

The Calderón projector: pseudo-differential methods in boundary value problem

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Abstract

In 1963, A.P. Calderón suggested the idea of using pseudodifferential operators to study elliptic boundary value problem. This idea was further developed by Seeley, Boutet de Monvel, Grubb, Hörmander and others. In this talk, we will discuss the Calderón projector and its applications.

1 Introduction

1.1 Motivation: boundary value for harmonic functions

Suppose we want to solve the boundary problem

$$\Delta u = 0 \text{ in } X, b_0 u_0 + b_1 u_1 = f \text{ on } \partial X$$

where X is an open set in \mathbb{R}^n with smooth boundary ∂X , Δ is the Laplacian, b_0, b_1 are differential operators in ∂X acting respectively on the boundary value $u_0 = u|_{\partial X}$ and the normal derivative $u_1 = \frac{du}{dn}|_{\partial X}$ of u . Let $E(x)$ be the fundamental solution of the Laplacian on \mathbb{R}^n :

$$E(x) = \begin{cases} \frac{1}{2\pi} \log|x| & \text{if } n = 2 \\ -\frac{1}{(n-2)c_n} |x|^{2-n} & \text{if } n > 2 \end{cases}$$

Then for smooth u , we obtain from Green's formula

$$u(x) = Dl(u_0)(x) - Sl(u_1)(x) = \int_{\partial X} \frac{dE(x-y)}{dn} u_0(y) dS(y) - \int_{\partial X} E(x-y) u_1(y) dS(y), x \in X$$

where the single/double layer potentials are defined as

$$Sl(f)(x) = \int_{\partial X} f(y)E(x-y)dy, x \in \mathbb{R}^n \setminus \partial X$$

$$Dl(f)(x) = \int_{\partial X} f(y)\frac{\partial E}{\partial n}(x-y)dy, x \in \mathbb{R}^n \setminus \partial X$$

They have the following properties: denote the limit of $v(z)$ as $z \rightarrow x$ from $z \in X$ and $z \in \mathbb{R}^n \setminus \bar{X}$, $v_+(x), v_-(x)$, respectively(if the limit exists). Then

$$Sl(f)_+(x) = Sl(f)_-(x) = Sf(x) = \int_{\partial X} f(y)E(x-y)dy$$

$$Dl(f)_\pm(x) = \pm \frac{1}{2}f(x) + Nf(x), Nf(x) = \int_{\partial X} f(y)\frac{\partial E}{\partial n}(x-y)dS(y)$$

So the question reduces to find out u_0 and u_1 . Although the formula above always determines a harmonic function, but it may not have boundary value u_0 and normal derivative u_1 for general u_0, u_1 . To see this, let $x \in X$ approach ∂X , we have

$$u_0 = k_0 u_0 + k_1 u_1$$

where k_0, k_1 are pseudodifferential operators of order -1 on ∂X since convolution by E is a pseudodifferential operator satisfying the transmission condition. k_0, k_1 only depend on X , so if u_0, u_1 are boundary conditions for a harmonic function, then they must satisfy the above relation.

Conversely, if u_0, u_1 satisfy the above relation, and u is defined from u_0, u_1 as above, then $u|_{\partial X} = u_0$. Green's formula also gives

$$u(x) = \int \frac{dE(x-y)}{dn} u_0(y) dS(y) - \int E(x-y) \frac{\partial u}{\partial n}(y) dS(y), x \in X$$

Therefore

$$\int E(x-y) \left(u_1(y) - \frac{\partial u}{\partial n} \right) dS(y) = 0, x \in X$$

The integral on the left-hand side is a continuous function of x which is harmonic outside \bar{X} and vanishes on ∂X and infinity. Therefore it is identically zero on \mathbb{R}^n . Therefore $\frac{\partial u}{\partial n}|_{\partial X} = u_1$.

Now to solve the boundary problem is thus equivalent to solving the system of pseudodifferential equations

$$(1 - k_0)u_0 - k_1 u_1 = 0, b_0 u_0 + b_1 u_1 = f$$

Notice that k_1 is an injective elliptic pseudo-differential operator of index 0, hence it has a pseudodifferential index k_1^{-1} . Therefore we obtain a pseudodifferential equation

$$(b_0 + b_1 k_1^{-1}(1 - k_0))u_0 = f$$

When $b_0 + b_1 k_1^{-1}(1 - k_0)$ is an elliptic pseudodifferential operator, we can apply Fredholm-type results for elliptic equations on a compact manifold. We call such boundary problem elliptic.

We hope to apply the ideas above to general boundary problems for elliptic operators on a manifold. Let A be an elliptic differential operator on X , B a differential operator on the Cauchy data of functions ρu on ∂X . The questions are as follows:

1. For what kinds of Cauchy data f , there exists solutions to

$$Au = 0 \text{ in } X, \rho u = f \text{ on } \partial X$$

2. For what kinds of boundary differential operator B , the boundary value problem

$$Au = v \text{ in } X, B\rho u = f \text{ on } \partial X$$

is well-posed, (elliptic, Fredholm or having a parametrix in some sense)?

In this talk, we will follow [G2] to answer the questions above using the Calderón projector. We will start by two basic examples, which are from [G2] and [S]. Then we will review pseudodifferential operators and introduce pseudodifferential boundary operators. Finally we will see how to define the Calderón projector and how the Calderón projector helps answering the questions.

1.2 History

The theory of pseudodifferential operators developed from the theory of singular integral operators. The application to elliptic differential operators on compact manifolds leads to the Atiyah-Singer index theory, which generalized the classical Riemann-Roch theorem.

The application to elliptic boundary value problem was first suggested by A.P. Calderón [C] in 1963. In this two-page paper, Calderón considered the problem of compatible boundary value for systems of elliptic equations. He claimed that the space of all compatible boundary values is the range of a projector on the whole space of boundary values and the

projector is actually a matrix of singular integral operators (which are in fact pseudodifferential operators whose symbols can be calculated explicitly in terms of the symbol of the differential operators). This projector is now called the Calderón projector. The idea of Calderón was fully developed by his student R. Seeley in [S]. Another formulation using pseudodifferential boundary operators is given by L. Boutet de Monvel [B].

Motivated by the $\bar{\partial}$ -Neumann problem in complex analysis of several variables, Hörmander [H1] studied the Calderón projector in non-elliptic boundary problem for over-determined elliptic systems. Another development was carried out by G. Grubb [G1]. He considered the system of partial differential operators with mixed orders and studied the Calderón projector in this case. His recent book [G2] gave the first detailed treatment for the Calderón projector in the textbook form. A detailed study of the Calderón projector and the index theory for boundary value problem is given in [H2].

2 Two basic examples

In this section, we will see how the Calderón projector appears in two classical problems.

2.1 Second order ordinary differential equations

Let $\lambda > 0$. Consider the differential operator $Au = D_x^2 u + \lambda^2 u$. The solution of $Au = 0$ in $\mathcal{D}'(\mathbb{R})$ is given by

$$u = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

The only solution in $L^2(\mathbb{R})$ is zero. The non-homogeneous equation $Au = f, f \in L^2(\mathbb{R})$ is uniquely solved in $L^2(\mathbb{R})$ by

$$u(x) = \mathcal{F}^{-1}((\lambda^2 + \xi^2)^{-1} \mathcal{F}u(\xi))(x) \in H^2(\mathbb{R})$$

We define the initial value or the Cauchy data for general $u \in H_{loc}^2(\mathbb{R})$ to be the pair

$$\rho u = \begin{pmatrix} u(0) \\ Du(0) \end{pmatrix} \in \mathbb{C}^2$$

Now we consider the same differential operator $Au = D_x^2 u + \lambda^2 u$ on the half-lines \mathbb{R}_\pm . Write the restriction of $v \in H_{loc}^2(\mathbb{R})$ to \mathbb{R}_\pm as $r^\pm v$. For $u \in H^2(\mathbb{R}_\pm)$ only defined on the half-lines, we denote the Cauchy data as $\rho^\pm u$. The solution space of A in $L^2(\mathbb{R}_\pm)$ is

$$Z_\pm = \{u \in L^2(\mathbb{R}_\pm) | Au = 0\} = \mathbb{C}(r^\pm e^{\mp \lambda x}) \subset \mathcal{S}'(\overline{\mathbb{R}_\pm})$$

Then we can define $N_{\pm} = \rho^{\pm}(Z_{\pm})$ to be the space of all possible Cauchy data for solutions of $Au = 0$ on \mathbb{R}_{\pm} , we have

$$N_{\pm} = \mathbb{C} \begin{pmatrix} 1 \\ \pm\lambda i \end{pmatrix} \subset \mathbb{C}^2$$

Then $\mathbb{C}^2 = N_+ \oplus N_-$ and the Calderón projectors are defined as the projections $C^{\pm} : \mathbb{C}^2 \rightarrow N_{\pm}$. We can calculate the explicit expression for C^{\pm} :

$$C^+ = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\lambda i} \\ \frac{1}{2}\lambda i & \frac{1}{2} \end{pmatrix}, C^- = \begin{pmatrix} \frac{1}{2} & \frac{i}{2\lambda} \\ \frac{1}{2i}\lambda & \frac{1}{2} \end{pmatrix}$$

We can find linear operators $K^{\pm} : \mathbb{C}^2 \rightarrow Z_{\pm}$, such that $C^{\pm} = \rho^{\pm} \circ K^{\pm}$. The construction is as follows: For $(\phi, \psi) \in \mathbb{C}^2$, there is a unique solution $u = c_+ e^{\lambda x} + c_- e^{-\lambda x}$ of $Au = 0$ with Cauchy data (ϕ, ψ) . Then $K^{\pm}(\phi, \psi) = c_{\mp} e^{\mp \lambda x}$.

2.2 Cauchy-Riemann equations in the unit disk

A function u defined on the complex plane \mathbb{C} is holomorphic if and only if it satisfies the Cauchy-Riemann equation $Au = 0$ where $A = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Let \mathbb{D} be the unit disk $\{z \in \mathbb{C} : |z| < 1\}$. Then we can define the following spaces of holomorphic functions

$$Z_+ = \{u \in H^1(\mathbb{D}) : Au = 0\}; Z_- = \{u \in H_{loc}^1(\mathbb{C} - \bar{\mathbb{D}}) : Au = 0, u \text{ vanishes at } \infty\}$$

Also, from the properties of trace operator, we need to consider the following function space on $\partial\mathbb{D}$:

$$H^{\frac{1}{2}}(\partial\mathbb{D}) = \{f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} : \sum_{n=-\infty}^{\infty} (1 + |n|)|\hat{f}(n)|^2 < \infty\}$$

Let $\rho : H_{loc}^1(\mathbb{C}) \rightarrow H^{\frac{1}{2}}(\partial\mathbb{D})$ be the trace operator. Similarly, we have $\rho^{\pm} : Z_{\pm} \rightarrow H^{\frac{1}{2}}(\partial\mathbb{D})$. Write $N_{\pm} = \rho^{\pm}(Z_{\pm})$. Then the Taylor/Laurent expansion gives that

$$N_+ = \{f(e^{i\theta}) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta} \in H^{\frac{1}{2}}(\partial\mathbb{D})\}$$

$$N_- = \{f(e^{i\theta}) = \sum_{n=-\infty}^{-1} \hat{f}(n)e^{in\theta} \in H^{\frac{1}{2}}(\partial\mathbb{D})\}$$

Therefore $H^{\frac{1}{2}}(\partial\mathbb{D}) = N_+ \oplus N_-$ and the Calderón projector is given by $C^\pm : H^{\frac{1}{2}}(\partial\mathbb{D}) \rightarrow N_\pm$.

$$C^+(f) = \sum_{n=0}^{\infty} \hat{f}(n)e^{in\theta}, C^-(f) = \sum_{n=-\infty}^{-1} \hat{f}(n)e^{in\theta}, \forall f = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{in\theta} \in H^{\frac{1}{2}}(\partial\mathbb{D})$$

They are basic pseudodifferential operators on the unit circle, i.e. the discrete analogue for the pseudodifferential operator with symbol $\chi_\pm = \chi_{(0,\infty)}(\xi), \chi_{(-\infty,0)}(\xi)$. Clearly, these symbols are projections: $\chi_\pm^2 = \chi_\pm$, so we also have $\text{Op}(\chi_\pm)^2 = \text{Op}(\chi_\pm)$.

Again, we can find linear operators $K^\pm : H^{\frac{1}{2}}(\partial\mathbb{D}) \rightarrow Z_\pm$ such that $C^\pm = \rho^\pm \circ K^\pm$, namely

$$K^+f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n, K^-f(z) = \sum_{n=-\infty}^{-1} \hat{f}(n)z^n, \forall f = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in H^{\frac{1}{2}}(\partial\mathbb{D})$$

Another way to interpret the Cauchy-Riemann equation is to write it as a system of two real partial differential equations under the identification $\mathbb{C} \simeq \mathbb{R}^2, z \mapsto x + iy$.

$$\tilde{A}\tilde{u} = 0, \tilde{A} = \frac{1}{2} \begin{pmatrix} \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix}, \tilde{u} = \begin{pmatrix} \text{Re}(u) \\ \text{Im}(u) \end{pmatrix}$$

Then A corresponds to \tilde{A} ; u to \tilde{u} ; Z_\pm to \tilde{Z}_\pm ; the complex $H_{\mathbb{C}}^{\frac{1}{2}}(\partial\mathbb{D})$ to real $H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D}) \oplus H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D})$; the Calderón projector C^\pm to \tilde{C}^\pm ; K^\pm to \tilde{K}^\pm .

Now suppose $B : H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D}) \oplus H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D}) \rightarrow H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D})$ is a bounded linear map. Then we say B is well-posed for \tilde{A} on \mathbb{D} if

- (i) $\ker(B\tilde{C}^+)/\ker(\tilde{C}^+)$ is finite dimensional;
- (ii) $H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D})/B\tilde{C}^+(H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D}) \oplus H_{\mathbb{R}}^{\frac{1}{2}}(\partial\mathbb{D}))$ is finite dimensional.

EXAMPLE: Dirichlet boundary condition $B(f, g) = f$ satisfies the conditions.

When B is well-posed, then

- (a) Consider the map $(\tilde{A}, B) : H^1(\mathbb{D}) \oplus H^1(\mathbb{D}) \rightarrow L^2(\mathbb{D}) \oplus L^2(\mathbb{D}) \oplus H^{\frac{1}{2}}(\partial\mathbb{D})$,

$$\tilde{u} \mapsto (\tilde{A}\tilde{u}, B(\rho^+\tilde{u}))$$

has finite dimensional kernel and cokernel.

- (b) Consider the map $\tilde{A}_B : \{\tilde{u} \in H^1(\mathbb{D}) \oplus H^1(\mathbb{D}) : B(\rho^+\tilde{u}) = 0\} \rightarrow L^2(\mathbb{D}) \oplus L^2(\mathbb{D}), \tilde{u} \mapsto \tilde{A}\tilde{u}$

has finite dimensional kernel and cokernel.

Conversely, either (a) or (b) implies the well-posedness of B for \tilde{A} . Therefore we can determine whether the boundary value problem

$$\tilde{A}\tilde{u} = \tilde{v}, B(\rho^+\tilde{u}) = \tilde{f}$$

is a suitable elliptic boundary problem (having the property of Fredholm alternative) from the Calderón projector C^+ .

3 Pseudodifferential operators

In this section, we review basic concepts of pseudodifferential operators on manifolds and introduce the pseudodifferential boundary operators. All factors of powers of 2π are omitted.

3.1 A brief review of pseudodifferential operators on manifolds

A pseudodifferential operator P on a subset U of \mathbb{R}^n is an operator from $C_0^\infty(U)$ to $C^\infty(U)$ with the form

$$Pu(x) = \int_{\mathbb{R}^n} p(x, \xi) \hat{u}(\xi) e^{ix \cdot \xi} d\xi$$

where $p : T^*U \rightarrow \mathbb{C}$ is called the symbol of P . The symbol class S^d contains all $p \in C^\infty$ such that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leq c_{\alpha, \beta, K} \langle \xi \rangle^{d - |\alpha|}$$

Another way to write this is

$$Pu(x) = \int_{\mathbb{R}^{2n}} p_c(x, y, \xi) u(y) e^{i(x-y) \cdot \xi} dy d\xi$$

If $p \sim \sum_{l \geq 0} p_{d-l}$ in $S^d(U)$, where p_{d-l} are homogeneous of degree $d-l$ in ξ for $|\xi| \geq 1$, then we say p is a classical symbol of order d with principal symbol p_d . The principal symbol is canonical, i.e. invariant under the change of coordinates. We can generalize this definition to a vector-valued operator with symbol $p \in S^d(T^*U) \otimes \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$.

Let X be a manifold, an operator $P : C_0^\infty(X) \rightarrow C^\infty(X)$ is said to be a pseudodifferential operator of order d if in any local coordinates, P is a pseudodifferential operator of order d . As above, if E, F are two vector bundles over X , we can define pseudodifferential operators $P : C^\infty(X, E) \rightarrow C^\infty(X, F)$ with symbols $p \in S^d(T^*(X \setminus 0)) \otimes \mathcal{L}(E, F)$.

THEOREM(Boundedness on Sobolev spaces): Suppose X is a compact manifold, P is a pseudodifferential operator on X of order d . Then P is continuous from $H^s(X, E)$ to $H^{s-d}(X, F)$ for all $s \in \mathbb{R}$.

THEOREM(Ellipticity) If P is an elliptic pseudodifferential operator of order d , i.e. its principal symbol is invertible everywhere. Then P has a parametrix Q , i.e. an elliptic pseudodifferential operator of order $-d$ such that $PQ - I$ and $QP - I$ are both of order $-\infty$.

REMARK: We should also introduce the half-density bundle to deal with the change of coordinates. But this will bring more complicated notations.

3.2 Pseudodifferential boundary operators

When we consider boundary value problem, we hope to treat the function u and its Cauchy data on the boundary simultaneously. This leads to the system of pseudodifferential boundary operators (what Boutet de Monvel called Green operators). We will only give the ideas in local models where the manifold is \mathbb{R}_+^n with boundary $\partial\mathbb{R}^{n-1}$. Suppose P is an $N' \times N$ matrix of pseudodifferential operators in \mathbb{R}^n satisfying the transmission condition. r^\pm the restriction operator to \mathbb{R}_\pm^n , e^\pm the extension operator from \mathbb{R}_\pm^n to \mathbb{R}^n by zero on the complement. $P^+ = r^+Pe^+$. Then the system of pseudodifferential boundary operators is

$$\mathcal{A} = \begin{pmatrix} P^+ + G & K \\ T & S \end{pmatrix} : C_0^\infty(\overline{\mathbb{R}_+^n})^N \times C_0^\infty(\mathbb{R}^{n-1})^M \rightarrow C^\infty(\overline{\mathbb{R}_+^n})^{N'} \times C^\infty(\mathbb{R}^{n-1})^{M'}$$

Here T is the trace operator going from \mathbb{R}_+^n to \mathbb{R}^{n-1} ; K is the Poisson operator (analogue to the layer potential), going from \mathbb{R}^{n-1} to \mathbb{R}_+^n ; S is a pseudodifferential operator on \mathbb{R}^{n-1} ; and G is an operator on \mathbb{R}_+^n , called a singular Green operator, a non-pseudodifferential term to make composition rules valid.

3.2.1 Trace operator

A trace operator of order d and class r is

$$Tu = \sum_{0 \leq j \leq r-1} S_j \gamma_j + T'$$

where γ_j is the standard trace operator

$$(\gamma_j u)(x') = D_{x_n}^j u(x', 0)$$

S_j are pseudodifferential operators in \mathbb{R}^{n-1} of order $d-j$, and T' is an operator of the form

$$(T'u)(x') = \int_{\mathbb{R}^{(n-1)}} e^{ix' \cdot \xi'} \int_0^\infty \tilde{t}'(x', x_n, \xi') \mathcal{F}_{x' \rightarrow \xi'} u(\xi', x_n) dx_n d\xi'$$

or

$$(T'u)(x') = \int_{\mathbb{R}^{2(n-1)}} e^{i(x'-y') \cdot \xi'} \int_0^\infty \tilde{t}'_c(x', y', y_n, \xi') u(y', x_n) dx_n d\xi'$$

The function

$$\tilde{t}(x', x_n, \xi') = \sum_{0 \leq j \leq r-1} s_j(x', \xi') D_{x_n}^j \delta(x_n) + \tilde{t}'(x', x_n, \xi')$$

is called the symbol-kernel of T and

$$t(x', \xi) = \mathcal{F}_{x_n \rightarrow \xi_n}^{-1} e^+ \tilde{t}(x, \xi') = \sum_{0 \leq j \leq r-1} s_j(x', \xi') \xi_n^j + t'(x', \xi)$$

is the symbol of T . Here $t' = \mathcal{F}_{x_n \rightarrow \xi_n}^{-1} e^+ \tilde{t}'(x, \xi')$.

3.2.2 Poisson operator

A Poisson operator of order d is an operator defined by

$$(Kv)(x', x_n) = \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \tilde{k}(x', x_n, \xi') \mathcal{F}_{x' \rightarrow \xi'} v(\xi') d\xi'$$

or

$$(Kv)(x', x_n) = \int_{\mathbb{R}^{2(n-1)}} e^{i(x'-y') \cdot \xi'} \tilde{k}_c(x', y', x_n, \xi') v(y') dy' d\xi'$$

The symbol is $k(x', \xi) = \mathcal{F}_{x_n \rightarrow \xi_n} e^+ \tilde{k}(x, \xi')$.

3.2.3 Singular Green operator

A singular Green operator G of order d and class r is an operator

$$G = \sum_{0 \leq j \leq r-1} K_j \gamma_j + G'$$

where K_j are Poisson operators of order $d-j$, γ_j the standard trace operator, G' is an operator of the form

$$(G'u)(x) = \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \int_0^\infty \tilde{g}'(x', x_n, y_n, \xi') \mathcal{F}_{x' \rightarrow \xi'} u(\xi', y_n) dy_n d\xi'$$

The symbol of G' is

$$g'(x', \xi', \xi_n, \eta_n) = \mathcal{F}_{x_n \rightarrow \xi_n} \mathcal{F}_{y_n \rightarrow \eta_n}^{-1} e_{x_n}^+ e_{y_n}^+ \tilde{g}'(x', x_n, y_n, \xi')$$

The symbol-kernel of G is

$$\tilde{g}(x', x_n, y_n, \xi) = \sum_{0 \leq j \leq r-1} \tilde{k}_j(x', x_n, \xi') D_{y_n}^j \delta(y_n) + \tilde{g}'(x', x_n, y_n, \xi')$$

The symbol of G is

$$g(x', \xi', \xi_n, \eta_n) = \sum_{0 \leq j \leq r-1} k_j(x', \xi) \eta_n^j + g'(x', \xi', \xi_n, \eta_n)$$

3.2.4 Properties

All pseudodifferential boundary operators form an algebra and have certain composition rules. They can also be defined on vector bundles.

Let \mathcal{A} be a pseudodifferential boundary operator with symbol $p(x, \xi), g(x', \xi, \eta_n), t(x', \xi), k(x', \xi), s(x', \xi')$.

(1) The principal interior symbol is $p^0(x, \xi)$. The principal boundary symbol is the operator

$$\mathfrak{a}_0(x', \xi', D_n) = \begin{pmatrix} p^0(x', 0, \xi', D_n) + g^0(x', \xi', D_n) & k^0(x', \xi', D_n) \\ t^0(x', \xi', D_n) & s^0(x', \xi') \end{pmatrix} : \mathcal{S}(\overline{\mathbb{R}_+})^N \times \mathbb{C}^M \rightarrow \mathcal{S}(\overline{\mathbb{R}_+})^{N'} \times \mathbb{C}^{M'}$$

(2) \mathcal{A} is said to be elliptic if p^0 is bijective for all $|\xi| \geq 1$ and all x , and all \mathfrak{a}_0 is bijective for all $|\xi'| \geq 1$ and all x' .

THEOREM: If \mathcal{A} is elliptic, then there exists a pseudodifferential boundary operator \mathcal{B} which is a parametrix with principal boundary symbol operator \mathfrak{a}_0^{-1} .

REMARK: The pseudodifferential boundary operators can be formulated in Fourier integral operators.

4 The Calderón projector

4.1 The setup and the main theorem

Let X be a compact smooth n -dimensional manifold, with two (hermitian) vector bundles E_1, E_2 . We assume X can be smoothly embedded in an n -dimensional manifold

\tilde{X} which has no boundary such that $X' = \partial X$ is an $(n-1)$ -dimensional hypersurface in \tilde{X} (This can be achieved by take the double of X). E_1, E_2 are restriction of some bundles \tilde{E}_1, \tilde{E}_2 over \tilde{X} . Write $X_+ = X^o, X_- = \tilde{X} - X, \tilde{E}_i|_{\overline{X_\pm}} = E_{i,\pm}, \tilde{E}_i|_{X'} = \tilde{E}'_i$.

We can assume that near the boundary, the manifold and bundles are described as a product situation, with a chosen normal coordinates x_n : there exists a neighborhood of $(x', x_n) : U \simeq X' \times [-1, 1]$ such that $x_n = 0$ on $X', x_n > 0$ in $X_+, x_n < 0$ in X_- .

If A extends to an elliptic operator of order d from $C^\infty(\tilde{E}_1)$ to $C^\infty(\tilde{E}_2)$, let Q be a parametrix of A on \tilde{X} . When \tilde{X} can be chosen to be compact, $\tilde{E}_1, \tilde{E}_2, A$ can be chosen such that A is invertible, then let Q be the inverse. In this case, the formulation is simpler.

Let $\rho = (\gamma_0, \dots, \gamma_{d-1}) = \gamma_0(1, D_n, \dots, D_n^{d-1})^t$ be the Cauchy data map on X' . Again, the map can be regarded on \tilde{X}, X_+, X_- , and denoted as $\tilde{\rho}, \rho^+, \rho^-$.

When $F = F_0 \oplus \dots \oplus F_{d-1}$ are vector bundle over X' , write

$$\mathcal{H}^s(F) = \prod_{0 \leq j \leq d-1} H^{s-j-\frac{1}{2}}(F_j)$$

Denote $E'_j{}^d = \bigoplus_{0 \leq j \leq d-1} E'_j$, then for $s > d - \frac{1}{2}$,

$$\rho^\pm : H^s(E_{i,\pm}) \rightarrow \mathcal{H}^s(E'_i{}^d), \tilde{\rho} : H^s(\tilde{E}_i) \rightarrow \mathcal{H}^s(E'_i{}^d)$$

and they are surjective.

The adjoint of $\tilde{\rho}$ is $\tilde{\rho}^* = (1, D_n, \dots, D_n^{d-1})\tilde{\gamma}_0^* : \mathcal{H}^s(E'_i{}^d)^* \rightarrow H^{-s}(\tilde{E}_i)$ where

$$\tilde{\gamma}_0^* v(x) = v(x') \otimes \delta(x_n)$$

Let

$$Z_\pm^s = \{z \in H^s(E_{1,\pm}) | Az = 0 \text{ on } X_\pm\}, N_\pm^s = \rho^\pm Z_\pm^s \subset \mathcal{H}^s(E'_1{}^d)$$

The question (1) is to characterize N_\pm^s which is answered by the following theorem.

THEOREM: Assume A has the inverse Q on \tilde{X} . Then

$$\mathcal{H}^s(E'_1{}^d) = N_+^s \oplus N_-^s$$

There exists Poisson operators $K^\pm : \mathcal{H}^s(E'_1{}^d) \rightarrow H^s(E_{1,\pm})$ with range equal to Z_\pm^s . $K^\pm : N_\pm^s \xrightarrow{\simeq} Z_\pm^s$ is the inverse of $\rho^\pm : Z_\pm^s \rightarrow N_\pm^s$. This gives a parametrization of the null space Z_\pm^s by its Cauchy data.

Also $C^\pm = \rho^\pm \circ K^\pm$ are the projections $\mathcal{H}^s(E'_1{}^d)$ onto N_\pm^s along N_\mp^s . In particular,

$$C^+ + C^- = I, (C^+)^2 = C^+, (C^-)^2 = C^-, C^+C^- = 0$$

If Q is only a parametrix, then we define K^+, C^+ as above modulo a smooth operator, then they have the mapping properties modulo a smooth operator. Also $(C^+)^2 - C^+$ is a smooth operator. C^+ is called the Calderón projector of A .

4.2 Construction of the Calderón projector

We shall assume that Q is the inverse of A . All the argument can be modified to modulo a smooth operator when Q is only a parametrix. Also we omit the argument on orders of the Sobolev spaces, which holds for any real number and need to be treated carefully.

4.2.1 Green's formula

Locally near X' , A can be written as $A = \sum_{l=0}^d S_l(x', x_n, D') D_n^l$ where S_l is a differential operator of order $d-l$ acting on X' for each $x_n \in (-1, 1)$. The for any $u \in H^d(E_{1,+}), v \in H^d(E_{2,+})$, we have Green's formula

$$(Au, v)_{X_+} - (u, A^*v)_{X_+} = (\mathfrak{A}\rho^+u, \rho^+v)_{X'}$$

where \mathfrak{A} is a (uniquely determined) matrix $\mathfrak{A} = (\mathfrak{A}_{jk})_{j,k=0,\dots,d-1}$ of differential operators \mathfrak{A}_{jk} from E'_1 to E'_2 of order $d-j-k-1$ with

$$\mathfrak{A}_{jk}(x', D') = iS_{j+k+1}(x', 0, D') + \text{lower-order terms}$$

zero if $j+k+1 > d$. Here \mathfrak{A} maps $\mathcal{H}^s(E_1^d)$ continuously into $\mathcal{H}^{s-d}(E_2^d)^*$ for all s . If S_d is bijective at $x_n = 0$, then \mathfrak{A} is bijective.

Since $(Au, v)_{\tilde{X}} - (u, A^*v)_{\tilde{X}} = 0$, we have for any $u \in H^d(E_{1,-}), v \in H^d(E_{2,-})$,

$$(Au, v)_{X_-} - (u, A^*v)_{X_-} = -(\mathfrak{A}\rho^-u, \rho^-v)$$

4.2.2 The Poisson operator

Let

$$K = Q\tilde{\rho}^*\mathfrak{A}, K^\pm = \mp r^\pm K = \mp r^\pm Q\tilde{\rho}^*\mathfrak{A}$$

then

$$K : \mathcal{H}^s(E_1^d) \rightarrow H^s(\tilde{E}_1), K^\pm : \mathcal{H}^s(E_1^d) \rightarrow H^s(E_{1,\pm})$$

Since $AK\phi = \tilde{\rho}^*\mathfrak{A}\phi$ is supported in X' , K^\pm map into Z_\pm^s . We have

$$K^\pm \rho^\pm z = z, \forall z \in Z_\pm^s, \rho^\pm K^\pm \phi = \phi, \forall \phi \in N_\pm^s$$

In fact, for any $w \in L^2(E_{1,+})$, let $v = Q^*e^+w \in H^d(\tilde{E}_2)$, then $r^+A^*v = w$. By Green's formula

$$\begin{aligned} -(z, w)_{X_+} &= (Az, r^+v)_{X_+} - (z, r^+A^*v)_{X_+} = (\mathfrak{A}\rho^+z, \rho^+v)_{X'} \\ &= (\mathfrak{A}\rho^+, \tilde{\rho}v)_{X'} = (\tilde{\rho}^*\mathfrak{A}\rho^+z, Q^*e^+w)_{\tilde{X}} \\ &= (Q\tilde{\rho}^*\mathfrak{A}\rho^+z, e^+w)_{\tilde{X}} = -(K^+\rho^+z, w)_{X_+} \end{aligned}$$

4.2.3 The Calderón projector

Now let $C^\pm = \rho^\pm \circ K^\pm$, it is easy to check that $C^\pm : \mathcal{H}^s(E_1^{td}) \rightarrow \mathcal{H}^s(E_1^{td})$ are projections into N_\pm^s for all $s \in \mathbb{R}$, $(C^+)^2 = \rho^+K^+\rho^+K^+ = \rho^+K^+ = C^+$, $(C^-)^2 = C^-$. Now we need to check $C^+ + C^- = I$:

Let $\phi \in \mathcal{H}^s(E_1^{td})$, $z^\pm = K^\pm\phi$. For each $\psi \in C^\infty(E_1^{td})$, choose a section $v \in C^\infty(E)$ with $\tilde{\rho}v = \psi$. Then by Green's formula and the fact that $z^\pm \in Z_\pm^d$.

$$\begin{aligned} (\mathfrak{A}(C^+ + C^-)\phi, \psi) &= (\mathfrak{A}\rho^+z^+, \rho^+r^+v)_{X'} + (\mathfrak{A}\rho^-z^-, \rho^-r^-v)_{X'} \\ &= (Az^+, r^+v)_{X_+} - (z^+, r^+A^*v)_{X_+} - (Az^-, r^-v)_{X_-} + (z^-, r^-A^*v)_{X_-} \\ &= -(K^+\phi, r^+A^*v)_{X_+} + (K^-\phi, r^-A^*v)_{X_-} = (Q\tilde{\rho}^*\mathfrak{A}\phi, A^*v)_{\tilde{X}} = (\tilde{\rho}^*\mathfrak{A}\phi, v)_{\tilde{X}} = (\mathfrak{A}\phi, \psi)_{X'} \end{aligned}$$

Since ϕ, ψ are arbitrary, \mathfrak{A} is invertible, it follows that $C^+ + C^- = I$. and $C^+C^- = C^+(I - C^+) = 0$.

4.2.4 Symbols of C^\pm

The symbol of C^\pm can be computed explicitly from the symbols of A : For fixed $x', \xi' \neq 0$, consider the model operator

$$a^0(x', \xi', D_n) = \text{Op}_n(a^0(x', 0, \xi', \xi_n))$$

acting on N -vector functions on \mathbb{R} . The solutions of $a^0u = 0$ that are bounded on \mathbb{R}_\pm form the vector spaces

$$Z_\pm(x', \xi') = \{u(x_n) \in \mathcal{S}(\overline{\mathbb{R}_\pm})^N \mid a^0(x', \xi', D_n)u = 0 \text{ on } \mathbb{R}_\pm\}$$

Let

$$N_\pm(x', \xi') = \rho^\pm Z_\pm(x', \xi')$$

The theory of ODE gives the inverse operator $K^\pm(x', \xi') : N_\pm(x', \xi') \rightarrow Z_\pm(x', \xi')$ which can be defined as follows:

$$K(x', \xi') = q^0(x', \xi', D_n) \tilde{\rho}^* \mathbf{a}^0(x', \xi'), K^\pm(x', \xi') = \mp r^\pm K(x', \xi')$$

where the matrix $\mathbf{a}^0(x', \xi')$ (the principal symbol of \mathfrak{A} satisfies

$$(a^0(D_n)u, v)_{\mathbb{R}^+} - (u, a^0(D_n)^*v)_{\mathbb{R}^+} = (\mathbf{a}^0 \rho^+ u, \rho^+ v)_{\mathbb{C}^{Nd}}$$

$q^0(x', \xi', D_n)$ is the principal symbol of Q : $q^0 = (a^0)^{-1}$, $q^0(x', \xi', D_n) = \text{Op}_n(q^0(x', 0, \xi))$; $\tilde{\rho}^*$ is the adjoint of $\tilde{\rho} : H^s(\mathbb{R})^N \rightarrow \mathbb{C}^{Nd}$:

$$\tilde{\rho}^* v = (1, D_n, \dots, D_n^{d-1}) \tilde{\gamma}_0^* v, \tilde{\gamma}_0^* v = v \delta(x_n), v \in \mathbb{C}^{Nd}$$

Then the principal boundary symbol operators for K^\pm are

$$k^{\pm,0}(x', \xi, D_n) = K^\pm(x', \xi') : N_\pm(x', \xi') \xrightarrow{\cong} Z_\pm(x', \xi')$$

The principal symbol for C^\pm are

$$c^{\pm,0}(x', \xi') = \rho^\pm k^{\pm,0}(x', \xi', D_n)$$

projecting \mathbb{C}^{Nd} onto its complementary subspaces $N_\pm(x', \xi')$, respectively.

5 Application to boundary value problem

Now we answer the question (2) using the Calderón projector. Consider $(A, B\rho)$ as a mapping from $H^s(E_1)$ to $H^{s-d}(E_2) \times \mathcal{H}^s(F)$ for some $s > d - \frac{1}{2}$; S as a mapping from $\mathcal{H}^s(E_1^{d'})$ to $\mathcal{H}^s(F)$ and C^\pm as mapping from $\mathcal{H}^s(E_1^{d'})$ to itself.

THEOREM: If A is invertible on \tilde{X} with inverse Q , then

(1) $\begin{pmatrix} A \\ B\rho \end{pmatrix}$ has the right inverse (R_B, K_B) if and only if BC^+ has a right inverse B_1 where

$$(R_B, K_B) = (Q_+ - K^+ B_1 B \rho Q_+, K^+ B_1), B_1 = \rho K_B$$

(2) $\begin{pmatrix} A \\ B\rho \end{pmatrix}$ has the left inverse (R_B, K_B) if and only if $\begin{pmatrix} B \\ C^- \end{pmatrix}$ has the left inverse (B_1, B_2) where

$$(R_B, K_B) = (Q_+ - K^+ B_1 B \rho Q_+, K^+ B_1), (B_1, B_2) = (\rho K_B, I - \rho K_B B)$$

When Q is only a parametrix of A , then these are also parametrices. These also hold on the level of symbols.

The proof is straightforward.

6 References

[B] L. Boutet de Monvel, Comportement d'un opérateur pseudo-différentiel sur une variété à bord, I. La Propriété de transmission; II. Pseudo-noyaux de Poisson, *Journal d'Analyse Mathématique* 17 (1966) 241-253; 255-304.

[C] A. P. Calderón, Boundary value problems for elliptic equations. 1963 Outlines of the Joint Symposium on Partial Differential Equations (Novosibirsk, 1963) pp. 303-304 Academy of Science of the USSR, Siberian Branch, Moscow.

[G1] G. Grubb, Boundary problems for systems of partial differential operators of mixed order, *Journal of Functional Analysis* 26 (1977), 131-165.

[G2] G. Grubb, *Distributions and operators*. Graduate Texts in Mathematics, 252, Springer, 2009.

[H1] L. Hörmander, Pseudo-differential operators and non-elliptic boundary problems, *Annals of Mathematics* 83 (1966), 129-209.

[H2] L. Hörmander, *The analysis of linear partial differential operators*, Volume III, Springer, 1985.

[S] R. T. Seeley, Singular integrals and boundary value problems, *American Journal of Mathematics* 88 (1966), 781-809.

[T] M. E. Taylor, *Partial differential equations*, Volume II, *Qualitative Studies of Linear Equations*, Springer, 1998.