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On Partial Regularity of Suitable Weak Solutions to the Three-Dimensional Navier–Stokes equations

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Abstract. We prove a criterion of local Hölder continuity for suitable weak solutions to the Navier–Stokes equations. One of the main part of the proof, based on a blow-up procedure, has quite general nature and can be applied to other problems in spaces of solenoidal vector fields.

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1. Inroduction

In the present paper we deal with weak solutions to the three-dimensional Navier–Stokes equations

$$\partial_t v + \operatorname{div}(v \otimes v) - \Delta v = f - \nabla p, \quad \operatorname{div} v = 0$$

in a bounded domain $Q_T \equiv \Omega \times]0,T[$ of the space $\mathbb{R}^4$. In [4] E. Hopf proved the global existence at least one weak solution v to the first initial boundary value problem with boundary condition $v|_{\partial\Omega\times[0,T]} = 0$ under quite general assumptions on domain Ω, external force f and initial data $v|_{t=0} = a$. In [5] his results were discribed and the class of Hopf's solutions was introduced. Corresponding definition includes all main properties that essentially were proved by E. Hopf. More precisely, a velocity field v is called Hopf's solution if it belongs to $L_\infty(0,T;\overset{\circ}{J}(\Omega)) \cap L_2(0,T;\overset{\circ}{J}{}^1_2(\Omega))$, is continuous in $t \in [0,T]$ in the weak topology of $L_2(\Omega;\mathbb{R}^3)$ and satisfies the integral identity

$$\int\limits_\Omega (v(t)-a)\cdot w\,dx + \int\limits_{Q_t=\Omega\times]0,t[} (\nabla v : \nabla w - v\otimes v : \nabla w - f\cdot w)\,dxds = 0$$

for all $t \in [0,T]$ and for all $w \in \overset{\circ}{J}{}^1_2(\Omega)$. No information on the pressure p is given. Here $\overset{\circ}{J}(\Omega)$ is the $L_2(\Omega;\mathbb{R}^3)$-closure of the set of all smooth solenoidal fields vanishing near $\partial\Omega$ and $\overset{\circ}{J}{}_2^1(\Omega)$ is the closure of the same set with respect to the Dirichlet integral. These solutions have two other properties: they are continuous at $t = 0$ with respect to the strong topology of $L_2(\Omega; \mathbb{R}^3)$ and for them the energy inequality

$$\int_\Omega |v(x,t)|^2\,dx + 2\int_{Q_t} |\nabla v|^2\,dxds \le \int_\Omega |a|^2\,dx + 2\int_{Q_t} f\cdot v\,dxds$$

holds for all $t \in [0, T]$. Here and in what follows it is assumed that $a \in \overset{\circ}{J}(\Omega)$ and $f \in L_2(0, T; \overset{\circ}{J}{}_2^{-1}(\Omega))$.

In [5] (more detailly in [6]) the author expressed her confidence that the class of Hopf's solutions is too wide in the sense that the uniqueness theorem is not valid in it. In [7] this was confirmed by examples. In [5], [6], [16] and others various conditional theorems on uniqueness and smoothness of Hopf's solutions were proved if some additional information about them is known. For example, it was proved that finiteness of the quantity

$$\operatorname{ess}\max_{z=(x,t)\in Q'\subset Q_T} |v(z)|$$

implies smoothness of v in spatial variables (but not in t!).

In [5] (see the last section of Chapter VI) the following unconditional result was proved. Supposed that $\Omega = \mathbb{R}^3$, i.e. the Cauchy problem is considered, and $a = 0$. Then any Hopf's solution has derivatives $\partial_t v$ and $\nabla^2 v$ in $L_{\frac{5}{4}}(Q_T)$. Moreover, there is a pressure field p with ∇p from $L_{\frac{5}{4}}(Q_T)$ such that the Navier–Stokes equations are satisfied a.e. in Q_T. It was remarked there that this statement is valid for the first initial boundary value problem as well (of course, under relevant initial data). The proof is based on L_p-estimates for solutions to the nonstationary Stokes equations. Such estimates were obtained in [5] for the Cauchy problem and formulated there for the first boundary value problem as the result of K. K. Golovkin and V. A. Solonnikov. A proof was published in [3], [17]. However, as it was remarked in [5] this unconditional result is too weak to get uniqueness in the class of Hopf's solutions.

Now, let us explain how the exponent $\frac{5}{4}$ occurs. For Hopf's solutions, the energy norm

$$|v|_{Q_T} \equiv \operatorname{ess}\max_{[0,T]} \|v(t)\|_{2,\Omega} + \|\nabla v\|_{2,Q_T}$$

is finite. On the other hand, for any functions $u \in \overset{\circ}{W}{}_2^1(\Omega)$, $\Omega \subset \mathbb{R}^3$, or, for any functions $u \in W_2^1(\Omega)$ with $\int_\Omega u\,dx = 0$, the multiplicative inequalities

$$\|u\|_{q,\Omega} \le \beta(q)\|\nabla u\|_{2,\Omega}^{\alpha}\|u\|_{2,\Omega}^{1-\alpha} \tag{1.1}$$

are valid with $q \in [2, 6]$ and $\alpha = 3(\frac{1}{2} - \frac{1}{q})$. They are very important for investigation of smoothness of solutions to PDE's. The first of them were found out specially for the investigation of solutions to the Navier–Stokes equations (see [8], [5], [9] and others). In what follows we often use some consequence of (1.1) for functions u, belonging to the Sobolev class W_2^1 on balls $B(x_0,R)$,

$$\|u\|_{q,B(x_0,R)} \le \beta_1(q)\Big(\|\nabla u\|^{\alpha}_{2,B(x_0,R)}\|u\|^{1-\alpha}_{2,B(x_0,R)} + \frac{1}{R^{\alpha}}\|u\|_{2,B(x_0,R)}\Big) \tag{1.2}$$

It is valid for the same values q and α as in inequality (1.1).

With the help of (1.1) it is proved that for all functions v, vanishing on $\partial\Omega\times[0,T]$ or satisfying the identity $\int\limits_{\Omega} v(x,t)\,dx = 0$ for $t \in]0,T[$, the inequality

$$\|v\|_{q,r,Q_T} \equiv \Big(\int\limits_0^T \|v(t)\|^r_{q,\Omega}\,dt\Big)^{\frac{1}{r}} \le C_1|v|_{Q_T} \tag{1.3}$$

holds if

$$\frac{1}{r} + \frac{3}{2q} \ge \frac{3}{4}, \quad r\in[2,\infty], \quad q\in[2,6].$$

In turn, with the help of (1.3) one can estimate the nonlinear term $(\nabla v)v$ in the Navier–Stokes equations. More precisely, we have

$$\|(\nabla v)v\|_{s,l,Q_T} \le C_2\|\nabla v\|_{2,Q_T}\|v\|_{\frac{2s}{2-s},\frac{2l}{2-l},Q_T} \le C_3|v|^2_{Q_T}, \tag{1.4}$$

if s and l satisfy the conditions

$$\frac{1}{l} + \frac{3}{2s} \ge 2, \quad l\in[1,2], \quad s\in\Big[1,\frac{3}{2}\Big]. \tag{1.5}$$

In particular, exponents $s = l = \frac{5}{4}$ satisfy the last conditions. For this reason they were taken in [5] to prove that functions $|\partial_t v|$, $|\nabla^2 v|$ and $|\nabla p|$ belong to $L_{\frac{5}{4}}(Q_T)$.

In [2], [11] the theorem on unique solvability of the first initial boundary value problem for the Stokes system was extended to the case of spaces $W^{2,1}_{s,l}(Q_T)\times W^{1,0}_{s,l}(Q_T)$ with two different exponents $s,l \in]1,\infty[$. Using this theorem, bound (1.4) and arguing as in [5], one can make the following conclusion.

Theorem 1.1. *Assume that our bounded domain Ω is of class C^2. Suppose, in addition, that $f \in L_{s,l}(Q_T;\mathbb{R}^3)$ with numbers $s>1$ and $l>1$, satisfying conditions* (1.5). *Then any Hopf's solution to the first initial boundary value problem for the Navier–Stokes equations has derivatives $\partial_t v$, $\nabla^2 v$, belonging to the space $L_{s,l}(Q_{\delta,T})$, where $Q_{\delta,T}\equiv\Omega\times]\delta,T[$ with any $\delta\in]0,T[$. Moreover, there is a locally summable pressure field p with $\nabla p \in L_{s,l}(Q_{\delta,T})$ such that the Navier–Stokes equations hold a.e. in Q_T. The pressure field p itself is an element of $L_{\frac{3s}{3-s},l}(Q_{\delta,T})$ for any $\delta\in]0,T[$ provided that*

$$\int\limits_{\Omega} p(x,t)\,dx = 0, \quad t\in]0,T[.$$

Described here additional information about weak Hopf's solutions is still unsufficient to prove the uniqueness theorem but it turns out to be helpful for the analysis of their partial regularity.

V. Scheffer began to study such regularity for some classes of weak solutions to the Navier–Stokes equations. He assumed that weak solutions possess some additional properties. The main one is the so-called local energy inequality. Considering the case $\Omega = \mathbb{R}^3$ and $f = 0$, he proved in [13] that for any solution v and p, satisfying his additional assumptions, the velocity field v is continuous on an open subset of Q_T and the two-dimensional Hausdorff measure of its compliment is finite. He also showed that among of weak solutions to the Cauchy problem there exists at least one solution with these additional properties. In [12] Scheffer studied Leray's weak solutions v of the three-dimensional Cauchy problem for the Navier–Stokes equations. He proved that the one-dimensional Hausdorff measure of the sets $S(t_k) \subset \mathbb{R}^3$ of discontinuities of v is finite. Here t_k belongs to so called Leray's set of time moments when $\liminf_{t\to t_k-0} \|\nabla v(t)\|_{2,\mathbb{R}^3} = \infty$. In [12] the author used the invariance of the homogeneous Navier–Stokes equations with respect to a certain changing variables x, t and functions v, p. In [14] V. Scheffer considered an other class of weak solutions to the first initial boundary value problem for the Navier–Stokes equations in a bounded domain Ω, assuming that $f = 0$. He proved that among of such weak solutions there exists at least one solution v such that $\operatorname{curl} v$ is continuous on an open subset of Q_T. He also showed that the Hausdorff dimension of the compliment of this subset is not more that $\frac{5}{3}$.

These investigations were continuied by L. Caffarelli, R.-V. Kohn and L. Nirenberg. In [1] they introduced the notion of *suitable weak* solutions. They call a pair v and p a *suitable weak* solution to the Navier–Stokes equations if v has the finite energy norm, p belongs to the space $L_{\frac{5}{4}}(Q_T)$, v and p are weak solution to the Navier–Stokes equations and satisfy the local energy inequality. It was proved by them that in fact $p \in L_{\frac{5}{3},\frac{5}{4}}(Q_T)$ at least locally in Q_T. They showed that if $f \in L_q(Q_T;\mathbb{R}^3)$ with $q > \frac{5}{2}$, then, for any suitable weak solution v and p, there is an open subset of Q_T such that v is locally bounded on it and the one-dimensional parabolic Hausdorff measure of the compliment of this subset is equal to zero. In the same work they proved that among of weak Hopf's solutions of the first initial boundary value problem there exists at least one suitable weak solution v and p with $p \in L_{\frac{5}{3},\frac{5}{4}}(Q_T)$. This corresponds to $s = l = \frac{5}{4}$ in Theorem 1.1.

In contrast to Scheffer's method their analysis of regularity is local, i.e. L. Caffarelli, R.-V. Kohn and L. Nirenberg proved some criteria for local regularity of v (more precisely, local boundness). Having them in hands, they established partial regularity and estimated the Hausdorff dimension of the singular set.

In [10], for $f = 0$, F.-H. Lin defined suitable weak solutions, assuming that p belongs to $L_{\frac{3}{2}}(Q_T)$. This corresponds to $s = \frac{9}{8}$, $l = \frac{3}{2}$ in Theorem 1.1.

To guarantee the existence of sutable weak solutions among of weak Hopf's solutions one should choose a proper class for the pressure. The choice is described by Theorem 1.1. It remains to show that in selected class there exists at least one solution satisfying the local energy inequality. This could be done with the help of an appropriate regularization of the initial boundary value problem (for example, as in [1]).

We are going to use the same class of suitable weak solutions as F.-H. Lin, i.e., in our considerations it is assumed that $p \in L_{\frac{3}{2}}(Q_T)$. In the case calculations become shorter. But in the last section we describe changes in our constructions for the class of suitable weak solutions studied in [1].

The main purpose of the present paper is to prove that, for any suitable solution v and p, there is an open subset of Q_T, where the velocity field v is Hölder continuous in $z = (x, t)$ and show that the one-dimensional Hausdorff measure of the compliment of this subset is equal to zero. We assume that the external force f belongs to some parabolic variant of the Morrey space, containig $L_q(Q_T; \mathbb{R}^3)$ with $q > \frac{5}{2}$. Our considerations are local. More precisely, as in [1] we prove various criteria whether a point $z \in Q_T$ is regular or not. *We call a point $z \in Q_T$ regular if the velocity feild v is Hölder continuous in some neighborhood of the point z.* It differs from the definition given in [1], where boundness of v is required for regularity of z. The main criterion coincides with the main one in [1] for local boundness.

Our proof of the main criterion is splitted into three parts. In the first part we give a criterion for local Hölder continuity of v, using a blow-up procedure for a proper excess. This part is similar to the approach developed for investigations of partial regularity for generalized solutions to elliptic and parabolic systems, but the equation $\operatorname{div} v = 0$ and the presence of the pressure have required some special corrections. The methods of this part are applicable to other systems for divergence free fields, in particular, to the three-dimensional modified Navier–Stokes equations (see [15]).

In the second part we improve our preliminary criterion with the help of a special scaling. It was also used in [12], [1], [10] but in a different way. Then, in the third part, we get the final criterion of local regularity of suitable weak solutions. In this part we make use of some interesting observations made in [1] and [10]. It seems to us that our way is shorter than in [1] and more transparent than in [10].

2. Notation and main results

We denote by $\mathbb{M}^3$ the space of all real 3×3 matrices. Adopting summation over repeated Latin indices, running from 1 to 3, we shall use the following notation

$$u \cdot v = u_i v_i, \quad |u| = \sqrt{u \cdot u}, \quad u = (u_i) \in \mathbb{R}^3,\, v = (v_i) \in \mathbb{R}^3;$$
$$A : B = \mathrm{tr} A^* B = A_{ij} B_{ij}, \quad |A| = \sqrt{A : A},$$
$$A^* = (A_{ji}), \quad \mathrm{tr} A = A_{ii}, \quad A = (A_{ij}) \in \mathbb{M}^3,\, B = (B_{ij}) \in \mathbb{M}^3;$$
$$u \otimes v = (u_i v_j) \in \mathbb{M}^3, \quad Au = (A_{ij} u_j) \in \mathbb{R}^3, \quad u, v \in \mathbb{R}^3,\, A \in \mathbb{M}^3.$$

Let ω be a domain in some finite-dimensional space. We denote by $L_m(\omega;\mathbb{R}^n)$ and $W^l_m(\omega;\mathbb{R}^n)$ the known Lebesgue and Sobolev spaces of functions from ω into $\mathbb{R}^n$. The norm of the space $L_m(\omega;\mathbb{R}^n)$ is denoted by $\|\cdot\|_{m,\omega}$. If $m=2$, then we use the abbreviation $\|\cdot\|_\omega \equiv \|\cdot\|_{m,\omega}$.

Let T be a positive parameter, Ω be a domain in $\mathbb{R}^n$. We denote by $Q_T \equiv \Omega\times]0,T[$ the space-time cylinder. Space-time points are denoted by $z=(x,t)$, $z_0=(x_0,t_0)$ and etc.

For summable in Q_T scalar-valued, vector-valued and tensor-valued functions, we shall use the following differential operators

$$\partial_t v = \frac{\partial v}{\partial t}, \quad v_{,i} = \frac{\partial v}{\partial x_i}, \quad \nabla p = (p_{,i}), \quad \nabla u = (u_{i,j}),$$

$$\operatorname{div} v = v_{i,i}, \quad \operatorname{div}\tau = (\tau_{ij,j}), \quad \Delta u = \operatorname{div}\nabla u,$$

which are understood in the sense of distributions. Here x_i, $i=1,2,3$, are the Cartesian coordinates of a point $x\in\mathbb{R}^3$, and $t\in]0,T[$ is a moment of time.

For balls and parabolic cylinders, we shall use the notation

$$B(x_0,R)\equiv\{x\in\mathbb{R}^3 \parallel |x-x_0|<R\}, \quad B(\theta)\equiv B(0,\theta), \quad B\equiv B(1);$$

$$Q(z_0,R)\equiv B(x_0,R)\times]t_0-R^2,t_0[, \qquad Q(\theta)\equiv Q(0,\theta), \quad Q\equiv Q(1).$$

Various mean values of summable functions h, p and v are denoted as follows

$$⨍_{t_0-R^2}^{t_0} h\,dt \equiv \frac{1}{R^2}\int\limits_{t_0-R^2}^{t_0} h\,dt,$$

$$[p]_{x_0,R}(t)\equiv ⨍_{B(x_0,R)} p(x,t)\,dx \equiv \frac{1}{|B(R)|}\int\limits_{B(x_0,R)} p(x,t)\,dx,$$

$$(v)_{z_0,R}\equiv ⨍_{Q(z_0,R)} v\,dz \equiv \frac{1}{Q(R)}\int\limits_{Q(z_0,R)} v\,dz.$$

We are going to use a "parabolic" variant of Morrey's spaces. Given domain ω in $\mathbb{R}^3\times\mathbb{R}$ and positive number γ, we define the space

$$M_{2,\gamma}(\omega;\mathbb{R}^3)\equiv\big\{f\in L_{2,\mathrm{loc}}(\omega;\mathbb{R}^3) \parallel c_\gamma(f;\omega)<+\infty\big\}.$$

Here

$$c_\gamma(f;\omega)\equiv\sup\left\{\frac{1}{R^{\gamma-2}}\Big(⨍_{Q(z,R)}|f|^2\,dz'\Big)^{\frac12} \parallel Q(z,R)\Subset\omega,\ R>0\right\}.$$

Definition 2.1. Let Ω be a domain in $\mathbb{R}^3$ and T be a positive parameter. Suppose that a function f satisfies the condition

$$f\in M_{2,\gamma}(Q_T;\mathbb{R}^3) \tag{2.1}$$

for some positive γ. We say that a pair of functions v and p is a suitable weak solution to the Navier–Stokes equations in Q_T if the following three conditions hold. Fuctions v and p have the properties

$$\left.\begin{aligned} v \in L\infty(0,T;L_2(\Omega;\mathbb{R}^3)) \cap L_2(0,T;W^1_2(\Omega;\mathbb{R}^3)), \\ p \in L_{\frac{3}{2}}(Q_T), \end{aligned}\right\} \tag{2.2}$$

meet the Navier–Stokes equations

$$\left.\begin{aligned} \partial_t v + \operatorname{div}(v\otimes v) - \Delta\, v &= f - \nabla\, p, \\ \operatorname{div} v &= 0, \end{aligned}\right\} \tag{2.3}$$

in Q_T (in the sense of distributions) and satisfy the inequality

$$\left.\begin{aligned} &\int\limits_{\Omega} |v(x,t)|^2\phi(x,t)\,dx + 2\int\limits_{Q_t} |\nabla\, v|^2\phi\,dx\,dt' \le \\ &\le \int\limits_{Q_t} \big\{|v|^2(\partial_t\phi + \Delta\,\phi) + (|v|^2+2p)v\cdot\nabla\,\phi + 2f\cdot v\phi\big\}\,dx\,dt' \end{aligned}\right\} \tag{2.4}$$

for a.a. $t \in [0,T]$ and for all non-negative functions $\phi \in C_0^\infty(Q_T)$.

Our aim is to prove the following fact.

Theorem 2.2. *Let γ be an arbitrary positive constant. Let $\{\Omega, T, f, v, p\}$ be an arbitrary collection, satisfying Definition 2.1 with this constant γ. There is a positive number ε_*, depending only on γ, with the following property. Assume that for a point $z_0 \in Q_T$ the inequality*

$$\lim_{R\to 0}\sup \frac{1}{R}\int\limits_{Q(z_0,R)} |\nabla v|^2\,dz < \varepsilon_*(\gamma) \tag{2.5}$$

holds. Then z_0 is a regular point, i.e. the function $z \mapsto v(z)$ is Hölder continuous in some neighborhood of the point z_0.

Remark 2.3. It follows from Theorem 2.2 that the one-dimensional parabolic Hausdorff measure of the set of singular points is equal to zero and thus its parabolic Hausdorff dimension is not greater than one.

For details we refer the reader to the paper [1].

We shall work with the following functionals:

$$Y(z_0,R;v,p) \equiv Y_1(z_0,R;v) + Y_2(z_0,R;p),$$

$$Y_1(z_0,R;v) \equiv \Big(\fint_{Q(z_0,R)} |v-(v)_{z_0,R}|^3\,dz\Big)^{\frac{1}{3}},$$

$$Y_2(z_0,R;p) \equiv R\left(\fint_{Q(z_0,R)} |p-[p]_{x_0,R}|^{\frac{3}{2}}\,dz\right)^{\frac{2}{3}},$$

$$\Psi(z_0,R;v) \equiv R\,|(v)_{z_0,R}|,$$

$$\overline{Y}(z_0,R;v,p) \equiv \left(\fint_{Q(z_0,R)} |v|^3\,dz\right)^{\frac{1}{3}} + R\left(\fint_{Q(z_0,R)} |p|^{\frac{3}{2}}\,dz\right)^{\frac{2}{3}}.$$

The proof of Theorem 2.2 is divided into three parts. At first, we prove Lemmata 2.4 and 2.5.

Lemma 2.4. *Suppose that numbers θ, M, γ, β are chosen so that*

$$0<\theta\le\frac{1}{2},\quad M\ge 3,\quad 0<\beta<\gamma \tag{2.6}$$

and fixed. There are positive numbers ε_1 and R_1, depending on θ, M, γ, β only and having the following property. For each collection $\{\Omega, T, f, v, p\}$, satisfying Definition 2.1 with fixed above number γ, and for each cylinder $Q(z_0,R)$, satisfying the conditions

$$\left.\begin{array}{c} Q(z_0,R)\Subset Q_T,\quad 0<R<R_1,\\ \Psi(z_0,R;v)<M,\\ Y(z_0,R;v,p)+c_\gamma\big(f;Q_T\big)R^\beta<\varepsilon_1, \end{array}\right\} \tag{2.7}$$

the decay estimate

$$Y(z_0,\theta R;v,p)\le c_1\theta^{\alpha_1}\big(Y(z_0,R;v,p)+c_\gamma\big(f;Q_T\big)R^\beta\big), \tag{2.8}$$

is valid. Here $\alpha_1=\frac{2}{3}$ and a positive constant c_1 depends on M only.

Lemma 2.5. *Let numbers $\theta, M, \gamma, \beta, \beta_1$ be taken so that*

$$M\ge 3,\quad 0<\beta_1\le\beta<\gamma,\quad 0<\beta_1<\alpha_1, \tag{2.9}$$

$$0<\theta\le\frac{1}{2},\quad c_1(M)\theta^{\frac{\alpha_1-\beta_1}{2}}\le 1. \tag{2.10}$$

and fixed. Assume that we are given an arbitrary collection $\{\Omega, T, f, v, p\}$, satisfying Definition 2.1 with above fixed γ, and an arbitrary cylinder $Q(z_0,R)$, satisfying the conditions

$$\left.\begin{array}{l} Q(z_0,R)\Subset Q_T,\quad 0<R<R_1(\theta,M,\gamma,\beta),\\ \qquad\Psi(z_0,R;v)<\frac{M}{2},\\ Y(z_0,R;v,p)+c_\gamma R^\beta<\overline{\varepsilon}_1=\overline{\varepsilon}_1(\theta,M,\gamma,\beta,\beta_1)=\\ \qquad =\big(1-\theta^{\frac{\alpha_1-\beta_1}{2}}\big)\min\Big\{\dfrac{\varepsilon_1}{2},\dfrac{\theta^{\frac{5}{3}}(1-\theta^{\beta_1})}{R_1}\dfrac{M}{2}\Big\}, \end{array}\right\} \tag{2.11}$$

where $c_\gamma \equiv c_\gamma(f;Q_T)$, *and* ε_1, R_1 *are numbers of Lemma 2.4. Then, for any* $k=0,1,2,\dots$, *the inequalities*

$$\left.\begin{aligned} \Psi(z_0,\theta^k R;v) < M, \\ Y(z_0,\theta^k R;v,p) + c_\gamma\,(\theta^k R)^\beta < \varepsilon_1, \\ Y(z_0,\theta^{k+1}R;v,p) \le \theta^{(k+1)\beta_1}(1-\theta^{\frac{\alpha_1-\beta_1}{2}})^{-1}\big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\big) \end{aligned}\right\} \tag{2.12}$$

hold.

From Lemma 2.5 we deduce some auxilary criterion of local regularity and complete the first part of the proof of Theorem 2.2.

Proposition 2.8. *Let* $\{\Omega,T,f,v,p\}$ *be an arbitrary collection, satisfying Definition 2.1 with a given number* $\gamma>0$. *There are numbers* ε_0 *and* R_0, *depending on* γ *only and having the following property. Suppose that for a point* z_0 *the conditions*

$$\left.\begin{aligned} Q(z_0,R)\Subset Q_T, \quad 0<R<R_0(\gamma), \\ \overline{Y}(z_0,R;v,p)+c_\gamma(f;Q_T)\,R^{\frac{\gamma}{2}} < \varepsilon_0(\gamma) \end{aligned}\right\} \tag{2.13}$$

hold. Then the function $z\mapsto v(z)$ *is Hölder continuous in some neighborhood of the point* z_0, *i.e.* z_0 *is a regular point.*

Remark 2.7. The exponent of the Hölder continuity with respect to the parabolic metrics

$$d(z,z')\equiv |x-x'|+|t-t'|^{\frac{1}{2}}$$

can be taken, for instance, as $\frac{1}{2}\min\{\alpha_1,\gamma\}$.

The second part of the proof of Theorem 2.2 is based upon a special invariant structure of the Navier–Stokes equations that leads to some improvements of Proposition 2.6. As a result, we have the following criterion of local regularity.

Proposition 2.8. *Let* $\{\Omega,T,f,v,p\}$ *be an arbitrary collection, satisfying Definition 2.1 with a given numbers* $\gamma>0$. *Suppose that* $z_0\in Q_T$ *and*

$$\liminf_{R\to 0} R\,\overline{Y}(z_0,R;v,p) < \overline{\varepsilon}_0(\gamma) \equiv \frac{\varepsilon_0(\gamma)R_0(\gamma)}{8}. \tag{2.14}$$

Then z_0 *is a regular point.*

In the last part of the proof of Theorem 2.2 we show how the main result follows from Proposition 2.8.

Proposition 2.9. *Let* $\{\Omega,T,f,v,p\}$ *be an arbitrary collection, satisfying Definition 2.1 with a given numbers* $\gamma>0$. *Then there is a number* ε_*, *depending on* γ *only, such that* (2.5) *implies* (2.14).

3. Proof of Theorem 2.2. Part I

Proof of Lemma 2.4. The lemma is proved by contradiction. So, assume that there are numbers θ, M, γ, β, satisfying conditions (2.7), sequences of collections $\{\Omega_m, T_m, f^m, v^m, p^m\}_{m=1}^{\infty}$, satisfying Definition 2.1 with the constant γ, and cylinders $\{Q(z^m, R_m)\}_{m=1}^{\infty}$ such that:

$$\left.\begin{aligned} Q(z^m, R_m) \Subset Q^m_{T_m} \equiv \Omega_m \times]0, T_m[, \quad R_m \to 0,\\ Y(z^m, R_m; v^m, p^m) + d_m\, R_m^\beta \equiv \varepsilon_m \to 0,\\ Y(z^m, \theta R_m; v^m, p^m) \geq c_1 \theta^{\alpha_1} \varepsilon_m, \end{aligned}\right\} \tag{3.1}$$

as $m \to +\infty$. Here $d_m \equiv c_\gamma(f^m; Q^m_{T_m})$ and constants c_1 and α_1 will be chosen in an appropriate way to obtain the contradiction. We now consider the scaling

$$\begin{aligned} x - x^m &= R_m y, \quad x \in B(x^m, R_m), \quad y \in B \equiv B(0,1);\\ t - t_m &= R_m^2 s, \quad t \in]t_m - R_m^2, t_m[, \quad s \in]-1, 0[;\\ w^m(e) &= (v^m(z) - (v^m)_{z^m, R_m})\varepsilon_m^{-1}, \quad q^m(e) = (p^m(z) - [p^m]_{x^m, R_m}(t))\varepsilon_m^{-1} R_m,\\ g^m(e) &= f^m(z), \quad e = (y, s) \in Q \equiv B \times]-1, 0[, \end{aligned}$$

and get after changing the variables

$$\begin{aligned} \varepsilon_m \nabla_y w^m(e) &= \nabla_x v^m(z) R_m, \quad \varepsilon_m \nabla_y^2 w^m(e) = \nabla_x^2 v^m(z) R_m^2,\\ \varepsilon_m \partial_s w^m(e) &= \partial_t v^m(z) R_m^2, \qquad \varepsilon_m \nabla_y q^m(e) = \nabla_x p^m(z) R_m^2,\\ (w^m)_{,1} &= 0, \qquad [q^m]_{,1}(s) = 0, \quad s \in]-1, 0[,\\ \frac{1}{\varepsilon_m} Y_1(z^m, \theta R_m; v^m) &= Z_1^m(\theta) \equiv \Big(\fint_{Q(\theta)} |w^m - (w^m)_{,\theta}|^3\, de \Big)^{\frac{1}{3}},\\ \frac{1}{\varepsilon_m} Y_2(z^m, \theta R_m; p^m) &= Z_2^m(\theta) \equiv \theta \Big(\fint_{Q(\theta)} |q^m - [q^m]_{,\theta}|^{\frac{3}{2}}\, de \Big)^{\frac{2}{3}}, \end{aligned}$$

$$\left.\begin{aligned} &Z^m(\theta) \equiv Z_1^m(\theta) + Z_2^m(\theta) \geq c_1 \theta^{\alpha_1},\\ &Z^m(1) = \Big(\fint_Q |w^m|^3\, de \Big)^{\frac{1}{3}} + \Big(\fint_Q |q^m|^{\frac{3}{2}}\, de \Big)^{\frac{2}{3}} + \frac{d_m R_m^\beta}{\varepsilon_m} = 1. \end{aligned}\right\} \tag{3.2}$$

Here and in what follows we abbreviate

$$(v)_{,\theta} = (v)_{0,\theta}, \quad [p]_{,\theta} = [p]_{0,\theta}.$$

Next, changing the varaibles in the integral identity, corresponding to (2.3), gives us the relation

$$\left.\begin{aligned} &\int\limits_Q \big(-w^m\cdot\partial_s u - w^m\cdot\Delta\, u\big)de = \\ &= \int\limits_Q \Big\{(w^m\otimes R_m a^m):\nabla\, u + \varepsilon_m R_m (w^m\otimes w^m):\nabla\, u + \\ &\quad + q^m \operatorname{div} u\Big\}de + \frac{R_m^2}{\varepsilon_m}\int\limits_Q g^m\cdot u\, de, \quad a^m \equiv (v^m)_{z_m,R_m}. \end{aligned}\right\} \tag{3.3}$$

It is valid for all $u \in C_0^\infty(Q;\mathbb{R}^3)$.

By (3.2), after passing to subsequences (still denoted by the same symbols) it may be assumed that:

$$\left.\begin{aligned} w^m \rightharpoonup w \quad &\text{in} \quad L_3(Q;\mathbb{R}^3),\\ q^m \rightharpoonup q \quad &\text{in} \quad L_{\frac{3}{2}}(Q;\mathbb{R}^3),\\ (w)_{,1} = 0, \qquad [q]_{,1}(s) &= 0, \quad s\in]-1,0[. \end{aligned}\right\} \tag{3.4}$$

Let us introduce functionals

$$Z(\theta) \equiv Z_1(\theta) + Z_2(\theta), \quad Z_1(\theta) \equiv \Big(\fint_{Q(\theta)} |w-(w)_{,\theta}|^3\, de\Big)^{\frac13},$$

$$Z_2(\theta) \equiv \theta\Big(\fint_{Q(\theta)} |q-[q]_{,\theta}|^{\frac32}\, de\Big)^{\frac23}.$$

We see now, by (3.2) and (3.4), that:

$$Z(1) = \Big(\fint_Q |w|^3\, de\Big)^{\frac13} + \Big(\fint_Q |q|^{\frac32}\, de\Big)^{\frac23} \le 1. \tag{3.5}$$

According to the definition of the quantity d_m and (3.2) we have

$$\left.\begin{aligned} \frac{R_m^2}{\varepsilon_m}\Big(\fint_Q |g^m|^2\, de\Big)^{\frac12} &= \frac{R_m^2}{\varepsilon_m}\Big(\fint_{Q(z^m,R_m)} |f^m|^2\, dz\Big)^{\frac12} \le \\ \le \frac{d_m R_m^{\gamma-2}}{\varepsilon_m} R_m^2 &= \frac{d_m R_m^{\beta}}{\varepsilon_m} R_m^{\gamma-\beta} \le R_m^{\gamma-\beta} \to 0 \end{aligned}\right\} \tag{3.6}$$

as $m\to+\infty$. Since

$$|R_m a_m| = \Psi(z^m, R_m; v^m) < M, \tag{3.7}$$

without loss of generality it may be assumed that:

$$R_m a_m \to b \quad \text{in} \quad \mathbb{R}^3 \quad \text{and} \quad |b| \le M. \tag{3.8}$$

So, it follows from (3.3)–(3.6) and (3.8) that functions w and p satisfy the linear system of PDE's with constant coefficients

$$\left.\begin{aligned} \partial_s\, w + \operatorname{div}(w\otimes b) - \Delta\, w &= -\nabla\, q,\\ \operatorname{div} w &= 0 \end{aligned}\right\} \text{ in } Q \tag{3.9}$$

in the sense of distributions. The same arguments as in the case of the Stokes system allow us to claim that w is Hölder continuous in the closure of the cylinder $Q(\frac{1}{2})$ and, moreover, the estimate

$$|w(e) - w(e')| \leq c_{31}(M)(|y - y'| + |s - s'|^{\frac{1}{3}})$$

holds for any $e \in Q(\frac{1}{2})$ and any $e' \in Q(\frac{1}{2})$. The latter leads to the bound

$$Z_1(\theta) \leq c_{32}(M)\theta^{\frac{2}{3}}. \tag{3.10}$$

Now let us discuss compactness of the sequence $\{w^m\}_{m=1}^{\infty}$. For any function $u \in C_0^1(0,T;\overset{\circ}{W}{}_2^2(B;\mathbb{R}^3))$, we derive from (3.3) the estimate (see (3.2), (3.7) and (3.6))

$$\begin{aligned} -\int\limits_Q w^m \cdot \partial_s u\, de = \int\limits_Q \Big\{ & w^m \cdot \Delta u + (w^m \otimes R_m a^m) : \nabla u + q^m \mathrm{div}\, u + \\ & + \varepsilon_m R_m (w^m \otimes w^m) : \nabla u + \frac{R_m^2}{\varepsilon_m} g^m \cdot u \Big\} de \leq \\ \leq\ & c_{33}(M)\|u\|_{L_3(0,T;W_2^2(B;\mathbb{R}^3))}. \end{aligned}$$

It says that:

$$\{\partial_s w^m\}_{m=1}^{\infty} \quad \text{is bounded in} \quad L_{\frac{3}{2}}(0,T;(\overset{\circ}{W}{}_2^2(B;\mathbb{R}^3))'). \tag{3.11}$$

Energy inequality (2.4) for v^m and q^m gives the following relation

$$\left.\begin{aligned} & \int\limits_B |w^m(y,s)|^2 \phi(y,s)\, dy + 2 \int\limits_{B\times]-1,s[} |\nabla w^m|^2 \phi\, dy\, ds' \leq \\ \leq & \int\limits_{B\times]-1,s[} \Big\{ |w^m|^2(\partial_s \phi + \Delta \phi) + R_m |w^m|^2 (a^m + \varepsilon_m w^m) \cdot \nabla \phi + \\ & + q^m w^m \cdot \nabla \phi + \frac{R_m^2}{\varepsilon_m} g^m \cdot w^m \phi \Big\} dy\, ds'. \end{aligned}\right\} \tag{3.12}$$

Inequality (3.12) holds for a.a. $s \in]-1,0[$ and for all non-negative functions $\phi \in C_0^\infty(Q)$. Recalling (3.2), (3.6) and (3.7), we deduce from (3.12) the estimate

$$\operatorname*{ess\ sup}_{s \in]-(\frac{3}{4})^2, 0[} \|w^m(s)\|^2_{B(\frac{3}{4})} + \|\nabla w^m\|^2_{Q(\frac{3}{4})} \leq c_{34}(M). \tag{3.13}$$

Bound (3.13) together with the known multiplicative inequality yields another important estimate

$$\|w^m\|_{\frac{10}{3}, Q(\frac{3}{4})} \leq c'_{34}(M). \tag{3.14}$$

Now, by (3.11), (3.13) and (3.14), a subsequence (still denoted by the same symbol) exists such that:

$$w^m \to w \quad \text{in} \quad L_3(Q(\frac{3}{4}); \mathbb{R}^3) \tag{3.15}$$

and, therefore,

$$Z_1^m(\theta) \to Z_1(\theta).$$

But then (3.2) and (3.10) imply the estimate

$$\left.\begin{aligned} c_1\theta^{\alpha_1} &\leq \lim_{m\to+\infty} Z_1^m(\theta) + \liminf_{m\to+\infty} Z_2^m(\theta) \leq \\ &\leq c_{32}(M)\theta^{\frac{2}{3}} + \liminf_{m\to+\infty} Z_2^m(\theta). \end{aligned}\right\} \tag{3.16}$$

Let us insert $u = \chi\nabla q$, where $q \in C_0^\infty(B)$ and $\chi \in C_0^\infty(0,1)$, into (3.3). Then, by arbitrariness of χ, by $\operatorname{div} w^m = 0$ and by the definition of the space $\overset{\circ}{W}{}^2_3(B)$, we obtain the identity

$$\left.\begin{aligned} \frac{R_m^2}{\varepsilon_m}\int\limits_B g^m(y,s) : \nabla q(y)dy + R_m\varepsilon_m\int\limits_B w^m(y,s)\otimes w^m(y,s) : \nabla^2 q(y)dy = \\ = -\int\limits_B q^m(y,s)\Delta\, q(y)\,dy. \end{aligned}\right\} \tag{3.17}$$

It is valid for a.a. $s \in]-1,0[$ and for all $q \in \overset{\circ}{W}{}^2_3(B)$.

One may represent the pressure q^m as the sum

$$q^m = q_1^m + q_2^m.$$

Here $q_1^m(s)$ satisfies the identity

$$\left.\begin{aligned} \frac{R_m^2}{\varepsilon_m}\int\limits_B g^m(y,s) : \nabla q(y)\,dy + R_m\varepsilon_m\int\limits_B w^m(y,s)\otimes w^m(y,s) : \nabla^2 q(y)\,dy = \\ = -\int\limits_B q_1^m(y,s)\Delta\, q(y)\,dy \end{aligned}\right\} \tag{3.18}$$

for all $q \in W_3^2(B)\cap \overset{\circ}{W}{}^1_3(B)$ and the function $y \mapsto q_2^m(y,s)$ is harmonic in B, i.e.

$$\Delta q_2^m(s) = 0 \quad \text{in} \quad B. \tag{3.19}$$

It is known that for the solution of the boundary value problem

$$\left.\begin{aligned} \Delta\, q(s) = |q_1^m(s)|^{\frac{1}{2}}\operatorname{sign}\{q_1^m(s)\} \text{ in } B, \\ q(s) = 0 \text{ on } \partial\, B, \end{aligned}\right\} \tag{3.20}$$

the estimate

$$\|q(s)\|_{3,B} + \|\nabla q(s)\|_{3,B} + \|\nabla^2 q(s)\|_{3,B} \leq c_{35}\|q_1^m(s)\|^{\frac{1}{2}}_{\frac{3}{2},B} \tag{3.21}$$

holds. Here c_{35} is an absolute positive constant. Now, let us insert this $q(s)$ into identity (3.18). Then relations (3.20) and bound (3.21) give the inequalities

$$\left.\begin{aligned} \|q_1^m\|_{\frac{3}{2},Q} &\le c'_{35}\Big[R_m\varepsilon_m\Big(⨍_Q|w^m|^3de\Big)^{\frac{2}{3}} + \frac{R_m^2}{\varepsilon_m}\Big(⨍_Q|g^m|^{\frac{3}{2}}de\Big)^{\frac{2}{3}}\Big] \\ &\le \Big(\text{see (3.2) and (3.6)}\Big) \le \\ &\le c'_{35}\Big[R_m\varepsilon_m + R_m^{\gamma-\beta}\Big] \to 0 \text{ as } m \to +\infty. \end{aligned}\right\} \qquad (3.22)$$

Next, we have

$$Z_2^m(\theta) \le \theta\Big[\Big(⨍_{Q(\theta)}|q_1^m - [q_1^m]_{,\theta}|^{\frac{3}{2}}de\Big)^{\frac{2}{3}} + \Big(⨍_{Q(\theta)}|q_2^m - [q_2^m]_{,\theta}|^{\frac{3}{2}}de\Big)^{\frac{2}{3}}\Big].$$

By (3.22),

$$\limsup_{m\to\infty} Z_2^m(\theta) \le \limsup_{m\to\infty} \theta\Big(⨍_{Q(\theta)}|q_2^m - [q_2^m]_{,\theta}|^{\frac{3}{2}}de\Big)^{\frac{2}{3}}.$$

Combining the last inequality with (3.16), we get

$$c_1\theta^{\alpha_1} \le c_{32}(M)\theta^{\frac{2}{3}} + \limsup_{m\to\infty} \theta\Big(⨍_{Q(\theta)}|q_2^m - [q_2^m]_{,\theta}|^{\frac{3}{2}}de\Big)^{\frac{2}{3}}. \qquad (3.23)$$

Harmonicity of the function $y \mapsto q_2^m(y,s)$ (see (3.19)) leads to the estimates

$$\begin{aligned} |q_2^m(y,s) - [q_2^m]_{,\theta}(s)| &= \Big|⨍_{B(\theta)}\Big(q_2^m(y,s) - q_2^m(y',s)\Big)dy'\Big| \le \\ &\le \theta\|\nabla q_2^m(s)\|_{\infty,B(\frac{1}{2})} \le c_{36}\theta\|q_2^m(s)\|_{\frac{3}{2},B}, \end{aligned}$$

where c_{36} is an absolute positive constant. So, we have

$$\left.\begin{aligned} \theta\Big(⨍_{Q(\theta)}|q_2^m - [q_2^m]_{,\theta}|^{\frac{3}{2}}de\Big)^{\frac{2}{3}} &\le c'_{36}\theta\Big(\frac{\theta^{3+\frac{3}{2}}}{\theta^5}\int_{-1}^{0}\|q_2^m(s)\|^{\frac{3}{2}}ds\Big)^{\frac{2}{3}} \le \\ &\le c'_{36}\theta^{\frac{2}{3}}\|q_2^m\|_{\frac{3}{2},Q} \le c''_{36}\theta^{\frac{2}{3}}\Big(\|q^m\|_{\frac{3}{2},Q} + \|q_1^m\|_{\frac{3}{2},Q}\Big) \le \\ &\le \Big(\text{see (3.2) and (3.22)}\Big) \le c_{37}\theta^{\frac{2}{3}}. \end{aligned}\right\} \qquad (3.24)$$

Here c_{37} is an absolute positive constant. Now from (3.23) and (3.24) we conclude that:

$$c_1\theta^{\alpha_1} \le (c_{32}(M) + c_{37})\theta^{\frac{2}{3}}.$$

To obtain the contradiction it is enough to take

$$c_1(M) = 2(c_{32}(M) + c_{37}), \qquad \alpha_1 = \frac{2}{3}.$$

Lemma 2.4 is proved.

Now let us establish a simple relation between functionals Ψ and Y_1. For $0<\theta<1$, we have

$$|(v)_{z_0,R}-(v)_{z_0,\theta R}|\leq \frac{1}{\theta^{\frac{5}{3}}}Y_1(z_0,R;v).$$

But then

$$|(v)_{z_0,\theta^k R}|\leq |(v)_{z_0,R}|+\frac{1}{\theta^{\frac{5}{3}}}\sum_{i=0}^{k-1}Y_1(z_0,\theta^i R;v)$$

and, therefore,

$$\Psi(z_0,\theta^k R;v)\leq \frac{\theta^k R}{\theta^{\frac{5}{3}}}\sum_{i=0}^{k-1}Y_1(z_0,\theta^i R;v)+\theta^k\Psi(z_0,R;v). \tag{3.25}$$

Proof of Lemma 2.5. The lemma is proved by induction on k. Suppose first that $k=0$. We have

$$Y(z_0,R;v,p)+c_\gamma R^\beta<\overline{\varepsilon}_1\leq (1-\theta^{\frac{1}{2}(\alpha_1-\beta_1)})\frac{\varepsilon_1}{2}<\varepsilon_1.$$

So, all conditions of Lemma 2.4 are fulfilled and thus we obtain

$$\begin{aligned}Y(z_0,\theta R;v,p)\;&\leq\;\theta^{\frac{1}{2}(\alpha_1+\beta_1)}(c_1\theta^{\frac{1}{2}(\alpha_1-\beta_1)})\big(Y(z_0,R;v,p)+c_\gamma R^\beta\big)\leq\\ &\leq\;\big(\text{see (2.10)}\big)\leq\theta^{\frac{1}{2}(\alpha_1+\beta_1)}\Big(Y(z_0,R;v,p)+c_\gamma R^\beta\Big).\end{aligned}$$

Hence, for $k=0$, all statements (2.12) of the lemma are proved.

Now we assume that, for $s=0,1,\dots,k$, the following inequalities hold

$$\left.\begin{aligned}&\Psi(z_0,\theta^s R;v)<M,\\ &Y(z_0,\theta^s R;v,p)+c_\gamma\,(\theta^s R)^\beta<\varepsilon_1,\\ &Y(z_0,\theta^{s+1}R;v,p)\leq\theta^{(s+1)\beta_1}\big(1-\theta^{\frac{\alpha_1-\beta_1}{2}}\big)^{-1}\big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\big).\end{aligned}\right\}\tag{3.26}$$

By (3.25),

$$\begin{aligned}\Psi(z_0,\theta^{k+1}R;v)\;&\leq\;\frac{\theta^{k+1}R}{\theta^{\frac{5}{3}}}\sum_{i=0}^{k}Y_1(z_0,\theta^i R;v)+\theta^{k+1}\Psi(z_0,R;v)<\\ &<\;\frac{R_1}{\theta^{\frac{5}{3}}}\frac{1}{1-\theta^{\beta_1}}\frac{\overline{\varepsilon}_1}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}+\frac{M}{2}\leq\\ &\leq\;\frac{R_1}{\theta^{\frac{5}{3}}}\frac{1}{1-\theta^{\beta_1}}\frac{1}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}\big(1-\theta^{\frac{\alpha_1-\beta_1}{2}}\big)\frac{\theta^{\frac{5}{3}}(1-\theta^{\beta_1})}{R_1}\frac{M}{2}+\frac{M}{2}=M.\end{aligned}$$

The third relation in (3.26) and (2.11) yield

$$\begin{aligned}
&Y\ (z_0,\theta^{k+1}R;v,p)+c_\gamma\,(\theta^{k+1}R)^\beta\le\\
&\le\ \frac{\theta^{(k+1)\beta_1}}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}\big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\big)+c_\gamma\,(\theta^{k+1}R)^\beta\le\\
&\le\ (\beta_1\le\beta,\quad 0<\theta<1)\le\\
&\le\ \theta^{(k+1)\beta_1}\left(\frac{Y(z_0,R;v,p)+c_\gamma\,R^\beta}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}+c_\gamma\,R^\beta\right)\le\\
&\le\ \theta^{(k+1)\beta_1}2\frac{Y(z_0,R;v,p)+c_\gamma\,R^\beta}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}<\varepsilon_1.
\end{aligned}$$

Taking into account the last two inequalities, we observe that the cylinder $Q(z_0,\theta^{k+1}R)$ satisfies all conditions of Lemma 2.4 and, therefore,

$$\begin{aligned}
Y(z_0,\theta^{k+2}R;v,p)\ &\le\ \theta^{\frac12(\alpha_1+\beta_1)}\Big(Y(z_0,\theta^{k+1}R;v,p)+c_\gamma\,(\theta^{k+1}R)^\beta\Big)\le\\
&\le\ \theta^{\frac12(\alpha_1+\beta_1)}\Big(\theta^{(k+1)\beta_1}\frac{Y(z_0,R;v,p)+c_\gamma\,R^\beta}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}+c_\gamma\,(\theta^{k+1}R)^\beta\Big)\le\\
&\le\ \theta^{\frac12(\alpha_1+\beta_1)}\theta^{(k+1)\beta_1}\Big(\frac{Y(z_0,R;v,p)+c_\gamma\,R^\beta}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}+c_\gamma\,R^\beta\Big)=\\
&=\ \theta^{(k+2)\beta_1}\theta^{\frac{\alpha_1-\beta_1}{2}}\Big(\frac{Y(z_0,R;v,p)+c_\gamma\,R^\beta}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}+c_\gamma\,R^\beta\Big)<\\
&<\ \theta^{(k+2)\beta_1}\frac{1}{1-\theta^{\frac{\alpha_1-\beta_1}{2}}}\Big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\Big).
\end{aligned}$$

Lemma 2.5 is proved.

Let us formulate two consequences of Lemma 2.5.

Lemma 3.1. *Suppose that all conditions of Lemma 2.5 hold. Then a positive constant $c_{38}=c_{38}(\theta,\beta_1)$ exists such that*

$$Y(z_0,\rho;v,p)\le c_{38}\Big(\frac{\rho}{R}\Big)^{\beta_1}\Big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\Big)\tag{3.27}$$

for all $\rho\in]0,R]$.

Proof. For an arbitrary number $\rho\in]0,R]$, we choose a non-negative integer k so that:

$$\theta^{k+1}<\frac{\rho}{R}\le\theta^k,$$

where θ is the number of Lemma 2.5. Then we have

$$\begin{aligned}Y(z_0,\rho;v,p) \le\ & \Big(\fint_{Q(z_0,\rho)} |v-(v)_{z_0,\theta^k R}|^3\,dz\Big)^{\frac13} + \\ & +|(v)_{z_0,\theta^k R}-(v)_{z_0,\rho}| + \rho\Big(\fint_{Q(z_0,\rho)}|p-[p]_{x_0,\theta^k R}|^{\frac32}\,dz\Big)^{\frac23} + \\ & +\rho\Big(\fint_{Q(z_0,\rho)}|[p]_{x_0,\theta^k R}-[p]_{x_0,\rho}|^{\frac32}\,dz\Big)^{\frac23} \le \\ \le\ & c'_{38}(\theta) Y(z_0,\theta^k R;v,p) \le \Big(\text{see (2.12)}\Big) \le \\ \le\ & c'_{38}(\theta)\theta^{k\beta_1}\frac{1}{1-\theta^{\frac12(\alpha_1-\beta_1)}}\Big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\Big) \le \\ \le\ & c'_{38}(\theta)\Big(\frac{1}{\theta}\frac{\rho}{R}\Big)^{\beta_1}\frac{1}{1-\theta^{\frac12(\alpha_1-\beta_1)}}\Big(Y(z_0,R;v,p)+c_\gamma\,R^\beta\Big).\end{aligned}$$

Lemma 3.1 is proved.

Lemma 3.2. *Let $\{\Omega,T,f,v,p\}$ be an arbitrary collection, satisfying Definition 2.1 with a given number $\gamma>0$. There are positive numbers ε_0, R_0 and c_{39}, depending on γ only and having the following property. Suppose that for a point z_0 conditions* (2.13) *are fulfilled. Then*

$$Y(z_0,\rho;v,p) \le c_{39}\Big(\frac{\rho}{R}\Big)^{\beta_1(\gamma)} \tag{3.28}$$

for all $\rho\in]0,R]$. Here $\beta_1(\gamma)=\frac12\min\{\alpha_1,\gamma\}$.

Proof. We let

$$M=3,\quad \beta=\beta(\gamma)=\frac{\gamma}{2},\quad \theta=\theta(\gamma)=\min\Big\{\frac12, c_1(3)^{-\frac{2}{\alpha_1-\beta_1}}\Big\}.$$

Obviously, numbers $\theta(\gamma), M=3, \beta(\gamma), \beta_1(\gamma)$ satisfy all conditions (2.9), (2.10) for each $\gamma>0$.

We also have two simple inequalities

$$\Psi(z_0,R;v) \le R\Big(\fint_{Q(z_0,R)}|v|^3\,dz\Big)^{\frac13} \le R\,\overline{Y}(z_0,R;v,p) \tag{3.29}$$

and

$$Y(z_0,R;v,p) \le 2\,\overline{Y}(z_0,R;v,p). \tag{3.30}$$

Now, let

$$\left.\begin{aligned} R_0(\gamma)\ &\equiv\ R_1(\theta(\gamma),3,\gamma,\beta(\gamma)),\\ \varepsilon_0(\gamma)\ &\equiv\ \frac12\min\Big\{\frac{3}{R_0(\gamma)}, \overline{\varepsilon}_1\big(\theta(\gamma),3,\gamma,\beta(\gamma),\beta_1(\gamma)\big)\Big\},\\ c_{39}(\gamma)\ &\equiv\ 2\,c_{38}\big(\theta(\gamma),\beta_1(\gamma)\big)\,\varepsilon_0(\gamma).\end{aligned}\right\} \tag{3.31}$$

Assume that conditions (2.13) hold. Then from (3.29)–(3.31) it follows that

$$\Psi(z_0,R;v) < R_0\,\overline{Y}(z_0,R;v,p) < R_0\,\varepsilon_0 < \frac{3}{2}$$

and

$$\begin{aligned} Y(z_0,R;v,p) + c_\gamma\,R^{\beta(\gamma)} &\le 2\Big(\overline{Y}(z_0,R;v,p) + c_\gamma\,R^{\beta(\gamma)}\Big) < \\ &< 2\,\varepsilon_0(\gamma) \le \overline{\varepsilon}_1(\theta(\gamma),3,\gamma,\beta(\gamma),\beta_1(\gamma)). \end{aligned}$$

So, the set $\{\Omega,T,f,v,p\}$, numbers γ, $\theta(\gamma)$, $M=3$, $\beta(\gamma)$ and $\beta_1(\gamma)$ satisfy all conditions of Lemma 3.1 and, by (3.27), we get

$$\begin{aligned} Y(z_0,\rho;v,p) &\le c_{38}(\theta(\gamma),\beta_1(\gamma))\left(\frac{\rho}{R}\right)^{\beta_1(\gamma)}\big(Y(z_0,R;v,p)+c_\gamma\,R^{\beta(\gamma)}\big) \le \\ &\le c_{38}(\theta(\gamma),\beta_1(\gamma))\left(\frac{\rho}{R}\right)^{\beta_1(\gamma)}\Big(2\,\overline{Y}(z_0,R;v,p)+c_\gamma\,R^{\beta(\gamma)}\Big) \le \\ &\le c_{38}(\theta(\gamma),\beta_1(\gamma))\left(\frac{\rho}{R}\right)^{\beta_1(\gamma)} 2\,\varepsilon_0(\gamma). \end{aligned}$$

Lemma 3.2 is proved.

Proof of Proposition 2.6. Since the function $z \mapsto \overline{Y}(z_0,R;v,p)$ is continuous, a neighborhood $\mathcal{O}(z_0)$ of the point z_0 exists such that

$$Q(z,R) \Subset Q_T, \quad 0 < R < R_0(\gamma),$$
$$\overline{Y}(z,R;v,p) + c_\gamma(f;Q_T)\,R^{\beta(\gamma)} < \varepsilon_0(\gamma)$$

for all $z \in \mathcal{O}(z_0)$. By Lemma 3.2,

$$Y(z,\rho;v,p) \le c_{39}(\gamma)\Big(\frac{\rho}{R}\Big)^{\beta_1(\gamma)}$$

for all $\rho \in]0,R]$ and for all $z \in \mathcal{O}(z_0)$. A parabolic version of the Campanato criterion completes the proof of Proposition 2.6. Proposition 2.6 is proved.

4. Proof of Theorem 2.2. Part II

It is easy to see that Proposition 2.8 follows from Proposition 4.1.

Proposition 4.1. *Let $\{\Omega,T,f,v,p\}$ be an arbitrary collection, satisfying Definition 2.1 with a given number $\gamma > 0$. Let $\varepsilon_0(\gamma)$ and $R_0(\gamma)$ be numbers of Proposition 2.6. Suppose that, for a point z_0, the conditions*

$$\left.\begin{aligned} \widetilde{Q}(z_0,2R) \equiv B(x_0,2R)\times]t_0-(2R)^2,t_0+(2R)^2[&\Subset Q_T, \\ 0 < R < \frac{1}{2}R_0(\gamma)&, \\ \frac{2R}{R_0(\gamma)}\big[\overline{Y}(z,R;v,p)+c_\gamma(f;Q_T)\,R^{\frac{\gamma}{2}}\big] &< \varepsilon_0(\gamma) \end{aligned}\right\} \tag{4.1}$$

hold. Then z_0 is a regular point.

Proof. Let us change the variables in the following way

$$y = \frac{x - x_0}{\tau}, \quad s = \frac{t - t_0}{\tau^2} + R_0^2(\gamma), \quad \tau \equiv \frac{2R}{R_0(\gamma)} < 1,$$

$$v^\tau(e) = \tau v(z), \quad p^\tau(e) = \tau^2 p(z), \quad f^\tau(e) = \tau^3 f(z),$$

where $z = (x, t)$, $e = (y, s)$. As a result, we have:

$$\left.\begin{array}{c} z_0 \leftrightarrow e_0, \quad B(x_0, 2R) \leftrightarrow B(R_0(\gamma)), \quad Q(z_0, R) \leftrightarrow Q(e_0, \frac{1}{2} R_0(\gamma)), \\ \widetilde{Q}(z_0, 2R) \leftrightarrow \widehat{Q}(\gamma) \equiv B(R_0(\gamma)) \times]0, 2\,R_0^2(\gamma)[, \end{array}\right\} \tag{4.2}$$

where $e_0 = (0, R_0^2(\gamma))$;

$$\begin{array}{ll} \partial_s\, v^\tau(e) = \tau^3 \partial_t v(z), & \nabla_y\, v^\tau(e) = \tau^2 \nabla_x\, v(z), \\ \nabla_y^2\, v^\tau(e) = \tau^3 \nabla_x^2\, v(z), & \nabla_y\, p^\tau(e) = \tau^3 \nabla_x\, p(z). \end{array}$$

It is easy to see that v^τ and p^τ have the properties

$$\left.\begin{array}{c} v^\tau \in L_\infty(0, 2R_0^2(\gamma); L_2(B(R_0(\gamma)); \mathbb{R}^3)), \\ v^\tau \in L_2(0, 2R_0^2(\gamma); W_2^1(B(R_0(\gamma)); \mathbb{R}^3)), \\ p^\tau \in L_{\frac{3}{2}}(\widehat{Q}(\gamma)), \end{array}\right\} \tag{4.3}$$

satisfy the equations

$$\left.\begin{array}{rl} \partial_s\, v^\tau(e) + \mathrm{div}_y\, v^\tau \otimes v^\tau - \Delta_y\, v^\tau &= f^\tau - \nabla_y\, p^\tau, \\ \mathrm{div}_y\, v^\tau &= 0, \end{array}\right\} \tag{4.4}$$

in $\widehat{Q}(\gamma)$ and the local energy inequality

$$\left.\begin{array}{l} \displaystyle\int\limits_{B(R_0(\gamma))} |v^\tau(y,s)|^2\, \phi(y,s)\, dy + 2 \int\limits_{B(R_0(\gamma)) \times]0,s[} \phi\, |\nabla_y\, v^\tau|^2\, dy\, ds' \leq \\ \displaystyle\leq \int\limits_{B(R_0(\gamma)) \times]0,s[} \Big\{ |v^\tau|^2 (\partial_s\, \phi + \Delta\, \phi) + (|v^\tau|^2 + 2\, p^\tau) v^\tau \cdot \nabla_y\, \phi + \\ \qquad\qquad + 2 f^\tau \cdot v^\tau \phi \Big\}\, dy\, ds' \end{array}\right\} \tag{4.5}$$

for a.a. $s \in]0, 2R_0^2(\gamma)[$ and for all non-negative functions $\phi \in C_0^\infty(\widehat{Q}(\gamma))$.

Now, let us show that:

$$c_\gamma(f^\tau; \widehat{Q}(\gamma)) \leq \tau^{\gamma+1} c_\gamma(f; Q_T). \tag{4.6}$$

By the definition,

$$c_\gamma(f^\tau; \widehat{Q}(\gamma)) \equiv \sup \left\{ \frac{1}{r^{\gamma-2}} \Big(\fint_{Q(e,r)} |f^\tau|^2\, de' \Big)^{\frac{1}{2}} \;\|\; \quad Q(e,r) \Subset \widehat{Q}(\gamma),\ r > 0 \right\},$$

where

$$Q(e,r) = \Big\{ e' = (y', s') \parallel \quad |y - y'| < r,\ -r^2 < s' - s < 0 \Big\}.$$

For

$$Q(z,\tau r) = \Big\{ z' = (x', t') \parallel \quad |x - x'| < \tau r,\ -\tau^2 r^2 < t' - t < 0 \Big\},$$

we have

$$Q(z,\tau r) \leftrightarrow Q(e,r), \quad z = (x,t),\ e = (y,s), \quad Q(z,\tau r) \Subset \widetilde{Q}(z_0, 2R),$$

and, therefore,

$$\frac{1}{r^{\gamma-2}} \Big(\fint_{Q(e,r)} |f^\tau|^2 \, de' \Big)^{\frac{1}{2}} = \frac{1}{r^{\gamma-2}} \Big(\fint_{Q(z,\tau r)} |\tau^3 f|^2 \, dz' \Big)^{\frac{1}{2}} \le$$
$$\le \tau^{\gamma+1} \frac{1}{(\tau r)^{\gamma-2}} \Big(\fint_{Q(z,\tau r)} |f|^2 \, dz' \Big)^{\frac{1}{2}} \le \tau^{\gamma+1} c_\gamma(f; Q_T).$$

So, as it follows from (4.3)–(4.6), the set $\{B(R_0(\gamma)), 2\, R_0^2(\gamma), f^\tau, v^\tau, p^\tau\}$ satisfies Definition 2.1 with the given positive $\gamma > 0$.

It is easy to check that:

$$\overline{Y}\Big(e_0, \frac{1}{2} R_0(\gamma); v^\tau, p^\tau\Big) = \tau\, \overline{Y}(z_0, R; v, p).$$

The latter together with (4.6) implies the inequalities

$$\overline{Y}\ \Big(e_0, \frac{1}{2} R_0(\gamma); v^\tau, p^\tau\Big) + c_\gamma(f^\tau; \widehat{Q}(\gamma)) \Big(\frac{R_0(\gamma)}{2}\Big)^{\frac{\gamma}{2}} \le$$
$$\le \tau \Big[\overline{Y}(z_0, R; v, p) + \tau^\gamma c_\gamma(f; Q_T) \Big(\frac{R_0(\gamma)}{2}\Big)^{\frac{\gamma}{2}} \Big] =$$
$$= \tau \Big[\overline{Y}(z_0, R; v, p) + \tau^{\frac{\gamma}{2}} c_\gamma(f; Q_T)\, R^{\frac{\gamma}{2}} \Big] < \varepsilon_0(\gamma).$$

Since $Q(e_0, \frac{1}{2} R_0(\gamma)) \Subset \widehat{Q}(\gamma)$, $\frac{1}{2} R_0(\gamma) < R_0(\gamma)$, we see that the set $\{B(R_0(\gamma)),$ $2\, R_0^2(\gamma), f^\tau, v^\tau, p^\tau\}$, the point e_0 and the number $\frac{1}{2} R_0(\gamma)$ satisfy conditions (2.13). But then Proposition 2.6 says that the function $e \mapsto v^\tau(e)$ is Hölder continuous in some neighborhood of the point e_0. Making inverse changing the variables, we obtain that z_0 is a regular point. Proposition 4.1 is proved.

5. Proof of Theorem 2.2. Part III

The general line of our considerations in this section is the same as in Section 3 of [10]. It is based on bounds (5.1), (5.2) and (5.5). Inequality (5.1) concerns arbitrary functions and is proved with the help of well known embedding theorems only. It is in [1] and in [10] as well. We give here its proof only for the reader convenience. For $f = 0$, inequality (5.2) and estimate (5.5) of the present paper follow from inequalities of (3.18) and (3.19), presented in [10]. Here we give the complete proof of (5.2) and (5.5). We would like to remark that our proof of (5.5) uses a decomposition of the pressure which differs from decompositions proposed in [1] and [10]. It seems to us that our decomposition of p is more convenient. The proof of Propostion 2.9 is standard but requires some calculations.

As in [10], for suitable weak solution v and p, we define the following functionals

$$A(r) \equiv \sup_{t_0 - r^2 \le t \le t_0} \frac{1}{r} \int\limits_{B(x_0,r)} |v(x,t)|^2\, dx, \quad E(r) \equiv \frac{1}{r} \int\limits_{Q(z_0,r)} |\nabla v|^2\, dz,$$

$$C(r) \equiv \frac{1}{r^2} \int\limits_{Q(z_0,r)} |v|^3\, dz, \quad D(r) \equiv \frac{1}{r^2} \int\limits_{Q(z_0,r)} |p|^{\frac{3}{2}}\, dz.$$

We have assumed that $Q(z_0, r) \Subset Q_T$.

We start with the proof of two auxilary lemmata.

Lemma 5.1. *Assume that $Q(z_0,\rho) \Subset Q_T$. Then*

$$C(r) \le c_{51}\Big[\Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac{3}{4}}(\rho) E^{\frac{3}{4}}(\rho)\Big] \tag{5.1}$$

for all $0 < r \le \rho$. Here c_{51} is an absolute positive constant.

Lemma 5.2. *Assume that $Q(z_0,R) \Subset Q_T$. Then*

$$\left.\begin{aligned} A(\tfrac{R}{2}) + E(\tfrac{R}{2}) \le\ & c_{52}\Big[C^{\frac{2}{3}}(R) + C^{\frac{1}{3}}(R) D^{\frac{2}{3}}(R) + \\ & + A^{\frac{1}{2}}(R) C^{\frac{1}{3}}(R) E^{\frac{1}{2}}(R) + c_\gamma^2 R^{2(\gamma+1)}\Big], \end{aligned}\right\} \tag{5.2}$$

where $c_\gamma \equiv c_\gamma(f; Q_T)$ and c_{52} is an absolute positive constant.

Proof of Lemma 5.1. We have

$$\begin{aligned} \int\limits_{B(x_0,r)} |v|^2\, dx =& \int\limits_{B(x_0,r)} \Big(|v|^2 - [|v|^2]_{x_0,\rho}\Big)\, dx + \int\limits_{B(x_0,r)} [|v|^2]_{x_0,\rho}\, dx \le \\ \le& \int\limits_{B(x_0,\rho)} \Big||v|^2 - [|v|^2]_{x_0,\rho}\Big|\, dx + \Big(\frac{r}{\rho}\Big)^3 \int\limits_{B(x_0,\rho)} |v|^2\, dx. \end{aligned}$$

By the Poincaré–Sobolev inequality,

$$\int\limits_{B(x_0,\rho)} \Big||v|^2 - [|v|^2]_{x_0,\rho}\Big|\, dx \le c_{53}\rho \int\limits_{B(x_0,\rho)} |\nabla v|\, |v|\, dx,$$

where c_{53} is an absolute positive constant. So, we get

$$\left.\begin{aligned}\int\limits_{B(x_0,r)} |v|^2\,dx \le\; & c_{53}\rho\Big(\int\limits_{B(x_0,\rho)} |\nabla v|^2\,dx\Big)^{\frac12}\Big(\int\limits_{B(x_0,\rho)} |v|^2\,dx\Big)^{\frac12}+\\ &+\Big(\frac{r}{\rho}\Big)^3\int\limits_{B(x_0,\rho)} |v|^2\,dx \le\\ &\le c_{53}\rho^{\frac32}A^{\frac12}(\rho)\Big(\int\limits_{B(x_0,\rho)} |v|^2\,dx\Big)^{\frac12}+\Big(\frac{r}{\rho}\Big)^3\rho A(\rho).\end{aligned}\right\}\qquad(5.3)$$

Using the known multiplicative inequality, one can obtain

$$\begin{aligned}\int\limits_{B(x_0,r)} |v|^3\,dx \le\; & c_{54}\Big[\Big(\int\limits_{B(x_0,r)} |\nabla v|^2\,dx\Big)^{\frac34}\Big(\int\limits_{B(x_0,r)} |v|^2\,dx\Big)^{\frac34}+\\ &+\frac{1}{r^{\frac32}}\Big(\int\limits_{B(x_0,r)} |v|^2\,dx\Big)^{\frac32}\Big] \le \Big(\text{see } (5.3)\Big) \le\\ &\le c_{54}\Big\{\rho^{\frac34}A^{\frac34}(\rho)\Big(\int\limits_{B(x_0,r)} |\nabla v|^2\,dx\Big)^{\frac34}+\\ &+\frac{1}{r^{\frac32}}\Big[c_{53}\rho^{\frac32}A^{\frac12}(\rho)\Big(\int\limits_{B(x_0,\rho)} |\nabla v|^2\,dx\Big)^{\frac12}+\\ &+\Big(\frac{r}{\rho}\Big)^3\rho A(\rho)\Big]^{\frac32}\Big\} \le\\ &\le c_{55}\Big\{\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+\Big(\int\limits_{B(x_0,\rho)} |\nabla v|^2\,dx\Big)^{\frac34}\Big[\rho^{\frac34}+\frac{\rho^{\frac94}}{r^{\frac32}}\Big]A^{\frac34}(\rho)\Big\}.\end{aligned}$$

Next, we integrate the last relation in t on $]t_0-r^2,t_0[$ and establish

$$\int\limits_{Q(z_0,r)} |v|^3\,dz \le c_{55}\Big\{r^2\Big(\frac{r}{\rho}\Big)^3A^{\frac32}(\rho)+\Big[\rho^{\frac34}+\frac{\rho^{\frac94}}{r^{\frac32}}\Big]A^{\frac34}(\rho)\int\limits_{t_0-r^2}^{t_0} dt\Big(\int\limits_{B(x_0,\rho)} |\nabla v|^2\,dx\Big)^{\frac34}\Big\} \le$$

$$\leq c_{55}\Big\{r^2\Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho)+$$

$$+\Big[\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\Big]A^{\frac{3}{4}}(\rho)r^{\frac{1}{2}}\Big(\int\limits_{Q(z_0,\rho)}|\nabla\, v|^2\,dz\Big)^{\frac{3}{4}}\Big\}\leq$$

$$\leq c_{55}\Big\{r^2\Big(\frac{r}{\rho}\Big)^3 A^{\frac{3}{2}}(\rho)+\Big[\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\Big]A^{\frac{3}{4}}(\rho)r^{\frac{1}{2}}E^{\frac{3}{4}}(\rho)\,\rho^{\frac{3}{4}}\Big\}.$$

It remains to remark that

$$\Big[\rho^{\frac{3}{4}}+\frac{\rho^{\frac{9}{4}}}{r^{\frac{3}{2}}}\Big]r^{\frac{1}{2}}\rho^{\frac{3}{4}}=\Big[\Big(\frac{\rho}{r}\Big)^{\frac{3}{2}}+\Big(\frac{\rho}{r}\Big)^3\Big]r^2\leq 2\Big(\frac{\rho}{r}\Big)^3 r^2$$

and complete the proof of Lemma 5.1.

Proof of Lemma 5.2. Let us consider (2.4) for $t=t_0$ and for a cut-off function ϕ, having the properties:

$$\phi\equiv 0 \quad \text{in} \quad Q_{t_0}\setminus Q(z_0,R),$$

$$0\leq\phi\leq 1 \quad \text{in} \quad Q_T, \quad \phi\equiv 1 \quad \text{in} \quad Q(z_0,\frac{R}{2}),$$

$$|\nabla\,\phi|<\frac{c_{56}}{R},\ |\partial_t\,\phi|+|\nabla^2\,\phi|<\frac{c_{56}}{R^2} \quad \text{in} \quad Q_{t_0}.$$

Then (2.4) gives:

$$A\Big(\frac{R}{2}\Big)+2\,E\Big(\frac{R}{2}\Big)\leq c_{57}\Big\{\frac{1}{R^3}\int\limits_{Q(z_0,R)}|v|^2\,dz+\frac{1}{R^2}\int\limits_{Q(z_0,R)}\Big||v|^2-[|v|^2]_{x_0,R}\Big|\,|v|\,dz+$$

$$+\frac{1}{R^2}\Big(\int\limits_{Q(z_0,R)}|p|^{\frac{3}{2}}\,dz\Big)^{\frac{2}{3}}\Big(\int\limits_{Q(z_0,R)}|v|^3\,dz\Big)^{\frac{1}{3}}+$$

$$+\frac{1}{R}\Big(\int\limits_{Q(z_0,R)}|f|^2\,dz\Big)^{\frac{1}{2}}\Big(\int\limits_{Q(z_0,R)}|v|^2\,dz\Big)^{\frac{1}{2}}\Big\}.$$

Since

$$\frac{1}{R^3}\int\limits_{Q(z_0,R)}|v|^2\,dz\leq c'_{57}C^{\frac{2}{3}}(R),$$

we obtain

$$\left.\begin{aligned} A\Big(\frac{R}{2}\Big)+2\,E\Big(\frac{R}{2}\Big) \le\; & c''_{57}\Big\{C^{\frac{2}{3}}(R)+C^{\frac{1}{3}}(R)\,D^{\frac{2}{3}}(R)+R\int\limits_{Q(z_0,R)}|f|^2\,dz+ \\ & +\frac{1}{R^2}\int\limits_{Q(z_0,R)}\Big||v|^2-[|v|^2]_{x_0,R}\Big|\,|v|\,dz\Big\}\le \\ \le\; & c_{58}\Big\{C^{\frac{2}{3}}(R)+C^{\frac{1}{3}}(R)\,D^{\frac{2}{3}}(R)+c_\gamma^2R^{2(\gamma+1)}+ \\ & +\tfrac{1}{R^2}\int\limits_{Q(z_0,R)}\Big||v|^2-[|v|^2]_{x_0,R}\Big|\,|v|\,dz\Big\}. \end{aligned}\right\} \qquad (5.4)$$

For the last term on the right-hand side of (5.4), one can get

$$\begin{aligned} S \equiv & \int\limits_{Q(z_0,R)}\Big||v|^2-[|v|^2]_{x_0,R}\Big|\,|v|\,dz\le \\ \le & \int\limits_{t_0-R^2}^{t_0}dt\Big(\int\limits_{B(x_0,R)}\Big||v|^2-[|v|^2]_{x_0,R}\Big|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}}\Big(\int\limits_{B(x_0,R)}|v|^3\,dx\Big)^{\frac{1}{3}}. \end{aligned}$$

By the Poincaré–Sobolev inequality,

$$\Big(\int\limits_{B(x_0,R)}\Big||v|^2-[|v|^2]_{x_0,R}\Big|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}}\le c_{59}\int\limits_{B(x_0,R)}|\nabla v|\,|v|\,dx$$

and, therefore,

$$\begin{aligned} S \le\; & c_{59}\int\limits_{t_0-R^2}^{t_0}dt\Big(\int\limits_{B(x_0,R)}|\nabla v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,R)}|v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,R)}|v|^3\,dx\Big)^{\frac{1}{3}}\le \\ \le\; & c_{59}\,R^{\frac{1}{2}}A^{\frac{1}{2}}(R)\int\limits_{t_0-R^2}^{t_0}dt\Big(\int\limits_{B(x_0,R)}|\nabla v|^2\,dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,R)}|v|^3\,dx\Big)^{\frac{1}{3}}\le \\ \le\; & c_{59}\,R^{\frac{1}{2}}A^{\frac{1}{2}}(R)\Big(\int\limits_{Q(z_0,R)}|v|^3\,dz\Big)^{\frac{1}{3}}\Big(\int\limits_{t_0-R^2}^{t_0}dt\Big(\int\limits_{B(x_0,R)}|\nabla v|^2\,dx\Big)^{\frac{3}{4}}\Big)^{\frac{2}{3}}\le \\ \le\; & c_{59}\,R^{\frac{1}{2}+\frac{2}{3}}A^{\frac{1}{2}}(R)C^{\frac{1}{3}}(R)R^{\frac{1}{3}}\Big(\int\limits_{Q(z_0,R)}|\nabla v|^2\,dz\Big)^{\frac{1}{2}}\le \\ \le\; & c_{59}\,R^2A^{\frac{1}{2}}(R)C^{\frac{1}{3}}(R)E^{\frac{1}{2}}(R). \end{aligned}$$

Now, (5.2) follows from (5.4), Hölder's inequality and from the last relation. Lemma 5.2 is proved.

It remains to find out an estimate for the pressure.

Lemma 5.3. *Suppose that $Q(z_0,\rho) \Subset Q_T$. Then*

$$D(r) \leq c_{510}\Big[\frac{r}{\rho}D(\rho) + \Big(\frac{\rho}{r}\Big)^2\Big(A^{\frac{3}{4}}(\rho)E^{\frac{3}{4}}(\rho) + c_\gamma^{\frac{3}{2}}\rho^{\frac{3}{2}(\gamma+1)}\Big)\Big] \tag{5.5}$$

for all $r \in]0,\rho]$. Here c_{510} is an absolute positive constant.

Proof. Arguing as in the proof of Lemma 2.4, we obtain the following identity for the pressure

$$\int\limits_{B(x_0,\rho)} p(x,s)\Delta\, q(x)\,dx = \int\limits_{B(x_0,\rho)} \Big(v(x,s)\otimes v(x,s) - \tau(s)\Big) : \nabla^2\, q(x)\,dx + \\ + \int\limits_{B(x_0,\rho)} f(x,s)\cdot\nabla\, q(x)\,dx$$

for all $q \in \overset{\circ}{W}{}^2_3(B(x_0,\rho))$ and for a.a. $s \in]0,T[$. Here

$$\tau(s) \equiv [v\otimes v]_{x_0,\rho}(s).$$

We consider the decomposition

$$p = p_1 + p_2, \tag{5.6}$$

where

$$\int\limits_{B(x_0,\rho)} p_1\Delta\, q\,dx = \int\limits_{B(x_0,\rho)} \Big(v\otimes v - \tau\Big) : \nabla^2\, q\,dx + \int\limits_{B(x_0,\rho)} f\cdot\nabla\, q\,dx \tag{5.7}$$

for all $q \in W^2_3(B(x_0,\rho)) \cap \overset{\circ}{W}{}^1_3(B(x_0,\rho))$, and

$$\Delta\, p_2 = 0 \quad \text{in} \quad B(x_0,\rho). \tag{5.8}$$

Let us choose a test function $q = q_0 \in W^2_3(B(x_0,\rho)) \cap \overset{\circ}{W}{}^1_3(B(x_0,\rho))$ in (5.8) so that:

$$\Delta\, q_0 = |p_1|^{\frac{1}{2}}\mathrm{sign}\, p_1 \quad \text{in} \quad B(x_0,\rho).$$

As it is known the function q_0 satisfies the following estimate

$$\Big(\int\limits_{B(x_0,\rho)} |\nabla^2\, q_0|^3\,dx\Big)^{\frac{1}{3}} + \frac{1}{\rho}\Big(\int\limits_{B(x_0,\rho)} |\nabla\, q_0|^3\,dx\Big)^{\frac{1}{3}} + \\ + \frac{1}{\rho^2}\Big(\int\limits_{B(x_0,\rho)} |q_0|^3\,dx\Big)^{\frac{1}{3}} \leq c_{511}\Big(\int\limits_{B(x_0,\rho)} |p_1|^{\frac{3}{2}}\,dx\Big)^{\frac{1}{3}}$$

with an absolute positive constant c_{511}. From the last two relations and from (5.7) it is easy to obtain the inequality

$$\Big(\int\limits_{B(x_0,\rho)}|p_1|^{\frac32}\,dx\Big)^{\frac23}\le c_{512}\Big[\Big(\int\limits_{B(x_0,\rho)}|v\otimes v-\tau|^{\frac32}\,dx\Big)^{\frac23}+\rho\Big(\int\limits_{B(x_0,\rho)}|f|^{\frac32}\,dx\Big)^{\frac23}\Big].$$

It yields

$$\begin{aligned}\Big(\int\limits_{B(x_0,\rho)}|p_1|^{\frac32}\,dx\Big)^{\frac23}&\le c'_{512}\Big[\int\limits_{B(x_0,\rho)}|\nabla v|\,|v|\,dx+\rho\Big(\int\limits_{B(x_0,\rho)}|f|^{\frac32}\,dx\Big)^{\frac23}\Big]\le\\&\le c''_{512}\Big[\Big(\int\limits_{B(x_0,\rho)}|v|^2\,dx\Big)^{\frac12}\Big(\int\limits_{B(x_0,\rho)}|\nabla v|^2\,dx\Big)^{\frac12}+\\&\quad+\rho^3\Big(\fint_{B(x_0,\rho)}|f|^{\frac32}\,dx\Big)^{\frac23}\Big]\le\\&\le c''_{512}\Big[\rho^{\frac12}A^{\frac12}(\rho)\Big(\int\limits_{B(x_0,\rho)}|\nabla v|^2\,dx\Big)^{\frac12}+\rho^3\Big(\fint_{B(x_0,\rho)}|f|^2\,dx\Big)^{\frac12}\Big].\end{aligned}$$

The integration in t gives:

$$\left.\begin{aligned}\int\limits_{Q(z_0,\rho)}|p_1|^{\frac32}\,dz&\le c_{513}\Big[\rho^{\frac34}A^{\frac34}(\rho)\int\limits_{t_0-\rho^2}^{t_0}dt\Big(\int\limits_{B(x_0,\rho)}|\nabla v|^2\,dx\Big)^{\frac34}+\\&\quad+\rho^{\frac92}\int\limits_{t_0-\rho^2}^{t_0}dt\Big(\fint_{B(x_0,\rho)}|f|^2\,dx\Big)^{\frac34}\Big]\le\\&\le c'_{513}\Big[\rho^2A^{\frac34}(\rho)E^{\frac34}(\rho)+\rho^{\frac92+2}\Big(\fint_{Q(z_0,\rho)}|f|^2\,dz\Big)^{\frac34}\Big]\le\\&\le c'_{513}\rho^2\Big[A^{\frac34}(\rho)E^{\frac34}(\rho)+c_\gamma^{\frac32}\rho^{\frac32(\gamma+1)}\Big].\end{aligned}\right\}\qquad(5.9)$$

From (5.6) and (5.9) it follows that:

$$\int\limits_{Q(z_0,\rho)}|p_2|^{\frac32}\,dz\le c_{514}\rho^2\Big[D(\rho)+A^{\frac34}(\rho)E^{\frac34}(\rho)+c_\gamma^{\frac32}\rho^{\frac32(\gamma+1)}\Big].\qquad(5.10)$$

Since p_2 is harmonic in $B(x_0,\rho)$, we have the estimate

$$\frac{1}{r^3}\int\limits_{B(x_0,r)}|p_2|^{\frac32}\,dx\le c_{515}\frac{1}{\rho^3}\int\limits_{B(x_0,\rho)}|p_2|^{\frac32}\,dx$$

with an absolute positive constant c_{515} and, therefore,

$$\frac{1}{r^3}\int\limits_{Q(z_0,r)}|p_2|^{\frac32}\,dz\le c_{515}\frac{1}{\rho^3}\int\limits_{Q(z_0,\rho)}|p_2|^{\frac32}\,dz.\qquad(5.11)$$

So,

$$\begin{aligned} D(r) &\le c_{516}\Big[\frac{1}{r^2}\int\limits_{Q(z_0,R)}|p_1|^{\frac32}\,dz+\frac{1}{r^2}\int\limits_{Q(z_0,R)}|p_2|^{\frac32}\,dz\Big]\le\\ &\le \big(\text{see (5.9)–(5.11)}\big)\le\\ &\le c'_{516}\Big[\Big(\frac{\rho}{r}\Big)^2\Big(A^{\frac34}(\rho)E^{\frac34}(\rho)+c_\gamma^{\frac32}\rho^{\frac32(\gamma+1)}\Big)+\\ &\quad+\frac{r}{\rho}\Big(D(\rho)+A^{\frac34}(\rho)E^{\frac34}(\rho)+c_\gamma^{\frac32}\rho^{\frac32(\gamma+1)}\Big)\Big]. \end{aligned}$$

The latter leads to (5.5). Lemma 5.3 is proved.

Proof of Proposition 2.9. First we introduce

$$\mathcal{E}(R)\equiv A^{\frac32}(R)+D^2(R).$$

Let $\theta\in]0,1[$ and $Q(z_0,\rho)\Subset Q_T$. We shall fix numbers θ and ρ later.

For $R=\theta\rho$, by (5.2),

$$\begin{aligned} A\Big(\frac12\theta\rho\Big) &\le c_{52}\Big[C^{\frac23}(\theta\rho)+C^{\frac13}(\theta\rho)D^{\frac23}(\theta\rho)+\\ &\quad+A^{\frac12}(\theta\rho)C^{\frac13}(\theta\rho)E^{\frac12}(\theta\rho)+c_\gamma^2(\theta\rho)^{2\gamma+2}\Big] \end{aligned}$$

and thus

$$\left.\begin{aligned} A^{\frac32}\Big(\frac12\theta\rho\Big) &\le c_{517}\Big[C(\theta\rho)+C^{\frac12}(\theta\rho)D(\theta\rho)+\\ &\quad+A^{\frac34}(\theta\rho)C^{\frac12}(\theta\rho)E^{\frac34}(\theta\rho)+c_\gamma^3(\theta\rho)^{3(\gamma+1)}\Big]\le\\ &\le c'_{517}\Big[C(\theta\rho)+A^{\frac32}(\theta\rho)E^{\frac32}(\theta\rho)+D^2(\theta\rho)+c_\gamma^3\rho^{3(\gamma+1)}\Big]. \end{aligned}\right\}\tag{5.12}$$

We also have from (5.1) and from (5.5) for $r=\theta\rho$ that:

$$C(\theta\rho)\le c_{51}\Big[\theta^3A^{\frac32}(\rho)+\frac{1}{\theta^3}A^{\frac34}(\rho)E^{\frac34}(\rho)\Big]\tag{5.13}$$

and

$$D(\theta\rho)\le c_{510}\Big[\theta D(\rho)+\frac{1}{\theta^2}A^{\frac34}(\rho)E^{\frac34}(\rho)+\frac{1}{\theta^2}c_\gamma^{\frac32}\rho^{\frac32(\gamma+1)}\Big].\tag{5.14}$$

Obviously,

$$A(\theta\rho)\le\frac1\theta A(\rho),\quad E(\theta\rho)\le\frac1\theta E(\rho).\tag{5.15}$$

Combining (5.12)–(5.15), we obtain the estimate

$$\left.\begin{aligned} A^{\frac32}\Big(\frac12\theta\rho\Big) &\le c_{518}\Big[\theta^2D^2(\rho)+\theta^3A^{\frac32}(\rho)+\frac{1}{\theta^3}A^{\frac34}(\rho)E^{\frac34}(\rho)+\\ &\quad+\Big(\frac{1}{\theta^3}+\frac{1}{\theta^4}\Big)A^{\frac32}(\rho)E^{\frac32}(\rho)+\Big(1+\frac{1}{\theta^4}\Big)c_\gamma^3\rho^{3(\gamma+1)}\Big]. \end{aligned}\right\}\tag{5.16}$$

On the other hand, relation (5.5) for $r = \frac{1}{2}\theta\rho$ implies

$$D\Big(\frac{1}{2}\theta\rho\Big) \leq c_{510}\Big[\frac{\theta}{2}D(\rho) + \Big(\frac{2}{\theta}\Big)^2 A^{\frac{3}{4}}(\rho)E^{\frac{3}{4}}(\rho) + \Big(\frac{2}{\theta}\Big)^2 c_\gamma^{\frac{3}{2}}\rho^{\frac{3}{2}(\gamma+1)}\Big]. \tag{5.17}$$

Taking into account that $\theta \in]0,1[$, we derive from (5.16) and (5.17) the inequality

$$\left.\begin{aligned} \mathcal{E}\Big(\frac{1}{2}\theta\rho\Big) \leq\ & c_{519}\Big[\theta^2 D^2(\rho) + \Big(\theta^3 + \frac{1}{\theta^4}E^{\frac{3}{2}}(\rho)\Big)A^{\frac{3}{2}}(\rho) + \\ & + \frac{1}{\theta^3}A^{\frac{3}{4}}(\rho)E^{\frac{3}{4}}(\rho) + \frac{1}{\theta^4}c_\gamma^3\rho^{3(\gamma+1)}\Big] \leq \\ \leq\ & c_{520}\Big[\theta^2 D^2(\rho) + \Big(\theta^3 + \frac{1}{\theta^4}E^{\frac{3}{2}}(\rho)\Big)A^{\frac{3}{2}}(\rho) + \\ & + \frac{1}{\theta^9}E^{\frac{3}{2}}(\rho) + \frac{1}{\theta^4}c_\gamma^3\rho^{3(\gamma+1)}\Big], \end{aligned}\right\} \tag{5.18}$$

where c_{520} is an absolute positive constant. Without loss of generality we may assume that $c_{520} \geq 1$. It is easy to check that

$$\rho\overline{Y}(z_0,\rho;v,p) \leq c_{521}\Big(C(\rho) + \mathcal{E}(\rho)\Big)^{\frac{1}{3}}, \tag{5.19}$$

where c_{521} is an absolute positive constant.

Now, let us choose a number $\theta \in]0,1[$ so that

$$c_{520}\theta^2 \leq \frac{1}{4} \tag{5.20}$$

and fix it. Having this number θ, we define $\varepsilon_*$by the identity

$$\varepsilon_*(\gamma) \equiv \frac{1}{2}\min\Big\{\Big(\frac{\theta^9}{4c_{520}}\Big)^{\frac{2}{3}}, \Big(\frac{\overline{\varepsilon}_0(\gamma)\theta^3}{2c_{521}(4c_{521})^{\frac{1}{3}}}\Big)^2\Big[2c_{51}\Big(\frac{2}{\theta}\Big)^3 + 1\Big]^{-\frac{2}{3}}\Big\}. \tag{5.21}$$

Then one may fix a positive number $\rho_0 = \rho_0(\theta, z_0, f, \gamma)$ via condition (2.5) so that

$$Q(z_0,\rho_0) \Subset Q_T, \quad E(\rho) \leq 2\varepsilon_*, \quad \frac{c_\gamma^3\rho^{3(\gamma+1)}}{\theta^4} \leq \frac{(2\varepsilon_*)^{\frac{3}{2}}}{\theta^9} \tag{5.22}$$

for all $\rho \in]0,\rho_0]$. By (5.18), (5.20)–(5.22), we have the inequality

$$\mathcal{E}\Big(\frac{1}{2}\theta\rho\Big) \leq \frac{1}{2}\mathcal{E}(\rho) + 2c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}}. \tag{5.23}$$

It is valid for some fixed $\theta \in]0,1[$ (see (5.20)) and for all $\rho \in]0,\rho_0(\theta,z_0,f,\gamma)]$. Let $\theta_1 = \frac{\theta}{2}$ and $\rho = \frac{\rho_0}{2}$. Iterating (5.23), we see that

$$\begin{aligned} \mathcal{E}(\theta_1^{k+1}\rho) \leq\ & \frac{1}{2^{k+1}}\mathcal{E}(\rho) + \frac{1}{2^k}\sum_{i=0}^{k} 2^i\, 2c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}} = \\ =\ & \frac{1}{2^{k+1}}\mathcal{E}(\rho) + \frac{2^{k+1}-1}{2^k}2c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}} \leq \\ \leq\ & \frac{1}{2^{k+1}}\mathcal{E}(\rho) + 4c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}} \end{aligned} \tag{5.24}$$

for all natural numbers k. By (5.1) and (5.24),

$$\begin{aligned} C(\theta_1^{k+1}\rho) \;&\leq\; c_{51}\Big[\theta_1^3 A^{\frac{3}{2}}(\theta_1^k\rho) + \frac{1}{\theta_1^3}A^{\frac{3}{4}}(\theta_1^k\rho)E^{\frac{3}{4}}(\theta_1^k\rho)\Big] \leq \\ &\leq\; c_{51}\Big[\theta_1^3\Big(\frac{1}{2^k}\mathcal{E}(\rho) + 4c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}}\Big) + \\ &\quad + \frac{1}{\theta_1^3}\Big(\frac{1}{2^k}\mathcal{E}(\rho) + 4c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}}\Big)^{\frac{1}{2}}(2\varepsilon_*)^{\frac{3}{4}}\Big]. \end{aligned} \tag{5.25}$$

But then, according to (5.19) and (5.25), we have

$$\begin{aligned} \limsup_{k\to\infty}\, \theta_1^k\rho\,\overline{Y}(z_0,\theta_1^k\rho;v,p) \;&\leq\; c_{521}\Big\{c_{51}\Big[\theta_1^3\Big(4c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}}\Big) + \\ &\quad + \frac{1}{\theta_1^3}\Big(4c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}}\Big)^{\frac{1}{2}}(2\varepsilon_*)^{\frac{3}{4}}\Big] + \\ &\quad + 4c_{520}\frac{1}{\theta^9}(2\varepsilon_*)^{\frac{3}{2}}\Big\}^{\frac{1}{3}} \leq \\ &\leq\; c_{521}\frac{(4c_{520})^{\frac{1}{3}}(2\varepsilon_*)^{\frac{1}{2}}}{\theta^3}\Big[c_{51}\Big(\theta_1^3 + \frac{1}{\theta_1^3}\Big) + 1\Big]^{\frac{1}{3}} \leq \\ &\leq\; c_{521}\frac{(4c_{520})^{\frac{1}{3}}(2\varepsilon_*)^{\frac{1}{2}}}{\theta^3}\Big[2c_{51}\Big(\frac{1}{\theta}\Big)^3 + 1\Big]^{\frac{1}{3}} \leq \\ &\leq\; \frac{\overline{\varepsilon}_0(\gamma)}{2} < \overline{\varepsilon}_0(\gamma). \end{aligned}$$

Proposition 2.9 is proved.

6. Other definitions of suitable weak solutions

Now we are going to explain how to work with suitable weak solutions introduced in [1].

Definition 6.1. Let Ω be a domain in $\mathbb{R}^3$ and T be a positive parameter. Suppose that a fuction f satisfies condition (2.1) for some positive γ. We say that a pair of functions v and p is a suitable weak solution to the Navier–Stokes equations in Q_T if they have the properties

$$v \in L\infty(0,T;L_2(\Omega;\mathbb{R}^3)) \cap L_2(0,T;W^1_2(\Omega;\mathbb{R}^3)),$$
$$p \in L_{\frac{5}{4}}(0,T;L_{\frac{5}{3}}(\Omega)),$$

and satisfy (2.3), (2.4).

For this solution, we have the same main result.

Theorem 6.2. *Let γ be an arbitrary positive constant. Let $\{\Omega, T, f, v, p\}$ be an* *arbitrary collection, satisfying Definition 6.1 with this constant γ. Assume that for a point $z_0 \in Q_T$ condition* (2.5) *holds. Then z_0 is a regular point.*

Theorem 6.2 is proved in the same way as Theorem 2.2. Corresponding changes to be made are as follows. First of all we introduce new functionals:

$$Y_1(z_0,R;v) \equiv Y_{11}(z_0,R;v) + Y_{12}(z_0,R;v),$$

$$Y_{11}(z_0,R;v) \equiv \Big(\fint_{Q(z_0,R)} |v-(v)_{z_0,R}|^3\,dz\Big)^{\frac13},$$

$$Y_{12}(z_0,R;v) \equiv \Big(\fint_{t_0-R^2}^{t_0} dt\Big(\fint_{B(x_0,R)} |v-(v)_{z_0,R}|^{\frac52}\,dx\Big)^2\Big)^{\frac15},$$

$$Y_2(z_0,R;p) \equiv R\,\Big(\fint_{t_0-R^2}^{t_0} dt\Big(\fint_{B(x_0,R)} |p-[p]_{z_0,R}|^{\frac53}\,dx\Big)^{\frac34}\Big)^{\frac45},$$

$$\overline{Y}(z_0,R;v,p) \equiv \Big(\fint_{Q(z_0,R)} |v|^3\,dz\Big)^{\frac13} + \Big(\fint_{t_0-R^2}^{t_0} dt\Big(\fint_{B(x_0,R)} |v|^{\frac52}\,dx\Big)^2\Big)^{\frac15} +$$

$$+R\,\Big(\fint_{t_0-R^2}^{t_0} dt\Big(\fint_{B(x_0,R)} |p|^{\frac53}\,dx\Big)^{\frac34}\Big)^{\frac45}.$$

$$\widetilde{D}(R) \equiv \frac{1}{R^{\frac74}} \int\limits_{t_0-R^2}^{t_0} \Big(\int\limits_{B(x_0,R)} |p|^{\frac53}\,dx\Big)^{\frac34},$$

$$G(R) \equiv \frac{1}{R^3} \int\limits_{t_0-R^2}^{t_0} \Big(\int\limits_{B(x_0,R)} |v|^{\frac52}\,dx\Big)^2,$$

$$\widetilde{\mathcal{E}}(R) \equiv A^{\frac54}(R) + \widetilde{D}^2(R).$$

Then Lemmata 2.4, 2.5, 3.1, 3.2 and Propositions 2.6, 2.8, 4.1 remain to be valid with $\alpha_1 = \frac25$ and with Definition 6.1 instead of Definition 2.1. To prove Proposition 2.9 we use the following statements instead of Lemmata 5.1, 5.2 and 5.3.

Lemma 6.3. *Assume that $Q(z_0,\rho) \Subset Q_T$. Then*

$$C(r) \leq c_{61}\Big[\Big(\frac{r}{\rho}\Big)^3 A^{\frac32}(\rho) + \Big(\frac{\rho}{r}\Big)^3 A^{\frac34}(\rho)E^{\frac34}(\rho)\Big],$$

$$G(r) \leq c_{61}\Big[\Big(\frac{r}{\rho}\Big)^{\frac72} A^{\frac52}(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac92}\Big(A^{\frac74}(\rho)E^{\frac34}(\rho) + A^{\frac32}(\rho)E(\rho)\Big)\Big]$$

for all $0 < r \leq \rho$. Here c_{61} is an absolute positive constant.

Lemma 6.4. *Assume that* $Q(z_0, R) \Subset Q_T$. *Then*

$$A\Big(\frac{R}{2}\Big) + E\Big(\frac{R}{2}\Big) \le c_{62}\big[C^{\frac{2}{3}}(R) + \widetilde{D}^{\frac{4}{5}}(R)G^{\frac{1}{5}}(R) + A(R)E(R) + c_\gamma^2 R^{2(\gamma+1)}\big],$$

where $c_\gamma \equiv c_\gamma(f; Q_T)$ *and* c_{62} *is an absolute positive constant.*

Lemma 6.5. *Suppose that* $Q(z_0, \rho) \Subset Q_T$. *Then*

$$\widetilde{D}(r) \le c_{63}\Big[\Big(\frac{r}{\rho}\Big)^{\frac{1}{2}}\widetilde{D}(\rho) + \Big(\frac{\rho}{r}\Big)^{\frac{7}{4}}\Big(A^{\frac{1}{2}}(\rho)E^{\frac{3}{4}}(\rho) + A^{\frac{5}{8}}(\rho)E^{\frac{5}{8}}(\rho) + c_\gamma^{\frac{5}{4}}\rho^{\frac{5}{4}(\gamma+1)}\Big)\Big]$$

for all $r \in]0, \rho]$. *Here* c_{63} *is an absolute positive constant.*

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