ONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
er. Meth. Fluids 2006; 50:673–682
nline 30 September 2005 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/fld.1062

Pressure boundary condition for the time-dependent incompressible Navier–Stokes equations

R. L. Sani[1,*,†], J. Shen[2], O. Pironneau[3] and P. M. Gresho[4]

[1] *Department of Chemical and Biological Engineering, University of Colorado, Boulder, CO, U.S.A.*
[2] *Department of Mathematics, Purdue University, West Lafayette, IN, U.S.A.*
[3] *Laboratoire Jacques-Louis Lions, Université Paris VI, France*
[4] *Lawrence Livermore National Laboratory, Livermore, CA, U.S.A.*

SUMMARY

In Gresho and Sani (*Int. J. Numer. Methods Fluids* 1987; **7**:1111–1145; *Incompressible Flow and the Finite Element Method*, vol. 2. Wiley: New York, 2000) was proposed an important hypothesis regarding the pressure Poisson equation (PPE) for incompressible flow: Stated there but not proven was a so-called equivalence theorem (assertion) that stated/asserted that if the Navier–Stokes momentum equation is solved simultaneously with the PPE whose boundary condition (BC) is the Neumann condition obtained by applying the normal component of the momentum equation on the boundary on which the normal component of velocity is specified as a Dirichlet BC, the solution (u, p) would be exactly the same as if the 'primitive' equations, in which the PPE plus Neumann BC is replaced by the usual divergence-free constraint ($\nabla \cdot u = 0$), were solved instead.

This issue is explored in sufficient detail in this paper so as to actually prove the theorem for at least some situations. Additionally, like the original/primitive equations that require *no BC* for the pressure, the new results *establish the same thing* when the PPE approach is employed. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: pressure boundary conditions; pressure Poisson equation; incompressible flow

1. INTRODUCTION

One of the most misunderstood aspects of incompressible flow has been the boundary condition (BC), if any, for the pressure. While the pressure in an incompressible flow has long been recognized as the Lagrangian constraint variable that enforces the divergence-free constraint [1], i.e. $\nabla \cdot u = 0$, and that the pressure Poisson equation (PPE) is a consequence of the constraint within the domain, there has been a great deal of confusion as to the appropriate BC for the PPE. In Reference [2], two versions of PPE are considered: one is the

*Correspondence to: R. L. Sani, Department of Chemical and Biological Engineering, 424 UCB, University of Colorado, Boulder, CO 80309-0424, U.S.A.
†E-mail: robert.sani@colorado.edu

Received 24 August 2004
Revised 5 July 2005
Accepted 11 July 2005

so-called consistent pressure Poisson equation (CPPE) where the term $\nu\Delta(\nabla\cdot u)$ is retained, and the other is the simplified pressure Poisson equation (SPPE) where this term is dropped. It has been almost universally recognized that the BC for SPPE is related to the fact that $\nabla\cdot u$ must vanish on the boundary if the solution and boundary are sufficiently smooth. However, the appropriate BC for the CPPE has been elusive and has taken various forms in the literature that have mainly been obtained without a detailed analysis. It has been described as a 'primary difficulty' in Reference [3], an 'open question' in Reference [4] while others [2, 5, 6] have made more definitive statements by resorting to heuristic or semi-rigorous arguments. The issue is an important one since oftentimes the continuity constraint is replaced by the derived PPE to effect either an analytical, or more often, a numerical solution of the transient Navier–Stokes equations. For example, this is the case in numerical solutions employing a projection or fractional step method.

A recent article by Rempfer [7] illustrates the confusion in the literature on the pressure BC issue. In assessing the PPE with Neumann BC he states, '...the resulting set of differential equations plus BCs represent an ill-posed problem'. This statement is proven herein to be absolutely incorrect.

The purpose of this paper is to explore this issue in sufficient detail and rigor to actually prove the theorem stated herein for at least some situations. In the course of the analysis, the difficulties in assigning the proper BC to the pressure field will become apparent and also some insight into the generation of seemingly good numerical simulations that utilize an improper pressure BC will be obtained. One surprising result of this new analysis is that, like the original/primitive equation approach that requires no BC for the pressure, the new results show the *same* thing when the CPPE approach is employed!

In order to simplify the analysis, the incompressible Stokes equations will be considered. The same conclusions hold for the Navier–Stokes equations but in general the analysis is more restricted and technical. Consider the time dependent, incompressible Stokes equations with Cauchy and Dirichlet data in a d-dimensional domain Ω with boundary Γ over the time interval $(0,T)$:

$$\frac{\partial u}{\partial t}+\nabla p-\nu\Delta u=f \quad \text{in } \Omega\times(0,T) \tag{1}$$

$$\nabla\cdot u=0 \quad \text{in } \Omega\times(0,T) \tag{2}$$

$$u=u^0 \quad \text{in } \Omega\times\{0\} \tag{3}$$

$$u=u_\Gamma \quad \text{on } \Gamma\times(0,T) \tag{4}$$

Here the external force f, the initial velocity field u_0 and the velocity at the boundary u_Γ are given data. Here it is assumed that $\partial u_\Gamma/\partial t$ has the same smoothness required of f and that $\nabla\cdot u^0=0$.

If u_Γ has zero flux on each connected component of the boundary Γ_i, i.e. $\int_{\Gamma_i} n\cdot u_{\Gamma_i}=0$, and if it is in $H^{1/2}(\Gamma)^d$, it can be extended into $\tilde{u}_\Gamma\in H^1(\Omega)^d$ with $\nabla\cdot u=0$ and by changing u into $u-\tilde{u}_\Gamma$, we come to the case of zero Dirichlet BC. So we can assume that $u_\Gamma=0$ without loss of generality.

The variational problem associated with (1)–(4) is to find $u \in L^2(0,T;V)$ satisfying (3) and such that

$$\int_{\Omega\times(0,T)} \left[\frac{\partial u}{\partial t}\cdot v + \nu\nabla u : \nabla v - f\cdot v\right] = 0 \quad \forall v \in V \tag{5}$$

where $A:B$ stands for $\Sigma_{ij}A_{ij}B_{ij}$ and

$$V = \{v \in H_0^1(\Omega)^d;\nabla\cdot v = 0\} \tag{6}$$

If $f \in L^2(0,T:H^{-1}(\Omega)^d)$ and Γ is smooth, the problem has a unique solution and there is a unique pressure $p \in L^2(\Omega\times(0,T))/R$ (i.e. up to a constant) such that

$$\int_{\Omega\times(0,T)} \left[\frac{\partial u}{\partial t}\cdot v + \nu\nabla u:\nabla v - p\nabla\cdot v - f\cdot v\right] = 0 \quad \forall v \in L^2\left(0,T;H_0^1(\Omega)^d\right) \tag{7}$$

and

$$\int_\Omega q\nabla\cdot u = 0 \quad \forall q \in L^2(\Omega) \tag{8}$$

are in $(0,T)$. (Herein we employ (5).) Problem (7)–(8) can also be studied directly with the same result.

Remark 1
If $f \in L^2(\Omega\times(0,T))^d$ and $u^0 \in V$, and Γ is smooth or if Ω is a convex polygon or polyhedron, then in addition (see References [8,9]) $u \in L^2(0,T;H^2(\Omega)^d)$ and $p \in L^2(0,T;H^1(\Omega))/R$.

2. CONSISTENT PRESSURE POISSON EQUATION FORMULATIONS

For numerical reasons, we would like to study the problem where the divergence equation is replaced by an equation for the pressure, namely we consider the following CPPE formulation (cf. References [2,5]):

$$\frac{\partial u}{\partial t} + \nabla p - \nu\Delta u = f \tag{9}$$

$$\Delta p - \nu\nabla\cdot(\Delta u) = \nabla\cdot f \tag{10}$$

$$u|_{t=0} = u^0 \tag{11}$$

$$u|_\Gamma = 0 \tag{12}$$

However, there were no rigorous mathematical analyses for this formulation and it was not completely clear how (10) should be interpreted mathematically and implemented numerically. The main purpose of this paper is to give a solid mathematical footing to this formulation, which in turn will guide its proper discretization.

2.1. A first weak formulation

We consider first the following weak formulation: with $f \in L^2(\Omega \times (0,T))^d$ *and* $u^0 \in V$, *and if* Γ *is smooth or if* Ω *is a convex polygon or polyhedron* find $u \in L^2(0,T;H_0^1(\Omega)^d)$ and $p \in L^2(\Omega \times (0,T))/R$ satisfying (11) and (12) such that

$$\int_{\Omega\times(0,T)} \left[\frac{\partial u}{\partial t}\cdot v + \nu\nabla u : \nabla v - p\nabla\cdot v - f\cdot v\right] = 0 \quad \forall v \in L^2(0,T;H_0^1(\Omega)^d) \tag{13}$$

$$\int_\Omega [(p - \nu\nabla\cdot u)\Delta q - q\nabla\cdot f] = 0 \quad \forall q \in H_0^2(\Omega) \tag{14}$$

a.e. in $(0,T)$.

Theorem 1
Problem (13)–(14) is equivalent to problem (7)–(8).

Proof
Note that all integrals are legitimate and so the formulation makes sense. Let us show first that the solution of (7)–(8) is solution of (13)–(14). We only need to show that (14) is satisfied. For this we choose $v = \nabla\phi$, $\phi \in H_0^2(\Omega)$ in (7). It becomes

$$\int_{\Omega\times(0,T)} \left[\frac{\partial u}{\partial t}\cdot\nabla\phi + \nu\nabla u : \nabla\nabla\phi + \nabla p\cdot\nabla\phi - f\cdot\nabla\phi\right] = 0 \quad \forall\phi \in H_0^2(\Omega) \tag{15}$$

The first term vanishes because of (2) after integration by parts. The second term is also zero by (2) because $\nu\nabla u : \nabla\nabla\phi = \nu[(\nabla\cdot u)\Delta\phi - \nabla\cdot(u\Delta\phi) + \nabla\cdot(\nabla\nabla\phi\cdot u)]$. The last two terms are integrated by parts and (14) is found.

Let us show now that the solution of (13)–(14) is solution of (7)–(8). To do this, we use (15) again but integrated by parts in the other direction. Then it is found that

$$\int_{\Omega\times(0,T)} \left[\frac{\partial u}{\partial t}\cdot\nabla\phi - (p - \nu\nabla\cdot u)\Delta\phi + \phi\nabla\cdot f\right] = 0 \quad \forall\phi \in H_0^2(\Omega) \tag{16}$$

Now by (14) and an integration by parts in the first term of (16) we find

$$\frac{\partial}{\partial t}\int_\Omega (\nabla\cdot u)\phi = 0 \quad \forall\phi \in H_0^2(\Omega) \tag{17}$$

which implies $\int_\Omega (\nabla\cdot u)q = \text{const.}$ a.e. in $(0,T)$. Finally, since it is zero at $t = 0$, $\nabla\cdot u = 0$. □

Remark 2
It is probably possible to prove directly that (13)–(14) has a solution; however, it is simpler to show, as we have done, that it has a solution because (7)–(8) has and any solution of one is solution of the other, and that it has only one solution because if it had more than one, then that would contradict the uniqueness of solution of (7)–(8).

Remark 3
If the term $\partial_t u$ is discretized by $(u^{n+1} - u^n)/\delta t$ and the scheme is made implicit (Euler or Crank–Nicolson/trapezoid rule) the same proof works on the semi-discrete problem.

2.2. An alternative weak formulation

We now present an alternative weak formulation of (9)–(12), which is more suitable for numerical implementation. Let $(\cdot,\cdot)$ be the inner product of $L^2(\Omega)^d$. We define

$$X := \{(u,p) : u \in H_0^1(\Omega)^d,\ p \in L_0^2(\Omega), -\nu\Delta u + \nabla p \in L^2(\Omega)^d\} \tag{18}$$

with the inner product

$$((u,p);(v,q))_X = (\nabla u, \nabla v) + (p,q) + (-\nu\Delta u + \nabla p, -\nu\Delta v + \nabla q) \tag{19}$$

It is clear that

$$\|(u,p)\|_X := (\|\nabla u\|_{L^2}^2 + \|p\|_{L^2}^2 + \|-\nu\Delta u + \nabla p\|_{L^2}^2)^{1/2} \tag{20}$$

is an induced norm in X. We first show that X is complete under the above norm, and therefore, X is a Hilbert space. Indeed, Let (u_m, p_m) be a Cauchy sequence in X. Thus, $u_m \to u$ in $H_0^1(\Omega)^d$, $p_m \to p$ in $L_0^2(\Omega)$ and $-\nu\Delta u_m + \nabla p_m \to w$ in $L^2(\Omega)^d$. On the other hand, $-\nu\Delta u_m \to -\nu\Delta u$ in $H^{-1}(\Omega)^d$ (the dual of $H_0^1(\Omega)^d$) and $\nabla p_m \to \nabla p$ in $H^{-1}(\Omega)^d$ and therefore,

$$-\nu\Delta u_m + \nabla p_m \to -\nu\Delta u + \nabla p \quad \text{in } H^{-1}(\Omega)^d$$

Since the limit is unique, we derive that $-\nu\Delta u + \nabla p = w$. Hence we have $(u_m, p_m) \to (u,p)$ in X.

Remark 4
Note that

$$\begin{aligned}\|\nabla p\|_{H^{-1}} &\leqslant \|-\nu\Delta u + \nabla p\|_{H^{-1}} + \|-\nu\Delta u\|_{H^{-1}} \\ &\leqslant C\,(\|-\nu\Delta u + \nabla p\|_{L^2} + \nu\,\|u\|_{H^1})\end{aligned}$$

and $\|p\|_{L^2} \leqslant C\|\nabla p\|_{H^{-1}}$ (see Reference [10]). Hence,

$$|||(u,p)||| := (\|\nabla u\|_{L^2}^2 + \|-\nu\Delta u + \nabla p\|_{L^2}^2)^{1/2}$$

is a norm of X equivalent to $\|(u,p)\|_X$

Thus, an alternative weak formulation for (9)–(12) is
For $f \in L^2(\Omega \times (0,T))^d$ and $u^0 \in V$, find $(u,p) \in L^2(0,T;X)$ such that

$$\begin{aligned}&\nu\frac{\mathrm{d}}{\mathrm{d}t}(\nabla u, \nabla v) + (-\nu\Delta u + \nabla p, -\nu\Delta v + \nabla q) = (f, -\nu\Delta v + \nabla q), \quad \forall (v,q) \in X \\ &u|_{t=0} = u^0\end{aligned} \tag{21}$$

Note that by taking $v = 0$ in the above formulation, we find

$$(-\nu\Delta u + \nabla p, \nabla q) = (f, \nabla q), \quad \forall q \in H^1(\Omega) \tag{22}$$

which is a weak form of (10).

Theorem 2
Problem (7)–(8) is equivalent to problem (21).

Proof
If $u \in H_0^1(\Omega)^d$ and $v \in H_0^1(\Omega)^d$ is such that there exists $q \in L_0^2(\Omega)$ with $(v, q) \in X$, then

$$\nu(\nabla u, \nabla v) = -\nu\langle u, \Delta v\rangle = \langle u, -\nu\Delta v + \nabla q\rangle - \langle u, \nabla q\rangle$$
$$= (u, -\nu\Delta v + \nabla q) + (\nabla \cdot u, q) \tag{23}$$

Now if (u, p) is a solution of (7)–(8), then if $f \in L^2(\Omega \times (0, T))$, then

$$\frac{\partial u}{\partial t} \in L^2(\Omega, (0, T))^d, \quad (u, p) \in X \tag{24}$$

almost everywhere in $(0, T)$. Therefore, taking the inner product of

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p - f = 0 \tag{25}$$

with $-\nu\Delta v - \nabla q$ for $(v, q) \in X$ yields

$$\left(\frac{\partial u}{\partial t}, -\nu\Delta v + \nabla q\right) + (-\nu\Delta u + \nabla p, -\nu\Delta v + \nabla q) = (f, -\nu\Delta v + \nabla q) \tag{26}$$

(The scalar products can be split because the three terms belong to $L^2(\Omega)$ almost everywhere in $(0, T)$.) But

$$\left(\frac{\partial u}{\partial t}, -\nu\Delta v + \nabla q\right) = \frac{d}{dt}(u, -\nu\Delta v + \nabla q) = \frac{d}{dt}[(u, -\nu\Delta v + \nabla q) + (\nabla \cdot u, q)]$$
$$= \nu\frac{d}{dt}(\nabla u, \nabla v) \tag{27}$$

since $\nabla \cdot u = 0$. Therefore, (21) is recovered.

Conversely, if for any $q \in L_0(\Omega)$ one chooses $v \in H_0^1(\Omega)^d$ such that

$$-\Delta v + \nabla q = 0 \tag{28}$$

i.e. $(v, q) \in X$, then

$$\frac{d}{dt}(\nabla \cdot u, q) = 0 \tag{29}$$

since $\nabla \cdot u = 0$. Then

$$\frac{d}{dt}(u, \nu\Delta v + \nabla q) + (-\nu\Delta u + \nabla p, -\nu\Delta v + \nabla q) = (f, -\nu\Delta v + \nabla q) \tag{30}$$

for $\forall(v, q) \in X$. But when (v, q) spans X, $-\nu\Delta v + \nabla q$ spans all $L^2(\Omega)^d$. Therefore, if $g \equiv \nu\Delta v$

$$\frac{d}{dt}(u, g) + (-\nu\Delta u + \nabla p, g) = (f, g) \quad \forall g \in L^2(\Omega)^d \tag{31}$$

or

$$\left(-\Delta v, \frac{\delta u}{\delta t} - v\Delta u + \nabla p - f\right) = 0$$

which leads to (7) and (8). □

Remark 5
One may prove the existence and uniqueness of the solution for problem (21) directly. Indeed, considering the following time-discretized backward Euler scheme ($n \geqslant 0$):

$$v\left(\nabla\frac{u^{n+1} - u^n}{\delta t}, \nabla v\right) + (-v\Delta u^{n+1} + \nabla p^{n+1}, -v\Delta v + \nabla q)$$
$$= (f^{n+1}, -v\Delta v + \nabla q), \quad \forall (v, q) \in X \tag{32}$$

By taking $(v, q) = (u^{n+1}, p^{n+1})$ in (25), we have the following energy identity:

$$\frac{1}{2\delta t}\left(\|\nabla u^{n+1}\|_{L^2}^2 - \|\nabla u^n\|_{L^2}^2 + \|\nabla(u^{n+1} - u^n)\|_{L^2}^2\right)$$
$$+ \|-v\Delta u^{n+1} + \nabla p^{n+1}\|_{L^2}^2 = (f, -v\Delta u^{n+1} + \nabla p^{n+1}) \tag{33}$$

Hence, the existence and uniqueness of (u^{n+1}, p^{n+1}) satisfying (32) is a direct consequence of the Lax–Milgram Theorem. The corresponding result for the time-continuous problem can be established by letting $\Delta t \to 0$. We leave the details to the interested readers.

Remark 6
The scheme (32) provides a stabilized formulation for the generalized Stokes problem (with $\partial u/\partial t$ replaced by αu in (1)), i.e. the inf–sup condition is satisfied for any pair of velocity–pressure approximation space such that (32) makes sense. Thus, (32) is very suitable in practice for solving time-dependent Stokes (or Navier–Stokes) equations or generalized Stokes equations. However, it cannot be applied to standard stationary Stokes or Navier–Stokes equations since it will not lead to a divergence-free solution.

3. DISCUSSION

In the above, we have focused on the consistent pressure Poisson formulation, i.e. (10). In practice, the following simplified pressure Poisson formulation (SPPE) has also been frequently used:

$$\frac{\partial u}{\partial t} + \nabla p - v\Delta u = f \tag{34}$$

$$\Delta p = \nabla \cdot f \tag{35}$$

$$u|_{t=0} = u^0 \tag{36}$$

$$\nabla \cdot u|_\Gamma = 0 \quad (37)$$

$$u|_\Gamma = 0 \quad (38)$$

It can be easily shown that the above formulation is equivalent to (1)–(2), with or without the term $\partial u/\partial t$. Note that the BC $\nabla \cdot u|_\Gamma = 0$ is essential but is difficult to implement in practice (e.g. cf. the 'influence matrix' technique [5]).

Remark 7
This PPE problem helps to explain the so-called 'PPE paradox' in Reference [2, p. 500].

3.1. Pressure boundary conditions

Now the question of the pressure BC for the CPPE can be addressed rigorously. In particular, is it

$$\nabla \cdot u|_\Gamma = 0 \quad (39)$$

or (where n denotes the outward pointing normal on Γ)

$$\left.\frac{\partial p}{\partial n}\right|_\Gamma = (f + \nu\Delta u) \cdot n|_\Gamma \quad (40)$$

or another since these and other alternatives have been proposed in References [2, 3, 5, 11, 12]. The answer now is clear that *none* of these BC is necessary, contrary to the SPPE formulation (34)–(38). (Both of the proofs presented above establish that a unique solution exists without specifying any pressure BC; in particular, the formulation (9)–(12), shown to be equivalent to (1)–(4), does not contain any pressure BC.)

Remark 8
In the more general case, using (4), the term $-n \cdot \nabla(\partial u_\Gamma/\partial t)$ must be added to the right-hand side of (40).

We now focus on (39) and (40) which often appear in the literature; whether either or both are satisfied is simply a matter of the regularity of the solution. Such regularity would require the boundary, the initial condition, and the forcing function to be sufficiently smooth (see, e.g. Reference [11]). While some have argued that the Neumann-type BC (40) is naturally contained in the basic continuum formulation, and imposing it in the continuum formulation (9)–(12) may impose a regularity that is not possible and thus favour (39), it is clear from our analysis that both will be satisfied by a smooth enough solution. For example, that (40) is satisfied by a smooth enough solution follows directly from (22) by integration by parts. Hence, either can preserve the divergence-free condition if the solution is smooth enough, and in this case, the one used depends on which is more convenient for the solution technique utilized. *The overriding result is that a unique solution to* (9)–(12) *exists without imposing any pressure BC.* This general result was obtained herein by using both an ultra weak formulation and a weak formulation, but it must remain valid for any solution; smoother solutions can possess additional properties such as (39) or (40). Another interesting question that arises is whether one can employ a different Neumann condition, i.e. a different right-hand side in (40) such as, for example, zero or even an arbitrary Dirichlet condition. These conditions have been proposed/utilized in certain numerical solution techniques in the literature and have yielded seemingly good results (see, for e.g. Reference [13]). Our analysis suggests that in fact it is possible for a discrete solution to seem reasonable but in general a non-smooth pressure field exhibiting a boundary layer would probably develop as the mesh is refined if, for example, a pressure BC other than (39) or (40) is used.

3.2. *Form of the pressure Poisson equation*

In many numerical schemes for the transient Navier–Stokes equations, the governing equations employ a PPE as the basis of the algorithm. For example, projection schemes or fractional step schemes are such schemes [2, 7, 12, 13]. Then the question arises as to whether terms such as $v\Delta(\nabla \cdot u)$ containing $\nabla \cdot u$ should be retained since $\nabla \cdot u = 0$ is included in the original continuum formulation, and thus vanishes. However, in the formulation of numerical algorithms for solving such problems, this question must be carefully addressed. Contrary to some numerical schemes, it is shown here that it is essential to *retain* the term. Unless the velocity is projected onto the space of divergence-free velocity fields at each time step, the solution of the SPPE scheme will not lead to a divergence-free velocity field and therefore will not be a solution of the incompressible Navier–Stokes equations. This projection is built-in to the CPPE form but not the SPPE form of the PPE. Thus, the SPPE formulation without the BC $\nabla \cdot u|_\Gamma = 0$ can lead to solutions which are not divergence free, i.e. do not represent incompressible flows [2]. Finally, it is noteworthy that the continuum formulation (9)–(12) can be recast into the following form by setting $q = p - v\nabla \cdot u$:

$$\frac{\partial u}{\partial t} + \nabla q + v\nabla \wedge \nabla \wedge u = f \quad \text{in } \Omega \times (0, T) \tag{41}$$

$$\Delta q = \nabla \cdot f \quad \text{in } \Omega \times (0, T) \tag{42}$$

$$u = u^0 \quad \text{in } \Omega \times \{0\} \tag{43}$$

$$u = 0 \quad \text{on } \Gamma \times (0, T) \tag{44}$$

This formulation can also be shown to be equivalent to (7)–(8) by slightly modifying the proofs presented above. It also has the property that the velocity is projected onto the space of divergence-free velocity fields at each time step and hence this formulation is a form that is equivalent to the formulation employing the CPPE. Note again that for a smooth enough solution, the solution will satisfy (39) and (40).

4. CONCLUDING REMARKS

In spite of what has been shown above, we (PMG and RLS) feel somewhat 'obligated' to return briefly to the issue of the Neumann BC, (40), for the PPE—especially in light of the fact some do use this BC for the PPE—both at $t = 0$ (to determine the initial pressure) and for $t > 0$ (to augment their analysis) (see Reference [11]).

We do this by simply verbally summarizing what was presented in detail in References [2, 5], viz.: Whenever the discrete PPE is generated from a consistent (but low order, using C_0 approximations with finite elements or low-order finite differences) discretization of the

NSE's, (1) and (2), with BC (4), which *requires* (among other things) that the incompressibility constraint, (2), be applied on the boundary—as well of course inside the domain—the resulting (so-called) CPPE, when examined closely at any boundary point, will always converge ($h \to 0$) to (40). Thus, such a consistent approximation will actually enforce, discretely, *both* (39) and (40).

Finally, we take this opportunity to correct an error in References [2, 5], where it is stated that setting $n \cdot u = n \cdot u_\Gamma$ is equivalent to setting $\nabla \cdot u = 0$ on Γ. This is not true and now we believe and assert that a very important aspect of incompressible flow is the requirement that $\nabla \cdot u = 0$ in $\Omega + \Gamma$ and for all $t \geqslant 0$. (Incompressibility is omnipotent!)

ACKNOWLEDGEMENTS

We are grateful to V. Girault for some important contributions. Additional useful input was provided by D. Griffiths, W. Henshaw, A. Peterson and A. Hindmarsh.

REFERENCES

1. Aris R. *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Prentice-Hall Inc.: Englewood Cliffs, NJ, 1962.
2. Gresho PM, Sani RL. *Incompressible Flow and the Finite Element Method*, vol. 2. Wiley: New York, 2000.
3. Peyret R, Taylor TD. *Computational Methods for Fluid Flow*. Springer: New York, 1983.
4. Ferziger JH. Simulation of incompressible flows. *Journal of Computational Physics* 1987; **69**:1–48.
5. Gresho PM, Sani RL. On the pressure boundary conditions for the incompressible Navier–Stokes equations. *International Journal for Numerical Methods in Fluids* 1987; **7**:1111–1145.
6. Roache PJ. *Computational Fluid Dynamics*. Hermosa Publishers: Albuquerque, NM, 1999.
7. Rempfer D. *Annual Review of Fluid Mechanics* 2003; **35**:229–265.
8. Ladyzhenskaya O. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach: New York, 1969.
9. Lions JL. *Problemès aux Limites Non-homogènes*. Dunod: Paris, 1970.
10. Temam R. *Navier Stokes Equations*. North-Holland: Amsterdam, 1977.
11. Heywood J, Rannacher R. Finite element approximations of the nonstationary Navier–Stokes problem. I. Regularity of solutions and the second-order error estimates for spatial discretizations. *SIAM Journal on Numerical Analysis* 1982; **19**:275–311.
12. Moin P, Kim J. On the numerical solution of time-dependent viscous incompressible flow involving solid boundaries. *Journal of Computational Physics* 1980; **35**:381–392.
13. Chorin AJ. A numerical method of solving incompressible viscous flow problems. *Journal of Computational Physics* 1967; **2**:12–26.

