$L_\infty -$boundedness of the FEM Galerkin operator for parabolic problems

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In the paper [JNi5] Nitsche/Wheeler leveraged the proven optimal convergence with respect to the convergence factor $O(h^r)$ to also balanced norms (i.e. balanced regularity assumptions to the heat equation solution). The core piece of the paper is a "parabolic type" shift theorem for the solution of the heat equation with respect to the norm

$$|||w|||_k^2 := \int_0^T \|w\|_k^2 dt$$

in the form

$$|||w|||_{k+2}^2 \le c|||Aw|||_k^2 \; .$$

Enabled by this shift theorem $L_\infty -$FE approximation error estimates are derived, using Nitsche's weight functions technique in the (parabolic like modified) form

$$\mu(x,t) := |x - x_0|^2 + |t - t_0| \, ,$$

whereby

$$\|u\|_{L_\infty(L_\infty)} = u(x_0, t_0) \, .$$

The proof of the shift theorem is based on appropriate estimates of the generalized Fourier coefficients $w_i\,(t)$ of the heat equation

$$\dot{u} - \Delta u = f \;\;,\;\; u(0) = u_0 \,,\;\; u|_{\partial\Omega} = 0$$

with

$$w_i(t) = e^{-\lambda_i t} u_0 + \int_0^t e^{-\lambda_i (t-\tau)} f_i(\tau) d\tau \; .$$

The "trick" to go there is about changing the order of integration in the following form:

$$\int_0^T w_i^2(t)dt \le \int_0^T \left[\int_0^t e^{-\lambda_i (t-\tau)} d\tau \right] \left[\int_0^t e^{-\lambda_i (t-\tau)} f_i^2(\tau) d\tau \right] dt \, .$$

$$\le \lambda_i^{-1} \int_0^T f_i^2(\tau) \left[\int_\tau^T e^{-\lambda_i (t-\tau)} dt \right] d\tau \le \lambda_i^{-2} \int_0^T f_i^2(\tau) d\tau \; .$$