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ON THE NATURALITY OF THE EXTERIOR DIFFERENTIAL

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Abstract. We give sufficient conditions for the naturality of the exterior differential under Sobolev mappings. In other words we study the validity of the equation $d\,f^*\alpha = f^*\,d\alpha$ for a smooth form α and a Sobolev map f.

Résumé. Nous donnons des conditions suffisantes pour la validité de la naturalité de la différentielle extérieure par rapport à une application dans un espace de Sobolev. Autrement dit, nous étudions la validité de l'équation $d\,f^*\alpha = f^*\,d\alpha$ pour une forme différentielle lisse α et une application de Sobolev f.

1. **Introduction.** One of the main properties of calculus with differential forms is the *naturality* of the exterior derivative, that is the fact that for any smooth map $f\colon U \to \mathbb{R}^n$, where U is a bounded domain in $\mathbb{R}^m$, and any smooth differential form α in $\mathbb{R}^n$, we have

$$df^*\alpha = f^*d\alpha. \tag{1.1}$$

Note that this equation is just an avatar of the chain rule; its proof can be found in any textbook on differential forms.

For applications in the calculus of variation, non-linear elasticity or geometric analysis, it is important to extend this result to non-smooth situations. If the map f is smooth and α is a Sobolev differential form, then the pull back $f^*\alpha$ is also a locally Sobolev differential form and the naturality (1.1) can be proved by standard arguments. If both the differential form α and the map f belong to $W^{1,1}_{\mathrm{loc}}$, then the problem is not well posed and it is not clear under what conditions should the equation (1.1) make sense and be proved.

If the differential form α is smooth, then the situation is better, and it is our goal in this paper to give sufficient conditions for a Sobolev map $f\colon U \to \mathbb{R}^n$ to satisfy the naturality of the exterior derivative for smooth forms. Our main results are Theorems 6.3 and Theorem 7.1. As consequences of these theorems, we can formulate the following special results (Corollaries 6.4 and 7.2):

- *Let U be a bounded domain in $\mathbb{R}^m$ and $f \in W^{1,k+1}(U, \mathbb{R}^n)$. Then the chain rule* (1.1) *holds for any smooth k-forms α on $\mathbb{R}^n$.*

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- *Suppose that* $f \in W^{1,k}(U, \mathbb{R}^n)$. *If all the* $k \times k$ *minors of the Jacobian matrix* $(\frac{\partial f_\nu}{\partial x_\mu})$ *belong to the space* $L^{k/(k-1)}(U)$, *then the chain rule* (1.1) *holds for any smooth* k*-forms* α *on* $\mathbb{R}^m$.

REMARKS. 1. The first result says in particular that if $f \in W^{1,m}(U, \mathbb{R}^n)$, then the naturality (1.1) holds for a smooth form of any degree. See [5] for more on this case.

2. The case $m = n = k+1$ of the second result has been studied by J. Ball and V. Šverák [1], [7], see also [2, chap. 7]. In this special case, this result has also been improved by S. Müller, T. Qi and B. S. Yan. These authors proved in [6] that this result also holds for $k = n-1$, $f \in W^{1,n-1}(U; \mathbb{R}^n)$ and $|\Lambda^k(f)| \in L^q(U)$ for some $q \geq n/(n-1)$ (instead of $q \geq p/(p-1)$). See also [3, p. 256] for another proof in the context of the theory of Cartesian currents.

3. For convenience, we work with maps from a bounded domain into Euclidean space. However, the chain rule (1.1) is a local formula and our results also apply to the case of mappings between smooth manifolds.

2. Measurable differential forms.

Let $U \subset \mathbb{R}^m$ be a domain in m-dimensional Euclidean space. A *measurable differential form* of degree k in U is a measurable function $\theta \colon U \to \Lambda^k(\mathbb{R}^m)$. If $x_1, x_2, \ldots, x_m$ is a system of smooth coordinates in U, then any measurable differential k-form can be written as

$$\theta = \sum_{i_1 < i_2 < \cdots < i_k} h_{i_1 i_2 \cdots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge dx_{i_k},$$

where the coefficients $h_{i_1 i_2 \cdots i_k}$ are measurable functions on U. The form θ belongs to $L^p(U, \Lambda^k)$ if $h_{i_1 i_2 \cdots i_k} \in L^p(U)$ for all multiindices $i_1 i_2 \cdots i_k$ and similarly $\theta \in C^r(U, \Lambda^k)$ if all $h_{i_1 i_2 \cdots i_k} \in C^r(U)$. If the coefficients vanish outside a compact subset of U, then one writes $\theta \in C_0^r(U, \Lambda^k)$.

Any k-form $\theta \in L^p(U, \Lambda^k)$ defines a continuous linear form on the space $\omega \in C_0^1(U, \Lambda^{m-k})$ by the following formula:

$$\langle \theta, \omega \rangle = \int_U \theta \wedge \omega.$$

DEFINITION 2.1. A sequence $\{\theta_j\} \subset L^1(U, \Lambda^k)$ is said to *converge weakly* to $\theta \in L^1(U, \Lambda^k)$ if and only if for every $\omega \in C_0^1(U, \Lambda^{m-k})$, we have

$$\int_U \theta_j \wedge \omega \to \int_U \theta \wedge \omega.$$

It is clear that strong convergence in L^1 implies weak convergence. The converse is not true.

Definition 2.2. Let $\theta \in L^1_{\text{loc}}(U,\Lambda^k)$ be a $k-$form. If there exists a $(k+1)$-form $\psi \in L^1_{\text{loc}}(M,\Lambda^{k+1})$ for which the equality

$$\int_U \theta \wedge d\omega = (-1)^{k+1} \int_U \psi \wedge \omega$$

holds for any $\omega \in C^1_0(U,\Lambda^{m-k-1})$, then ψ is called the *weak exterior derivative* of θ (or the exterior derivative of θ in the sense of currents) and is denoted by $\psi = d\theta$. The form $\theta \in L^1_{\text{loc}}(M,\Lambda^k)$ is *weakly closed* if $d\theta = 0$ in the weak sense, that is if

$$\int_U \theta \wedge d\omega = 0$$

holds for any $\omega \in C^1_0(U,\Lambda^{m-k-1})$.

Lemma 2.1. *Let $\alpha \in L^1_{\text{loc}}(U,\Lambda^k)$ and $\beta \in L^1_{\text{loc}}(U,\Lambda^{k+1})$. If there exists a sequence $\{\alpha_j\} \subset C^1(U,\Lambda^k)$ such that $\alpha_j \to \alpha$ and $d\alpha_j \to \beta$ weakly, then $d\alpha = \beta$ in the weak sense.*

Proof. For any $\omega \in C^1_0(U,\Lambda^{m-k-1})$, we have

$$\int_U \alpha \wedge d\omega = \lim_{j\to\infty} \int_U \alpha_j \wedge d\omega = (-1)^{k+1} \int_U d\alpha_j \wedge \omega = (-1)^{k+1} \int_U \beta \wedge \omega. \quad \square$$

Lemma 2.2. *Let $h\colon U \to \mathbb{R}$ be a bounded function such that $dh \in L^{p'}(U)$ and $\beta \in L^p(U,\Lambda^k)$ such that $d\beta \in L^\infty(U,\Lambda^k)$ where $p' = p/(p-1)$. Then $h \cdot \beta \in L^p(U,\Lambda^k)$ and*

$$d(h\cdot\beta) = dh \wedge \beta + h \cdot d\beta.$$

Proof. The equation (2.2) is classic for smooth forms. Now use the density of smooth forms in L^p and the Hölder inequality to obtain the equality (2.2) in the general case. $\square$

3. Sobolev mappings.

Definition. A map $f\colon U \to \mathbb{R}^n$ is said to be *bounded* if $f(U) \subset \mathbb{R}^n$ is relatively compact. It belongs to $W^{1,p}(U,\mathbb{R}^n)$ if all its components $(f_1, f_2, \dots, f_n)$ belong to the Sobolev space $W^{1,p}(U,\mathbb{R})$.

Given a map $f \in W^{1,p}(U,\mathbb{R}^n)$, one defines the pullback of a smooth differential form $\alpha \in C^1(\mathbb{R}^n,\Lambda^k)$ by the following formula: if

$$\alpha = \sum_{i_1<i_2<\cdots<i_k} a_{i_1 i_2 \cdots i_k}(y)\, dy_{i_1} \wedge dy_{i_2} \wedge dy_{i_k},$$

then

$$f^*\alpha = \Lambda^k f(\alpha) = \sum_{i_1<i_2<\cdots<i_k} a_{i_1 i_2 \cdots i_k}\big(f(x)\big)\, df_{i_1} \wedge df_{i_2} \wedge df_{i_k},$$

where

$$df_\nu = \sum_{\mu=1}^{m} \frac{\partial f_\nu}{\partial x_\mu} dx_\mu.$$

Clearly, $f^*\alpha$ is a differential form with measurable coefficients in U for any $\alpha \in C^1(\mathbb{R}^n, \Lambda^k)$.

Let us denote by $Df(x)$ the formal Jacobian matrix of f at the point $x \in U$. This is the $n \times m$ matrix whose entries are the partial derivatives of f:

$$Df = \Big(\frac{\partial f_\nu}{\partial x_\mu}\Big),$$

it is defined almost everywhere in U.

The pullback operator $\Lambda^k f$ is represented by the matrix of $k \times k$ minor determinants of $Df(x)$. Indeed we have, by linear algebra,

$$\Lambda^k f(dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k}) = \sum_{j_1 < j_2 < \cdots < j_k} \frac{\partial(f_{i_1}, f_{i_2}, \ldots, f_{i_k})}{\partial(x_{j_1}, x_{j_2}, \ldots, x_{j_k})} dx_{j_1} \wedge dx_{j_2} \wedge \cdots \wedge dx_{j_k},$$

where we have used the old-fashioned but convenient notation $\frac{\partial(f_{i_1}, f_{i_2}, \ldots, f_{i_k})}{\partial(x_{j_1}, x_{j_2}, \ldots, x_{j_k})}$ to denote the entries of the $k \times k$ minor determinant of Df.

We will use the following norm for $\Lambda^k f$:

$$|\Lambda^k f| = \max\Big|\frac{\partial(f_{i_1}, f_{i_2}, \ldots, f_{i_k})}{\partial(x_{j_1}, x_{j_2}, \ldots, x_{j_k})}\Big|,$$

where the max is taken over all ordered k-tuples $i_1 < i_2 < \cdots < i_k$; $j_1 < j_2 < \cdots < j_k$. Observe that

$$|\Lambda f^k(\alpha)| \leq |\Lambda^k f|\,|\alpha|.$$

Observe finally that the map $f \mapsto \Lambda^k f$ is non-linear for $k \geq 2$.

4. The class $\mathcal{F}^k(U, \mathbb{R}^n)$, $U \subset \mathbb{R}^m$.

Definition. Let us denote by $\mathcal{F}^k(U, \mathbb{R}^n)$ the class of maps $f \colon U \to \mathbb{R}^n$ defined as follows:

$$f \in \mathcal{F}^k(U, \mathbb{R}^n) \Leftrightarrow f \in W^{1,1}(U, \mathbb{R}^n) \quad \text{and} \quad \Lambda^k(f) \in L^1(U).$$

This definition is motivated by the obvious fact that for any map $f \in \mathcal{F}^k(U, \mathbb{R}^n)$, the pull back $\alpha \mapsto f^*\alpha$ defines a bounded operator

$$\Lambda^k f = f^* \colon C^1_0(\mathbb{R}^n, \Lambda^k) \to L^1(U, \Lambda^k).$$

Observe that $\mathcal{F}^1(U,\mathbb{R}^n) = W^{1,1}(U,\mathbb{R}^n)$ and that $\mathcal{F}^k(U,\mathbb{R}^n)$ is not a vector space for $2 \le k \le m$.

We denote by τ^k the initial topology on $\mathcal{F}^k(U,\mathbb{R}^n)$ induced by the inclusion $\mathcal{F}^k(U,\mathbb{R}^n) \subset W^{1,1}(U,\mathbb{R}^n)$ and the family of functions

$$\lambda_{\alpha,\omega}\colon \mathcal{F}^k(U,\mathbb{R}^n) \to \mathbb{R}, \quad \lambda_{\alpha,\omega}(f) = \int_U f^*\alpha \wedge \omega$$

where $\alpha \in C^1(\mathbb{R}^n,\Lambda^k)$ and $\omega \in C^1_0(U,\Lambda^{m-k})$. In other words τ^k is the coarsest topology for which the inclusion $\mathcal{F}^k(U,\mathbb{R}^n) \subset W^{1,1}(U,\mathbb{R}^n)$ is continuous, as well as all functions $\lambda_{\alpha,\omega}$.

Observe that if a sequence $f_j \in \mathcal{F}^k(U,\mathbb{R}^n)$ converges to a map f in the τ^k topology, then $f_j^*\alpha$ converges weakly to $f^*\alpha$ by definition.

An explicit sufficient condition for the τ^k-convergence in $\mathcal{F}^k(U,\mathbb{R}^n)$ is given in the next result:

Lemma 4.1. *Let $\{f_j\} \subset W^{1,1}(U,\mathbb{R}^n)$ be a sequence of mappings which converges to a map $f \in \mathcal{F}^k(U,\mathbb{R}^n)$ in the $W^{1,1}$-topology. Assume that $\{|\Lambda^k f_j|\}$ is equi-integrable,* i.e., *there exists a function $w \in L^1(U,\mathbb{R})$ such that $|\Lambda^k f_j| \le w(x)$ a.e. $x \in U$ for any $j \in \mathbb{N}$. Then $f_j \to f$ in the τ^k topology.*

Proof. Let $\alpha \in C^1(\mathbb{R}^n,\Lambda^k)$ be an arbitrary smooth k-form on $\mathbb{R}^n$ and $\omega \in C^1_0(U,\Lambda^{m-k})$. Since $f_j \to f$ in $W^{1,1}$, we have

$$\lim_{j\to\infty} (f_j^*\alpha) \wedge \omega = \lim_{j\to\infty} (\Lambda^k f_j)\alpha \wedge \omega = f^*\alpha \wedge \omega$$

almost everywhere. Furthermore, we have at every point $x \in U$

$$|(f_j^*\alpha)_x \wedge \omega_x| \le |\Lambda^k f_j(x)|\, |\alpha_x|\, |\omega_x| \le Q \cdot |\Lambda^k f_j(x)| \le Q \cdot w(x)$$

for some constant Q. Because $w \in L^1(U,\mathbb{R})$, the Lebesgue-dominated convergence theorem implies that

$$\lim_{j\to\infty} \int_U (f_j^*\alpha) \wedge \omega = \int_U (f^*\alpha) \wedge \omega. \qquad \square$$

Proposition 4.2. *Let $f \in W^{1,1}(U,\mathbb{R}^n)$ be a map such that*

(a) *The m-dimensional Hausdorff measure of the image $f(U) \subset \mathbb{R}^n$ is finite;*

(b) *f has* essentially finite multiplicity, i.e., *there exists a constant $Q < \infty$ and a set $E \subset U$ with measure zero such that for every point $y \in \mathbb{R}^n$,*

$$\operatorname{Card}\{x \in U \setminus E \,\big|\, f(x) = y\} \le Q.$$

Then $f \in \mathcal{F}^m(U,\mathbb{R}^n)$.

This proposition applies, *e.g.*, if f is a homeomorphism onto a bounded domain.

Proof. In that case, $|\Lambda^k(f)|$ belongs to L^1 by the area formula (see, *e.g.*, [3, p. 220]). □

Remark. In [3, p. 229], Giaquinta introduced a class of maps $\mathcal{A}_1(U,\mathbb{R}^n)$ which is very similar to our class $\mathcal{F}^m(U,\mathbb{R}^n)$ (where $m = \dim(U)$). The main difference is that the condition $f \in W^{1,1}(U,\mathbb{R}^n)$ is relaxed to the assumption that f is approximately differentiable almost everywhere. In any case, we have a continuous embedding

$$\mathcal{F}^m(U,\mathbb{R}^n) \subset \mathcal{A}_1(U,\mathbb{R}^n).$$

5. k-stable maps in $\mathcal{F}^k(U,\mathbb{R}^n)$.

Definition. A map $f \in \mathcal{F}^k(U,\mathbb{R}^n)$ is said to be *k-stable* if it belongs to the closure of $C^1(U,\mathbb{R}^n)$ in the τ^k topology, *i.e.*, there exists a sequence of smooth maps converging to f in the τ^k topology. We denote by $\mathcal{S}^k(U,\mathbb{R}^n) \subset \mathcal{F}^k(U,\mathbb{R}^n)$ the set of k-stable maps:

$$\mathcal{S}^k(U,\mathbb{R}^n) = \overline{C^1(U,\mathbb{R}^n)}^{\tau^k} \subset \mathcal{F}^k(U,\mathbb{R}^n).$$

Observe that $W^{1,k}(U,\mathbb{R}^n) \subset \mathcal{S}^k(U,\mathbb{R}^n)$.

The pullback of a closed form by a stable map is again a closed form:

Proposition 5.1. *Let $f \in \mathcal{S}^k(U,\mathbb{R}^n)$ be k-stable map and $\alpha \in C^1(\mathbb{R}^n,\Lambda^k)$. If α is closed, then $f^*\alpha$ is weakly closed.*

Proof. Because $f \in \mathcal{S}^k(U,\mathbb{R}^n)$, there exists a sequence $\{f_j\}$ of smooth maps converging to f in the τ^k-topology. Assume that $d\alpha = 0$, then for any $\phi \in C^1_0(U,\Lambda^{m-k-1})$ we have

$$\int_U (f_j^*\alpha) \wedge d\phi = (-1)^{k+1} \int_U d(f_j^*\alpha) \wedge \phi = (-1)^{k+1} \int_U f_j^*\,(d\alpha) \wedge \phi = 0.$$

We thus have

$$\int_U (f^*\alpha) \wedge \, d\phi = \lim_{j\to\infty} \int_U (f_j^*\alpha) \wedge \, d\phi = 0,$$

for any $\phi \in C^1_0(U,\Lambda^{m-k-1})$; this means that $f^*\alpha$ is weakly closed. □

Proposition 5.2. *Let $f \in W^{1,1}(U,\mathbb{R}^n)$ be a map such that*

$$\inf_{\{f_j\}} \int_U (\sup_j |\Lambda^k f_j|) \, dx < \infty,$$

where the infimum is taken over the set of all sequences $\{f_j\}$ of smooth maps such that $\|f_j - f\|_{W^{1,1}} \to 0$. Then $f \in \mathcal{S}^k(U,\mathbb{R}^n)$.

PROOF. By mollification, we know that the set of sequences $\{f_j\}$ of smooth maps such that $\|f_j - f\|_{W^{1,1}} \to 0$ is not empty. We can then apply Lemma 4.1. □

6. $k^\dagger$-stable maps.

DEFINITION 6.1. We define the space $\mathcal{F}^{k^\dagger}(U,\mathbb{R}^n)$ by

$$\mathcal{F}^{k^\dagger}(U,\mathbb{R}^n) = \begin{cases} \mathcal{F}^n(U,\mathbb{R}^n) & \text{if } k = n, \\ \mathcal{F}^k(U,\mathbb{R}^n) \cap \mathcal{F}^{k+1}(U,\mathbb{R}^n) & \text{if } 0 \le k < n. \end{cases}$$

The $\tau^{k^\dagger}$ topology is defined for $k < n$ to be the initial topology for which both inclusions

$$\mathcal{F}^{k^\dagger}(U,\mathbb{R}^n) \subset \mathcal{F}^k(U,\mathbb{R}^n) \quad \text{and} \quad \mathcal{F}^{k^\dagger}(U,\mathbb{R}^n) \subset \mathcal{F}^{k+1}(U,\mathbb{R}^n)$$

are continuous. For $k = n$, we simply define $\tau^{k^\dagger} = \tau^k$.

We then say that a map $f\colon U \to \mathbb{R}^n$ is $k^\dagger$*-stable* if it belongs to the closure of $C^1(U,\mathbb{R}^n)$ in the space $\mathcal{F}^{k^\dagger}(U,\mathbb{R}^n)$ for the $\tau^{k^\dagger}$ topology, the class of $k^\dagger$-stable maps is denoted by $\mathcal{S}^{k^\dagger}(U,\mathbb{R}^n)$

Observe the following elementary result.

LEMMA 6.1. *A map $f\colon U \to \mathbb{R}^n$ is $k^\dagger$-stable if and only if there exists a sequence $\{f_j\} \subset C^1(U,\mathbb{R}^n)$ of smooth maps which weakly converges to f in both spaces $\mathcal{F}^k(U,\mathbb{R}^n)$ and $\mathcal{F}^{k+1}(U,\mathbb{R}^n)$.*

PROPOSITION 6.2. *Let $f \in W^{1,1}(U,\mathbb{R}^n)$ be a map such that for some $k < n$,*

$$\inf_{\{f_j\}} \int_U \left(\sup_j(|\Lambda^k f_j| + |\Lambda^{k+1} f_k|)\right) dx < \infty,$$

where the infimum is taken over all sequences $\{f_j\}$ of smooth maps such that $\|f_j - f\|_{W^{1,1}} \to 0$. Then $f \in \mathcal{S}^{k^\dagger}(U,\mathbb{R}^n)$.

PROOF. This follows directly from Proposition 5.2 and the previous lemma. □

One can rephrase this proposition as follows. Let $f \in W^{1,1}(U,\mathbb{R}^n)$, and assume that there exists a sequence of smooth maps $\{f_j\} \subset C^1(U,\mathbb{R}^n)$ such that $f_j \to f$ in $W^{1,1}(U,\mathbb{R}^n)$ and there exists a function $w \in L^1(U,\mathbb{R})$ such that

$$|\Lambda^k f_j(x)| + |\Lambda^{k+1} f_j(x)| \le w(x)$$

a.e. $x \in U$ for any $j \in \mathbb{N}$. Then f is $k^\dagger$-stable.

The naturality of the exterior differential holds for $k^\dagger$-stable maps:

Theorem 6.3. *Let $f \in \mathcal{S}^{k^\dagger}(U,\mathbb{R}^n)$ be $k^\dagger$-stable map, and let $\alpha \in C^1(\mathbb{R}^n,\Lambda^k)$ be a smooth k-form in $\mathbb{R}^m$, then $f^*\alpha \in L^1(U,\Lambda^k)$, $f^*d\alpha \in L^1(U,\Lambda^{k+1})$ and the equation*

$$df^*\alpha = f^*d\alpha$$

holds in the weak sense.

Proof. By hypothesis, there exists a sequence of smooth mappings $f_j \in C^1(U,\mathbb{R}^n)$ which converges to f in $\mathcal{F}^k(U,\mathbb{R}^n)$ and $\mathcal{F}^{k+1}(U,\mathbb{R}^n)$ for both the τ^k and τ^{k+1} topologies.

Let $\alpha \in C^1(\mathbb{R}^n,\Lambda^k)$ be an arbitrary smooth k-form on $\mathbb{R}^m$ and let $\theta \in C^1_0(U,\Lambda^{m-k})$. By hypothesis, we have

$$\lim_{j\to\infty}\int_U (f_j^*\alpha)\wedge\theta = \int_U (f^*\alpha)\wedge\theta. \tag{6.1}$$

We also have

$$\lim_{j\to\infty}\int_U (f_j^*\beta)\wedge\phi = \int_U (f^*\beta)\wedge\phi \tag{6.2}$$

for any $\beta \in C^1(\mathbb{R}^n,\Lambda^{k+1})$ and $\phi \in C^1_0(U,\Lambda^{m-k-1})$. Let us now choose $\beta = d\alpha$ and $\theta = d\phi$, we then have $df_j^*\alpha = f_j^*d\alpha$ for any $j \in \mathbb{N}$ because both α and f_j are of class C^1, this implies that

$$\int_U (f_j^*\alpha)\wedge d\phi = (-1)^{k+1}\int_U d(f_j^*\alpha)\wedge\phi = (-1)^{k+1}\int_U (f_j^*d\alpha)\wedge\phi.$$

Applying (6.1) and (6.2) one then gets

$$\begin{aligned}\int_U (f^*\alpha)\wedge d\phi &= \lim_{j\to\infty}\int_U (f_j^*\alpha)\wedge\, d\phi\\ &= \lim_{j\to\infty}(-1)^k\int_U f_j^*\,(d\alpha)\wedge\phi\\ &= (-1)^k\int_U f^*\,(d\alpha)\wedge\phi\end{aligned}$$

for any $\phi \in C^1_0(U,\Lambda^{n-k-1})$, this means precisely that $d(f^*\alpha) = f^*(d\alpha)$ in the weak sense. □

Corollary 6.4. *Let U be a domain in $\mathbb{R}^m$ and $f \in W^{1,k+1}(U,\mathbb{R}^n)$. Then the naturality* (1.1) *holds for any smooth k-forms α on $\mathbb{R}^n$.*

Proof. This follows from the fact that $W^{1,k+1}(U,\mathbb{R}^n) \subset \mathcal{S}^{k^\dagger}(U,\mathbb{R}^n)$. □

7. Another class of maps. We denote by $\mathcal{S}^k_{q,p}(U,\mathbb{R}^n)$ the class of maps $f \in \mathcal{S}^k(U,\mathbb{R}^n)$ such that

$$|df| \in L^p(U) \quad \text{and} \quad |\Lambda^k(f)| \in L^q(U).$$

Observe that $\mathcal{S}^k_{q,p}(U,\mathbb{R}^n) \subset W^{1,p}(U,\mathbb{R}^n)$.

THEOREM 7.1. *Let U be a bounded domain in $\mathbb{R}^n$ and $f \in \mathcal{S}^k_{q,p}(U,\mathbb{R}^n)$, and assume $1 \leq p \leq \infty$, $q = p/(p-1)$.*

Let $\alpha \in C^1(\mathbb{R}^n, \Lambda^k)$ be a smooth k-form in $\mathbb{R}^n$, then $f^*\alpha \in L^1(U, \Lambda^k)$, $f^*d\alpha \in L^1(U, \Lambda^{k+1})$ and the chain rule

$$df^*\alpha = f^*d\alpha$$

holds in the weak sense.

PROOF. Observe that by Proposition 5.1 $f^*\gamma$ is weakly closed for any closed k-form $\gamma \in C^1(\mathbb{R}^m, \Lambda^k)$. Suppose first that $\alpha = a \cdot \gamma$, where $\gamma \in C^1(\mathbb{R}^m, \Lambda^k)$ is a closed $k-$form and that $a \in C^1(\mathbb{R}^n)$ is a function. Then $f^*a = a \circ f \in W^{1,1}(U)$ and $df^*a = f^*da$ (see, *e.g.*, [4, Theorem 7.8]). Because $f \in \mathcal{S}^k_{q,p}(U,\mathbb{R}^n)$, we have in fact $|df^*a| \in L^p(U)$ and $|f^*(\gamma)| \leq \ |\Lambda^k f_j(x)| \cdot |\gamma| \in L^q(U)$. Since $q = p/(p-1)$, we have by Lemma 2.2:

$$\begin{aligned} df^*\alpha &= df^*(a \cdot \gamma) \\ &= d(f^*a \cdot f^*\gamma) \\ &= d(f^*a) \wedge f^*\gamma + (f^*a) \cdot \underbrace{(df^*\gamma)}_{=0} \\ &= d(f^*a) \wedge f^*\gamma \\ &= (f^*da) \wedge f^*\gamma \\ &= f^*(da \wedge \gamma) \\ &= f^*(d\alpha). \end{aligned}$$

Consider now an arbitrary smooth k-form on $\mathbb{R}^n$. It can be written as a sum

$$\alpha = \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1 i_2 \cdots i_k}(x)\, dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k},$$

where $a_{i_1 i_2 \cdots i_k}(x)$ is an element in $C^1(\mathbb{R}^n)$. Since $dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_k}$ is a closed (in fact exact) form, the proof is complete. □

COROLLARY 7.2. *Suppose that $f \in W^{1,k}(U,\mathbb{R}^m)$ and $\Lambda^k(f) \in L^{k/(k-1)}(U)$, then the chain rule* (1.1) *holds for any smooth k-forms α on $\mathbb{R}^m$.*

Proof. The hypothesis implies that $f \in \mathcal{S}^k_{q,p}(U, \mathbb{R}^n)$. □

References

1. J. Ball, *Convexity conditions and existence theorems in nonlinear elasticity.* Arch. Rational Mech. Anal. **63**(1976/77), 337–403.
2. P. G. Ciarlet, *Mathematical elasticity. Volume I: Three-dimensional elasticity.* North-Holland, 1988.
3. M. Giaquinta, G. Modica and J. Souček, *Cartesian currents in the calculus of variations. I.* Springer-Verlag, 1998.
4. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order.* Second edition. Grundlehren Math. Wiss. **224**, Springer-Verlag, Berlin, 1983.
5. V. Gol'dshtein and M. Troyanov, *A Conformal de Rham Complex.* arXiv:0711.1286
6. S. Müller, T. Qi and B. S. Yan, *On a new class of elastic deformations not allowing for cavitation.* Ann. Inst. Henri Poincaré **11**(1994), 217–243.
7. V. Šverák, *Regularity Properties of Deformations with Finite Energy.* Arch. Rational Mach. Anal. **100**(1988), 105–127

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