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On the lowest eigenvalue of a pseudo-differential operator

(sharp Gårding inequalities/uncertainty principle/subelliptic estimates/commutators of vector fields)

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ABSTRACT **Positive lower bounds for pseudo-differential operators with nonnegative symbols are derived; the bounds in particular yield subelliptic estimates for operators arising as sums of squares of vector fields.**

Let $p(x,\xi)$ be a nonnegative symbol satisfying the estimates

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \le C_{\alpha\beta} M^{2-|\beta|}. \qquad [1]$$

We shall outline an algorithm to determine the order of magnitude of the lowest eigenvalue of the corresponding pseudo-differential operator $p(x,D)$. This is closely related to earlier work on conditions ensuring the estimate

$$\mathrm{Re}\langle p(x,D)u,u\rangle + C\|u\|^2 \ge 0 \qquad u \in L^2(\mathbf{R}^n). \qquad [2]$$

The sharpest known sufficient conditions for inequality **2** are the following:

(*i*) $p \in S^2(\mathbf{R}^n \times \mathbf{R}^n)$, $p \ge 0$ (see ref. 1)

(*ii*) $p \in S^{6/5}(\mathbf{R}^n \times \mathbf{R}^n)$, $p + Tr^+p \ge 0$, in which

Tr^+p is a nonnegative quantity defined in terms of the Hessian of p [see Hörmander (2) and also Melin (3)].

Our first main result on the eigenvalue problem, motivated by the uncertainty principle of quantum mechanics, is the following:

Let $Q_0 = \{(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x|,\ |\xi| \le 1\}$; say that a canonical transformation Φ: $(x,\xi) \rightarrow (y,\eta)$ mapping Q_0 into $\mathbf{R}^n \times \mathbf{R}^n$ is a testing map if $y - y_0$ and $(\eta - \eta_0)/M$ are C^α functions of (x,ξ) with norms bounded by a fixed constant. Here (y_0,η_0) denotes $\Phi(0,0)$, and α is an integer that depends on ε below.

THEOREM 1. *If* $p(x,\xi) \ge 0$ *satisfies inequality* **1**, *and* $K \ge C_\varepsilon M^\varepsilon$ *is a constant such that*

$$\|p\circ\Phi\|_{C^0(Q_0)} \ge K \textit{ for any testing map } \Phi,$$

then

$$Re\langle p(x,D)u,u\rangle \ge c_\varepsilon K\|u\|^2 \qquad u \in L^2(\mathbf{R}^n). \qquad [3]$$

From *Theorem 1*, one can easily read off the following special case of the theorem of Hörmander (4) on commutators of vector fields:

COROLLARY. *Let* $X_1, \ldots, X_m$ *be vector fields on* $\mathbf{R}^n$ *whose Lie brackets up to order* k *generate the Lie algebra at each point. Then*

$$\sum_{j=1}^{m} \|X_j u\|^2 + C_\varepsilon\|u\|^2 \ge c_\varepsilon\|u\|^2_{-\varepsilon+1/(k+1)} \qquad \varepsilon > 0 \qquad [4]$$

for $u \in C^\infty$ *supported in the unit ball in* $\mathbf{R}^n$.

In fact, inequality **4** holds for $\varepsilon = 0$, as was proved by Rothschild and Stein (5) (together with estimates in norms other than L^2); we shall also derive that result from a refinement of *Theorem 1* to be given below.

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Proof of the Corollary. First observe that

$$\|f\|_{C^k(Q_0)} \le C_\varepsilon M^\varepsilon\|f\|_{C^0(Q_0)} + C'_\varepsilon M^{-2}\|f\|_{C^\beta(Q_0)} \qquad [5]$$

if $\beta > [3k/\varepsilon] + 1$, as can be seen by comparing f with its Taylor expansion of order β in a ball of radius $M^{-\varepsilon/k}$. Next let $p_1, \ldots, p_m$ be the symbols of $X_1, \ldots, X_m$ localized to a conic neighborhood U in the region $|\xi| \approx M$, and assume that

$$|\{p_{i_1},\{p_{i_2},\ldots\{p_{i_k},p_{i_{k+1}}\}\ldots\}\}| \approx M \qquad \text{throughout } U. \qquad [6]$$

If Φ is a testing map then expression **6** will still hold with p_{i_j} replaced by $p_{i_j}\circ\Phi$. The commutator is of course a polynomial homogeneous of degree $k + 1$ in derivatives (of order up to k) of the $p_{i_j}\circ\Phi$, and therefore expression **6** yields

$$\|p_{i_j}\circ\Phi\|_{C^k(Q_0)} \ge cM^{1/k+1}$$

for some i_j, which in turn implies

$$\max_j \|p_{i_j}\circ\Phi\|_{C^0(Q_0)} \ge c_\varepsilon M^{-\varepsilon+1/(k+1)}$$

in view of inequality **5**. The corollary is now an immediate consequence of *Theorem 1*.

The proof of *Theorem 1* proceeds by induction on the number of variables; a key tool in the induction is the following theorem, which also provides an algorithm for computing the size of the lowest eigenvalue:

THEOREM 2. *Suppose* $p(x,\xi) \ge 0$ *satisfying inequality* **1** *in* $\{|\xi_n| < M\}$ *is of the form*

$$p(x,\xi) = \xi_n^2 + a(x,\xi'), \textit{ in which } \xi' = (\xi_1, \ldots, \xi_{n-1}). \qquad [7]$$

We define its derived symbol to be

$$p^*(x',x_n,\xi') = \sum_{j=0}^{N} a(x',x_n + jK^{-1/2}/N,\xi'),$$

in which $N = [3/\varepsilon] + 1$. *Then inequality* **3** *is equivalent to*

$$Re\langle p^*(x',x_n,D')v,v\rangle_{L^2(\mathbf{R}^{n-1})} \ge cK\|v\|^2_{L^2(\mathbf{R}^{n-1})} \qquad [8]$$

for each x_n *in* $\mathbf{R}$ *and* v *in* $L^2(\mathbf{R}^{n-1})$.

We now present the promised algorithm, which will either determine the lowest eigenvalue λ of $p(x,D)$ up to a bounded factor or else show that $\lambda \le C_\varepsilon M^\varepsilon$. Assume we already know how to carry out our algorithm in $(n - 1)$ dimensions $(n \ge 1)$; given a symbol $p \ge 0$ on $\mathbf{R}^n \times \mathbf{R}^n$ satisfying inequality **1**, and a constant $K \ge C_\varepsilon M^\varepsilon$, our task is to decide whether the estimate **3** holds. Perform a Calderón–Zygmund decomposition of the basic $1 \times M$ blocks in $\mathbf{R}^n \times \mathbf{R}^n$, stopping at Q_s if one of the following occurs:

(*i*) $(\min_{Q_s^*} p)(\mathrm{diam}_x Q_s)^{-2}(\mathrm{diam}_\xi Q_s)^{-2} \ge A,$

(*ii*) $\max_{|\alpha|+|\beta|=2} \|D_x^\alpha D_\xi^\beta p\|_{L^\infty(Q_s^*)} \times (\mathrm{diam}_x Q_s)^{|\alpha|-2}(\mathrm{diam}_\xi Q_s)^{|\beta|-2} \ge A,$

(*iii*) $(\mathrm{diam}_x Q_s)(\mathrm{diam}_\xi Q_s) \leq K^{1/2}/A$.

Here A is a large constant, and Q_s^* is the dilate of Q_s by a large constant factor. In view of the $S^{M,m}_{\Phi,\varphi}$ calculus (see ref. 6), inequality **3** holds for $p(x,\xi)$ if and only if localized estimates hold for $p|Q_s$. Thus inequality **3** is evidently false if there is any Q_s satisfying **iii**. Otherwise, because the localized estimate is obviously true for Q_s satisfying **i**, the only delicate case is **ii**. However, a suitable canonical transformation carries the symbol $p|Q_s$ to a symbol of the form **7**, so that *Theorem 2* reduces the problem to an eigenvalue computation in fewer variables.

The estimate **4** with $\varepsilon = 0$ can be obtained from our algorithm, which in fact shows that if $p = \Sigma p_j^2$ and

$$|\{p_{i_1},\{p_{i_2},\ldots,\{p_{i_l},p_{i_{l+1}}\}\ldots\}\}| \geq K^{(l+1)/2} \text{ for some } l, \quad [9]$$

then inequality **3** holds, the reason being essentially that the derived symbol of p arising from a cube Q_s of type **ii** is again a sum of squares satisfying hypotheses analogous to inequality **9**.

Theorem 2 in turn can be deduced from the following result on the spectral decomposition of psuedo-differential operators, which may be of intrinsic interest:

THEOREM 3. *Given* $p(x,\xi) \geq 0$ *and a constant* K, *let*

$$p_K(u) = \langle \min\{K, p^w(x,D)\}u,u\rangle,$$

in which $p^w(x,D)$ *is defined by the Weyl calculus as in Hörmander (2), and the minimum is taken in the sense of spectral theory. Then if* p,q *are nonnegative symbols satisfying inequality* **1**, *we have*

$$(p + q)_K(u) \leq C_\varepsilon[p_K(u) + q_K(u) + M^\varepsilon\|u\|^2].$$

The proofs of the results announced here will appear in a forthcoming article.

It would be interesting to know whether the lower bound for the least eigenvalue of $p(x,D)$ given by *Theorem 1* is sharp.

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