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ON THE POSSIBLE EXCEPTIONS FOR THE TRANSCENDENCE OF THE LOG-GAMMA FUNCTION AT RATIONAL ENTRIES

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Abstract. In a very recent work [JNT **129**, 2154 (2009)], Gun and co-workers have claimed that the number $\log\Gamma(x)+\log\Gamma(1-x)$, x being a rational number between 0 and 1, is transcendental with at most *one* possible exception, but the proof presented there in that work is *incorrect*. Here in this paper, I point out the mistake they committed and I present a theorem that establishes the transcendence of those numbers with at most *two* possible exceptions. As a consequence, I make use of the reflection property of this function to establish a criteria for the transcendence of $\log\pi$, a number whose irrationality is not proved yet. I also show that each pair $\{\log\left[\pi/\sin(\pi x)\right], \log\left[\pi/\sin(\pi y)\right]\}$, x and y being rational numbers between 0 and 1, contains at least one transcendental number. This has an interesting consequence for the transcendence of the product $\pi\cdot e$, another number whose irrationality is not proved.

1. Introduction

The gamma function, defined as $\Gamma(x) := \int_0^\infty e^{-t}\, t^{x-1}\, dt$, $x > 0$, has attracted much interest since its introduction by Euler, appearing frequently in both mathematics and natural sciences problems. The transcendental nature of this function at rational values of x in the open interval $(0,1)$, to which we shall restrict our attention hereafter, is enigmatic, just a few special values having their transcendence established. Such special values are: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, whose transcendence follows from the Lindemann's proof that π is transcendental (1882) [1], $\Gamma(\frac{1}{4})$, as shown by Chudnovsky (1976) [2], $\Gamma(\frac{1}{3})$, as proved by Le Lionnais (1983) [3], and $\Gamma(\frac{1}{6})$, as can be deduced from a theorem of Schneider (1941) on the transcendence of the beta function at rational entries [4]. The most recent result in this line was obtained by Grinspan (2002), who showed that at least two of the numbers $\Gamma(\frac{1}{5})$, $\Gamma(\frac{2}{5})$ and π are algebraically independent [5]. For other rational values of x in the interval $(0,1)$, not even irrationality was established for $\Gamma(x)$.

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The function $\log\Gamma(x)$, known as the log-gamma function, on the other hand, received less attention with respect to the transcendence at rational points. In a recent work, however, Gun, Murty and Rath (GMR) have presented a "theorem" asserting that [6]:

Conjecture 1. *The number* $\log\Gamma(x)+\log\Gamma(1-x)$ *is transcendental for any rational value of* x*,* $0<x<1$*, with at most* ***one*** *possible exception.*

This has some interesting consequences. For a better discussion of these consequences, let us define a function $f\colon(0,1)\to\mathbb{R}_+$ as follows:

$$f(x):=\log\Gamma(x)+\log\Gamma(1-x)\,. \tag{1.1}$$

Note that $f(1-x)=f(x)$, which implies that $f(x)$ is symmetric with respect to $x=\frac{1}{2}$. By taking into account the well-known *reflection property* of the gamma function

$$\Gamma(x)\cdot\Gamma(1-x)=\frac{\pi}{\sin{(\pi\,x)}}\,, \tag{1.2}$$

valid for all $x\notin\mathbb{Z}$, and being $\log\left[\,\Gamma(x)\cdot\Gamma(1-x)\right]=\log\Gamma(x)+\log\Gamma(1-x)$, one easily deduces that

$$f(x)=\log\left[\frac{\pi}{\sin{(\pi\,x)}}\right]=\log\pi-\log\sin{(\pi\,x)}\,. \tag{1.3}$$

From this logarithmic expression, one promptly deduces that $f(x)$ is differentiable (hence continuous) in the interval $(0,1)$, its derivative being $f\,'(x)=-\,\pi\,\cot{(\pi x)}$. The symmetry of $f(x)$ around $x=\frac{1}{2}$ can be taken into account for proofing that, being Conjec. 1 true, the only exception (if there is one) has to take place for $x=\frac{1}{2}$ (see the Appendix). From Eq. (1.3), we promptly deduce that $\log\pi-\log\sin{(\pi\,x)}$ is transcendental for all rational x in $(0,1)$, the only possible exception being $f(\frac{1}{2})=\log\pi=1.1447298858\ldots$ [1] All these consequences would be impressive, but the proof presented in Ref. [6] for Conjec. 1 is *incorrect*. This is because those authors implicitly assume that $f(x_1)\neq f(x_2)$ for every pair of distinct rational numbers x_1,x_2 in $(0,1)$, which is not true, as may be seen in Fig. 1, where the symmetry of $f(x)$ around $x=\frac{1}{2}$ can be appreciated. To be explicit, let me exhibit a simple counterexample: for the pair $x_1=\frac{1}{4}$ and $x_2=\frac{3}{4}$, Eq. (1.3) yields $f(x_1)=f(x_2)=\log\pi+\log\sqrt{2}$ and then $f(x_1)-f(x_2)=0$.[2] This *null* result clearly makes it invalid their conclusion that $f(x_1)-f(x_2)$ is a *non-null* Baker period.

[1] This is an interesting number whose irrationality is not yet established.

[2] In fact, null results are found for every pair of rational numbers $x_1,x_2\in(0,1)$ with $x_1+x_2=1$ (i.e., symmetric with respect to $x=\frac{1}{2}$).

Here in this short paper, I take Conjec. 1 on the transcendence of $f(x) = \log\Gamma(x) + \log\Gamma(1-x)$ into account for setting up a theorem establishing that there are at most *two* possible exceptions for the transcendence of $f(x)$, x being a rational in $(0,1)$. This theorem is proved here based upon a careful analysis of the monotonicity of $f(x)$, taking also into account its obvious symmetry with respect to $x = \frac{1}{2}$. Interestingly, this yields a criteria for the transcendence of $\log\pi$, an important number in the study of the algebraic nature of special values of a general class of L–functions [7]. This reformulation of the GMR "theorem" allows us to exhibit an infinity of pairs of logarithms of certain algebraic multiples of π whose elements are not both algebraic.

2. Transcendence of $\log\Gamma(x) + \log\Gamma(1-x)$ and exceptions

For simplicity, let us define $\mathbb{Q}_{(0,1)}$ as $\mathbb{Q} \bigcap (0,1)$, i.e. the set of all rational numbers in the real open interval $(0,1)$, which is a countable infinite set. My theorem on the transcendence of $\log\Gamma(x) + \log\Gamma(1-x)$ depends upon the fundamental theorem of Baker (1966) on the transcendence of linear forms in logarithms. We record this as:

Lemma 2.1 (Baker)**.** *Let $\alpha_1, \ldots, \alpha_n$ be nonzero algebraic numbers and $\beta_1, \ldots, \beta_n$ be algebraic numbers. Then the number*

$$\beta_1 \log\alpha_1 + \ldots + \beta_n \log\alpha_n$$

is either zero or transcendental. The latter case arises if $\log\alpha_1, \ldots, \log\alpha_n$ are linearly independent over $\mathbb{Q}$ and $\beta_1, \ldots, \beta_n$ are not all zero.

Proof. See theorems 2.1 and 2.2 of Ref. [8]. □

Now, let us define a *Baker period* according to Refs. [9, 10].

Definition 2.2 (Baker period)**.** A Baker period is any linear combination in the form $\beta_1 \log\alpha_1 + \ldots + \beta_n \log\alpha_n$, with $\alpha_1, \ldots, \alpha_n$ nonzero algebraic numbers and $\beta_1, \ldots, \beta_n$ algebraic numbers.

From Baker's theorem, it follows that

Corollary 2.3. *Any* non-null *Baker period is necessarily a transcendental number.*

Now, let us demonstrate the following theorem, which comprises the main result of this paper.

Theorem 2.4 (Main result)**.** *The number* $\log\Gamma(x) + \log\Gamma(1-x)$ *is transcendental for all $x \in \mathbb{Q}_{(0,1)}$, with at most* ***two*** *possible exceptions.*

Proof. Let $f(x)$ be the function defined in Eq. (1.1). From Eq. (1.3), $f(x) = \log \pi - \log \sin(\pi\, x)$ for all real $x \in (0,1)$. Let us divide the open interval $(0,1)$ into two adjacent subintervals by doing $(0,1) \equiv (0,\frac{1}{2}] \bigcup [\frac{1}{2},1)$. Note that $\sin(\pi\, x)$ — and thus $f(x)$ — is either a monotonically increasing or decreasing function in each subinterval. Now, suppose that $f(x_1)$ and $f(x_2)$ are both algebraic numbers, for some pair of distinct real numbers x_1 and x_2 in $(0,\frac{1}{2}]$. Then, the difference

$$f(x_2) - f(x_1) = \log \sin(\pi\, x_1) - \log \sin(\pi\, x_2) \tag{2.1}$$

will, itself, be an algebraic number. However, as the sine of any rational multiple of π is an algebraic number [11, 12], then Lemma 2.1 guarantees that, being $x_1, x_2 \in \mathbb{Q}$, then $\log \sin(\pi\, x_1) - \log \sin(\pi\, x_2)$ is either null or transcendental. Since $\sin(\pi x)$ is a continuous, monotonically increasing function in $(0,\frac{1}{2})$, then $\sin \pi x_1 \neq \sin \pi x_2$ for all $x_1 \neq x_2$ in $(0,\frac{1}{2}]$. Therefore, $\log \sin(\pi\, x_1) \neq \log \sin(\pi\, x_2)$ and then $\log \sin(\pi\, x_1) - \log \sin(\pi\, x_2)$ is a *non-null* Baker period. From Corol. 2.3, we know that non-null Baker periods are transcendental numbers, which contradicts our initial assumption. Then, there is at most one exception for the transcendence of $f(x)$, $x \in \mathbb{Q} \bigcap (0,\frac{1}{2}]$. Clearly, as $\sin(\pi x)$ is a continuous and monotonically decreasing function for $x \in [\frac{1}{2},1)$, an analogue assertion applies to this complementary subinterval, which yields another possible exception for the transcendence of $f(x)$, $x \in \mathbb{Q} \bigcap [\frac{1}{2},1)$. □

It is most likely that not even one exception takes place for the transcendence of $\log \Gamma(x) + \log \Gamma(1-x)$ with $x \in \mathbb{Q}_{(0,1)}$. If this is true, it can be deduced that $\log \pi$ is transcendental. If there are exceptions, however, then their number — either one or two, according to Theorem 2.4 — will determine the transcendence of $\log \pi$. The next theorem summarizes these connections between the existence of exceptions to the transcendence of $f(x)$, $x \in \mathbb{Q}_{(0,1)}$, and the transcendence of $\log \pi$.

Theorem 2.5 (Exceptions)**.** *With respect to the possible exceptions to the transcendence of* $\log \Gamma(x) + \log \Gamma(1-x)$*,* $x \in \mathbb{Q}_{(0,1)}$*, exactly one of the following statements is true:*

(i) *There are no exceptions, hence* $\log \pi$ *is a transcendental number;*
(ii) *There is only one exception and it has to be for* $x = \frac{1}{2}$*, hence* $\log \pi$ *is an algebraic number;*
(iii) *There are exactly two exceptions for some* $x \neq \frac{1}{2}$*, hence* $\log \pi$ *is a transcendental number.*

Proof. If $f(x) = \log\Gamma(x) + \log\Gamma(1-x)$ is a transcendental number for every $x \in \mathbb{Q}_{(0,1)}$, item(i), it suffices to put $x = \frac{1}{2}$ in Eq. (1.3) for finding that $f(\frac{1}{2}) = \log\pi$ is transcendental. If there is *exactly one* exception, item (ii), then it has to take place for $x = \frac{1}{2}$, otherwise (i.e., for $x \neq \frac{1}{2}$) the symmetry property $f(1-x) = f(x)$ would yield algebraic values for *two* distinct arguments, namely x and $1-x$. Therefore, $f(\frac{1}{2}) = \log\pi$ is the only exception, thus it is an algebraic number. If there are two exceptions, item (iii), both for $x \neq \frac{1}{2}$, then they have to be symmetric with respect to $x = \frac{1}{2}$, otherwise, by the property $f(1-x) = f(x)$, we would find more than two exceptions, which is prohibited by Theorem 2.4. Indeed, if one of the two exceptions is for $x = \frac{1}{2}$, then the other, for $x \neq \frac{1}{2}$, would yield a third exception, corresponding to $1-x \neq \frac{1}{2}$, which is again prohibited by Theorem 2.4. Then the two exceptions are for values of the argument distinct from $\frac{1}{2}$ and then $f(\frac{1}{2}) = \log\pi$ is a transcendental number. □

From this theorem, it is straightforward to conclude that

Criteria 1 (Algebraicity of $\log\pi$)**.** *The number* $\log\pi$ *is algebraic if and only if* $\log\Gamma(x) + \log\Gamma(1-x)$ *is a transcendental number for every* $x \in \mathbb{Q}_{(0,1)}$*, except* $x = \frac{1}{2}$*.*

The symmetry of the possible exceptions for the transcendence of $\log\Gamma(x) + \log\Gamma(1-x)$ around $x = \frac{1}{2}$ yields the following conclusion.

Corollary 2.6 (Pairs of logarithms)**.** *Every pair* $\{\log[\pi/\sin(\pi x)],\, \log[\pi/\sin(\pi y)]\}$*, with both* x *and* y *rational numbers in the interval* $(0,1)$*,* $y \neq 1-x$*, contains at least one transcendental number.*

By fixing $x = \frac{1}{2}$ in this corollary, one has

Corollary 2.7 (Pairs containing $\log\pi$)**.** *Every pair* $\{\log\pi,\, \log[\pi/\sin(\pi y)]\}$*,* y *being a rational in* $(0,1)$*,* $y \neq \frac{1}{2}$*, contains at least one transcendental number.*

3. The log-gamma function and the transcendence of $\pi \cdot e$

An interesting consequence of Corol. 2.7, together the famous Hermite-Lindemann (HL) theorem, is that the algebraicity of $\log\Gamma(y) + \log\Gamma(1-y)$ for some $y \in \mathbb{Q}_{(0,1)}$ implies the transcendence of $\pi \cdot e = 8.5397342226\ldots$, another number whose irrationality is not established yet. Let me proof this assertion based upon a logarithmic version of the HL theorem.

Lemma 3.1 (HL)**.** *For any non-zero complex number* w*, one at least of the two numbers* w *and* $\exp(w)$ *is transcendental.*

Proof. See Ref. [13] and references therein. □

Lemma 3.2 (HL, logarithmic version). *For any positive real number z, $z \neq 1$, one at least of the real numbers z and $\log z$ is transcendental.*

Proof. It is enough to put $w = \log z$, z being a non-negative real number, in Lemma 3.1 and to exclude the singularity of $\log z$ at $z = 0$. □

Theorem 3.3 (Transcendence of $\pi\, e$). *If the number $\log\Gamma(y) + \log\Gamma(1-y)$ is algebraic for some $y \in \mathbb{Q}_{(0,1)}$, then the number $\pi \cdot e$ is transcendental.*

Proof. Let us denote by $\overline{\mathbb{Q}}$ the set of all algebraic numbers and by $\overline{\mathbb{Q}}^*$ the set of all non-null algebraic numbers. First, note that $k(y) := 1/\sin(\pi y) \in \overline{\mathbb{Q}}^*$ for every $y \in \mathbb{Q}_{(0,1)}$ and that, from Eq. (1.3), $\log\Gamma(y) + \log\Gamma(1-y) = \log[k(y)\,\pi]$. Now, note that if $\log[k(y)\,\pi] \in \overline{\mathbb{Q}}$ for some $y = \tilde{y}$, then $1 + \log[k(\tilde{y})\,\pi]$ is also an algebraic number. Therefore, $\log e + \log[k(\tilde{y})\,\pi] = \log[k(\tilde{y})\,\pi\, e] \in \overline{\mathbb{Q}}$ and, by Lemma 3.2, the number $k(\tilde{y})\,\pi\, e$ has to be either transcendental or 1. However, it cannot be equal to 1 because this would imply that $k(y) = 1/(\pi\, e) < 1$, which is not possible because $0 < \sin(\pi\, y) \leq 1$ implies that $k(y) \geq 1$. Therefore, the product $k(\tilde{y})\,\pi\, e$ is a transcendental number. Since $k(\tilde{y}) \in \overline{\mathbb{Q}}^*$, then $\pi \cdot e$ has to be transcendental. □

4. Summary

In this work, the transcendental nature of $\log\Gamma(x) + \log\Gamma(1-x)$ for rational values of x in the interval $(0,1)$ has been investigated. I have first shown that the proof presented in Ref. [6] for the assertion that $\log\Gamma(x) + \log\Gamma(1-x)$ is transcendental for any rational value of x, $0 < x < 1$, with at most *one* possible exception is incorrect. I then reformulate their conjecture, presenting and proofing a theorem that establishes the transcendence of $\log\Gamma(x) + \log\Gamma(1-x)$, x being a rational in $(0,1)$, with at most *two* possible exceptions. The careful analysis of the number of possible exceptions has yielded a criteria for the number $\log\pi$ to be algebraic. I have also shown that each pair $\{\log[\pi/\sin(\pi x)],\, \log[\pi/\sin(\pi y)]\}$, $x, y \in \mathbb{Q}$, $y \neq 1-x$, contains at least one transcendental number. This occurs, in particular, with the pair $\{\log\pi,\, \log[\pi/\sin(\pi y)]\}$, $y \neq \frac{1}{2}$. At last, I have shown that if $\log[\pi/\sin(\pi y)]\}$ is algebraic for some $y \neq \frac{1}{2}$, then the product $\pi \cdot e$ has to be transcendental.

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Appendix

Let us show that the assumption that Conjec. 1 is true — i.e., that $f(x) = \log\Gamma(x) + \log\Gamma(1-x)$ is transcendental with at most *one* possible exception, x being a rational in $(0,1)$ — implies that if one exception exists then it has to be just $f(\frac{1}{2}) = \log\pi$.

The fact that $f(1-x) = f(x)$ for all $x \in (0,1)$ implies that, if the only exception would take place for some rational x distinct from $\frac{1}{2}$, then automatically there would be another rational $1-x$, distinct from x, at which the function would also assume an algebraic value (in fact, the same value obtained for x). However, Conjec. 1 restricts the number of exceptions to at most *one*. Then, we have to conclude that if an exception exists, it has to be for $x = \frac{1}{2}$, where $f(x)$ evaluates to $\log\pi$. □

References

[1] F. Lindemann, Sur le rapport de la circonference au diametre, et sur les logarithmes neperiens des nombres commensurables ou des irrationnelles algebriques, C. R. Acad. Sci. Paris **95** (1882) 72–74.
[2] G. V. Chudnovsky, Algebraic independence of constants connected with the exponential and elliptic functions, Dokl. Akad. Nauk Ukrain. SSR Ser. A **8** (1976) 698–701.
[3] F. Le Lionnais, Les nombres remarquables, Hermann, 1979, p. 46.
[4] T. Schneider, Zur Theorie der Abelschen Funktionen und Integrale, J. Reine Angew. Math. **183** (1941) 110–128.
[5] P. Grinspan, Measures of simultaneous approximation for quasi-periods of abelian varieties, J. Number Theory **94** (2002) 136–176.
[6] S. Gun, M. Ram Murty, P. Rath, Transcendence of the log gamma function and some discrete periods, J. Number Theory **129** (2009) 2154–2165.
[7] S. Gun, R. Murty, P. Rath, Transcendental nature of special values of L-functions, Canad. J. Math., in press.
[8] A. Baker, Transcendental Number Theory, Cambridge University Press, 1975, Chap. 2.
[9] M. Ram Murty, N. Saradha, Transcendental values of the digamma function, J. Number Theory **125** (2007) 298–318.
[10] M. Kontsevich, D. Zagier, Periods, in: *Mathematics Unlimited-2001 and Beyond*, Springer-Verlag, 2001, pp. 771–808.
[11] I. Niven, Irrational numbers, MAA, 2005, pp. 29–30.
[12] G. P. Dresden, A New Approach to Rational Values of Trigonometric Functions. Available at http://arxiv.org/abs/0904.0826
[13] A. Baker and G. Wüstholz, Logarithmic Forms and Diophantine Geometry, Cambridge University Press, 2007, Sec. 1.2.

Figures

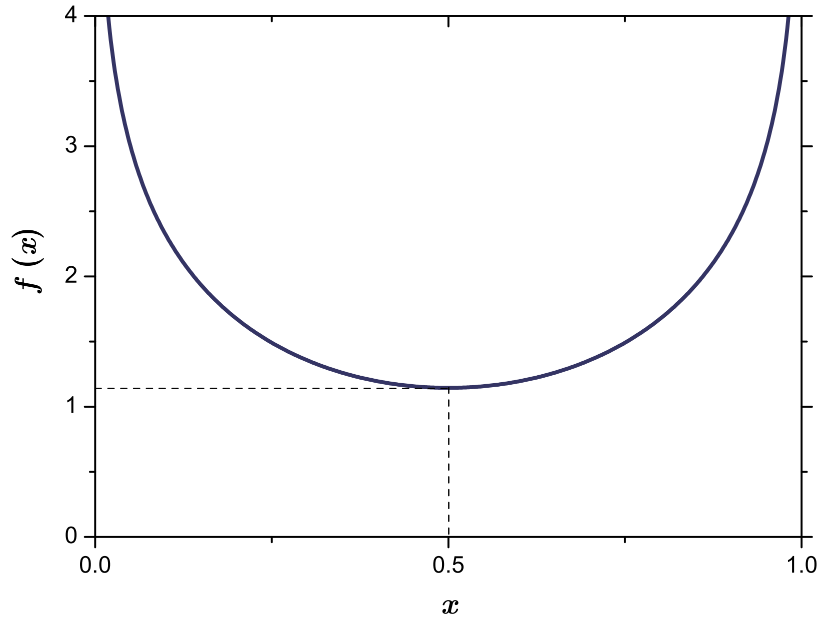

Figure 1. The graph of the function $f(x) = \log\Gamma(x) + \log\Gamma(1-x) = \log\pi - \log[\sin(\pi x)]$ in the interval $(0,1)$. Since $f(1-x) = f(x)$, the graph is symmetric with respect to $x = \frac{1}{2}$. Note that, as $0 < \sin(\pi\,x) \leq 1$ for all $x \in (0,1)$, then $\log\sin(\pi x) \leq 0$, and then $f(x) \geq \log\pi$ and the minimum of $f(x)$, x being in the interval $(0,1)$, is attained just for $x = \frac{1}{2}$, where $f(x)$ evaluates to $\log\pi$. The dashed lines highlight the coordinates of this point.

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