On Boundary Integral Operators for the Laplace and the Helmholtz Equations and Their Discretisations

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Abstract

Boundary integral operators for the solutions of the Laplace and the Helmholtz equations are considered. These are classical strongly elliptic pseudodifferential operators of integer orders α, mapping the Sobolev space $\mathcal{H}^r$ to $\mathcal{H}^{r-\alpha}$. We study the spectral properties of the single layer Laplacian potential operator and its tangential derivative and also the double layer Laplacian potential operator and its normal derivative, the hypersingular operator, over a circle boundary. We extend the analysis to the Helmholtz potential operators. We derive important analytical results for the elements of the discrete operators and their eigenvalues and eigenvectors.

Keywords: Pseudodifferential operators, hypersingular operator, boundary integral equations, regularisation, Laplace potentials, Helmholtz potentials.

1 Introduction

Many boundary value problems of mathematical physics and engineering are now commonly reformulated as integral equations over the boundary of the domain of interest; see [5, 8, 3, 16] and references therein. By deriving explicitly qualitative properties of a number of important boundary integral operators we hope to make many abstract concepts from the pseudodifferential operator theory accessible to the engineering community.

Over smooth closed curves Γ in 2 dimensional space most boundary integral equations can be written in the form $\mathcal{A}\phi(\mathbf{p}) = f(\mathbf{p})$ for $\mathbf{p} \in \Gamma$, where $\mathcal{A} : \mathcal{H}^r(\Gamma) \to \mathcal{H}^{r-\alpha}(\Gamma)$ is a linear strongly elliptic pseudodifferential operator of order α; see [21]. This implies that $\lambda_m(\mathcal{A})$, the m-th eigenvalue of the pseudodifferential operator $\mathcal{A}$, behaves as $\mathcal{O}(m^\alpha)$ for large m. The pseudodifferential concept allows the study of the differential and integral operators within the same algebra of operators.

In connection with the solution of Laplace's equation in the interior or the exterior of Γ, we encounter the single layer and the double layer potential operators

$$(\mathcal{L}\sigma)(\mathbf{p}) = \int_\Gamma G(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})d\Gamma_\mathbf{q} \tag{1}$$

and

$$(\mathcal{M}\sigma)(\mathbf{p}) = \int_\Gamma \frac{\partial G}{\partial n_\mathbf{q}}(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})d\Gamma_\mathbf{q} \tag{2}$$

where

$$G(\mathbf{p},\mathbf{q}) = -\frac{1}{2\pi}\ln|\mathbf{p}-\mathbf{q}|$$

is the fundamental solution for the Laplacian operator, $\sigma \in \mathcal{H}^r(\Gamma)$ is a boundary density function and $n_{\mathbf{q}}$ represents the outward normal to Γ at $\mathbf{q}$. Other operators of interest are the tangential derivative of $\mathcal{L}$, a Cauchy singular operator, and the normal derivative of $\mathcal{M}$, an integro-differential or a hypersingular operator.

In this paper we study the qualitative properties of boundary integral operators and their discretisations arising in the solution of the Laplace and the Helmholtz equations. In Section 2, for the case where Γ is a circle, we present a brief and unified derivation of the spectral properties of the Laplace potential operators. In each case we show explicitly their order as pseudodifferential operators. A compact perturbation argument generally allows us to utilise these results in the study of the qualitative behaviour of the operators on general smooth boundaries [20]. The extent to which these qualitative results carry over to the case of Helmholtz potential operators, arising from time harmonic wave motions, is discussed in Section 3. In Section 4, by considering piecewise constant approximations on a circle, we are able to derive analytical expressions for the elements of the discrete form of these operators for the Laplace case and in the process arrive at many new and interesting identities. Furthermore, we derive new and exact expressions for the eigenvalues and eigenvectors of these boundary element matrices. Such results can be used for preconditioning boundary element systems for the solutions of the first kind or the hypersingular equations [15].

Briefly, to fix ideas, let $\mathcal{K}$ denote a generic boundary integral operator defined as

$$(\mathcal{K}\sigma)(\mathbf{p}) = \int_\Gamma K(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})d\Gamma_{\mathbf{q}}, \quad \mathbf{p}\in\Gamma.$$

Assume a smooth 2π-periodic parameterization of Γ in the form $\mathbf{z}:[0,2\pi]\to\Gamma$ with $\mathbf{p}=(p_1,p_2)^T = (z_1(s),z_2(s))^T$ and $\mathbf{q}=(q_1,q_2)^T = (z_1(t),z_2(t))^T$ where

$$d\Gamma_{\mathbf{q}} = \sqrt{\left(\frac{dz_1}{dt}\right)^2 + \left(\frac{dz_2}{dt}\right)^2}\, dt = |\mathbf{z}'(t)|dt$$

and the Jacobian $|\mathbf{z}'| \neq 0$. With this parameterization we may write

$$(\mathcal{K}\sigma)(\mathbf{p}) = \int_0^{2\pi} K(\mathbf{z}(s),\mathbf{z}(t))\sigma(\mathbf{z}(t))|\mathbf{z}'(t)|dt = \int_0^{2\pi} K_{\mathbf{z}}(s,t)\sigma_{\mathbf{z}}(t)dt, \quad s\in[0,2\pi], \tag{3}$$

with obvious definitions for $K_{\mathbf{z}}(s,t)$ and $\sigma_{\mathbf{z}}(t) = \sigma(\mathbf{z}(t))|\mathbf{z}'(t)|$. Where no confusion arises we drop the subscript $\mathbf{z}$ from the kernel $K_{\mathbf{z}}$ and the density function $\sigma_{\mathbf{z}}$.

2 Laplace Integral Operators on a Circle

The crucial observation here is that for the case where Γ is a circle the resulting operators have 2π-periodic kernels of convolution type $K(s-t)$ which are invariant under rotations. Consequently, $\mathcal{K}v_m = \lambda_m v_m$ for all $m\in\mathbb{Z}$, where the eigenfunctions are $v_m(s) = e^{ims}$ with $i=\sqrt{-1}$ and the eigenvalues are given by

$$\lambda_m = \int_0^{2\pi} e^{-imt}K(t)dt.$$

Therefore, in this section we are essentially developing Fourier series representations for the action of these operators. To this end, let us expand the 2π-periodic function v in its complex Fourier series

$$v(t) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbf{Z}} \widehat{v}(m)e^{imt}, \tag{4}$$

where the Fourier coefficients are given by

$$\widehat{v}(m) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} v(s)e^{-ims}ds.$$

Let us recall that the Sobolev space $\mathcal{H}^r[0,2\pi]$ of 2π-periodic functions consists of those functions v for which

$$||v||^2_{\mathcal{H}^r} = |\widehat{v}(0)|^2 + \sum_{m\neq 0} |m|^{2r}|\widehat{v}(m)|^2 \tag{5}$$

is finite [13]. The Sobolev space $\mathcal{H}^0[0,2\pi]$ is simply $\mathcal{L}^2[0,2\pi]$, the space of square-integrable functions. For $r<0$ functions in $\mathcal{H}^r$ should be seen as distributions or generalised functions as the series representation (4) does not in general converge in the classical sense. We come across examples of Sobolev spaces of distributions in later sections.

2.1 Single Layer Potential

Let $\mathbf{p} = (p_1, p_2) = (a\cos\theta_p, a\sin\theta_p)$ be the parametrization of the circle Γ. Then $|\mathbf{z}'(t)| = a$ and

$$r = |\mathbf{p}-\mathbf{q}| = 2a\left|\sin\left(\frac{\theta_p-\theta_q}{2}\right)\right|.$$

The single layer operator can be written as

$$(\mathcal{L}v)(\mathbf{p}) \equiv (\mathcal{L}v)(\theta_p) = -\frac{a}{2\pi}\int_0^{2\pi} \ln\left\{2a\left|\sin\frac{\theta_p-\theta_q}{2}\right|\right\} v(\theta_q)d\theta_q. \tag{6}$$

Substituting from (4) into (6) we find that

$$(\mathcal{L}v)(t) \quad = \quad -\frac{a}{4\pi}\frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbf{Z}} \widehat{v}(m)e^{imt}\int_0^{2\pi} \ln\left\{4a^2\sin^2\frac{\theta}{2}\right\} e^{im\theta}d\theta. \tag{7}$$

These integrals may be calculated using the Fourier cosine series expansion of the even 2π-periodic function $\phi_1(s)$ $(0<s<2\pi)$ where

$$\phi_1(s) = -\frac{1}{2}\ln\left\{4\sin^2\frac{s}{2}\right\} = -\frac{1}{2}\ln\{2(1-\cos s)\} = \sum_{n=1}^{\infty}\frac{1}{n}\cos(ns) = \sum_{n=1}^{\infty}\frac{1}{2n}\left(e^{ins}+e^{-ins}\right), \tag{8}$$

(see for example [10, §1.44]). Indeed

$$\int_0^{2\pi}\phi_1(\theta)e^{im\theta}d\theta = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{2\pi}\left(e^{i(n+m)\theta}+e^{i(m-n)\theta}\right)d\theta = \frac{\pi}{|m|}, \qquad m\neq 0,$$

where changing the order of integration and summation is valid in this series as the effect of integration is to place an additional power of n in the denominator, leading to a faster converging series than the original one (see [11]). Now we obtain

$$\int_0^{2\pi}\ln\left(4\sin^2\frac{\theta}{2}\right)e^{im\theta}d\theta = \begin{cases} 0 & m=0 \\ -\frac{2\pi}{|m|} & m\neq 0. \end{cases} \tag{9}$$

From (9) and (7) we obtain the required Fourier expansion

$$(\mathcal{L}v)(t) = \frac{1}{\sqrt{2\pi}} \left\{ \sum_{m\neq 0} \frac{a\,\widehat{v}(m)}{2\,|m|} e^{imt} - a\ln(a)\,\widehat{v}(0) \right\} \tag{10}$$

(see also [22, 20, 14]). From (10) it follows that:

Theorem 2.1 *The eigenvalues of $\mathcal{L}$, corresponding to the eigenfunctions $e^{\pm imt}$ with $m = 0, 1, 2, \ldots$ are given by:*

$$\lambda_m(\mathcal{L}) = \begin{cases} -a\ln a & m = 0 \\ \frac{a}{2|m|} & m \neq 0. \end{cases} \tag{11}$$

The single layer operator is singular for $a = 1$. In this case the transfinite diameter of Γ is equal to 1 [20, p.307]. The $\mathcal{O}(m^{-1})$ behaviour of the eigenvalues $\lambda_m(\mathcal{L})$ shows that $\mathcal{L}$ essentially acts as an integration operator. Thus, $\mathcal{L} : \mathcal{H}^r(\Gamma) \longrightarrow \mathcal{H}^{r+1}(\Gamma), \quad \forall r \in \mathbb{R}$, is a once smoothing pseudodifferential operator of order $\alpha = -1$.

2.2 Double Layer Potential

The double layer potential is defined by (2) where

$$\frac{\partial G}{\partial n_{\mathbf{q}}} = \frac{\partial G}{\partial r}\frac{\partial r}{\partial n_{\mathbf{q}}} \quad \text{and} \quad \frac{\partial r}{\partial n_{\mathbf{q}}} = \nabla r.n_{\mathbf{q}} = -\frac{\mathbf{r}.n_{\mathbf{q}}}{r}.$$

On a circle (2) becomes

$$\begin{aligned} (\mathcal{M}v)(\theta_p) &= \int_0^{2\pi} \left(-\frac{1}{2\pi r}\right)\left(\frac{r}{2a}\right) av(\theta_q)d\theta_q \\ &= \int_0^{2\pi} -\frac{1}{4\pi} v(\theta_q)d\theta_q = -\frac{\widehat{v}(0)}{2\sqrt{2\pi}}. \end{aligned} \tag{12}$$

From (12) it follows that:

Theorem 2.2 *The eigenvalues of $\mathcal{M}$, corresponding to the eigenfunctions e^{imt} with $m \in \mathbb{Z}$, are:*

$$\lambda_m(\mathcal{M}) = \begin{cases} -\frac{1}{2} & m = 0 \\ 0 & m \neq 0. \end{cases} \tag{13}$$

The result on the non-zero eigenvalue of $\mathcal{M}$ corresponding to the constant eigenfunction is simply the well known Gauss' Integral written as $(\mathcal{M}1)(\theta_p) = -\frac{1}{2}$, $\theta_p \in \Gamma$, a result which is valid for all smooth Γ (see for example [18, p.383]).

Equation (12) or (13) indicates that $\mathcal{M}$ is an infinitely smoothing operator, i.e. $\mathcal{M} : \mathcal{H}^r(\Gamma) \longrightarrow \mathcal{C}^\infty(\Gamma), \ \forall r \in \mathbb{R}$. This latter result remains true for all closed $\mathcal{C}^\infty$ boundaries [8, p.250].

2.3 Hilbert Operator

On a circle, the tangential derivative of the single layer operator satisfies $\frac{d}{d\theta}(\mathcal{L}v)(\theta) = \frac{a}{2}(\mathcal{P}v)(\theta)$, where $\mathcal{P}$ is the Hilbert operator [13, §7.2], [18, pp.393–397]

$$\begin{aligned} (\mathcal{P}v)\,(\theta) &= \frac{1}{2\pi} \int\!\!\!\!\!\!-_{-\pi}^{\pi} v(\omega)\cot\frac{\omega-\theta}{2}d\omega \\ &= \lim_{\varepsilon\to 0}\left\{ \frac{1}{2\pi}\int_{-\pi}^{\theta-\varepsilon} v(\omega)\cot\frac{\omega-\theta}{2}d\omega + \frac{1}{2\pi}\int_{\theta+\varepsilon}^{\pi} v(\omega)\cot\frac{\omega-\theta}{2}d\omega \right\}, \end{aligned} \tag{14}$$

where $⨍$ indicates that the singular integral is to be understood in the *Cauchy principal value* sense.

Replacing the Fourier expansion of v from (4) into (14) and noting that $\cot(t/2) = d/dt(\ln \sin^2(t/2))$ we can show using (9) that

$$(\mathcal{P}v)(\theta) = -\frac{i}{\sqrt{2\pi}} \sum_{m=-\infty}^{-1} \widehat{v}(m)e^{im\theta} + \frac{i}{\sqrt{2\pi}} \sum_{m=1}^{\infty} \widehat{v}(m)e^{im\theta}. \tag{15}$$

Equation (15) is essentially obtained from term by term differentiation of (10), which is valid as functions are viewed in the sense of distribution (see [23]). Note that if $v(\theta)$ is an even function, that is $\widehat{v}(m) = \widehat{v}(-m)$ for $m = 1, 2, \ldots$, then $(\mathcal{P}v)(\theta)$ will be odd and if $v(\theta)$ is odd then $(\mathcal{P}v)(\theta)$ will be even. From (15) it follows that:

Theorem 2.3 *The eigenvalues of $\mathcal{P}$, corresponding to the eigenfunctions e^{imt} are:*

$$\lambda_m(\mathcal{P}) = \begin{cases} -i & m < 0 \\ 0 & m = 0 \\ +i & m > 0. \end{cases} \tag{16}$$

Equation (16) implies that $\mathcal{P} : \mathcal{H}^r(\Gamma) \to \mathcal{H}^r(\Gamma)$, is a pseudodifferential operator of order $\alpha = 0$.

2.4 Hypersingular Operator

The hypersingular operator $\mathcal{N}$, the normal derivative of the double layer potential, is defined by

$$(\mathcal{N}v)(\mathbf{p}) = \frac{\partial}{\partial n_{\mathbf{p}}} \int_\Gamma \frac{\partial G}{\partial n_{\mathbf{q}}}(\mathbf{p}, \mathbf{q})v(\mathbf{q})d\Gamma_{\mathbf{q}} = ⨎_\Gamma \frac{\partial^2 G}{\partial n_{\mathbf{p}} \partial n_{\mathbf{q}}}(\mathbf{p}, \mathbf{q})v(\mathbf{q})d\Gamma_{\mathbf{q}},$$

where $⨎$ indicates that the integral should be understood in the Hadamard *finite part* sense since the kernel now has strong singularity of the form $\mathcal{O}(r^{-2})$. To be precise,

$$⨎_\Gamma \frac{\partial^2 G}{\partial n_{\mathbf{p}} \partial n_{\mathbf{q}}}(\mathbf{p}, \mathbf{q})v(\mathbf{q})d\Gamma_{\mathbf{q}} = \lim_{\epsilon \to 0} \{ \int_{\Gamma - B_{\mathbf{p},\epsilon}} \frac{\partial^2 G}{\partial n_{\mathbf{p}} \partial n_{\mathbf{q}}}(\mathbf{p}, \mathbf{q})v(\mathbf{q})d\Gamma_{\mathbf{q}} - \frac{v(p)}{\pi\epsilon} \}, \tag{17}$$

where $B_{\mathbf{p},\epsilon}$ is an interval of length ϵ on Γ about $\mathbf{p}$. Alternative definitions for (14) and (17), not requiring ' limit as $\epsilon \to 0$', suitable for computation, are given in [4]. Maue [17] and Mitzner [19] (see also [7, 16]) rewrite $\mathcal{N}$ in the following form involving tangential rather than normal derivatives:

$$(\mathcal{N}v)(\mathbf{p}) = -\frac{\partial}{\partial t_{\mathbf{p}}} ⨍_\Gamma \frac{\partial G}{\partial t_{\mathbf{q}}}(\mathbf{p}, \mathbf{q})v(\mathbf{q})d\Gamma_{\mathbf{q}},$$

which shows that $\mathcal{N}$ acts as the derivative of a Cauchy principal value integral.

In the case of a circle of radius a we have $dt \equiv ds = a\, d\theta$ where t denotes tangential and s arc length, hence

$$(\mathcal{N}v)(\theta_p) = -\frac{1}{4\pi a} \frac{d}{d\theta_p} ⨍_0^{2\pi} \cot\left(\frac{\theta_p - \theta_q}{2}\right) v(\theta_q)d\theta_q. \tag{18}$$

That is, on a circle, $\mathcal{N}$ behaves as the derivative of the Hilbert operator, given by

$$(\mathcal{N}v)(\theta) = \frac{1}{2a} \frac{d}{d\theta}(\mathcal{P}v)(\theta). \tag{19}$$

Taking the derivative under the integral sign in (18) we obtain

$$(\mathcal{N}v)(\theta_p) = \frac{1}{8\pi a} ⨎_0^{2\pi} \operatorname{cosec}^2\left(\frac{\theta_p - \theta_q}{2}\right) v(\theta_q)d\theta_q, \tag{20}$$

the hypersingular form of the operator. Using (15) in (19) we obtain the required expansion

$$(\mathcal{N}v)\,(t) = -\frac{1}{\sqrt{2\pi}} \sum_{m\neq 0} \frac{|m|}{2a} \widehat{v}(m) e^{imt}. \tag{21}$$

Again term by term differentiation of (15) is valid when functions are viewed in the sense of distribution (see [23]). From (21) it follows that:

Theorem 2.4 *The eigenvalues of* $\mathcal{N}$*, corresponding to the eigenfunctions* $e^{\pm imt}$*,*

$$\lambda_m(\mathcal{N}) = \begin{cases} 0 & m = 0 \\ -\frac{|m|}{2a} & m \neq 0. \end{cases} \tag{22}$$

Equation (22) implies that $\mathcal{N}$ is essentially a once differentiation operator, satisfying $\mathcal{N} : \mathcal{H}^r(\Gamma) \longrightarrow \mathcal{H}^{r-1}(\Gamma),\ \forall r \in \mathbb{R}$.

3 Helmholtz Integral Operators on a Circle

The boundary integral solution of the Helmholtz equation, $(\nabla^2 + k^2)\phi = 0$, governing time harmonic wave motion, is of considerable practical interest; see [3] and the many references therein. The principal part of this operator is the Laplacian operator. We show explicitly the extent to which the qualitative behaviour of the Helmholtz potential operators, namely the single layer, the double layer and the hypersingular operator follow those of the corresponding Laplacian case.

The fundamental solution for the Helmholtz operator is given by

$$G_k(\mathbf{p}, \mathbf{q}) \equiv G_k(r) = \frac{i}{4} H_0^{(1)}(kr) \tag{23}$$

where $r = |\mathbf{p} - \mathbf{q}|$ and $k \in \mathbb{R}$ is the wavenumber. $H_n^{(1)}(z) = J_n(z) + iY_n(z)$ denotes the Hankel function of integer order, with J_n and Y_n the Bessel functions of first and second kind respectively. We will require the following properties of the Hankel functions [1, §9]

$$\begin{cases} \frac{d}{dz} H_0^{(1)}(z) &=& -H_1^{(1)}(z) \\ \frac{d}{dz} H_1^{(1)}(z) &=& H_0^{(1)}(z) - \frac{H_1^{(1)}(z)}{z} \\ iH_0^{(1)}(z) &=& -\frac{2}{\pi} \ln z + C + \frac{z^2}{4\pi} \ln z^2 + \mathcal{O}(z^2), \quad z \to 0 \\ iH_1^{(1)}(z) &=& \frac{2}{\pi z} - \frac{z}{\pi} \ln z + \mathcal{O}(z), \quad z \to 0 \end{cases} \tag{24}$$

where $C = \frac{2}{\pi}(\ln 2 - \gamma) + i$ and $\gamma = 0.5772\ldots$ is Euler's constant.

3.1 Single Layer Helmholtz Potential

The single layer Helmholtz potential operator is defined by

$$(\mathcal{L}_k \sigma)(\mathbf{p}) = \int_\Gamma G_k(\mathbf{p}, \mathbf{q}) \sigma(\mathbf{q}) d\Gamma_{\mathbf{q}} \quad \mathbf{p} \in \Gamma.$$

From (23) and (24) we obtain the expansion

$$G_k(r) = \frac{i}{4} H_0^{(1)}(kr) = -\frac{1}{2\pi} \ln r + D + \frac{k^2}{16\pi} r^2 \ln r^2 + \mathcal{O}(r^2), \tag{25}$$

where the constant D depends only on k. The first term on the right hand side of (25) is the Green's function for the Laplace equation. The constant kernel function defines an infinitely smoothing operator and as we shall see shortly, an operator, $\mathcal{R}$ say, with kernel $r^2 \ln r^2$ maps $\mathcal{H}^s \to \mathcal{H}^{s+3}$, indicating that $\lambda_m(\mathcal{R}) = \mathcal{O}(m^{-3})$. It follows from (25) that $\mathcal{L}_k : \mathcal{H}^s \to \mathcal{H}^{s+1}$ and that qualitatively the behaviour of $\mathcal{L}_k$ is similar to $\mathcal{L}$.

When the boundary Γ is a circle the Helmholtz potential operators also have e^{imt} as their eigenfunctions, with $m \in \mathbb{Z}$. In this case we can find [12] explicitly the eigenvalues of the $\mathcal{L}_k$ as $\lambda_m(\mathcal{L}_k) = \{\frac{i\pi}{2} J_m(ka) H_m(ka)\}$ where J_m and H_m are the Bessel and Hankel functions of order m. See also [2] where appropriate expansions of the J_m and H_m explicitly reveals that $\lambda_m(\mathcal{L}_k) = \lambda_m(\mathcal{L}) + \mathcal{O}(\frac{k^2}{m^3})$, in agreement with results here.

3.2 Double Layer Helmholtz Potential

As in equation (2) the Helmholtz double layer potential operator is defined as

$$(\mathcal{M}_k\sigma)(\mathbf{p}) = \int_\Gamma \frac{\partial G_k}{\partial n_\mathbf{q}}(\mathbf{p}, \mathbf{q})\sigma(\mathbf{q}) d\Gamma_\mathbf{q} \quad \mathbf{p} \in \Gamma.$$

On a circle the kernel is given by

$$\frac{\partial G_k}{\partial n_\mathbf{q}} = \frac{\partial r}{\partial n_\mathbf{q}} \frac{\partial G_k}{\partial r} = \left(\frac{r}{2a}\right)\left(-\frac{ik}{4} H_1^{(1)}(kr)\right) = -\frac{ikr}{8a} H_1^{(1)}(kr). \tag{26}$$

Hence, it follows from (24) that

$$\frac{\partial G_k}{\partial n_\mathbf{q}} = -\frac{1}{4\pi} + \frac{k^2}{16\pi} r^2 \ln r^2 + \mathcal{O}(r^2) \tag{27}$$

where the constant term is identified as the kernel of the operator $\mathcal{M}$, an infinitely smoothing operator. Therefore $\mathcal{M}_k$ is dominated by the operator $\mathcal{R}$. In order to demonstrate the mapping properties of $\mathcal{R}$ we need to consider integrals

$$\begin{aligned} \mathcal{I} &= \int_0^{2\pi} r^2 \ln r^2 e^{imt} dt, \qquad \text{with } r = 2a \sin\frac{t}{2} \\ &= 4a^2(2\ln a + 1)\int_0^{2\pi} \sin^2\frac{t}{2} e^{imt} dt + 4a^2 \int_0^{2\pi} \sin^2\frac{t}{2} \ln\left(\frac{4}{e}\sin^2\frac{t}{2}\right) e^{imt} dt. \end{aligned}$$

Writing $\sin^2\frac{t}{2} = \frac{1}{4}\left(-e^{it} + 2 - e^{-it}\right)$ we can show easily that

$$\int_0^{2\pi} \sin^2\frac{t}{2} e^{imt} dt = \begin{cases} \pi & m = 0 \\ -\pi/2 & m = \pm 1 \\ 0 & \text{otherwise,} \end{cases} \tag{28}$$

and also that

$$\gamma_m := \int_0^{2\pi} \sin^2\frac{t}{2} \ln\left(\frac{4}{e}\sin^2\frac{t}{2}\right) e^{imt} dt = \frac{1}{4}(-c_{m-1} + 2c_m - c_{m+1}),$$

where

$$c_m := \int_0^{2\pi} \ln\left(\frac{4}{e}\sin^2\frac{t}{2}\right) e^{imt} dt = -\frac{2\pi}{\max(1, |m|)}. \tag{29}$$

Equation (29) follows from (9). We now find that

$$\gamma_0 = 0 \quad \gamma_1 = \gamma_{-1} = -1/8 \quad \text{and} \quad \gamma_m = \frac{1}{2\,|m|\,(m^2-1)} \qquad |m| \geq 2, \tag{30}$$

and therefore deduce the required result [14]

$$\mathcal{I} = \int_0^{2\pi} r^2 \ln r^2 e^{imt} dt = \begin{cases} 4\pi a^2(2\ln a + 1) & m = 0 \\ -\pi a^2(4\ln a + 3) & m = \pm 1 \\ \dfrac{4a^2\pi}{|m|\,(m^2-1)} & \text{otherwise,} \end{cases}$$

With equation (27) this shows that for fixed k

$$\lambda_m(\mathcal{M}_k) \simeq \frac{a^2k^2}{4}\frac{1}{|m|\,(m^2-1)} = \mathcal{O}(m^{-3}) \quad \text{as } m \to \pm\infty. \tag{31}$$

This implies that $\mathcal{R}$ and hence $\mathcal{M}_k$ on a circle, map $\mathcal{H}^s \longrightarrow \mathcal{H}^{s+3}$, $\forall s \in \mathbb{R}$. Let us remark here that the operator $\mathcal{M}_k$ on a circle is smoother than that on a general smooth boundary where its order is -1. From [12], the eigenvalues of $\mathcal{M}_k$ are given by $\lambda_m(\mathcal{M}_k) = -\frac{1}{2} + \frac{i\pi}{2} k J'_m(ka) H_m(ka)$ where appropriate asymptotic expansions J'_m and H_m in [2] gives results similar to (31).

3.3 Hypersingular Helmholtz Operator

The normal derivative of $\mathcal{M}_k$ is denoted by $\mathcal{N}_k$ and

$$(\mathcal{N}_k\sigma)(\mathbf{p}) = \frac{\partial}{\partial n_\mathbf{p}} \int_\Gamma \frac{\partial G_k}{\partial n_\mathbf{q}}(\mathbf{p},\mathbf{q})\sigma(\mathbf{q})d\Gamma_\mathbf{q} \quad \mathbf{p} \in \Gamma.$$

As in the Laplacian case, we may take the derivative inside the integral provided that we interpret the resulting hypersingular integral in the sense of Hadamard's finite part. The kernel then may be written as

$$\frac{\partial^2 G_k}{\partial n_\mathbf{p} \partial n_\mathbf{q}} = \frac{\partial G_k}{\partial r}\frac{\partial^2 r}{\partial n_\mathbf{p} \partial n_\mathbf{q}} + \frac{\partial^2 G_k}{\partial r^2}\frac{\partial r}{\partial n_\mathbf{p}}\frac{\partial r}{\partial n_\mathbf{q}}$$

where

$$\frac{\partial^2 r}{\partial n_\mathbf{p} \partial n_\mathbf{q}} = -\frac{1}{r}\left(\mathbf{n}_p.\mathbf{n}_q + \frac{\partial r}{\partial n_\mathbf{p}}\frac{\partial r}{\partial n_\mathbf{q}}\right).$$

On a circle this becomes

$$\frac{\partial^2 r}{\partial n_\mathbf{p} \partial n_\mathbf{q}} = -\frac{1}{r}\left(1 - \frac{r^2}{4a^2}\right),$$

and using (24) we find

$$\frac{\partial^2 G_k}{\partial n_\mathbf{p} \partial n_\mathbf{q}} = \frac{ik}{4r}H_1^{(1)}(kr) - \frac{ik^2r^2}{16a^2}H_0^{(1)}(kr) = \frac{1}{2\pi r^2} - \frac{k^2}{4\pi}\ln r + \mathcal{O}(1), \quad \text{as } r \to 0. \tag{32}$$

Thus $\mathcal{N}_k$ is dominated by the order $+1$ operator $\mathcal{N}$ (see equation (20)) whilst the second term on the right hand side corresponds to the single layer Laplace operator with order -1. This also demonstrates that in practice we ought to calculate $\mathcal{N}_k\sigma$ in the form

$$(\mathcal{N}_k\sigma)(\mathbf{p}) = (\mathcal{N}_k - \mathcal{N} - \frac{k^2}{2}\mathcal{L})\sigma(\mathbf{p}) + (\mathcal{N}\sigma)(\mathbf{p}) + \frac{k^2}{2}(\mathcal{L}\sigma)(\mathbf{p}),$$

as recommended in [6].

The eigenvalues of $\mathcal{N}_k$ are given in [12] as $\lambda_m(\mathcal{N}_k) = \frac{i\pi}{2}(ka)^2 J'_m(ka)H'_m(ka)$ and appropriate expansions of J'_m and H'_m in [2] yield results in agreement with our analysis here.

4 Discrete Boundary Integral Operators

In boundary element methods the operator equation $\mathcal{A}\phi = f$ is often discretised using collocation methods based on piecewise polynomial approximation of ϕ, to yield a full linear system of equations $A_n\phi_n = f_n$. In general the elements of the boundary element matrix are given by

$$(A_n)_{i,j} = \int_\Gamma K(\mathbf{p}_i, \mathbf{q}) v_j(\mathbf{q}) d\Gamma_{\mathbf{q}},$$

where $\{\mathbf{p}_i\}$ are the collocation points and $\{v_j\}$ are the compactly supported polynomial basis functions. In this section, in order to carry out the computation of the matrix elements analytically we will consider the case of piecewise constant basis functions only. The collocation points will be taken at the mid-points of each element.

We limit our analysis to the Laplacian operators as the results will be qualitatively the same for the Helmholtz operators. Note that because of the special geometry of the boundary and the equally spaced nature of our piecewise constant approximation space, the boundary element matrices L, M and N are all symmetric and circulant whilst P will be circulant. Recall that a Toeplitz matrix is characterised by having constant entries along each diagonal, i.e. $A_{i,j} = \alpha_{j-i}$ for some scalars $\alpha_{-n+1}, \alpha_{-n+2}, \cdots, \alpha_0, \cdots, \alpha_{n-1}$. A circulant matrix is a Toeplitz matrix in which the rows 'wrap around'. That is, a Toeplitz matrix where $\alpha_{-1} = \alpha_{n-1}, \alpha_{-2} = \alpha_{n-2}, \cdots, \alpha_{-n+1} = \alpha_1$. Therefore, a circulant matrix is fully described by its first row $(A_{11}, A_{12}, \cdots, A_{1n})$.

4.1 Piecewise Constant Basis Functions

We divide the circle into n equal elements and define the piecewise constant basis functions by

$$v_i(s) = \begin{cases} 1 & s \in [\theta_{i-1}, \theta_i] \\ 0 & \text{otherwise,} \end{cases}$$

$\theta_i = i\Delta\theta, \quad i = 1, \ldots, n$ and $\Delta\theta = \frac{2\pi}{n}$. In this section we write v_i in a form suitable for spectral analysis of discrete operators and derive some useful identities.

The complex Fourier coefficients of v_i are given by

$$\widehat{v}_i(m) = \frac{1}{\sqrt{2\pi}} \int_{\theta_{i-1}}^{\theta_i} e^{-ims} ds = \frac{1}{\sqrt{2\pi}} \frac{i}{m} \left(e^{-im\theta_i} - e^{-im\theta_{i-1}} \right) = \frac{1}{\sqrt{2\pi}} \frac{i}{m} e^{-im\theta_i} \left(1 - e^{iq} \right), \tag{33}$$

where $m \neq 0$, $q = m\Delta\theta$ and $\widehat{v}_i(0) = \sqrt{2\pi}/n$. It follows from (33) that

$$|\widehat{v}_i(m)|^2 = \frac{2}{\pi} |m|^{-2} \sin^2\left(\frac{m\pi}{n}\right) \tag{34}$$

and using (5) we find

$$\|v_i\|^2_{\mathcal{H}^r} = \frac{2\pi}{n^2} + \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{1}{m^{2(1-r)}} \sin^2\left(\frac{m\pi}{n}\right), \tag{35}$$

which is finite if $r < 1/2$ by comparison with $\sum_{m=1}^{\infty} m^{-2(1-r)}$. In other words, $v_i \in \mathcal{H}^r$, for all $r < \frac{1}{2}$. To demonstrate that $v_i \notin \mathcal{H}^{\frac{1}{2}}$, we put $r = \frac{1}{2}$ in (35) and show that the series diverges. This is because

$$\sum_{m=1}^{\infty} \frac{1}{m} \sin^2\left(\frac{m\pi}{n}\right) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m} \cos\left(\frac{2m\pi}{n}\right), \tag{36}$$

where the first summation on the right hand side of (36) is the unbounded harmonic series, and the second summation converges to $\phi_1\left(\frac{2\pi}{n}\right)$, with ϕ_1 defined in (8). In particular for $r=0$ equation (35) becomes

$$\|v_i\|^2_{\mathcal{H}^0} = \frac{2\pi}{n^2} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin^2\left(\frac{m\pi}{n}\right). \tag{37}$$

Note however that from the standard definition of the norms on the $\mathcal{L}^2 = \mathcal{H}^0$ space

$$\|v_i\|^2_{\mathcal{L}^2} = \int_{\theta_{i-1}}^{\theta_i} |v_i|^2\, d\theta = \theta_i - \theta_{i-1} = \Delta\theta = \frac{2\pi}{n}.$$

Using this result in (37) we find the value of the series

$$S := \sum_{m=1}^{\infty}\frac{1}{m^2}\sin^2\left(\frac{m\pi}{n}\right) = \frac{\pi^2}{2}\left(\frac{1}{n} - \frac{1}{n^2}\right), \tag{38}$$

which will be needed in later sections.

We now calculate the Fourier series for v_i in a form suitable for our analysis in this paper.

$$\begin{aligned}
v_i(s) &= \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\widehat{v_i}(m)e^{ims} = \frac{1}{n} + \frac{i}{2\pi}\sum_{m\neq 0}\frac{1}{m}\left(e^{-im\theta_i} - e^{-im\theta_{i-1}}\right)e^{ims} \\
&= \frac{1}{n} + \frac{i}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\left(e^{im(s-\theta_i)} - e^{-im(s-\theta_i)}\right) - \left(e^{im(s-\theta_i+\Delta\theta)} - e^{-im(s-\theta_i+\Delta\theta)}\right)\right] \\
&= \frac{1}{n} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\sin m(s-\theta_i+\Delta\theta) - \sin m(s-\theta_i)\right].
\end{aligned} \tag{39}$$

Clearly v_i is an even function about $s_{1/2} = \frac{1}{2}(\theta_i + \theta_{i-1}) = \frac{\pi}{n}(2i-1)$, so letting $s = s_{1/2} + \theta$ we find that

$$\begin{aligned}
v_i(s_{1/2}+\theta) &= \frac{1}{n} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left[\sin m\left(\theta + \frac{\pi}{n}\right) - \sin m\left(\theta - \frac{\pi}{n}\right)\right] \\
&= \frac{1}{n} + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\frac{m\pi}{n}\right)\cos\left(m\theta\right).
\end{aligned} \tag{40}$$

4.2 Discrete Single and Double Layer Operators

To find the elements of the boundary element matrix L, discretising the single layer operator $\mathcal{L}$, we need $(\mathcal{L}v_i)(s)$ at the collocation points. From (10) we find

$$(\mathcal{L}v_i)(s) \equiv w_i(s) = \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\widehat{w_i}(m)e^{ims}, \tag{41}$$

with

$$\widehat{w_i}(0) = -a\ln a\,\widehat{v_i}(0) \quad\text{and}\quad \widehat{w_i}(m) = \frac{a\widehat{v_i}(m)}{2|m|} \quad m\neq 0. \tag{42}$$

Hence it now follows from (5) and (34) that

$$\begin{aligned}
\|w_i\|^2_{\mathcal{H}^r} &= |\widehat{w_i}(0)|^2 + \sum_{m\neq 0}|m|^{2r}|\widehat{w_i}(m)|^2 \\
&= \frac{2\pi}{n^2}(a\ln a)^2 + \frac{a^2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^{2(2-r)}}\sin^2\left(\frac{m\pi}{n}\right),
\end{aligned}$$

which converges for $r < 3/2$ and is consistent with the result $\mathcal{L} : \mathcal{H}^r \to \mathcal{H}^{r+1}$.
In particular for $r = 1$

$$\|w_i\|^2_{\mathcal{H}^1} = \frac{2\pi}{n^2}(a \ln a)^2 + \frac{a^2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin^2\left(\frac{m\pi}{n}\right) = \frac{a^2\pi}{2n^2}\left\{4(\ln a)^2 - 1 + n\right\}, \tag{43}$$

using the summation (38).

We shall now write $(\mathcal{L}v_i)(s)$ in a form suitable for the eigenvalue computations in Section 5. Using (33), (41) and (42) and some simple manipulation we find

$$\begin{aligned}
(\mathcal{L}v_i)(s) &= -\frac{a \ln a}{n} + \frac{ai}{4\pi} \sum_{m \neq 0} \frac{1}{|m|} \frac{1}{m} \left[e^{-im\theta_i} - e^{-im\theta_{i-1}}\right] e^{ims} \\
&= -\frac{a \ln a}{n} + \frac{ai}{4\pi} \sum_{m=1}^{\infty} \frac{1}{m^2} \left[\left(e^{im(s-\theta_i)} - e^{im(s-\theta_{i-1})}\right) - \left(e^{-im(s-\theta_i)} - e^{-im(s-\theta_{i-1})}\right)\right] \\
&= -\frac{a \ln a}{n} + \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\left(\frac{m\pi}{n}\right) \cos m\left(\frac{\pi}{n}(2i-1) - s\right).
\end{aligned}$$

$(\mathcal{L}v_i)(s)$ has a maximum at $s = s_{1/2}$ with the value

$$(\mathcal{L}v_i)(s_{1/2}) = -\frac{a \ln a}{n} + \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\left(\frac{m\pi}{n}\right) \tag{44}$$

and is continuous at $s = \theta_{i-1}$ and $s = \theta_i$, with

$$(\mathcal{L}v_i)(\theta_i) = -\frac{a \ln a}{n} + \frac{a}{2\pi} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\left(\frac{2m\pi}{n}\right).$$

The circular matrix L is completely described by its first row, corresponding to $(\mathcal{L}v_i)(s)$ with $s = \frac{\pi}{n}$, the first collocation point. We have

$$(\mathcal{L}v_i)\left(\frac{\pi}{n}\right) = -\frac{a \ln a}{n} + \frac{a}{\pi} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\left(\frac{m\pi}{n}\right) \cos\left(\frac{2m\pi}{n}(i-1)\right). \tag{45}$$

Finally in this section, we point out that from (12) it follows that

$$(\mathcal{M}v_i)(s) = -\frac{1}{2n}, \tag{46}$$

a result which could also be deduced from Gauss' Integral by dividing $-\frac{1}{2}$ equally between the n elements. In this case the boundary element matrix M, the discretisation of the double layer potential operator, is a rank one matrix with all elements equal to $-\frac{1}{2n}$.

4.3 Discrete Hilbert Operator

Using the definition (15) and (33) we find

$$(\mathcal{P}v_i)(s) = p_i(s) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} \widehat{p}_i(m) e^{ims},$$

where $\widehat{p}_i(0) = 0$ and for $m \neq 0$

$$\widehat{p}_i(m) = -\frac{1}{\sqrt{2\pi}} \frac{1}{|m|} \left(e^{-im\theta_i} - e^{-im\theta_{i-1}}\right).$$

We note that $|\widehat{p}_i(m)|^2 = |\widehat{v}_i(m)|^2$, given in (34). Hence

$$\|p_i\|^2_{\mathcal{H}^r} = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^{2(1-r)}}\sin^2\left(\frac{m\pi}{n}\right)$$

which is finite for $r < 1/2$, is in agreement with $\mathcal{P} : \mathcal{H}^r \to \mathcal{H}^r$. In particular for $r = 0$, using (38), we have

$$\|p_i\|^2_{\mathcal{L}^2} = \frac{4}{\pi}S = \frac{2\pi}{n^2}(n-1).$$

Furthermore we can write

$$\begin{aligned}(\mathcal{P}v_i)(s) &= -\frac{1}{2\pi}\left\{\sum_{m=1}^{\infty}\frac{1}{m}\left[e^{-im\theta_i} - e^{-im\theta_{i-1}}\right]e^{ims} - \sum_{m=-\infty}^{-1}\frac{1}{m}\left[e^{-im\theta_i} - e^{-im\theta_{i-1}}\right]e^{ims}\right\} \\ &= -\frac{1}{2\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\left(e^{im(s-\theta_i)} + e^{-im(s-\theta_i)}\right) - \left(e^{im(s-\theta_i+\Delta\theta)} + e^{-im(s-\theta_i+\Delta\theta)}\right)\right\} \\ &= -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\cos m(s-\theta_i) - \cos m(s-\theta_i+\Delta\theta)\right\}. \qquad (47)\end{aligned}$$

Equation (47) may also be written as

$$(\mathcal{P}v_i)(s) = \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\frac{m\pi}{n}\right)\sin m\left(\frac{\pi}{n}(2i-1) - s\right). \qquad (48)$$

Since

$$(\mathcal{P}v_i)(s) = \frac{1}{2\pi}\int_{\theta_{i-1}}^{\theta_i}\cot\left(\frac{q-s}{2}\right)dq = \frac{1}{\pi}\ln\left|\frac{\sin\left(\frac{s-\theta_i}{2}\right)}{\sin\left(\frac{s-\theta_{i-1}}{2}\right)}\right| \qquad (49)$$

this provides the value for the summation in (48). In particular when $s = \frac{\pi}{n}$, the first collocation point, equations (47), (48) and (49) become

$$\begin{aligned}(\mathcal{P}v_i)\left(\frac{\pi}{n}\right) &= -\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\cos\frac{m\pi}{n}(2i-1) - \cos\frac{m\pi}{n}(2i-3)\right\} \\ &= \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{m}\sin\left(\frac{m\pi}{n}\right)\sin\frac{2m\pi}{n}(i-1) \qquad (50) \\ &= \frac{1}{\pi}\ln\left|\frac{\sin\frac{\pi}{2n}(2i-1)}{\sin\frac{\pi}{2n}(2i-3)}\right|.\end{aligned}$$

4.4 Discrete Hypersingular Operator

Letting

$$(\mathcal{N}v_i)(s) \equiv z_i(s) = \frac{1}{\sqrt{2\pi}}\sum_{m\in\mathbb{Z}}\widehat{z}_i(m)e^{ims}$$

we find from (21) that

$$\widehat{z}_i(m) = \begin{cases} 0 & m = 0 \\ -\frac{|m|}{2a}\widehat{v}_i(m) & \text{otherwise.}\end{cases}$$

Using (34) it follows that

$$|\widehat{z}_i(m)|^2 = \frac{1}{2\pi a^2}\sin^2\left(\frac{m\pi}{n}\right),$$

giving

$$\|z_i\|^2_{\mathcal{H}^r} = \frac{1}{\pi a^2}\sum_{m=1}^{\infty}\frac{1}{m^{-2r}}\sin^2\left(\frac{m\pi}{n}\right),$$

which converges for $r < -1/2$, in agreement with $\mathcal{N}:\mathcal{H}^r \to \mathcal{H}^{r-1}$.
It is interesting to note that the Dirac delta function, as a 2π-periodic distribution belongs to the Sobolev space $\mathcal{H}^r[0,2\pi]$ with $r < -\frac{1}{2}$. If $\delta_i(s) = \delta(s-\theta_i)$ for $0 \le s \le 2\pi$ then

$$\widehat{\delta}_i(m) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi}\delta(s-\theta_i)e^{-imt}dt = \frac{1}{\sqrt{2\pi}}e^{-im\theta_i},$$

which using (5) gives

$$\|\delta_i\|^2_{\mathcal{H}^r} = \frac{1}{2\pi}\left(1+\sum_{m\neq 0}|m|^{2r}\left|e^{-im\theta_i}\right|^2\right) = \frac{1}{2\pi}(1+\sum_{m\neq 0}|m|^{2r}),$$

which converges for $r < -\frac{1}{2}$. Not surprisingly, $\mathcal{N}v_i$ is in the same Sobolev space as the Dirac delta functionals because of the discontinuity of v_i at θ_{i-1} and θ_i and the differentiation nature of $\mathcal{N}$.

From (20) and after finite part evaluation, we find the value of the hypersingular operator as

$$\begin{aligned}(\mathcal{N}v_i)(s) &= \frac{1}{8\pi a}\,\neq\!\!\!\!\!\!\int_{\theta_{i-1}}^{\theta_i}\operatorname{cosec}^2\left(\frac{q-s}{2}\right)dq \\ &= \frac{1}{4\pi a}\left\{\cot\left(\frac{\theta_{i-1}-s}{2}\right)-\cot\left(\frac{\theta_i-s}{2}\right)\right\} \qquad (51)\end{aligned}$$

which is well defined and smooth except at the points θ_{i-1} and θ_i. However, the Fourier series representation of $\mathcal{N}v_i$ does not converge in the usual $\mathcal{L}^2$ sense. Indeed we find from equations (19), (47), (48) and (49) that

$$\begin{aligned}(\mathcal{N}v_i)(s) &= \frac{1}{2a}\frac{d}{ds}(\mathcal{P}v_i)(s) \quad 0<s<2\pi, \quad s\neq\theta_{i-1},\theta_i \\ &= \frac{1}{2a\pi}\,\mathrm{FP}\sum_{m=1}^{\infty}\{\sin m(s-\theta_i)-\sin m(s-\theta_i+\Delta\theta)\} \qquad (52) \\ &= -\frac{1}{a\pi}\,\mathrm{FP}\sum_{m=1}^{\infty}\sin\left(\frac{m\pi}{n}\right)\cos m\left(\frac{\pi}{n}(2i-1)-s\right) \qquad (53) \\ &= \frac{1}{4\pi a}\left\{\cot\left(\frac{\theta_{i-1}-s}{2}\right)-\cot\left(\frac{\theta_i-s}{2}\right)\right\},\end{aligned}$$

where the FP notation is employed to remind us that the series represents a generalised function and may not converge in the usual sense and that we may take as the finite part of this divergent series the value from (51).

When $s=\pi/n$, the first collocation point, equations (52) and (51) become

$$\begin{aligned}(\mathcal{N}v_i)\left(\frac{\pi}{n}\right) &= \frac{1}{2a\pi}\,\mathrm{FP}\sum_{m=1}^{\infty}\left\{\sin\frac{m\pi}{n}(2i-3)-\sin\frac{m\pi}{n}(2i-1)\right\} \\ &= \frac{1}{4\pi a}\left\{\cot\frac{\pi}{2n}(2i-3)-\cot\frac{\pi}{2n}(2i-1)\right\}. \qquad (54)\end{aligned}$$

To write (54) in a form suitable for our matrix eigenvalue computation we need the following technical lemma.

Lemma 4.1 *For $\ell, n \in \mathbb{N}$ with $\ell \notin n\mathbb{Z}$*

$$\sum_{m=1}^{n} \sin\left(\frac{\ell m\pi}{n}\right) = \begin{cases} 0 & \ell \text{ even} \\ \cot\left(\frac{\ell\pi}{2n}\right) & \ell \text{ odd,} \end{cases}$$

Proof The proof follows easily by noting that the required sum is the imaginary part of the geometric sum $\sum_{m=1}^{n} e^{i\left(\frac{\ell m\pi}{n}\right)}$. □

Now, putting $\ell = 2i - r$ with $r = 1$ or 3 in Lemma 4.1, it follows from (54) that

$$\begin{aligned}(\mathcal{N}v_i)\left(\frac{\pi}{n}\right) &= \frac{1}{4a\pi}\sum_{m=1}^{n}\left\{\sin\frac{m\pi}{n}(2i-3) - \sin\frac{m\pi}{n}(2i-1)\right\} \\ &= -\frac{1}{2\pi a}\sum_{m=1}^{n}\sin\left(\frac{m\pi}{n}\right)\cos\frac{2m\pi}{n}(i-1). \qquad (55)\end{aligned}$$

5 Eigenvalues of the Discrete Operators

Finally in this section we derive the eigenvalues and eigenvectors of the discrete operators. The eigenvalues, $\lambda_k(A)$, of a circulant matrix A are given by the inner product of $\mathbf{v}$, the first row of the matrix, with the eigenvectors $\mathbf{w}_k = (1\ t_k\ t_k^2\ \cdots\ t_k^{n-1})^T$ where $t_k = e^{\frac{2ik\pi}{n}}$ (an n-th root of unity) for $k = 0, 1, \ldots, n-1$, [9].

5.1 Eigenvalues of the Matrix M

The $n \times n$ matrix M, the discrete counterpart of the operator $\mathcal{M}$, provides a simple example for using the above procedure. From equation (46) the first row of M is

$$\mathbf{v} = -\frac{1}{2n}(1\ 1\ \cdots\ 1)^T \in \mathbb{R}^n.$$

When $k = 0$, that is $t_0 = 1$ and $\mathbf{w}_0 = (1\ 1\ \cdots\ 1)^T$, we find $\lambda_0(M) = \mathbf{v}.\mathbf{w}_0 = -\frac{1}{2}$. When $k \neq 0$

$$\begin{aligned}\lambda_k(M) &= \mathbf{v}.\mathbf{w}_k = -\frac{1}{2n}\sum_{i=1}^{n} t_k^{i-1} \\ &= -\frac{1}{2n}\left(\frac{1-t_k^n}{1-t_k}\right) = 0 \quad \text{since } t_k^n = 1.\end{aligned}$$

Therefore, the rank one matrix M has $-\frac{1}{2}$ as its only non-zero eigenvalue. Comparison with (13) shows this to be in exact agreement with the continuous operator $\mathcal{M}$.

5.2 Eigenvalues of the Matrix L

Equation (45) gives the elements of the first row of the matrix L. Taking the dot product with the eigenvectors $\mathbf{w}_k$ gives the eigenvalues in the form

$$\lambda_k(L) = \sum_{i=1}^{n}\left\{-\frac{a\ln a}{n} + \frac{a}{\pi}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\left(\frac{m\pi}{n}\right)\cos\left(\frac{2m\pi}{n}(i-1)\right)\right\}.t_k^{i-1}.$$

If $k = 0$ with $t_0 = 1$ then $\lambda_0(L) = \sum(\mathcal{L}v_i)(\frac{\pi}{n})$, is simply the matrix row sum which corresponds to $\mathcal{L}$ acting on the constant eigenfunction. Hence it follows from (11) that $\lambda_0(L) = -a\ln a$, which, not surprisingly, is identical to the corresponding eigenvalue of the continuous operator $\mathcal{L}$. Note that if $a = 1$ the matrix is singular as is the single layer operator $\mathcal{L}$.

For $k = 1, \ldots, n-1$ we have

$$\begin{aligned}\lambda_k(L) &= \frac{a}{\pi} \sum_{i=1}^{n} \sum_{m=1}^{\infty} \frac{1}{m^2} \sin\left(\frac{m\pi}{n}\right) \cos\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1} \quad \text{since } \textstyle\sum_{i=1}^{n} t_k^{i-1} = 0 \\ &= \frac{a}{\pi} \sum_{m=1}^{\infty} \left\{ \frac{1}{m^2} \sin\left(\frac{m\pi}{n}\right) \sum_{i=1}^{n} \cos\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1} \right\}.\end{aligned}$$

To simplify this expression we require the following technical lemma.

Lemma 5.1

$$\sum_{i=1}^{n} \cos\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1} = \begin{cases} \frac{n}{2} & m \equiv k \ (mod\ n) \ or\ m \equiv n-k \ (mod\ n) \\ 0 & otherwise. \end{cases} \tag{56}$$

Proof The summation may be written

$$\begin{aligned} 2\sum_{i=1}^{n} \cos\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1} &= \sum_{i=1}^{n} \left(e^{\frac{2im\pi}{n}(i-1)} + e^{-\frac{2im\pi}{n}(i-1)}\right) e^{\frac{2ik\pi}{n}(i-1)} \\ &= \sum_{i=1}^{n} \left\{ e^{\frac{2i\pi}{n}(i-1)(m+k)} + e^{\frac{2i\pi}{n}(i-1)(k-m)} \right\} \\ &= \Sigma_1 + \Sigma_2. \end{aligned}$$

If $m \equiv n-k \pmod n$ then $\Sigma_1 = n$, otherwise

$$\Sigma_1 = \frac{1 - e^{2\pi i(m+k)}}{1 - e^{\frac{2i\pi}{n}(m+k)}} = 0.$$

Similarly if $m \equiv k \pmod n$ then $\Sigma_2 = n$, otherwise

$$\Sigma_2 = \frac{1 - e^{2\pi i(k-m)}}{1 - e^{\frac{2i\pi}{n}(k-m)}} = 0.$$

The results now follows. □

It now follows that

$$\begin{aligned} \lambda_k(L) &= \frac{a}{\pi}\left(\frac{n}{2}\right) \sum_{r=0}^{\infty} \left\{ \frac{1}{(k+rn)^2} \sin\left(\frac{(k+rn)\pi}{n}\right) + \frac{1}{(n-k+rn)^2} \sin\left(\frac{(n-k+rn)\pi}{n}\right)\right\} \\ &= \frac{an}{2\pi} \sin\left(\frac{k\pi}{n}\right) \sum_{r=0}^{\infty} (-1)^r \left\{ \frac{1}{(k+rn)^2} + \frac{1}{(n-k+rn)^2} \right\}. \end{aligned} \tag{57}$$

We note that the early eigenvalues with $1 \leq k \ll n$ behave as

$$\lambda_k(L) \approx \frac{an}{2\pi}\frac{k\pi}{n} \left\{ \frac{1}{k^2} + \mathcal{O}(n^{-2}) \right\} \to \frac{a}{2k}, \quad n \to \infty$$

implying that the low order eigenvalues converge to the operator eigenvalues with errors of order n^{-2}. The maximum (in magnitude) eigenvalue occurs for $k = 0$ or 1. Also note that since $\lambda_k(L) = \lambda_{n-k}(L)$ for $k = 1, \ldots, k_{\max}$ (where $k_{\max} = \frac{n}{2} - 1$ if n is even; $k_{\max} = \frac{n-1}{2}$, n odd) the eigenvalues occur in pairs, except for the simple eigenvalue(s) $\lambda_0(L)$ (and $\lambda_{n/2}(L)$ when n is even).

5.3 Eigenvalues of the Matrix P

It can be seen from (49) that the matrix P is circulant but not symmetric. Similar to computation of the eigenvalues of L and M we find

$$\lambda_k(P) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{m\pi}{n}\right) \sum_{i=1}^{n} \sin\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1}.$$

To simplify this we need the result

$$\sum_{i=1}^{n} \sin\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1} = \begin{cases} \frac{in}{2} & m \equiv k \pmod{n} \\ -\frac{in}{2} & m \equiv n-k \pmod{n} \\ 0 & \text{otherwise} \end{cases} \tag{58}$$

the proof of which is simple and follows along the lines of that of Lemma 5.1. It now follows that

$$\lambda_k(P) = \frac{ni}{\pi} \sin\left(\frac{k\pi}{n}\right) \sum_{r=0}^{\infty} (-1)^r \left\{\frac{1}{k+rn} - \frac{1}{n-k+rn}\right\}. \tag{59}$$

For $1 \leq k \ll n$, the early eigenvalues of P behave as

$$\lambda_k(P) \approx \frac{ni}{\pi} \frac{k\pi}{n} \left(\frac{1}{k}\right) = i$$

whilst the later eigenvalues behave as

$$\lambda_{n-k}(P) \approx \frac{ni}{\pi} \frac{k\pi}{n} \left(-\frac{1}{k}\right) = -i$$

where we have used the fact that $\sin\left(\frac{n-k}{n}\pi\right) = \sin\left(\frac{k\pi}{n}\right)$. These results are in agreement with the eigenvalues of the continuous operator given by (16).

5.4 Eigenvalues of the Matrix N

Using equation (55) and the summation (56) we find the eigenvalues of the matrix N as

$$\begin{aligned} \lambda_k(N) &= -\frac{1}{2\pi a} \sum_{m=1}^{n} \sin\left(\frac{m\pi}{n}\right) \sum_{i=1}^{n} \cos\left(\frac{2m\pi}{n}(i-1)\right) t_k^{i-1} \\ &= -\frac{1}{2\pi a} \left(\frac{n}{2}\right) \left\{\sin\left(\frac{k\pi}{n}\right) + \sin\left(\frac{(n-k)\pi}{n}\right)\right\} \\ &= -\frac{n}{2\pi a} \sin\left(\frac{k\pi}{n}\right). \end{aligned}$$

In particular $\lambda_0(N) = 0$; that is the matrix N is singular for all a. This corresponds to the zero eigenvalue of the continuous operator $\mathcal{N}$ with constant eigenfunction. Once again the eigenvalues occur in pairs similar to that for the matrix L.

For small values of k

$$\lambda_k(N) \approx -\frac{n}{2\pi a} \left(\frac{k\pi}{n}\right) \to -\frac{k}{2a} \quad \text{as } n \to \infty,$$

converging to their corresponding eigenvalues $\lambda_k(\mathcal{N})$ as given in (22). For n even and $k = n/2$ we find a largest (in magnitude) eigenvalue equal to

$$\lambda_{\max}(N) = -\frac{n}{2\pi a}.$$

6 Conclusions

For the boundary integral operators such as the single layer, the double layer, the Cauchy singular and the hypersingular operator arising in the solution of Laplace's equation, we have carried out a complete qualitative analysis for the case where the boundary is a circle. We have shown explicitly the extent to which these results carry forward to the interesting case of Helmholtz potential operators. Many original and interesting results have been derived in Section 4, where we obtain explicit analytical expressions for the elements of the discrete operators based on piecewise constant approximation. Circulant matrices can be extremely efficient preconditioners. This is because, using FFT, multiplication of a vector by a cirulant matrix can be carried out in $\mathcal{O}(n \ln n)$ rather than $\mathcal{O}(n^2)$ operations [9]. Analytical expressions for the various discrete operators together with a knowledge of their spectral properties, developed in Section 5, are useful in designing preconditioners to deal with linear systems arising in the solution of the first kind and hypersingular boundary integral equations [15].

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