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The Structure of Stationary One Dimensional Varifolds with Positive Density*

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0. Introduction

Suppose on a smooth Riemannian manifold M one has a graph G such that each edge of G is a geodesic segment. Additionally, suppose to each edge is associated a positive density or "tension" so that at each vertex of G the tangent vector sum of the tension forces acting there is zero. Such a geodesic graph with densities is a heuristic model for stationary 1-dimensional varifolds in M. In this paper we examine the analytic and geometric structure of quite general stationary 1-dimensional varifolds and seek, among other things, conditions sufficient to imply such a graph structure.

In [PJ] it was shown, for example, that the stationary 1-dimensional integral varifolds arising from variational methods in the large have such structure with positive integer densities. On the other hand [AA, Example 2, p. 256] illustrates a stationary 1-dimensional varifold in $\mathbb{R}^2$ which has such graph structure except at one point of infinite complexity.

In general, a stationary 1-dimensional varifold V need not lie on a 1-dimensional set (since, for example, stationarity is preserved under convolution with smoothing functions [BK 4.3]). However, if the 1-dimensional densities of V are bounded away from 0, then V must lie on a 1-dimensional set [FH 2.10.19(3)]; moreover, as we show in Section 3, both geometrically and measure theoretically V is then the sum of geodesic segments with densities, and in case the set of densities which occurs is discrete this sum is locally finite. The example constructed in Section 4 shows that without the discreteness assumption the sum can be locally infinite (the example in [AA] above did not have a positive lower density bound). Even without the discreteness assumption we are able to show in Section 5 that at every point (even those of infinite local complexity) V admits a *unique* varifold tangent consisting of a *finite* number of half lines with densities.

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Section 1 contains the preliminary material necessary to make this paper self-contained while Section 2 contains various estimates including formulas for mass inside geodesic balls.

1. Preliminaries

Suppose M is a smooth connected m-dimensional Riemannian manifold. Let $T(M)$ be the bundle whose fiber $T_a(M)$ at $a \in M$ is the tangent space to M at a and let $P(M)$ be the bundle whose fiber $P_a(M)$ at $a \in M$ consists of the lines through the origin in $T_a(M)$. Let π: $P(M) \to M$ be the projection. Let $V(M)$ be the weakly topologized space of (nonnegative) Radon measures on $P(M)$. For $V \in V(M)$, let $\|V\|(A) = V(\pi^{-1}(A))$ for $A \subset M$. In the terminology of [AW] the members of $V(M)$ would be called *1-dimensional varifolds in M*. Given a continuously differentiable 1-dimensional submanifold I of M of locally finite length, we let $|I|$ be the member of $V(M)$ which assigns to each open subset B of $P(M)$ the length of $\{x: T_x(I) \in B\}$. Thus the members of $V(M)$ could be considered generalized curves in M. Suppose N is a smooth connected Riemannian manifold and F: $M \to N$ is an imbedding. Let $J(F)$: $P(M) \to \{s: 0 < s < \infty\}$ have at $\gamma \in P(M)$ the value $|T_a(F)(u)|$ where $a = \pi(\gamma)$, u is a unit vector in γ and $T_a(F)$: $T_a(M) \to T_{F(a)}(N)$ is the tangent map of F at a. We can then define $F_\#$: $V(M) \to V(N)$ by the condition that

$$F_\#(V)(B) = \int_{\{\gamma: T_{\pi(\gamma)}(F)(\gamma) \in B\}} J(F)\, dV$$

for each $V \in V(M)$ and each open subset B of $P(N)$. It is elementary that if I is as above, $F_\#(|I|) = |F(I)|$.

Let $\mathscr{X}(M)$ be the vectorspace of smooth vectorfields on M with compact support. For each $X \in \mathscr{X}(M)$, let δ_X: $P(M) \to \mathbb{R}$ have at $\gamma \in P(M)$ the value $\nabla_u X \cdot u$, where u is a unit vector in γ and ∇ is covariant differentiation with respect to the Levi-Civita connection. For each $V \in V(M)$ we define δV: $\mathscr{X}(M) \to \mathbb{R}$ by letting $\delta V(X) = \int \delta_X \, dV$ for $X \in \mathscr{X}(M)$; note that δV is linear. We call δV the *first variation distribution* of V because if $V \in V(M)$, $\|V\|(M) < \infty$, $X \in \mathscr{X}(M)$ and ϕ_t is the flow of X we may easily calculate that

$$\frac{d}{dt} \|\phi_{t\#}(V)\|(M) \Big|_{t=0} = \delta V(X).$$

We call $V \in V(M)$ *stationary* if $\delta V = 0$. Suppose I is a smooth continuously differentiable 1-dimensional submanifold of M of locally finite length. We say I is an *interval in M* if I is connected and there is an open neighborhood U of I such that $\delta |I|(X) = 0$ for every $X \in \mathscr{X}(U)$, which is equivalent to the tangent space to I being parallel along I. Suppose I is an interval; let $\mathbf{n}$ assign to each point of (Closure $I) \sim I$ the unit vector at that point which points out of I. It is elementary that domain $\mathbf{n}$ has at most two points and that

$$(1) \quad \delta |I|(X) = \sum_{a \in \text{domain}\, \mathbf{n}} X_a \cdot \mathbf{n}_a \quad \text{for } X \in \mathscr{X}(M).$$

It is well known that for each $a \in M$ and each $\gamma \in P_a(M)$ there is an interval I in M such that $a \in I$ and $\gamma = T_a(I)$, and that any two such intervals have the same intersection with some neighborhood of a.

For each $(x, y) \in M \times M$ let $\rho(x, y)$, the *distance from x to y*, be the infimum of the set of lengths of the piecewise smooth curves in M joining x to y. It is well known that ρ is a metric whose topology is the same as the given topology on M. Now fix $a \in M$. Let $\rho_a(x) = \rho(x, a)$ for $x \in M$ and let $U_a(r) = \{x : \rho_a(x) < r\}$ whenever $0 < r \leqq \infty$. Let $\exp_a$ be the exponential map at a and let S_a be the supremum of the set of s such that $U_a(s)$ has compact closure in M and $\exp_a$ restricted to $T_a(s) = \{v \in T_a(M) : |v| < s\}$ is a diffeomorphism. Let $\log_a$ be the inverse of $\exp_a | T_a(S_a)$ and let E_a be the vectorfield on $U_a(S_a)$ which is the image under $\exp_a$ of the restriction to $T_a(S_a)$ of the vector field on $T_a(M)$ whose flow is $\phi_t(v) = e^t v$ for $(t, v) \in \mathbb{R} \times T_a(M)$. It is well known that

(2) $(E_x)_y$ and $\rho_x(y)^2$ are smooth in there dependence on (x, y) in a neighborhood of $\{a\} \times U_a(S_a)$;

(3) $|E_a| = \rho_a \quad$ on $\; U_a(S_a)$;

(4) $\nabla_{E_a} E_a = E_a$;

(5) ∇E_a is self-adjoint;

(6) $(\nabla E_a)_a$ is the identity map of $T_a(M)$; note that (3), (4), (5) imply

(7) $E_a = \frac{1}{2} \nabla \rho_a^2$; Furthermore, it is well known that

(8) $(\nabla \nabla (w \circ \log_a))_a = 0$ whenever $w : T_a(M) \to \mathbb{R}$ is linear.

(If f is a smooth function, ∇f is its gradient vector field and $\nabla \nabla f$ is the covariant differential of ∇f.)

Let R_a be the largest of the $r \leqq S_a$ such that ∇E_a is non-negative definite on $U_a(r)$. Because $R_a \leqq S_a$, $R_a > 0$ and R_a is lowersemicontinuous in its dependence on a,

(9) $0 < \inf\{R_a : a \in K\} \leqq \inf\{S_a : a \in K\}$

whenever K is a compact subset of M. For each $\gamma \in \pi^{-1}(U_a(R_a) \sim \{a\})$, we set

$$\Psi_a(\gamma) = |u \cdot (\nabla \rho_a)_b|^2$$

where $b = \pi(\gamma)$ and u is a unit vector in γ. Owing to (3), (4) and (5) and the definition of R_a we have

(10) $\Psi_a \leqq \delta_{E_a} \quad$ on $\; \pi^{-1}(U_a(R_a) \sim \{a\})$.

Furthermore, if I is an interval in $U_a(R_a) \sim \{a\}$ and $\Psi_a(T_x(I)) = 1$ for some $x \in I$,

(11) I is contained in a interval passing through a.

(12) We say that the ordered pair (U, B) is *admissible* if (a)–(e) below hold.

(a) U is an open subset of M and $1 \leqq B < \infty$.

(b) U is convex; that is, any pair of points of U is contained in the closure of one and only one interval contained in U.

(c) Whenever $b \in U$, $u \in T_b(M)$, $|u|=1$ and $\beta=\{tu: t\in\mathbb{R}\}$, there is a unique maximal interval I_β in U passing through b with $T_b(I_\beta)=\beta$ together with a smooth orthonormal frame field $(X_1, \ldots, X_m)$ on U with $(X_1)_b=u$ which is parallel along I_β as well as along any interval in U which meets I_β orthogonally.

(d) If $b, u, (X_1, \ldots, X_m)$ are as in (c) and f is the set of $(x, y)\in U\times\mathbb{R}^m$ such that x occurs at time 1 along the integral curve of $\sum_{i=2}^{m} y_i X_i$ starting at the point which occurs at time 1 along the integral curve of $y_1 X_1$ starting at b, then f is a smooth coordinate system on U.

(e) Let $\sigma=\left(\sum_{i=2}^{m} f_i^2\right)^{1/2}$; for each $i, j=1, \ldots, m$,

$$|\nabla f_i - X_i| \leqq B\sigma; \quad |\nabla_{X_j} X_i| \leqq B\sigma; \quad |\nabla_{X_j} \nabla f_i| \leqq B\sigma.$$

For any $a\in M$ it is well known that for some $s>0$ and some B,

(13) $(U_a(r), B)$ is admissible whenever $0<r\leqq s$.

Suppose (U, B) is admissible; adopting the notation above, letting $F=\sum_{i=2}^{m} f_i X_i$ and letting ν be orthogonal projection on the span of $\{X_i: i=2, \ldots, m\}$ we remark that it is well known that

$$\sigma(x)=\inf\{\rho(x, y): y\in I_\beta\} \quad \text{for } x\in U,$$
$$F=\tfrac{1}{2}\nabla\sigma^2.$$

Furthermore,

$$\nabla_{X_1} F=\sum_{i=2}^{m} [(\nabla f_i - X_i)\cdot X_1 X_i + f_i \nabla_{X_1} X_i],$$

$$\nabla_{X_j} F - X_j=\sum_{i=2}^{m} [(\nabla f_i - X_i)\cdot X_j X_i + f_i \nabla_{X_j} X_i], \quad j=2, \ldots, m,$$

so that for any unit vector field $X=\sum_{i=1}^{m} \xi_i X_i$ on U

$$\begin{aligned}(14)\quad & |\nabla_X F\cdot X - |\nu(X)|^2| \\ &= \left|\sum_{i=2}^{m} [(\nabla f_i - X_i)\cdot X(X_i\cdot X) + f_i \nabla_X X_i\cdot X]\right| \\ &\leqq (m-1)B\sigma\,|\nu(X)| + B\sigma^2 \\ &\leqq |\nu(X)|^2/4 + (mB\sigma)^2.\end{aligned}$$

2. Monotonicity

Theorem. *Suppose* $a \in M$, $V \in V(M)$, *V is stationary,* $Q_a = \inf\{\rho_a(x): x \in \operatorname{spt}\|V\|\}$ *and, whenever* $0 < r \leqq R_a$,

$$m(r) = \int_{\pi^{-1}(U_a(r))} \delta_{E_a}\, dV,$$

$$n(r) = \int_{\pi^{-1}(U_a(r) \sim \{a\})} (\rho_a \circ \pi)(\delta_{E_a} - \Psi_a)\, dV.$$

Then

(1) $m(r) = r \lim_{h \downarrow 0} \int_{\pi^{-1}(U_a(r+h) \sim U_a(r))} \Psi_a\, dV$

whenever $0 < r < R_a$;

(2) $\dfrac{m(s)}{s} = \dfrac{m(r)}{r} \exp \int_r^s \dfrac{dn(t)}{t\, m(t)}$

whenever $Q_a < r < s \leqq R_a$;

(3) $\dfrac{m(s)}{s} = \dfrac{m(r)}{r} + \int_r^s t^{-2}\, dn(t)$

whenever $Q_a < r < s \leqq R_a$;

(4) $\dfrac{m(r)}{r}$ *is nondecreasing on* $0 < r \leqq R_a$;

(5) $0 \leqq \Theta_V(a) = \lim_{r \downarrow 0} \dfrac{\|V\|\, U_a(r)}{2r} \leqq \dfrac{m(r)}{2r}$

whenever $0 < r \leqq R_a$.

Proof. (3) follows from differentiating (2) with respect to s; (4) follows from (2); (5) follows from (4) and 1(6).

Given a smooth function $\phi: \mathbb{R} \to \mathbb{R}$ with $\operatorname{spt} \phi \subset (-\infty, R_a)$ we set $Y = (\phi \circ \rho_a) E_a$ and calculate

$$\delta_Y = (\phi' \circ \rho_a \circ \pi)(\rho_a \circ \pi)\, \Psi_a + (\phi \circ \rho_a \circ \pi)\, \delta_{E_a} \quad \text{on } \pi^{-1}(U_a(R_a) \sim \{a\});$$

inasmuch as Y is Lipschitzian and vanishes at a we infer that

(6) $0 = \int (\phi' \circ \rho_a \circ \pi)(\rho_a \circ \pi)\, \Psi_a + (\phi \circ \rho_a \circ \pi)\, \delta_{E_a}\, dV.$

Taking $r \in (0, R_a)$ and $h \in (0, R_a - r)$; letting

$$f_h(t) = \begin{cases} 1 & \text{if } t \leqq r, \\ 1 - (t-r)/h & \text{if } r < t \leqq r+h, \\ 0 & \text{if } r+h < t; \end{cases}$$

letting ϕ approximate f_h and then letting $h \downarrow 0$, we deduce (1) from (6). On the other hand, we can write $\Psi_a = \delta_{E_a} - (\delta_{E_a} - \Psi_a)$ in (6) and integrate by parts to obtain

$$0 = \int \phi'(t)\, t\, dm(t) - \int \phi'(t)\, dn(t) - \int \phi'(t)\, m(t)\, dt.$$

Given any smooth ζ with compact support contained in $(-\infty, R_a)$ we can set $\phi(r) = \int_r^\infty \zeta(t)\, dt$ and conclude that, in the sense of distributions,

$$0 = t\, dm(t) - dn(t) - m(t)\, dt.$$

We divide by $t\, m(t)$ and antidifferentiate to obtain (2).

Theorem. *Suppose $V_1, V_2, \ldots \in V(M)$ are stationary and $V = \lim_{i\to\infty} V_i \in V(M)$. Then*

(7) *V is stationary;*

(8) *if $a_1, a_2, \ldots \in M$ and $a = \lim_{i\to\infty} a_i \in M$ then*

$$\Theta_V(a) \geqq \limsup_{i\to\infty} \Theta_{V_i}(a_i);$$

(9) *if $0 < c < \infty$ and for $i = 1, 2, \ldots$, $\Theta_{V_i}(x) \geqq c$ for $\|V_i\|$ almost all $x \in M$ then*

$$\sup\{\inf\{\rho(x,y) : y \in \operatorname{spt}\|V\|\} : x \in K \cap \operatorname{spt}\|V_i\|\}$$
$$+ \sup\{\inf\{\rho(x,y) : x \in \operatorname{spt}\|V_i\|\} : y \in K \cap \operatorname{spt}\|V\|\}$$

tends to zero as i tends to ∞ for any compact subset K of M.

Proof. (7) is trivial and (9) is an elementary consequence of (8). To prove (8), note that $R_a \leqq \liminf_{i\to\infty} R_{a_i}$ by 1(9), let r be such that $0 < r < R_a$ and infer from (5) and the fact that ρ is a metric that for sufficiently large i

$$0 < r_i = r - \rho_a(a_i), \qquad U_{a_i}(r_i) \subset U_a(r),$$
$$2 r_i\, \Theta_{V_i}(a_i) \leqq m_i = \int_{\pi^{-1}(U_{a_i}(R_{a_i}))} \delta_{E_{a_i}}\, dV_i,$$

(10) $$\Theta_{V_i}(a_i) \frac{r_i}{r} \frac{\|V\|\, U_{a_i}(r_i)}{m_i} \leqq \frac{\|V\|\, U_a(r)}{2r}$$

Keeping in mind 1(2)(6), we deduce (8) from (10) and (5).

3. A Structure Theorem

(1) **Lemma.** *Every point of M is contained in an open set U with the following property: If $V \in V(U) \sim \{0\}$ and $\operatorname{spt}\|V\|$ is compact then*

(a) *$\operatorname{spt}\|\delta V\|$ contains at least two points;*

(b) *if $\operatorname{spt}\|\delta V\|$ contains precisely two points, $V = c\,|I|$ for some c with $0 < c < \infty$ and some interval I.*

Proof. Suppose $a \in M$. By 1(13) we may choose $s \in (0, R_a)$ and B so that $(U_a(r), B)$ is admissible whenever $0 < r \leqq s$. Choose $r \in (0, \inf\{\frac{1}{2}, s\})$ and $D > 0$ so that $\nabla_X E_a \cdot X \geqq \frac{1}{2}$

whenever X is a unit vectorfield on $U_a(r)$ and $m^2 B^2 + 4D^2 r^2 \leqq D/4$; this is possible by 1(6). Suppose $V \in V(U_a(r))$, spt $\|V\|$ is a compact subset of $U_a(r)$ and spt $\|\delta V\| \subset \{b, c\} \subset U_a(r)$. Adopting the notation of 1(12), we let $u \in T_b(M)$ be such that $|u|=1$ and $\{b, c\} \subset I_\beta$ and consider the vectorfield $Y = F + D\sigma^2 E_a$ on $U_a(r)$. Let X be a unit vectorfield on $U_a(r)$. We estimate

$$|D\, d\sigma^2(X)(E_a \cdot X)| = 2D\, |(F \cdot X)(E_a \cdot X)|$$
$$\leqq 2D\sigma\, |v(X)|\, r \leqq 4D^2 r^2 \sigma^2 + |v(X)|^2/4.$$

Thus, by 1(14),

$$\begin{aligned} \nabla_X Y \cdot X &= |v(X)|^2 + (\nabla_X F - v(X)) \cdot X + D\, d\sigma^2(X)\, E_a \cdot X + D\sigma^2\, \nabla_X E_a \cdot X \\ &\geqq |v(X)|^2 - |v(X)|^2/4 - m^2 B^2 \sigma^2 - 4D^2 r^2 \sigma^2 - |v(X)|^2/4 + D\sigma^2/2 \\ &\geqq |v(X)|^2/2 + D\sigma^2/4. \end{aligned}$$

We conclude that spt $\|V\| \subset I_\beta$ and that $X_1 \in \gamma$ for V almost all γ. Inasmuch as $\delta V(\phi X_1) = 0$ whenever $\phi \colon M \to \mathbb{R}$ is smooth with compact support contained in $U_a(r) \sim \{b, c\}$ we see that for some $c \geqq 0$, $V = c\,|I|$ where I is the component of $I_\beta \sim \{b, c\}$ whose closure is compact in $U_a(r)$. Thus the lemma is proved.

Theorem. *Suppose $V \in V(M)$, V is stationary, $0 < c < \infty$ and $\Theta_V(x) \geqq c$ for $\|V\|$ almost all points x of M. Let S_V (the singular set) be the set of points of M near which Θ_V, restricted to* spt $\|V\|$, *is not constant; let $\mathscr{I}$ be the family of connected components of* spt $\|V\| \sim S_V$; *and for each $I \in \mathscr{I}$ let $\Theta(I)$ be the unique member of the range of Θ_V restricted to I. Then*

(2) *each $I \in \mathscr{I}$ is an interval;*

(3) *each $I \in \mathscr{I}$ is open relative to* spt $\|V\|$;

(4) *$V = \Sigma\{\Theta_V(I)\,|I| \colon I \in \mathscr{I}\}$ and $\|V\|$ equals 1-dimensional Hausdorff measure on M times Θ_V. Furthermore, if $\{\Theta_V(I) \colon I \in \mathscr{I}\}$ is discrete,*

(5) *$\mathscr{I} \cap \{I \colon I \cap K \neq \phi\}$ is finite for every compact subset K of M.*

Remark. Since $S_V \subset \operatorname{spt} \|V\|$, we see from (3) that S_V is closed and from (4) that $\|V\|(S_V) = 0$ and that S_V has no interior relative to spt $\|V\|$.

We know of no example where S_V is uncountable. In 4 we give an example that shows the hypothesis of (5) is necessary.

Proof. Let μ be the indefinite integral with respect to $\|V\|$ of $1/\Theta_V$ and let A be the set of $a \in M$ such that $\lim_{r \downarrow 0} (2r)^{-1}\, \mu(U_a(r)) = 1$.

Now fix $a \in M$ and for each $r > 0$ let $c(r)$ be the cardinality of

$$B_r = \{x \colon \rho_a(x) = r\} \cap \operatorname{spt} \|V\|.$$

We claim that whenever $b_1, \ldots, b_l$ are distinct points of B_r,

$$\begin{aligned} l &= \lim_{h \downarrow 0} (2h)^{-1} \sum_{i=1}^{l} \|V\|\, U_{b_i}(h)/\Theta_V(b_i) \\ &\leqq \liminf_{h \downarrow 0} (2h)^{-1}\, \mu\left(\bigcup_{i=1}^{l} U_{b_i}(h)\right) \\ &\leqq \liminf_{h \downarrow 0} (2h)^{-1}\, \mu(U_a(r+h) \sim U_a(r-h)); \end{aligned}$$

the first inequality is a consequence of 2(5), the second is a consequence of the uppersemicontinuity of Θ_V which is implied by 2(8) and the third holds because ρ is a metric. We conclude

$$(6)\quad \int_0^s c(r)\,dr \leqq \mu(U_a(s)), \qquad 0<s<\infty.$$

Now suppose $a\in A$; by (1a) we see that $c(r)\geqq 2$ for r sufficiently small so that (6) implies

$$\lim_{s\downarrow 0} s^{-1}\mathscr{L}^1\{r: 0<r<s \text{ and } c(r)=2\}=1$$

where $\mathscr{L}^1$ is the Lebesgue measure on $\mathbb{R}$. But then (1b) implies that V near a equals $\Theta_V(a)$ times $|I|$ for some interval I passing through a.

Applying the principle that a measurable function is approximately continuous almost everywhere ([FH, 2.9.13]) in conjunction with the Besicovitch Covering Lemma ([FH, 2.8.18]) we infer that $\|V\|(M\sim A)=0$; (2), (3) and the formula for V in (4) should now be clear. The formula for $\|V\|$ in (4) follows immediately from [FH 2.10.19(3) and 3.2.5].

To prove (5), we fix $a\in\operatorname{spt}\|V\|$ and, for each $x\in U_a(R_a)\cap\bigcup\mathscr{I}$ we let $C(x)=|u\cdot(\nabla\rho_a)_x|$ and let $E(x)=(\nabla_u E_a)_x\cdot u$ where $x\in I\in\mathscr{I}$, $u\in T_x(I)$ and $|u|=1$; note that $(C\circ\pi)^2$ is V essentially equal Ψ_a and that $E\geqq C^2$; in particular 2(1) says

$$(7)\quad \frac{m(r)}{r}=\Sigma\{\Theta_V(x)\,C(x): x\in B_r\}$$

whenever $0<r<R_a$, $c(r)<\infty$ and $B_r\cap S_V=\phi$. On the other hand, 2(3) implies that

$$\lim_{r\downarrow 0}\sum_{I\in\mathscr{I}}\int_{I\cap U_a(R_a)}\rho_a^{-1}(E-C^2)\,d\|V\|=0.$$

Furthermore, $\mathscr{L}^1\{r: B_r\cap S_V\neq\phi\}=0$ since the one dimensional Hausdorff measure of S_V equals 0 by (4) and ρ_a is Lipschitzian. We may therefore choose a sequence of radii $r_i<R_a$ with limit 0 in such a way that $c(r_i)<\infty$, $B_{r_i}\cap S_V=\phi$ and

$$\lim_{i\to\infty}\inf\{C(x): x\in B_{r_i}\}=1.$$

Together with (7), this implies

$$2\,\Theta_V(a)=\lim_{i\to\infty}\Sigma\{\Theta_V(x): x\in B_{r_i}\}.$$

Assuming $\{\Theta(I): I\in\mathscr{I}\}$ is discrete, we infer that for some N, $i\geqq N$ implies

$$2\,\Theta_V(a)=\Sigma\{\Theta_V(x): x\in B_{r_i}\}$$

But now 2(5) and (7) imply

$$\frac{m(r_i)}{r_i}=2\,\Theta_V(a), \qquad i\geqq N$$

so that 2(3) implies

$$C(x)=1 \quad \text{on } U_a(r_i)\cap\bigcup\mathscr{I},\ i\geqq N.$$

We use 1(11) to complete the proof of (5).

4. An Example

Let e_1, e_2 be the standard basis vectors of $\mathbb{R}^2$. Suppose $0<\alpha<\sqrt{5}-2$, $0<\beta<\infty$ and A is a point of $\mathbb{R}^2$ for which $A\cdot e_2>0$.

Let $\theta=(1-\alpha)/2$ and for each $m=0,1,2,\ldots$ let

$$\begin{aligned}
\lambda_m&=e_1+\beta\theta^m e_2,\\
\mu_m&=(1+\alpha\theta^m)e_1+\beta\theta^m e_2,\\
v_m&=(1-\alpha\theta^m)e_1+\beta\theta^m e_2,\\
\xi_m&=(1-\alpha\theta^m)e_1-\beta\alpha\theta^m e_2,\\
\zeta_m&=\beta\alpha\theta^m e_1+(1-\alpha\theta^m)e_2.
\end{aligned}$$

Note that

(1) $$\frac{v_m\cdot e_1}{\lambda_m\cdot e_1}<\frac{v_m\cdot e_2}{\lambda_m\cdot e_2}=1,$$

(2) $$\frac{1-\alpha}{1+\alpha}<\frac{\lambda_m\cdot e_1}{\mu_m\cdot e_1}<\frac{\lambda_m\cdot\zeta_m}{\mu_m\cdot\zeta_m}<\frac{\lambda_m\cdot e_2}{\mu_m\cdot e_2}=1,\qquad m=0,1,2,\ldots.$$

Let

$$\begin{aligned}
A_0&=A,\\
B_0&=A_0-\frac{A_0\cdot e_2}{v_0\cdot e_2}v_0,\\
C_0&=A_0-\frac{A_0\cdot e_2}{\lambda_0\cdot e_2}\lambda_0,\\
D_0&=A_0-\frac{A_0\cdot e_2}{\mu_0\cdot e_2}\mu_0,\\
E_0&=A_0-\frac{A_0\cdot e_2}{\lambda_0\cdot e_2}\frac{\lambda_0\cdot\zeta_0}{\mu_0\cdot\zeta_0}\mu_0
\end{aligned}$$

and for each $m=1,2,3,\ldots$ let

$$\begin{aligned}
A_m&=D_{m-1}\\
B_m&=A_m-\frac{A_m\cdot e_2}{v_m\cdot e_2}v_m,\\
C_m&=A_m-\frac{A_m\cdot e_2}{\lambda_m\cdot e_2}\lambda_m,\\
E_m&=A_m-\frac{A_m\cdot e_2}{\mu_m\cdot e_2}\mu_m,\\
D_m&=A_m-\frac{A_m\cdot e_2}{\lambda_m\cdot e_2}\frac{\lambda_m\cdot\zeta_m}{\mu_m\cdot\zeta_m}\mu_m.
\end{aligned}$$

With the help of (1) and (2) we see that

$$(3)\quad A_m \cdot e_2 = (A_0 \cdot e_2) \prod_{0 \le i < m} \left(1 - \frac{\lambda_i \cdot \zeta_i}{\mu_i \cdot \zeta_i}\right) < (A_0 \cdot e_2) \left(\frac{2\alpha}{1+\alpha}\right)^m,$$

$$(4)\quad (A_m - A_{m+1}) \cdot e_1 = \frac{A_m \cdot e_2}{\lambda_m \cdot e_2} \frac{\lambda_m \cdot \zeta_m}{\mu_m \cdot \zeta_m} \mu_m \cdot e_1 < \frac{A_0 \cdot e_2}{\beta} \left(\frac{4\alpha}{1-\alpha^2}\right)^m (1+\alpha)$$

$$= (A_0 - C_0) \cdot e_1 \left(\frac{4\alpha}{1-\alpha^2}\right)^m (1+\alpha)$$

$$(5)\quad E_m \cdot e_1 < D_m \cdot e_1 < C_m \cdot e_1 < B_m \cdot e_1 < A_m \cdot e_1,$$

$$(6)\quad 0 = E_m \cdot e_2 = C_m \cdot e_2 = B_m \cdot e_2 < D_m \cdot e_2 < A_m \cdot e_2, \qquad m = 0, 1, 2, \ldots.$$

For each $m = 0, 1, 2, \ldots$ let $X_m \in V(\mathbb{R}^2)$ be the sum of

$|\nu_m| \, |\{(1-t) A_m + t B_m : 0 < t < 1\}|,$

$(1 - \alpha\theta^m) \, |\{(1-t) B_m + t C_m : 0 < t < 1\}|,$

$|\xi_m| \, |\{(1-t) C_m + t D_m : 0 < t < 1\}|,$

$|\mu_m| \, |\{(1-t) D_m + t A_m : 0 < t < 1\}|.$

Thus δX_m is the sum of the point masses at A_m, B_m, C_m, D_m multiplied, respectively, by the vectors

$|\nu_m| \, |A_m - B_m|^{-1} (A_m - B_m) + |\mu_m| \, |A_m - D_m|^{-1} (A_m - D_m),$

$|\nu_m| \, |B_m - A_m|^{-1} (B_m - A_m) + (1 - \alpha\theta^m) |B_m - C_m|^{-1} (B_m - C_m),$

$(1 - \alpha\theta^m) |C_m - B_m|^{-1} (C_m - B_m) + |\xi_m| \, |C_m - D_m|^{-1} (C_m - D_m),$

$|\xi_m| \, |D_m - C_m|^{-1} (D_m - C_m) + |\mu_m| \, |D_m - A_m|^{-1} (D_m - A_m).$

Moreover

$$(7)\quad C_m - D_m \quad \text{is a multiple of } \xi_m$$

because $(C_m - D_m) \cdot \xi_m = 0$, $\xi_m \cdot \zeta_m = 0$, $m = 0, 1, 2, \ldots$.

The positions of A_m, B_m, C_m, D_m, E_m relative to one another are illustrated in Figure 1.

Inasmuch as $4\alpha < 1 - \alpha^2$,

$$A_\infty = \lim_{m \to \infty} A_m \in \mathbb{R}^2 \quad \text{and} \quad A_\infty \cdot e_2 = 0.$$

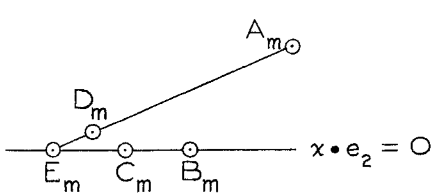

Fig. 1

Moreover, $A_\infty \to A - \frac{A \cdot e_2}{\beta}(e_1 + \beta e_2)$ as $\alpha \to 0$ for fixed β.

We assert that $A_m - B_m$, $A_m - D_m$, $B_m - C_m$ and $C_m - D_m$ are positive multiples of ν_m, μ_m, e_1 and ξ_m, respectively. In fact, $A_m \cdot e_2/\nu_m \cdot e_2$ and $(A_m \cdot e_2/\lambda_m \cdot e_2) \cdot (\lambda_m \cdot \zeta_m/\mu_m \cdot \zeta_m)$ are positive by (2) and (3), $(B_m - C_m) \cdot e_1 > 0$ and $(B_m - C_m) \cdot e_2 = 0$ by (5), (6); by (7) $C_m - D_m$ is a multiple of ξ_m, and this multiple is positive since $\xi_m \cdot e_1 > 0$ and, by (5), $(C_m - D_m) \cdot e_1 > 0$. Thus the four vectors above are respectively,

$$\nu_m + \mu_m = 2\lambda_m,$$
$$-\nu_m + (1 - \alpha\theta^m)e_1 = -\beta\theta^m e_2,$$
$$-(1 - \alpha\theta^m)e_1 + \xi_m = -\beta\alpha\theta^m e_2,$$
$$-\xi_m - \mu_m = -2\lambda_{m+1}.$$

Let W_m be X_m plus its reflection across the line $\{x: x \cdot e_2 = 0\}$, let $W = \sum_{i=0}^{\infty} W_i$ and let V be W plus $4|\{te_1: t < A_\infty \cdot e_1\}|$. It should be clear from our previous observations that δV is supported at A and its reflection across $\{x: x \cdot e_2 = 0\}$.

Thus the set S which occurs in the structure theorem of 3 need not be locally finite. It is easy to see that in V above one could remove a small ball about each B_m and perform the construction again to obtain an example where A_∞ is the accumulation point of points which are accumulation points of S. One can do this only finitely many times if one wants the density to stay bounded away from zero.

5. Tangent Cones

Lemma. *Suppose*

(1) *(U, B) is admissible and $b, u, I_\beta, (X_1, \ldots, X_m), (f_1, \ldots, f_m), F, \sigma, \nu$ are as in* 1(11);
(2) *$V \in \mathbf{V}(M)$, V is stationary, $0 < c < \infty$ and $\Theta_V(x) \geqq c$ for $\|V\|$ almost all $x \in M$;*
(3) *$\mathscr{I}, \Theta$ are as in* 3 (2), (3), (4) *and whenever $x \in I \in \mathscr{I}$ and $i = 2, \ldots, m$, $C_i(x)$ is the length of the projection of $(\nabla f_i)_x$ on $T_x(I)$;*
(4) *$0 < s < \infty$ and $U_b(3s) \subset U$;*
(5) *$0 < \varepsilon < \infty$ and $U \cap \operatorname{spt}\|V\| \subset \{x \in U: \sigma(x) \leqq \varepsilon\}$.*
Then for any $t \in \mathbb{R}$ and any $i = 2, \ldots, m$

$$(6)\quad \sum_{I \in \mathscr{I}} \sum_{x \in I \cap U_b(s) \cap \{y:\, f_i(y) = t\}} \Theta(I)\, C_i(x) \leqq D\, \varepsilon \|V\|\, U_b(3s)$$

where D is a constant depending on B, m and s.

Proof. (Part one) Let $Y = (\psi \circ \rho_b)^2 F$ where $\psi: \mathbb{R} \to \mathbb{R}$ is smooth, $\psi' \leqq 0$, $\psi(r) = 1$ if $r \leqq 2s$ and $\operatorname{spt}\psi \subset (-\infty, 3s)$. Inasmuch as

$$(\psi \circ \rho_b)^2 = \nabla Y - (\psi \circ \rho_b)^2(\nabla F - \nu) - 2(\psi' \circ \rho_b)(\psi \circ \rho_b)\, d\rho_b F$$

we see from 1(12), (14) and Schwarz's inequality that, with

$$\alpha = \left(\int (\psi \circ \rho_b \circ \pi)^2 \nu \cdot \gamma\, dV\gamma\right)^{1/2} \quad \text{and} \quad \mu = \|V\|\, U_b(3s),$$

we have

$$\alpha^2 \leqq \frac{\alpha^2}{4} + B^2 m^2 \varepsilon^2 \mu + 2(\sup \psi')\, \varepsilon \mu^{1/2} \alpha.$$

In case $\varepsilon \mu^{1/2} \leqq \alpha$, this implies

$$\tfrac{3}{4}\alpha \leqq (B^2 m^2 + 2(\sup \psi'))\, \varepsilon \mu^{1/2}$$

so that, in any case,

$$\alpha \leqq \tfrac{4}{3}(B^2 m^2 + 2(\sup \psi'))\, \varepsilon \mu^{1/2}.$$

considering the restrictions on ψ', we obtain

(7) $$\beta \leqq \tfrac{4}{3}\left(B^2 m^2 + \frac{2}{s}\right) \varepsilon \mu^{1/2}$$

where

$$\beta = \Big(\int_{\pi^{-1}(U_b(2r))} v \cdot \gamma \, d V \gamma\Big)^{1/2}.$$

(Part two) Fix i with $i=2, \ldots, m$. Suppose $\phi, \psi: \mathbb{R} \to \mathbb{R}$ are smooth, $\psi' \leqq 0$, $\psi(r)=1$ if $r \leqq s$, $\operatorname{spt} \psi \subset (-\infty, 2s)$, $0 \leqq \phi \leqq 1$ and $\phi' \geqq 0$. Set $Z_i = (\psi \circ \rho_b)(\phi \circ f_i) \nabla f_i$; inasmuch as

$$\begin{aligned}(\psi \circ \rho_b)(\phi' \circ f_i)\, d f_i \nabla f_i = \nabla Z_i - (\psi' \circ \rho_b)(\phi \circ f_i)\, d\rho_b (\nabla f_i - X_i) \\ - (\psi' \circ \rho_b)(\phi \circ f_i)\, d\rho_b X_i - (\psi \circ \rho_b)(\phi \circ f_i) \nabla \nabla f_i,\end{aligned}$$

we see with the help of Schwarz's inequality and the estimates of 1(12) that

$$\sum_{I \in \mathscr{I}} \int_{I \cap U_b(s)} (\phi' \circ f_i)\, C_i^2 \, d\|V\| \leqq (\sup \psi') B \varepsilon \mu + (\sup \psi') \beta \mu^{1/2} + B \varepsilon \mu.$$

We deduce (6) by letting ϕ approximate the characteristic function of (t, ∞), keeping in mind the restrictions on ψ and using (7).

(8) **Lemma.** *(Maximum principle.) Suppose*

(a) $h: M \to \mathbb{R}$ *is smooth and* $\nabla \nabla h$ *is positive definite;*

(b) $V \in \mathbf{V}(M)$ *and* $\operatorname{spt}\|V\|$ *is compact.*

Then

$$\sup\{h(x): x \in \operatorname{spt}\|V\|\} \leqq \sup\{h(x): x \in \operatorname{spt} \delta V\}.$$

Proof. Let $C = \sup\{h(x): x \in \operatorname{spt} \delta V\}$, let W be V restricted to $\pi^{-1}\{x: h(x) > C\}$ and let $Y = (h - C)\nabla h$. Then $\delta W(Y) = 0$ since $\operatorname{spt} \delta W \subset \{x: h(x) = C\}$ and Y vanishes there; moreover,

$$\delta_Y = dh \nabla h + (h - C) \circ \pi\, \delta_{\nabla h}$$

is positive on $\pi^{-1}\{x: h(x) > C\}$. Thus $W = 0$.

Suppose $V \in \mathbf{V}(M)$ and $a \in M$. We say $C \in \mathbf{V}(T_a(M))$ is a *varifold tangent to* V *at* a if

(9) $$C = \lim_{i \to \infty} (r_i \log_a)_\# V_{R_a} \quad \text{in } \mathbf{V}(T_a(M))$$

where $r_1, r_2, \ldots$ is a sequence of real numbers with limit ∞, $r_i \log_a$ is $\log_a$ followed by dilation of $T_a(M)$ by the factor r_i and V_{R_a} is the restriction of V to $\pi^{-1}(U_a(R_a))$.

Suppose V is stationary and $a \in \operatorname{spt}\|V\|$. Given a sequence $s_1, s_2, \ldots$ of positive real numbers we infer from the fact that $T_a(\log_a)$ is an isometry and that $0 \leqq \Theta_V(a) < \infty$ that there is a subsequence $r_1, r_2, \ldots$ such that C exists as in (9) and

(10) $\|C\|\, T_a(r) = \Theta_V(a)\, r, \quad 0 < r < \infty.$

Given $X \in X(T_a(M))$ and $0 < r < \infty$ with $\operatorname{spt} X \subset T_a(r R_a)$ we define $X^r \in X(M)$ by letting

$$(X^r)_b = T_{(r \log_a)(b)}(\exp_a)((X)_{(r \log_a)(b)})$$

for $b \in U_a(R_a)$ and letting it equal 0 elsewhere. Keeping in mind 1(8), which together with other observations made in 1 implies that the 1-jet of the metric of $T_a(M)$ at 0 equals the 1-jet of the image of the metric of M under $\log_a$ at 0, we infer that

$$\lim_{r \to \infty} \delta_{T_a(M)} X(T_{\pi(\gamma)}(r \log_a)(\gamma))\, J(r \log_a)(\gamma) - \delta_M X^r(\gamma) = 0$$

uniformly for γ in $\pi^{-1}(U_a(R_a))$; the subscripts on the "δ"'s indicate the space where δ is computed. Thus C is stationary. In view of 2(3), applied to C, we deduce that

(11) C almost all ζ in $P(T_a(M))$ contain the radial direction.

Now we suppose $\Theta_V(x) \geqq c > 0$ for $\|V\|$ almost all $x \in M$ and infer from the fact that $T_a(\log_a)$ is an isometry that

(12) $\Theta_C(y) \geqq c \quad$ for $\|C\|$ almost all $y \in T_a(M)$.

From (11), (12), 3(2) (3) (4) and 1(11) we see that there is $F \subset T_a(M)$ such that

F is finite, $\quad |v| \geqq c \quad$ for $v \in F$,

(13) $\sum_{v \in F} v = 0,$

$$C = \sum_{v \in F} |v|\, |\{t v \colon 0 < t < \infty\}|.$$

Now let d_i be the Hausdorff distance between the intersections of $\operatorname{spt}\|V\|$ and $\exp_a(\operatorname{spt}\|C\|)$, respectively, with the closure of $U_a(R_a/r_i)$. By 2(4) and 1(2), (6) we see that

(14) $\lim_{i \to \infty} r_i d_i = 0.$

We now apply (1)–(6) with $(b, u) = (a, |v|^{-1} v)$ for each $v \in F$ and conclude that

(15) $\lim_{i \to \infty} V \operatorname{ess\,inf} \Psi_a \,|\, \pi^{-1}(U_a(R_a/r_i) \sim U_a(R_a/2 r_i)) = 1.$

Owing to the lack of restriction on the sequence $s_1, s_2, \ldots$, we have proved the

Theorem. *Suppose $V \in V(M)$, V is stationary, $0 < c < \infty$ and $\Theta_V(x) \geqq c$ for $\|V\|$ almost all $x \in M$. Then for each $a \in M$,*

(16) $\lim_{r \to \infty} V \operatorname{ess\,inf} \Psi_a \,|\, U_a(r) = 1.$

Remark. In studying the rate of approach of Ψ_a to 1 one must keep in mind how Ψ_a behaves on varifolds of the form

$$V=\sum_{i=1}^{3}|v_i|\,|\{tv_i\colon 0<t<\infty\}|\in V(\mathbb{R}^2)$$

where v_1, v_2, v_3 are nonzero noncollinear points in $\mathbb{R}^2$ with sum zero and where $a\in \operatorname{spt}\|V\|$ is close to but not equal the origin.

We now analyze the dependence of C on the sequence $r_1, r_2, \ldots$. For this purpose, suppose $u\in T_a(M)$ is of unit length and $u\cdot v=0$ for no $v\in F$. Choose λ such that $0<\lambda<\inf\{|v|^{-1}|u\cdot v|\colon v\in F\}$ and let $U(x)=u\cdot \log_a(x)$ for $x\in U_a(R_a)$. Let

$$h^+=-U+\lambda(\rho_a+\rho_a^2/2),\qquad h^-=+U+\lambda(\rho_a+\rho_a^2/2),$$
$$h\;=-|U|+\lambda(\rho_a+\rho_a^2/2).$$

Thus, $h=h^+$ when $U\geqq 0$ and $h=h^-$ when $U\leqq 0$. We will prove that, for r sufficiently small,

(17) $\quad h(x)\leqq 0 \quad$ for $x\in U_a(r)\cap \operatorname{spt}\|V\|$.

To begin our demonstration of (17) we will show that, sufficiently near a, $h|\operatorname{spt}\|V\|\sim\{a\}$ never has a local maximum when $U=0$. To this end, choose η such that $(1+\eta)\eta<\lambda(1-\eta^2/2)^{1/2}$ and $t_1\in(0,R_a)$ such that

(a) for V almost all $\gamma\in\pi^{-1}(U_a(t_1))$,
$|w-(\nabla\rho_a)_b|\leqq\eta$ whenever $w\in\gamma$, $|w|=1$ and $b=\pi(\gamma)$;

(b) $|(\nabla U)_b|\leqq 1+\eta$, whenever $b\in U_a(t_1)$;

this is possible by (16) and the fact that $|(\nabla U)_a|=1$. Suppose there were $b\in\operatorname{spt}\|V\|\cap U_a(t_1)\sim\{a\}$ such that $U(b)=0$ and $h(x)\leqq h(b)$ for $x\in\operatorname{spt}\|V\|$ near b. Let C_b be a varifold tangent to V at b and let F_b be as in (13). Evidently,

$$(\nabla U)_b\cdot v=0\Rightarrow(\nabla\rho_a)_b\cdot v<0,$$
$$(\nabla U)_b\cdot v>0\Rightarrow(\nabla h^+)_b\cdot v\leqq 0,$$
$$(\nabla U)_b\cdot v<0\Rightarrow(\nabla h^-)_b\cdot v\leqq 0$$

for $v\in F_b$, which amounts to

(c) $|(\nabla U)_b\cdot v|\geqq\lambda(1+\rho_a(b))(\nabla\rho_a)_b\cdot v$ for $v\in F_b$.

Now (a) implies $||v|^{-1}v-(\nabla\rho_a)_b|\leqq\eta$ for $v\in F_b$ so that (b) together with the fact that $(\nabla U)_b\cdot(\nabla\rho_a)_b=0$ implies

(d) $|v|^{-1}|(\nabla U)_b\cdot v|=|(\nabla U)_b\cdot(|v|^{-1}v-(\nabla\rho_a)_b)|\leqq(1+\eta)\eta$;

moreover, $2-2|v|^{-2}((\nabla\rho_a)_b\cdot v)^2\leqq\eta^2$ so

(e) $|(\nabla\rho_a)_b\cdot v|\geqq(1-\eta^2/2)^{1/2}|v|$

for all $v\in F_b$. Now (c), (d), (e) together with the choice of η imply $(\nabla\rho_a)_b\cdot v<0$ for all $v\in F_b$ which contradicts $\sum\{v\colon v\in F_b\}=0$.

Now choose $t_2>0$ such that $\nabla\nabla h^+$ and $\nabla\nabla h^-$ are positive definite on $U_a(t_2)\sim\{a\}$; this is possible by 1(8). Next choose i so large that $R_a/r_i<\inf\{t_1,t_2\}$ and such that $h(x)<0$ for $x\in\operatorname{spt}\|V\|\cap\{y:\rho_a(y)=R_a/r_i\}$; this is possible by (14) and the choice of u. Let $b\in\operatorname{spt}\|V\|\cap\{x:\rho_a(x)\leqq R_a/r_i\}$ be such that $h(x)\leqq h(b)$ for $x\in\operatorname{spt}\|V\|\cap\{x:\rho_a(x)\leqq R_a/r_i\}$. Suppose $h(b)$ were positive. Then $0<\rho_a(b)<R_a/r_i$ and $U(b)\neq 0$. In case $U(b)>0$, we could apply (8) with V and h replaced by V restricted to $\pi^{-1}(U_a(R_a/r_i)\cap\{x: U(x)>0\})$ and h^+ and conclude thereby that $h(b)$ would not exceed the supremum of h on the union of $\operatorname{spt}\|V\|\cap\{x:\rho_a(x)=R/r_i\}$ with $\operatorname{spt}\|V\|\cap U_a(R/r_i)\cap\{x: U(x)=0\}$; this would be impossible. We argue similarly in case $U(b)<0$. This completes the proof of (17).

Suppose now that D is a conical neighborhood of $\operatorname{spt}\|C\|$. Applying (17) for all u as above, we deduce that, for r sufficiently small, the image of $\operatorname{spt}\|V\|\cap U_a(r)$ under $\log_a$ is contained in D. We have proved the

Theorem. *Suppose $V\in\mathbf{V}(M)$, V is stationary, $0<c<\infty$ and $\Theta_V(x)\geqq c$ for $\|V\|$ almost all $x\in M$. For each $a\in M$ there is a finite subset F_a of $T_a(M)$ such that*

$$\sum_{v\in F_a}|v|\,|\{tv: 0<t<\infty\}|$$

is the unique varifold tangent to V at a.

References

[AW] Allard, W. K.: On the first variation of a varifold. Ann. of Math. **95**, 417–491 (1972)

[AA] Allard, W.K., Almgren, Jr., F.J.: An introduction to regularity theory for parametric elliptic variational problems. Proc. Symp. Pure Math., XXIII, Amer. Math. Soc., 1973, 231–260

[BK] Brakke, K.A.: The motion of a surface by its mean curvature. Ph.D. Thesis, Princeton University, 1975

[FH] Federer, H.: Geometric measure theory. New York: Springer 1969

[PJ] Pitts, J.: Regularity and singularity of one dimensional stationary integral varifolds arising from variational methods in the large. Symp. Math., XIV, pp. 465–472. New York: Academic Press 1974

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