega) : r^{\alpha/2}\, u \in L^2(\Omega)\},$$

$$H^m_\alpha := \{u \in H^{m,2}_{\mathrm{loc}}(\Omega) : \nabla K u \in L^2_{\alpha - 2(m-k)},\ k = 0, \ldots, m\},\ m \in \mathbb{N}.$$

Here and below, (r, ϕ) denote the polar coordinates with respect to σ. The weighted Sobolev spaces are provided with the natural weighted norms

$$\| v \|^2_{m;\alpha} = \sum_{k=0}^{m} \| r^{\alpha/2 - m + k} \nabla^k v \|^2_{L^2(\Omega)}.$$

The norm of the usual Sobolev spaces $H^{m,p}(\Omega)$ are denoted by

$$\| v \|_{m,p} = \left(\sum_{k=0}^{m} \| \nabla^k v \|^p_{L^p(\Omega)} \right)^{1/p}.$$

We define the singular functions of problem (1.1),

$$s_i=\tau(r)\, r^{i\pi/\omega} \sin i\pi\phi/\omega, \quad i\in\mathbb{N},$$

and the singular functions of the adjoint operator,

$$s_{-i}=\tau(r)^{-i\pi/\omega} \sin i\pi\phi/\omega, \quad i\in\mathbb{N}.$$

Note that $\Delta s_i=\Delta s_{-i}=0$ in a neighbourhood of σ.

The results of Kondrat'ev [6], Grisvard [5], and Maz'ja-Plamenevskij [7] can be stated as follows:

Theorem 2.1: *Assume* $1+m-\alpha/2 \neq n\pi/\omega$ *for all* $n\in\mathbb{Z}$ *and* $f\in H^m_\alpha$, $1+m-\alpha/2>0$.

(*i*) *The weak solution* $u\in \mathring{H}^{1,2}(\Omega)$ *of* (1.1) *admits the singular expansion*

$$u=\sum_{i\in I\cap\mathbb{N}} k_i s_i+w, \tag{2.1}$$

with $I=\left(0, \frac{\omega}{\pi}(1+m-\alpha/2)\right)$ *and* $w\in H^{m+2}_\alpha$. *The remainder part* w *satisfies the estimate*

$$\|w\|_{m+2;\alpha}\le c\,\|f\|_{m;\alpha}.$$

(*ii*) *The constants* k_i *in* (2.1) *are determined by*

$$k_i=\frac{1}{i\pi}\,\{(f,s_{-i})+(u,\Delta s_{-i})\}. \tag{2.2}$$

Remark: The representation of the stress intensity factors in (2.2) is not meaningless although the right hand side depends on u. Since $\Delta s_{-i}=0$ in a neighbourhood of σ the second term in (2.2) can be regarded as a compact perturbation with respect to the first term.

Proof: We will only prove the second statement. By assumption, there exists a radius R such that

$$C_1=\Omega\cap B(\sigma,R)$$

is a bounded cone. Here $B(\sigma,R)$ denotes the ball with center σ and radius R. Choosing R such that $\Delta s_i=\Delta s_{-i}=0$ in C_1 and using the notation $C_2=\Omega\backslash C_1$ we have

$$\int_\Omega f s_{-i}\,dx=\int_{C_1} -\Delta w\, s_{-i}\,dx+\int_{C_2} -\Delta u\, s_{-i}\,dx.$$

By the results of Theorem 2.1 (i), all integrals in the above identity exist. From integration by parts we obtain

$$\int_\Omega \{f s_{-i}+u\Delta s_{-i}\}\,dx=\int_0^\omega \{\partial_r(u-w)\,s_{-i}-(u-w)\,\partial_r s_{-i}\}\,r\,d\phi_{|r=R}.$$

Using the representation for $u-w$ *in* (i) we conclude

$$\int_\Omega \{f s_{-i}+u\Delta s_{-i}\}\,dx=k_i\int_0^\omega \{\partial_r s_i\, s_{-i}-s_i\,\partial_r s_{-i}\}\,r\,d\phi_{|r=R}$$
$$=i\pi k_i.$$

Part (ii) of the theorem is proved. □

3. The Finite Element Method (FEM)

We want to approximate problem (1.1) by the usual finite element displacement method using piecewise polynomial shape functions of order $\leq m-1$. For a discretization parameter $h>0$ let Π_h be a triangulation of Ω into triangles Λ_h which satisfy the usual regularity condition:

$$\text{Every two triangles of } \Pi_h \text{ only meet in whole common sides or in vertices. Each } \Lambda_h \in \Pi_h \text{ contains a circle with radius } c\,h \text{ and is contained in a circle with radius } h. \tag{3.1}$$

Here and in the following, c denotes a positive generic constant which may not have the same value at different places but is independent of the parameter h.

The finite element spaces S^m on Π_h are defined by:

$$S^m = \{v_h \in \mathring{H}^{1,2}(\Omega)\text{: The restriction } v_{h|\Lambda^h} \text{ is a polynomial of degree } \leq m-1 \text{ on each } \Lambda_h \in \Pi_h.\}$$

In the above, $\mathring{H}^{1,2}(\Omega)$ denotes the space of functions $v \in H^{1,2}(\Omega)$ which vanish on $\partial\Omega$ in the generalized sense.

The spaces S^m satisfy the approximation properties:

$$\begin{gathered}\text{To each } v \in H^{k,p}(\Omega) \cap \mathring{H}^{1,2}(\Omega) \text{ there is a } I_h v \in S^m \text{ such that}\\ \| v - I_h v \|_{l,p;\Lambda^h} \leq c\, h^{k-l} \, \| \nabla^k v \|_{0,p;\Lambda^h},\\ 0 \leq l \leq k \leq m;\ 1 \leq p \leq \infty;\ \Lambda_h \in \Pi_h.\end{gathered} \tag{3.2}$$

The approximate of the FEM for equation (1.1) is given by the problem:

$$\begin{gathered}\text{Find } P\,u \in S^m \text{ such that}\\ a(P\,u, v_h) = (f, v_h) \qquad \forall\, v_h \in S^m.\end{gathered} \tag{3.3}$$

At first we will study the FEM in the case of "realistic" solution, i.e. solutions for which $k_1 \neq 0$ in (2.1). Obviously, it is sufficient to analyze the error $s_1 - P\,s_1$:

Theorem 3.1: *Assume that $\pi/\omega < 1$. Then the error estimates hold*

$$c\, h^{(2-l)\pi/\omega} \leq \| \nabla^l (s_1 - P\,s_1) \|_{0,2} \leq c\, h^{(2-l)\pi/\omega - \varepsilon},$$

for all $\varepsilon > 0$, $l = 0, 1$.

Proof: Since $s_1 \notin H^{1+\pi/\omega, 2}(\Omega)$ we have for every $v_h \in S^m$

$$c\, h^{\pi/\omega} \leq \| \nabla (s_1 - v_h) \|_{0,2}. \tag{3.4}$$

By using the projection property of the FEM the theorem is proved for $l=1$.

In order to estimate $\| s_1 - P\,s_1 \|_{0,2}$ we derive a simpler representation of the first stress intensity factor. Let $s \in \mathring{H}^{1,2}(\Omega)$ be the weak solution of the problem

$$a(s, v) = -(\Delta s_{-1}, v) \qquad \forall\, v \in \mathring{H}^{1,2}(\Omega)$$

and define

$$s'_{-1} = s_{-1} - s.$$

From Theorem 2.1 it follows that

$$k_1 = \frac{1}{\pi} \int_\Omega f s'_{-1}\, dx. \tag{3.5}$$

Since $s'_{-1} \in L^2(\Omega)$ there exists a unique solution $\phi \in \mathring{H}^{1,2}(\Omega)$ of the problem

$$a(\phi, v) = (s'_{-1}, v) \qquad \forall v \in \mathring{H}^{1,2}(\Omega).$$

Using the representation

$$\phi = k_\phi s_1 + \tilde{\phi}$$

and inserting $v = s_1 - P s_1$ yield

$$(s'_{-1}, s_1 - P s_1) = k_\phi a(s_1, s_1 - P s_1) + a(\tilde{\phi}, s_1 - P s_1),$$

and, by using (3.5),

$$(s'_{-1}, s_1 - P s_1) = c\, a(s_1 - P s_1, s_1 - P s_1) + a(\tilde{\phi} - I_h \tilde{\phi}, s_1 - P s_1).$$

From the estimate

$$|a(\tilde{\phi} - I_h \tilde{\phi}, s_1 - P s_1)| \leq c\, h^{1+\pi/\omega - \varepsilon} \qquad \forall \varepsilon > 0$$

we obtain

$$(s'_{-1}, s_1 - P s_1) \geq c\, h^{2\pi/\omega}. \tag{3.6}$$

For estimating $\| s_1 - P s_1 \|_{0,2}$ we define $\psi \in \mathring{H}^{1,2}(\Omega)$ by

$$a(\psi, v) = (s_1 - P s_1, v) \qquad \forall v \in \mathring{H}^{1,2}(\Omega)$$

and we get, using (3.5) and (3.6)

$$\begin{aligned} \| s_1 - P s_1 \|_{0,2}^2 &= k_\psi a(s_1 - P s_1, s_1 - P s_1) + a(\tilde{\psi} - I_h \tilde{\psi}, s_1 - P s_1) \\ &\geq c\, h^{4\pi/\omega} - c\, h^{1+\pi/\omega-\varepsilon} \| \tilde{\psi} \|_{2,2} \end{aligned}$$

for all $\varepsilon > 0$. By Theorem 2.1 the function $\tilde{\psi}$ satisfies the estimate

$$\| \tilde{\psi} \|_{2,2} \leq c \| s_1 - P s_1 \|_{0,2}$$

and the first term in the above identity dominates the second.

□

Denoting the dual space of $H^{l,1}(\Omega)$ by $H^{-l,\infty}(\Omega)$ we have the L^∞-estimate:

Theorem 3.2: Assume that the solution u *of* (1.1) *belongs to the space* $H^{m,\infty}(\Omega)$. *Then the error estimates hold*

$$\| u - P u \|_{-l,\infty} \leq c\, h^{m+1} |\ln h| \, \| \nabla^m u \|_{0,\infty}, \quad l = 0, 1, \ldots, m-2.$$

Proof: For $l = 0$, this result is proved by Schatz [8] via a discrete maximum principle. For the general case $0 \leq l \leq m-2$ we refer to the weighted norm estimates in Dobrowolski [3].

□

The Theorems 3.1 and 3.2 show the different behaviour of the FEM in L^2 and L^∞. The FEM is (nearly) quasi-optimal in $H^{-l,\infty}(\Omega)$ (Theorem 3.2) but fails to be quasi-optimal in $H^{k,2}(\Omega)$ (Theorem 3.1).

4. An Iterative Procedure

We want to apply the representation of the stress intensity factors in (2.2) to the solution of the FEM. For this purpose, we define the iterative procedure:

1. Set $k_i^0=0$, $i=1,\ldots,n$.
2. Determine approximate functions u_h^j, $w_h^j\in S^m$, $j\in\mathbb{N}$, to u, w in (2.1) by

$$a(w_h^j,v_h)=\left(f+\sum_{i=1}^{n}k_i^{j-1}\,\Delta s_i, v_h\right)\quad \forall v_h\in S^m, \tag{4.1}$$

$$u_h^j=w_h^j+\sum_{i=1}^{n}k_i^{j-1}\,s_i,$$

$$k_i^j=\frac{1}{i\pi}\int_{\Omega}\{f s_{-i}+u_h^j\,\Delta s_{-i}\}\,dx,\quad i=1,\ldots,n.$$

Obviously the method is well defined if the existence of the stress intensity factors k_i is guaranteed by Theorem 2.1. For the first iterate we have $u_h^1=w_h^1=Pu$. The behaviour for $j\to\infty$ will be studied in the next paragraph. For the iterates we have:

Theorem 4.1: *Assume that $\pi/\omega<1$. Then the iterates $k_i^j\in\mathbb{R}$ and u_h^j of (4.1) satisfy the estimates*

$$\|u-u_h^j\|_{1,2}+h^{-\pi/\omega}\sum_{i=1}^{n}|k_i-k_i^j|\le$$
$$\le c\,h^{\min\{(2j-1)\pi/\omega-\varepsilon,\,m-1\}}\,\|\nabla^m w\|_{0,2},\; j\in\mathbb{N}.$$

Proof: Subtracting the equation for k_i^j in (4.1) from (2.2) gives us

$$|k_i-k_i^j|\le c\,\|u-u_h^j\|_{0,2},\; j\in\mathbb{N}. \tag{4.2}$$

From the definition in (4.1) we obtain

$$w_h^j=Pu-\sum_{i=1}^{n}k_i^{j-1}\,Ps_i$$
$$=Pw+\sum_{i=1}^{n}(k_i-k_i^{j-1})\,Ps_i$$

and hence

$$u-u_h^j=w-Pw+\sum_{i=1}^{n}(k_i-k_i^{j-1})(s_i-Ps_i).$$

This identity leads to the estimates

$$\|u-u_h^j\|_{l,2}\le c\,h^{m-1+(1-l)\pi/\omega}+c\,h^{(2-l)\pi/\omega}\sum_{i=1}^{n}|k_i-k_i^{j-1}|,\; l=0,1,$$

and, by combining with (4.2), the theorem is proved. □

5. The Dual Singular Function Method (DSFM)

Using the notation of the last sections, we define the finite dimensional spaces

$$S_n^m = S^m \bigoplus_{i=1}^{n} s_i, \qquad S_{-n}^m = S^m \bigoplus_{i=1}^{n} s_{-i}, \qquad n \in \mathbb{N}.$$

We extend $a(.,.)$ to a bilinear form $a'(.,.)$ over $S_n^m \times S_{-n}^m$ by

$$\begin{aligned} a'(\phi_h, w_h) &= a(\phi_h, w_h) & \phi_h \in S_n^m,\ w_h \in S^m, \\ a'(v_h, s_{-j}) &= -(v_h, \Delta s_{-j}) & v_h \in S^m,\ j=1,\dots,n, \\ a'(s_i, s_{-j}) &= -(\Delta s_i, s_{-j}) & i,j=1,\dots,n. \end{aligned} \tag{5.1}$$

Because of the fact that $\Delta s_i = \Delta s_{-i} = 0$ in a neighbourhood of σ, all integrals on the right hand side of (5.1) exist. By linearity the form $a'(.,.)$ is well defined over $S_n^m \times S_{-n}^m$.

For h sufficiently small, the Dual Singular Function Method is defined by:

$$\begin{gathered} \text{Find } P_{-n} u \in S_n^m \text{ such that} \\ a'(P_{-n} u, v_h) = (f, v_h) \qquad \forall v_h \in S_{-n}^m. \end{gathered} \tag{5.2}$$

The DSFM can be stated in the following equivalent form which is more suitable for numerical purposes:

Determine appriximate stress intensity factors $k_1^h, \dots, k_n^h$ by solving the $n \times n$-system

$$\sum_{i=1}^{n} a_{ij} k_i^h = (f, s_{-j}) + (P u, \Delta s_{-j}),\ j=1,\dots,n,$$

where (5.3)

$$a_{ii} = i\pi + (P s_i - s_i, \Delta s_{-i}),$$

$$a_{ij} = (P s_i, \Delta s_{-j}),\ i \neq j,$$

and determine an approximate solution $P_{-n} u \in S_n^m$ by

$$P_{-n} u = P u + \sum_{i=1}^{n} k_i^h (s_i - P s_i).$$

For solving (5.3), the numerical computation of $n+1$ finite element approximations is required but good starting values are available for $P s_i$, for instance $I_h s_i$.

Another possibility for solving (5.2) or (5.3) is given by the following existence result:

Lemma 5.1: *Assume that the stress intensity factors $k_1, \dots, k_n$ of the solution u exist in Theorem* 2.1 *and that h is sufficiently small. Then the problems* (5.2) *and* (5.3) *are uniquely solvable and equivalent. Moreover, the function $P_{-n} u \in S_n^m$ is the limit of the sequence defined in* (4.1).

Proof: Using polar coordinates we see that

$$\int_\Omega \Delta s_i s_{-j} dx = 0 \quad \text{for} \quad i \neq j. \tag{5.4}$$

By inserting $u = s_i$ in formula (2.2) we get

$$\int_\Omega \Delta s_i s_{-i} dx = \int_\Omega s_i \Delta s_{-i} dx - i\pi. \tag{5.5}$$

By (5.4), the coefficients of the $n \times n$-matrix in (5.3) can be written in the form

$$a_{ij} = (P s_i - s_i, \Delta s_{-j}), \; i \neq j.$$

Since $P s_i \to s_i$ in $L^2(\Omega)$ for $h \to 0$ we have

$$a_{ij} \to 0, \; i \neq j; \; a_{ii} \to i\pi.$$

We conclude that the matrix is positive definite for h sufficiently small and that the approximate stress intensity factors $k_1^h, \ldots, k_n^h$ are uniquely determined by (5.3).

In order to prove the equivalence of (5.2) and (5.3), we assume that a solution $P_{-n} u$ of (5.2) exists. This solution can be written in the form

$$P_{-n} u = w_h + \sum_{i=1}^n k_i^h s_i$$

and satisfies

$$a\left(w_h + \sum_{i=1}^n k_i^h s_i, v_h\right) = (f, v_h) \qquad \forall v_h \in S^m.$$

It follows that

$$P_{-n} u = P u + \sum_{i=1}^n k_i^h (s_i - P s_i). \tag{5.5}$$

We insert $v_h = s_{-j}$ in (5.2) and we get

$$-\int_\Omega w_h \Delta s_{-j} dx - \sum_{i=1}^n k_i^h \int_\Omega \Delta s_i s_{-j} dx = (f, s_{-j}),$$

and by using (5.4) and (5.5)

$$k_j^h = \frac{1}{j\pi} \{(f, s_{-j}) + (P_{-n} u, \Delta s_{-j})\}. \tag{5.6}$$

This is the fundamental identity of the DSFM. It is the same formula which we have obtained for the continuous solution in Theorem 2.1. Inserting the representation of $P_{-n} u$ in (5.5) into (5.6) gives us the desired equivalence of the methods (5.2) and (5.3).

Finally, we have to prove that the DSFM can be obtained as the limit of the interative procedure described in the preceding section. The iterates u_h^j satisfy

$$a(u_h^j, v_h) = (f, v_h) \qquad \forall v_h \in S^m,$$

and therefore

$$\|\nabla(P_{-n}u-u_h^j)\|_{0,2}\le\sum_{i=1}^{n}|k_i^h-k_i^{j-1}|\;\|\nabla(s_i-I_h s_i)\|_{0,2}$$
$$\le c\,\|\nabla(P_{-n}u-u_h^{j-1})\|_{0,2}\max_i\|\nabla(s_i-I_h s_i)\|_{0,2}.$$

Hence we have proved the convergence $u_h^j\to P_{-n}u$ and $k_i^j\to k_i^h$. □

Using the various representations for $P_{-n}u$ and k_i^h in (5.5) and (5.6), we have the following

Lemma 5.2: *The DSFM satisfies the error relations*

$$(i)\;\; k_i-k_i^h=\frac{1}{i\pi}(u-P_{-n}u,\Delta s_{-i}),$$

$$(ii)\;\; u-P_{-n}u=w-Pw+\sum_{i=1}^{n}(k_i-k_i^h)(s_i-Ps_i).$$

By Lemma 5.2, it turns out that the error analysis of the DSFM is simple. If $\|\cdot\|_B$ is a Sobolev norm in which Ps_i converges to s_i for $h\to 0$, we obtain the error estimates

$$\|u-P_{-n}u\|_B+\sum_{i=1}^{n}|k_i-k_i^h|\le c\,\|w-Pw\|_B. \tag{5.7}$$

From this estimate it can be seen that the influence of the singular functions is completely ignored by the DSFM. The behaviour of the method is given by the behaviour of the FEM for approximating smooth solutions w of elliptic problems on domains with corners. Combining (5.7) and Theorem 3.2 gives the result:

Theorem 5.1: *Assume that the solution u of* (1.1) *admits the representation*

$$u=\sum_{i=1}^{n}k_i s_i+w \quad \textit{with} \quad w\in H^{m,\infty}(\Omega).$$

Then the error estimates hold

$$\|u-P_{-n}u\|_\infty\le c\,h^m|\ln h|,$$

$$\sum_{i=1}^{n}|k_i-k_i^h|\le c\,h^{2m-2}|\ln h|.$$

Remark: The estimates in the theorem remain true for remainders w which ly in certain weighted Sobolev spaces (see Dobrowolski [3]).

6. Numerical Results

Here we will consider the Laplace equation on a slit domain $\omega=2\pi$. Further numerical results can be found in Dobrowolski [3] for second order problems and in Blum [1] for the biharmonic equation.

The slit domain is a special case since the second singular function is regular. Hence we define

$$s_1=\tau(r)\,r^{1/2}\sin\phi/2,\quad s_2=\tau(r)\,r^{3/2}\sin 3\phi/2.$$

The dual singular functions are defined analogously.

Our results are compared with the Singular Function Method (SFM, see Fix-Gulati-Wakoff [4] and Strang-Fix [10]) which is defined by:

$$\begin{aligned} &\text{Find } P_n u \in S_n^m \text{ such that} \\ &a(P_n u, v_h) = (f, v_h) \quad \forall v_h \in S_n^m. \end{aligned} \tag{6.1}$$

The SFM also gives approximate stress intensity factors by expanding the solution in the form

$$P_n u = \sum_{i=1}^{n} k_i^h s_i + w_h, \quad w_h \in S^m.$$

It turns out that the numerical amount of the SFM and the DSFM is similar.

The relative errors of the various methods are listed in Table 6.1 for a piecewise constant right hand side and linear elements with two singular functions. The results of the methods using one singular function are similar and are omitted here. It turns out that the SFM and the DSFM give similar answers as far as the approximate solutions $P_2 u$ and $P_{-2} u$ are concerned. The errors $\| u - . \|_{\infty,\text{loc}}$ are measured on a 10×10 grid. Hence only local convergence rates can be obtained. The convergence of the stress intensity factors, however, is quite better for the DSFM. Since $k_2 \sim 1.8$ we have no convergence for the second stress intensity factor obtained by the SFM.

Table 6.1. *Comparison of the methods on* $\Omega = (0,1)^2 \setminus (\{.5\} \times [.5, 1))$

h^{-1}	FEM	SFM			DSFM		
	$\\|u - Pu\\|_{\infty,\text{loc}}$	$\\|u - P_2 u\\|_{\infty,\text{loc}}$	$\|k_1 - k_1^h\|$	k_2^h	$\\|u - P_{-2} u\\|_{\infty,\text{loc}}$	$\|k_1 - k_1^h\|$	$\|k_2 - k_2^h\|$
10	.1785	.1448	.8285	.1292	.3545(−1)	.6073(−1)	.2761(−2)
20	.8458(−1)	.4168(−1)	.4968	.2931	.1382(−1)	.2174(−1)	.1343(−3)
30	.5360(−1)	.2083(−1)	.3901	.3141	.6461(−2)	.9899(−2)	.3963(−4)
40	.3912(−1)	.1264(−1)	.3238	.3192	.3718(−2)	.5650(−2)	.1624(−4)
60	.2703(−1)	.6097(−2)	.2369	.3284	.1846(−2)	.2803(−2)	.6328(−5)
80	.1895(−1)	.3628(−2)	.1904	.3349	.9973(−3)	.1472(−2)	.2853(−5)

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