An Introduction to the Navier-Stokes Initial-Boundary Value Problem

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Rechts auf zwei hohen Felsen befinden sich Schlösser,
unten breitet sich die Stadt

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Introduction

The equations of motion of an incompressible, Newtonian fluid –usually called *Navier-Stokes equations*– have been written almost one hundred eighty years ago. In fact, they were proposed in 1822 by the French engineer C. M. L. H. Navier upon the basis of a suitable molecular model. It is interesting to observe, however, that the law of interaction between the molecules postulated by Navier were shortly recognized to be totally inconsistent from the physical point of view for several materials and, in particular, for liquids. It was only more than twenty years later that the same equations were rederived by the twenty-six year old G. H. Stokes (1845) in a quite general way, by means of the theory of continua.

In the case where the fluid is subject to the action of a body force $\boldsymbol{f}$, the Navier-Stokes equations can be written as follows

$$\begin{aligned}&\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = \nu\Delta\boldsymbol{v} + \nabla p + \boldsymbol{f}\\ &\operatorname{div}\boldsymbol{v} = 0\end{aligned} \tag{0.1}$$

where $\boldsymbol{v} = \boldsymbol{v}(x,t)$ is the velocity field evaluated at the point $x \in \Omega$ and at time $t \in [0,T]$, ρp is the pressure field, ρ is the constant density of the fluid, and ν (> 0) is the coefficient of kinematical viscosity. Finally, Ω denotes the relevant geometrical domain where the spatial variables are ranging. Therefore, it will coincide with the region of flow for three-dimensional motions ($i.e.$, $\Omega \subset \mathbb{R}^3$), while it will coincide with a two-dimensional region, in case of plane flows ($\Omega \subset \mathbb{R}^2$).

To the equations (0.1) we append the *initial condition*: [1]

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0, \quad x \in \Omega \tag{0.2}$$

and the *boundary condition*

$$\boldsymbol{v}(y,t) = 0, \quad y \in \partial\Omega, \; t > 0 \; ^2. \tag{0.3}$$

In the case where Ω extends to infinity, we should impose also convergence conditions on $\boldsymbol{v}(x,t)$ (and/or, possibly, on $p(x,t)$) when $|x| \to \infty$.

Several mathematical properties for system (0.1) have been deeply investigated over the years and are still the object of profound researches. However, after more than one hundred seventy years from their formulation, the *Fundamental Problem* (FP)related to them remains still unsolved, that is:

Given the body force $\boldsymbol{f}$ *and the initial distribution of velocity* $\boldsymbol{v}_0$ *(no matter how smooth), to determine a corresponding unique regular solution* $\boldsymbol{v}(x,t), p(x,t)$ *to* $(0.1) - (0.3)$ *for all times* $t > 0$*.*

So far, this problem is only partially solved, despite numerous efforts by mathematicians and despite being viewed as an "obvious truth" by engineers. All this adds more weight to the following profound consideration due to Sir Cyril Hinshelwood, see Lighthill (1956, p. 343)

> Fluid dynamicists were divided into hydraulic engineers who observe what cannot be explained and mathematicians who explain things that cannot be observed

One of the aims of this article is to furnish an elementary presentation of some of the basic results so far known for (FP). In Section 1, we shall discuss the main features of system (0.1) and describe the main difficulties related to

[1]Without loss of generality, we can take 0 as initial time.

[2]For simplicity, we shall consider the case of homogeneous *no-slip* conditions.

it. Successively, following the classical methods of Leray (1934a, 1934b) and Hopf (1951/1952), we introduce the definition of *weak solution* to (0.1)-(0.3) and study some of the related properties (Section 2). These solutions play a major role in the mathematical theory of Navier-Stokes equations, in that they are the only solutions, so far known, which exist for *all times* and *without restrictions on the size of the data*. In Section 3 we shall show the existence of a *weak solution* for all times $t > 0$. Uniqueness and regularity of Leray-Hopf solutions will be presented in Sections 4 and 5, respectively. Due to the particular form of the nonlinearity involved in the Navier-Stokes equations, this study will naturally lead to the functional class $L^{s,r} \equiv L^r(0,T;\boldsymbol{L}^s(\Omega))$, $n/s + 2/r = 1$, $s > n$, [3] such that any weak solution belonging to $L^{s,r}$ is unique and regular. In view of this result, we shall see that every weak solution in dimension two is unique within its class, and that it possesses as much space-time regularity as allowed by the data. Since it is not known if a weak solution in dimension three is in $L^{s,r}$, it is not known if these properties continue to hold for three-dimensional flows. However, "partial regularity" results are available. To show some of these latter, we begin to prove the existence of more regular solutions in Sections 6. This existence theory will lead to the celebrated "théorème de structure" of Leray, which, roughly speaking, states that every weak solution is regular in space and time, with the possible exception of a set of times I of zero 1/2-dimensional Hausdorff measure. Moreover, defining a finite time $t_1 \in I$ an *epoch of irregularity* for a weak solution $\boldsymbol{v}$, if $\boldsymbol{v}$ is regular in a left-neighborhood of t_1 but it can not be extended to a regular solution after t_1, we shall give blow-up estimates for the Dirichlet norm of $\boldsymbol{v}$ at any (possible) epoch of irregularity. In view of the relevance of the functional class $L^{s,r}$, in Section 7 we will investigate the existence of weak solutions in such a class. Specifically, we shall prove the existence of weak solutions in $L^{s,r}$, at least for small times, provided the initial data are given in Lebesgue spaces L^q, for a suitable q. To avoid technical difficulties, this study will be performed for the case $\Omega = \mathbb{R}^n$ (Cauchy problem). As a consequence of these results, we shall enlarge the class of uniqueness of weak solutions, to include the case $s = n$. In addition, we shall give partial regularity results of a weak solution belonging to $L^{n,\infty}$. The important question of whether a weak solution in $L^{n,\infty}$ is regular, is left open.

[3] n denotes the space dimension.

1 Some Considerations on the Structure of the Navier-Stokes Equations.

Before getting involved with weak solutions à la Leray-Hopf and with their regularity, we wish to emphasize the main mathematical difficulties relating to (FP). First of all, we should notice that the unknowns $\boldsymbol{v}, p$ do not appear in (0.1) in a "symmetric way". In other words, the equation of conservation of mass is *not* of the following form

$$\frac{\partial p}{\partial t} = G(p, \boldsymbol{v}).$$

This is due to the fact that, from the mechanical point of view, the pressure plays the role of *reaction force* (Lagrange multiplier) associated with the isochoricity constraint $\operatorname{div} \boldsymbol{v} = 0$. In these regards, it is worth noticing that, in a perfect analogy with problems of motion of constrained rigid bodies, the pressure field must be generally deduced in terms of the velocity field, once this latter has been determined. In particular, we recall that the field $p(x,t)$ can be formally obtained –by operating with "div" on both sides of (0.1_1)– as a solution of the following Neumann problem

$$\begin{aligned} &\Delta p = \operatorname{div}(\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \boldsymbol{f}) \quad \text{in } \Omega \\ &\frac{\partial p}{\partial n} = -(\nu \Delta \boldsymbol{v} + \boldsymbol{f}) \cdot \boldsymbol{n} \quad \text{at } \partial\Omega \end{aligned} \tag{1.1}$$

where $\boldsymbol{n}$ denotes the unit outer normal to $\partial\Omega$ [4].

Because of the mentioned lack of "symmetry" in $\boldsymbol{v}$ e p, the system (0.1) does not fall in any of the classical categories of equations, even though, in a sense, it could be considered close to a quasi-linear parabolic system. Nevertheless, the basic difficulty related to problem (0.1)–(0.3) does not arise from the lack of such a symmetry but, rather, from the *coupled* effect of the *lack* of symmetry and of the *presence* of the nonlinear term. In fact, the (FP) formulated for any of the following systems

$$\begin{aligned} &\frac{\partial \boldsymbol{v}}{\partial t} = \nu \Delta \boldsymbol{v} + \nabla p + \boldsymbol{f} \\ &\operatorname{div} \boldsymbol{v} = 0 \end{aligned} \tag{0.1'}$$

[4] From this it is clear that to prescribe the values of the pressure at the bounding walls or at the initial time *independently* of $\boldsymbol{v}$, could be incompatible with (1.1) and, therefore, could render the problem ill-posed.

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \nu \Delta \boldsymbol{v} + \boldsymbol{f} \tag{0.1$''$}$$

obtained from (0.1) by disregarding either the nonlinear term $[(0.1')]$ or the isochoricity condition $[(0.1'')]$ can be completely solved. While for $(0.1')$ this solvability will be clear when we shall consider the solvability of (FP) for (0.1), the solvability of $(0.1'')$ is a consequence of an interesting *a priori* estimate discovered by Kiselev and Ladyzhenskaya (1957) and based on a maximum principle that we would like to mention here. Setting

$$\boldsymbol{u}(x,t) = \boldsymbol{v}(x,t)e^{-\alpha t} \quad \alpha > 0$$

from $(0.1'')$ we obtain

$$\tfrac{1}{2}\frac{\partial \boldsymbol{u}^2}{\partial t} + \tfrac{1}{2}\boldsymbol{v} \cdot \nabla \boldsymbol{u}^2 + \alpha \boldsymbol{u}^2 = \nu \Delta \boldsymbol{u} \cdot \boldsymbol{u} + \boldsymbol{f} \cdot \boldsymbol{u} e^{-\alpha t}. \tag{1.2}$$

Consider a point $P = (\widetilde{x}, \widetilde{t})$ of the cylinder $\overline{\Omega} \times [0,T]$ where $\boldsymbol{u}^2$ assumes its maximum. If such a point lies either on the bottom face of the cylinder ($i.e.$, at $\widetilde{t} = 0$) or on its lateral surface ($i.e.$, at $\widetilde{x} \in \partial\Omega$) we have

$$\max_{x,t} \boldsymbol{u}^2(x,t) \le \boldsymbol{u}^2(\widetilde{x}, 0) \le \max_x \boldsymbol{v}_0^2(x). \tag{1.3}$$

If, on the contrary, P is an interior point of the cylinder or lies on its top face we find

$$\left.\begin{array}{l} \dfrac{\partial \boldsymbol{u}^2}{\partial t} \ge 0, \quad \nabla \boldsymbol{u}^2 = 0, \\ -\boldsymbol{u} \cdot \Delta \boldsymbol{u} = |\nabla \boldsymbol{u}|^2 - \tfrac{1}{2}\Delta \boldsymbol{u}^2 \ge 0 \end{array}\right\} \text{ evaluated at } \quad (\widetilde{x}, \widetilde{t}) = P.$$

Therefore, from (1.2) we deduce

$$\alpha \boldsymbol{u}^2(\widetilde{x}, \widetilde{t}) \le \boldsymbol{f}(\widetilde{x}, \widetilde{t}) \cdot \boldsymbol{u}(\widetilde{x}, \widetilde{t}) e^{-\alpha \widetilde{t}}. \tag{1.4}$$

As a consequence, from (1.3), (1.4) we prove the following *a priori* estimate holding for all (sufficiently regular) solutions to system $(0.1)''$

$$|\boldsymbol{v}(x,t)| \le e^{\alpha t} \left\{ \frac{1}{\alpha} \max_{x,t} [e^{-\alpha t} |\boldsymbol{f}(x,t)|] + \max_x |\boldsymbol{v}_0(x)|^2 \right\}. \tag{1.5}$$

Notice that (1.5) is *independent* of the spatial dimension. Unfortunately, nothing similar to (1.5) is so far known for system (0.1) in dimension 3. Nevertheless, as we shall show later on, in dimension 2 the global ($i.e.$, for all times) estimates that we are able to derive will suffice to ensure the existence and uniqueness of a regular solution for (0.1).

2 The Leray-Hopf Weak Solutions and Related Properties.

We shall begin by giving the definition of weak solution in the sense of Leray-Hopf. To this end, we need to introduce some notation. By $L^q(\Omega)$ and $W^{m,q}(\Omega)$, $1 \leq q \leq \infty$, $m = 0, 1, \ldots$, we denote the usual Lebesgue and Sobolev spaces, respectively. The norm in $W^{m,q}$ is indicated by $\| \cdot \|_{m,q}$. For $m = 0$, it is $W^{0,q} \equiv L^q$ and we set $\| \cdot \|_{0,q} \equiv \| \cdot \|_q$. Whenever we need to specify the domain D on which these norms are evaluated, we shall write $\| \cdot \|_{m,q,D}$. We denote by $W_0^{m,q}(\Omega)$ the completion in the norm $\| \cdot \|_{m,q,\Omega}$ of the space $C_0^\infty(\Omega)$ constituted by all infinitely differentiable functions with compact support in Ω. The dual space of $W_0^{m,q}$ will be denoted by $W^{-m,q'}$.

Let [5]

$$\mathcal{D}(\Omega) = \{\boldsymbol{\psi} \in \boldsymbol{C}_0^\infty(\Omega) : \ \operatorname{div} \boldsymbol{\psi} = 0 \text{ in } \Omega\}.$$

We define $H_q = H_q(\Omega)$ as the completion of $\mathcal{D}(\Omega)$ in the Lebesgue space $\boldsymbol{L}^q(\Omega)$. Moreover, we denote by $H_q^1(\Omega)$ the completion of $\mathcal{D}(\Omega)$ in the Sobolev space $\boldsymbol{W}^{1,q}(\Omega)$. For $q = 2$, we shall simply write H and H^1, respectively. It is well known, see Galdi (1994, Chapter III), that if Ω has a bounded boundary which is locally lipschitzian, or if Ω is a half-space, the following characterizations for H_q and H_q^1 hold, for $1 < q < \infty$:

$$H_q(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^q(\Omega) : \ \operatorname{div} \boldsymbol{u} = 0, \ \ \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\}$$

$$H_q^1(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{1,q}(\Omega) : \ \operatorname{div} \boldsymbol{u} = 0, \ \ \boldsymbol{u}|_{\partial\Omega} = 0\right\}$$

where the values at the boundary have to be understood in the trace sense. Furthermore, if Ω is of class C^1, the following *Helmholtz-Weyl decomposition* holds

$$\boldsymbol{L}^q(\Omega) = H_q(\Omega) \oplus G_q(\Omega) \tag{2.1}$$

where

$$G_q(\Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^q(\Omega) : \ \boldsymbol{u} = \nabla p, \text{ for some } p \in L^1_{loc}(\Omega) \ \text{ with } \nabla p \in \boldsymbol{L}^q(\Omega)\}$$

(we set $G = G_2$). The projection of $\boldsymbol{L}^q$ onto H_q is denoted by P_q ($\equiv P$, for $q = 2$). In the case $q = 2$, H and G are orthogonal subspaces of $\boldsymbol{L}^2$, and (2.1) holds for any open set Ω.

[5] If X is a space of scalar functions, we denote by $\mathbf{X}$ the space constituted by vector or tensor valued functions having components in X.

For $T \in (0, \infty]$ we set $\Omega_T = \Omega \times [0, T)$ and define

$$\mathcal{D}_T = \{\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\Omega_T) : \ \operatorname{div} \boldsymbol{\varphi}(x,t) = 0 \text{ in } \Omega_T\}$$

Notice that for $\boldsymbol{\varphi} \in \mathcal{D}_T$, $\boldsymbol{\varphi}(x, 0)$ need not be zero. For $\boldsymbol{a}, \boldsymbol{b}$ vector functions in Ω we put

$$(\boldsymbol{a}, \boldsymbol{b}) \equiv \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b}, \quad \|\boldsymbol{a}\|_2 = (\boldsymbol{a}, \boldsymbol{a})^{1/2}$$

$$(\nabla \boldsymbol{a}, \nabla \boldsymbol{b}) \equiv \sum_{i,j=1}^n \int_\Omega \frac{\partial a_i}{\partial x_j} \frac{\partial b_i}{\partial x_j}, \quad \|\nabla \boldsymbol{a}\|_2 = (\nabla \boldsymbol{a}, \nabla \boldsymbol{a})^{1/2}.$$

If we need to specify the domain D on which these quantities are evaluated, we shall write

$$(\cdot, \cdot)_D, \quad \|\cdot\|_{2,D}.$$

Moreover, for a given Banach space X, with associated norm $\|\cdot\|_X$, and a real interval (a, b), we denote by $L^q(a, b; X)$ the linear space of (equivalence classes of) functions $f : (a, b) \to X$ such that the functional

$$\|f\|_{L^q(a,b;X)} \equiv \begin{cases} \left(\displaystyle\int_a^b \|f(t)\|_X^q dt \right)^{1/q} & \text{if } 1 \le q < \infty \\ \operatorname*{ess\,sup}_{t \in (a,b)} \|f(t)\|_X & \text{if } q = \infty \end{cases}$$

is finite. It is known that this functional defines a norm with respect to which $L^q(a, b; X)$ becomes a Banach space (Hille and Phillips 1957, Chapter III). Likewise, for r a non-negative integer and I a real interval, we denote by $C^r(I; X)$ the class of continuous functions from I to X, which are differentiable in I up to the order r included. Finally, if I is open or semi-open, by $BC(I; X)$, we denote the subspace of $C(I; X)$ such that $\sup_{t \in I} \|u(t)\|_X < \infty$. Depending on X, these spaces might share several properties with the "usual" Lebesgue spaces $L^q(a, b)$ and spaces $C^r(I)$, and we refer to the monograph of Hille and Phillips for further information.

Assume now $\boldsymbol{v}(x, t), p(x, t)$ is a classical solution to (0.1)-(0.3). [6] Then,

[6] For instance, $\boldsymbol{v}$ is one time differentiable in t and twice differentiable in x, while p is one time differentiable in x. Moreover, $\boldsymbol{v}$ assumes continuously the initial and boundary data.

multiplying $(0.1)_1$ by $\boldsymbol{\varphi} \in \mathcal{D}_T$ and integrating over Ω_∞ we find

$$\int_0^\infty \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) \right\} dt$$

$$= -\int_0^\infty (\boldsymbol{f}, \boldsymbol{\varphi}) dt - (\boldsymbol{v}_0, \boldsymbol{\varphi}(0)), \tag{2.2}$$

$$\text{for all } \boldsymbol{\varphi} \in \mathcal{D}_T.$$

Conversely, if $\boldsymbol{v}(x,t)$ is a vector field satisfying (2.2), and having enough smoothness as to allow for integration by parts over Ω_∞ in some sense, [7] we easily obtain

$$\int_0^\infty \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nu \Delta \boldsymbol{v} - \boldsymbol{f}, h(t) \boldsymbol{\psi} \right) dt = 0$$

for all $h \in C_0^\infty((0,T))$ and $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$. Therefore, for every such $\boldsymbol{\psi}$

$$\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nu \Delta \boldsymbol{v} - \boldsymbol{f}, \boldsymbol{\psi} \right) = 0,$$

and by a well known result of Hopf (1950/1951), see Galdi (1994, Lemma III.1.1), we conclude the validity of (0.1) for some pressure field $p(x,t)$. However, it is clear that if $\boldsymbol{v}(x,t)$ is a solenoidal vector field that satisfies (2.2) but is *not* sufficiently differentiable, we can *not* go from (2.2) to $(0.1)_1$ and it is precisely in this sense that (2.2) has to be considered as the *weak formulation* of $(0.1)_1$.

Remark 2.1 It is simple to give examples of solenoidal vector fields which satisfy (2.2) but which do not have enough smoothness as to verify $(0.1)_1$, no matter how smooth $\boldsymbol{f}$ and $\boldsymbol{v}_0$ are. Take, for instance

$$\overline{\boldsymbol{v}}(x,t) = a(t) \nabla \sigma(x), \quad \Delta \sigma = 0 \text{ in } \Omega \tag{2.3}$$

where $a(t)$ has no more regularity than the local integrability in $(0,T)$ with $a(0)$ finite. Since

$$\overline{\boldsymbol{v}} \cdot \nabla \overline{\boldsymbol{v}} = \tfrac{1}{2} \nabla (\nabla \sigma)^2$$

and $\operatorname{div} \overline{\boldsymbol{v}} = 0$, we deduce that $\overline{\boldsymbol{v}}(x,t)$ is a non-smooth solenoidal vector field satisfying (2.2) with $\boldsymbol{f} = 0$ and $\boldsymbol{v}_0 = a(0) \nabla \psi$. Notice that, since σ is harmonic, $\boldsymbol{v}_0$ is analytic.

[7] For instance, in the sense of generalized differentiation.

We wish to give a generalized meaning to the solenoidality condition $(0.1)_2$ and to the boundary condition (0.4). This will be accomplished, for instance, if we require that, for almost all times $t \in [0,T]$, $\boldsymbol{v}(\cdot,t)$ belongs to $H^1(\Omega)$. Moreover, to ensure that all integrals in (2.2) are meaningful, we may require $\boldsymbol{v} \in L^2(0,T;H^1)$.

These considerations then lead to the following definition of weak solution, due to Leray (1934a, 1934b) and Hopf (1951/1952).

Definition 2.1 Let $\boldsymbol{v}_0 \in H(\Omega)$, $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega_T)$. A measurable function $\boldsymbol{v} : \Omega_T \to \mathbb{R}^n$, $n = 2, 3$, [8] is said a *weak solution* of the problem (0.1)-(0.4) in Ω_T if

a) $\boldsymbol{v} \in V_T \equiv L^2(0,T;H^1) \cap L^\infty(0,T;H)$;

b) $\boldsymbol{v}$ verifies (2.2).

If $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega_T)$ for all $T > 0$, $\boldsymbol{v}$ will be called a *global weak solution* if it is a weak solution in Ω_T for all $T > 0$.

Remark 2.2 In a) we have included the condition that $\boldsymbol{v} \in L^\infty(0,T;H)$ which, *a priori*, does not seem to be strictly necessary. However, on one hand, this condition ensures that the kinetic energy of a weak solution is essentially bounded in the time interval $[0,T]$, and this is a natural request from the physical point of view. On the other hand, excluding such a condition would result in a definition of weak solution too poor to allow for the development of any further relevant property. And last, but not least, we shall prove that the class of weak solutions is not empty, see Theorem 3.1.

Remark 2.3 Definition 2.1 is apparently silent about the pressure field. Later on (Theorem 2.1) we shall see that to every weak solution we can always associate a corresponding pressure field.

Our next objective is to collect a certain number of properties of weak solutions which will eventually lead, among other things, to a definition equivalent to Definition 1.1. The following result is due to Hopf (1951/1952, Satz 2.1); see also Prodi (1959, Lemma 1) and Serrin (1963, Theorem 4).

Lemma 2.1 *Let $\boldsymbol{v}$ be a weak solution in Ω_T. Then $\boldsymbol{v}$ can be redefined on a set of zero Lebesgue measure in such a way that $\boldsymbol{v}(t) \in \boldsymbol{L}^2(\Omega)$ for all $t \in [0,T)$*

[8] Of course, the definition of weak solution can be given for any spatial dimension $n \geq 2$, but we shall be mainly interested in the physical interesting cases of two and three dimensions.

and satisfies the identity

$$\int_s^t \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) \; -(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) \right\} d\tau$$
$$= -\int_s^t (\boldsymbol{f}, \boldsymbol{\varphi}) d\tau + (\boldsymbol{v}(t), \boldsymbol{\varphi}(t)) - (\boldsymbol{v}(s), \boldsymbol{\varphi}(s)), \tag{2.4}$$

for all $s \in [0,t]$*,* $t < T$*, and all* $\boldsymbol{\varphi} \in \mathcal{D}_T$*.*

Proof. It is clear that to show (2.4) for arbitrary $s \in [0,t]$, it is enough to prove it for $s = 0$. We begin to show that (2.4) holds for $s = 0$ and almost every $t \in [0,T)$. Let $\theta \in C^1(\mathbb{R})$ be a monotonic, non-negative function such that

$$\theta(\xi) = \begin{cases} 1 & \text{if } \xi \leq 1 \\ 0 & \text{if } \xi \geq 2 \end{cases}$$

For a fixed $t \in [0,T)$ and $h > 0$ with $t + h < T$ we set

$$\theta_h(\tau) = \theta\left(\frac{\tau - t + h}{h}\right).$$

Notice that

$$\left|\frac{d\theta_h}{d\tau}\right| \leq Ch^{-1}, \quad C > 0,$$
$$\int_t^{t+h} \frac{d\theta_h}{d\tau} d\tau = -1. \tag{2.5}$$

Choosing in (2.2) $\boldsymbol{\varphi}(x,t)$ as $\theta_h(t)\boldsymbol{\varphi}(x,t)$, we obtain

$$\int_0^{t+h} \theta_h(\tau) \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) \; -(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) + (\boldsymbol{f}, \boldsymbol{\varphi}) \right\} d\tau$$
$$= -\int_0^{t+h} \frac{d\theta_h(\tau)}{d\tau} (\boldsymbol{v}, \boldsymbol{\varphi}) d\tau - (\boldsymbol{v}_0, \boldsymbol{\varphi}). \tag{2.6}$$

Letting $h \to 0$ in this relation and recalling Definition 2.1, we easily deduce that the integral on the left-hand side of this relation tends to

$$\int_0^t \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\varphi}) + (\boldsymbol{f}, \boldsymbol{\varphi}) \right\} d\tau.$$

Let now investigate the behavior of the integral at the right-hand side of (2.6). In view of (2.5) and of a) of Definition 2.1, we have for each fixed t

$$\begin{aligned}
\ell(h,t) &\equiv \left| \int_0^{t+h} \frac{d\theta_h(\tau)}{d\tau} (\boldsymbol{v}(\tau), \boldsymbol{\varphi}(\tau)) d\tau + (\boldsymbol{v}(t), \boldsymbol{\varphi}(t)) \right| \\
&= \left| \int_0^{t+h} \frac{d\theta_h(\tau)}{d\tau} \left\{ (\boldsymbol{v}(\tau) - \boldsymbol{v}(t), \boldsymbol{\varphi}(t)) + (\boldsymbol{v}(\tau), \boldsymbol{\varphi}(\tau) - \boldsymbol{\varphi}(t)) \right\} d\tau \right| \\
&\le C \|\boldsymbol{\varphi}(t)\|_2 \left(h^{-1} \int_t^{t+h} \|\boldsymbol{v}(\tau) - \boldsymbol{v}(t)\|_2 d\tau \right) \\
&\qquad + \max_{\tau \in [t, t+h]} \|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(\tau)\|_2 \left(h^{-1} \int_t^{t+h} \|\boldsymbol{v}(\tau)\|_2 d\tau \right) \\
&\le C \|\boldsymbol{\varphi}(t)\|_2 \left(h^{-1} \int_t^{t+h} \|\boldsymbol{v}(\tau) - \boldsymbol{v}(t)\|_2 d\tau \right) \\
&\qquad + M \max_{\tau \in [t, t+h]} \|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(\tau)\|_2 .
\end{aligned}$$

Denote by $\mathcal{L}(\boldsymbol{v})$ the set of all those $t \in [0,T)$ for which

$$\lim_{h \to 0} h^{-1} \int_t^{t+h} \|\boldsymbol{v}(\tau) - \boldsymbol{v}(t)\|_2 d\tau = 0.$$

As is well known from the theory of Lebesgue integration (Titchmarsh, 1964, §11.6, Hille and Phillips, 1957, Theorem 38.5), $\mathcal{L}(\boldsymbol{v})$ can differ from $[0,T)$ only by a set of zero Lebesgue measure. Therefore, since

$$\lim_{h \to 0} \max_{\tau \in [t, t+h]} \|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(\tau)\|_2 = 0,$$

we obtain

$$\lim_{h \to 0} \ell(h,t) = 0, \quad \text{for all } t \in \mathcal{L}(\boldsymbol{v}),$$

and so identity (2.4) follows for $s = 0$ and all $t \in \mathcal{L}(\boldsymbol{v})$. We set $E_1 = [0,T) - \mathcal{L}(\boldsymbol{v})$. Moreover, by a) of Definition 2.1, there exists a constant $M > 0$ and a set $E_2 \subset [0,T)$ of zero Lebesgue measure such that

$$\|\boldsymbol{v}(t)\|_2 \le M, \quad \text{for all } t \in [0,T) - E_2. \tag{2.7}$$

Put $E = E_1 \cup E_2$ and pick $\bar{t} \in E$. Then, there exists a sequence $\{t_k\} \subset [0,T) - E$ converging to $\bar{t}$ as $k \to \infty$. By (2.7), $\|\boldsymbol{v}(t_k)\|_2 \le M$ and so, by the weak compactness of the spaces H we find $\boldsymbol{U}_{\bar{t}} \in H(\Omega)$ such that

$$\lim_{k\to\infty}(\boldsymbol{v}(t_k) - \boldsymbol{U}_{\bar{t}}, \boldsymbol{\psi}) = 0, \quad \text{for all } \boldsymbol{\psi} \in \mathcal{D}(\Omega).$$

Define

$$\boldsymbol{v}^*(x,t) = \begin{cases} \boldsymbol{v}(x,t) & \text{if } t \in [0,T) - E \\ \boldsymbol{U}_t(x) & \text{if } t \in E. \end{cases}$$

(Notice that $\boldsymbol{v}^*(x,0) = \boldsymbol{v}_0(x)$.) Clearly, $\boldsymbol{v}^* \in L^2(\Omega)$, for all $t \in [0,T)$. Furthermore, evaluating (2.4) along the sequence $\{t_k\}$ associated to $\boldsymbol{U}_t$ and letting $k \to \infty$ it is easy to verify the validity of the following statements:

1) $\boldsymbol{v}^*$ satisfies (2.4) for all $t \in [0,T)$;

2) $\boldsymbol{U}_t$ does not depend on the sequence $\{t_k\}$.

The lemma is then completely proved.

As a corollary to this result, we have

Lemma 2.2 *Let $\boldsymbol{v}$ be a weak solution in Ω_T. Then $\boldsymbol{v}$ can be redefined on a set of zero Lebesgue measure in such a way that it satisfies the identity*

$$\begin{aligned}\int_0^t \{-\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{\psi}) - (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\psi})\}\, ds \\ = -\int_0^t (\boldsymbol{f}, \boldsymbol{\psi})ds + (\boldsymbol{v}(t), \boldsymbol{\psi}) - (\boldsymbol{v}_0, \boldsymbol{\psi}),\end{aligned} \tag{2.8}$$

for all $t \in [0,T)$ and all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$. Furthermore, $\boldsymbol{v}$ is L^2 weakly continuous, that is,

$$\lim_{t\to t_0}(\boldsymbol{v}(t) - \boldsymbol{v}(t_0), \boldsymbol{u}) = 0,$$

for all $t_0 \in [0,T)$ and all $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$.

Proof. We put in (2.4) $s = 0$ and choose $\boldsymbol{\varphi}(x,t) = \theta_h(t)\boldsymbol{\psi}(x)$, where θ_h is the function introduced in the proof of the previous lemma and $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$. Noticing that $\boldsymbol{\varphi} = \boldsymbol{\psi}$ in $t \in [0,t]$, (2.8) follows at once. To show the L^2 weak continuity, we observe that for any fixed $t_0 \in [0,T)$ from (2.8) it follows that

$$\forall\varepsilon > 0,\ \exists\,\delta(\varepsilon) > 0 : |t - t_0| < \delta \Longrightarrow |(\boldsymbol{v}(t) - \boldsymbol{v}(t_0), \boldsymbol{\psi})| < \varepsilon,$$

for all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$. It is clear that this property continues to hold (by density) for all $\boldsymbol{w} \in H(\Omega)$. Let now $\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$. By the Helmholtz-Weyl decomposition (2.1) we may write

$$\boldsymbol{u} = \boldsymbol{w} + \nabla q, \quad \boldsymbol{w} \in H(\Omega), \ \nabla q \in G(\Omega)$$

and so, since $\boldsymbol{v} \in H(\Omega)$, we have

$$(\boldsymbol{v}(t) - \boldsymbol{v}(t_0), \boldsymbol{u}) = (\boldsymbol{v}(t) - \boldsymbol{v}(t_0), \boldsymbol{w}),$$

and the lemma follows.

Remark 2.4 Lemma 2.2 tells us, in particular, the way in which a weak solution assumes the initial data, namely, in the sense of the weak L^2 convergence.

Throughout the rest of these notes, we shall assume that every weak solution has been modified on a set of zero Lebesgue measure in such a way that it verifies the assertions of Lemma 2.1 *and* Lemma 2.2

Our next concern is to investigate if and in which sense, we can associate a "pressure field" to a weak solution. Let us first assume that $\boldsymbol{v}, p$ is a classical solution to (0.1)-(0.4). Then, multiplying $(0.1)_1$ by $\boldsymbol{\chi} \in \boldsymbol{C}_0^\infty(\Omega)$ and integrating by parts over Ω_t we formally obtain

$$\begin{aligned}\int_0^t \{-\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\chi}) \ -(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\chi}) + (\boldsymbol{f}, \boldsymbol{\chi})\}\, ds \\ = \int_0^t (p, \operatorname{div} \boldsymbol{\chi}) + (\boldsymbol{v}(t), \boldsymbol{\chi}) - (\boldsymbol{v}_0, \boldsymbol{\chi}).\end{aligned} \tag{2.9}$$

In the next theorem, we shall show that to any weak solution $\boldsymbol{v}$ in Ω_T we can associate a function $P(t) \in L^2(\omega)$, $t \in [0,T)$, $\omega \subset\subset \Omega$, such that

$$\begin{aligned}\int_0^t \{-\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\chi}) \ -(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\chi}) + (\boldsymbol{f}, \boldsymbol{\chi})\}\, ds \\ = (P(t), \operatorname{div} \boldsymbol{\chi}) + (\boldsymbol{v}(t), \boldsymbol{\chi}) - (\boldsymbol{v}_0, \boldsymbol{\chi}),\end{aligned} \tag{2.10}$$

for all $\boldsymbol{\chi} \in \boldsymbol{C}_0^\infty(\Omega)$. We wish to emphasize that, using only (2.8) and the local regularity property of weak solution, in general, we can *not* write

$$P(\cdot, t) = \int_0^t p(\cdot, s) ds, \quad \text{for some } p \in L^1_{loc}([0,T)), \tag{2.11}$$

due to the fact that a weak solution has *a priori* only a mild degree of regularity in time. To see this, let us consider the vector field $\overline{\boldsymbol{v}}$ given in (2.3) and choose $a(t) \in C([0,T))$ but $a' \notin L^1_{loc}([0,T))$. By a straightforward calculation we find that

$$\int_0^t \{-\nu(\nabla\overline{\boldsymbol{v}}, \nabla\boldsymbol{\chi}) \ -(\overline{\boldsymbol{v}}\cdot\nabla\overline{\boldsymbol{v}}, \boldsymbol{\chi})\}\, ds - (\overline{\boldsymbol{v}}(t), \boldsymbol{\chi}) + (\overline{\boldsymbol{v}}_0, \boldsymbol{\chi})$$
$$= \tfrac{1}{2}\int_0^t a^2(s)((\nabla\sigma)^2, \operatorname{div}\boldsymbol{\chi}) ds + (a(t)-a(0))(\sigma, \operatorname{div}\boldsymbol{\chi})$$

and therefore (2.10) is satisfied with $\boldsymbol{f} \equiv 0$ and

$$P = \tfrac{1}{2}\int_0^t a^2(s)(\nabla\sigma)^2 ds + (a(t)-a(0))\sigma.$$

Since $a' \notin L^1_{loc}([0,T))$, P does not verify (2.11); see also Remark 2.6.

Theorem 2.1 *Let $\boldsymbol{v}$ be a weak solution in Ω_T. Then, there exists a scalar field $P : \Omega_T \to \mathbb{R}$ with*

$$P(t) \in L^2(\omega), \quad \text{for all } t \in [0,T) \text{ and } \omega \subset\subset \Omega,$$

verifying (2.10) for all $\boldsymbol{\chi} \in \boldsymbol{C}_0^\infty(\Omega)$ and all $t \in [0,T)$. Moreover, if ω satisfies the cone condition, there exist $C = C(t,\omega) \in \mathbb{R}$ and $C_1 = C_1(\omega) > 0$ such that

$$\|P(t) - C\|_{2,\omega} \le C_1\left\{\int_0^t \left(\|\nabla\boldsymbol{v}(s)\|_{2,\omega} + M^\alpha\|\nabla\boldsymbol{v}(s)\|_{2,\omega}^\beta + \|\boldsymbol{f}(s)\|_{2,\omega}\right) ds + M\right\}$$

for all $t \in [0,T)$, where

$$M = \operatorname*{ess\,sup}_{s\in[0,t]} \|\boldsymbol{v}(s)\|_{2,\omega}$$

and

$$\alpha = \begin{cases} 1 & \text{if } n=2 \\ 1/2 & \text{if } n=3 \end{cases}$$

$$\beta = \begin{cases} 1 & \text{if } n=2 \\ 3/2 & \text{if } n=3. \end{cases}$$

Proof. Let us consider a sequence of bounded "invading domains" $\{\Omega_k\}$, that is, Ω_k is bounded for each k, and

$$\Omega_{k+1} \supset \Omega_k, \quad \Omega = \bigcup_{k=1}^{\infty} \Omega_k.$$

Without loss, we may assume that Ω_k satisfies the cone condition for each k. For fixed $t \in [0,T)$, consider the functional

$$\mathcal{F}(\boldsymbol{\chi}) = \int_0^t \{-\nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{\chi}) - (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\chi}) + (\boldsymbol{f}, \boldsymbol{\chi})\}\, ds - (\boldsymbol{v}(t), \boldsymbol{\chi}) + (\boldsymbol{v}_0, \boldsymbol{\chi}),$$
$$\boldsymbol{\chi} \in \boldsymbol{W}_0^{1,2}(\Omega_k).$$

It is clear that $\mathcal{F}$ is linear functional on $\boldsymbol{W}_0^{1,2}(\Omega_k)$. Moreover, using the Schwarz inequality and the following ones (see, $e.g.$, Galdi, 1994, Chapter II)

$$\begin{aligned} \|u\|_4 &\le 2^{-1/4}\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}, \quad n=2, \\ \|u\|_4 &\le \left(\tfrac{4}{3\sqrt{3}}\right)^{3/4}\|u\|_2^{1/4}\|\nabla u\|_2^{3/4}, \quad n=3, \end{aligned} \tag{2.12}$$

it is easy to see that

$$|\mathcal{F}(\boldsymbol{\chi})| \le c\|\boldsymbol{\chi}\|_2 \left\{\int_0^t \left(\|\nabla \boldsymbol{v}\|_2 + M_k^{\alpha}\|\nabla\boldsymbol{v}\|^{\beta}\|\boldsymbol{f}\|_2\right) ds + M_k\right\} \tag{2.13}$$

where

$$M_k = \text{ess}\sup_{s\in[0,t]} \|\boldsymbol{v}(s)\|_{2,\Omega_k}.$$

As a consequence, $\mathcal{F}$ is a continuous linear functional on $\boldsymbol{W}_0^{1,2}(\Omega_k)$ which, by Lemma 2.2, vanishes on $H^1(\Omega_k)$. Thus, since Ω_k is bounded for all k, by known results (Galdi, 1994, Corollary III.5.1) there exists $P_1 = P_1(t) \in L^2(\Omega_1)$ such that

$$\mathcal{F}(\boldsymbol{\chi}) = (P_1, \operatorname{div}\boldsymbol{\chi}), \quad \text{for all } \boldsymbol{\chi} \in \boldsymbol{W}_0^{1,2}(\Omega_1).$$

Likewise, we show that there exists $P_2 = P_2(t) \in L^2(\Omega_2)$ such that

$$\mathcal{F}(\boldsymbol{\chi}) = (P_2, \operatorname{div}\boldsymbol{\chi}), \quad \text{for all } \boldsymbol{\chi} \in \boldsymbol{W}_0^{1,2}(\Omega_2).$$

Since, for $x \in \Omega_1$, we have $P_2(x,t) = P_1(x,t) + c(\Omega_1,\Omega_2,t)$, $c(\Omega_1,\Omega_2,t) \in \mathbb{R}$, we can modify P_2 by a function of time so that $P_2 \equiv P_1$ in Ω_1. By means of an induction argument, we then prove the existence of a function $P : \Omega_T \to \mathbb{R}$ with $P \in L^2(\Omega_k)$, for all $k \in \mathbb{N}$. [9] Furthermore,

$$\mathcal{F}(\boldsymbol{\chi}) = (P, \operatorname{div} \boldsymbol{\chi}), \quad \text{for all } \boldsymbol{\chi} \in \boldsymbol{W}_0^{1,2}(\Omega_k).$$

and, again by Galdi (1994), Corollary III.5.1, and (2.13), we have

$$\|P(t)\|_{2,\Omega_k} \leq C_1 \left\{ \int_0^t \left(\|\nabla \boldsymbol{v}\|_{2,\Omega_k} + M_k^\alpha \|\nabla \boldsymbol{v}\|_{2,\Omega_k}^\beta \|\boldsymbol{f}\|_{2,\Omega_k} \right) ds + M_k \right\}$$

$$(P(t), 1)_{\Omega_k} = 0$$

which proves the theorem.

Remark 2.5 If Ω has a bounded boundary satisfying the cone condition, one can show that the field P introduced in the previous theorem can be chosen to belong to $L^\infty(0,T; L^2(\Omega))$. In fact, in such a case, assuming some more regularity on Ω one shows that relation (2.11) holds, see Sohr and von Wahl (1986).

Remark 2.6 In a recent paper, J. Simon (1999) has shown that, when Ω is bounded, there exists at least one weak solution satisfying (2.9), with corresponding $p \in W^{-1,\infty}(0,T; L^2_{loc}(\Omega))$, if Ω has no regularity, and with. $p \in W^{-1,\infty}(0,T; L^2(\Omega))$, if Ω is locally lipschitzian. For this result to hold it is sufficient to assume $\boldsymbol{f} \in L^2(0,T; W^{-1,2}(\Omega))$.

We wish now to prove a converse of Lemma 2.2, that is, any function $\boldsymbol{v} \in V_T$ (see Definition 2.1) which satisfies (2.8) for all $t \in [0,T)$ and all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$ must also satisfy (2.2). This will lead to an equivalent formulation of weak solution involving identity (2.8) instead of (2.2). We begin to show that if $\boldsymbol{v} \in V_T$ solves (2.8) for all $t \in [0,T)$ and all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$, then it also satisfies (2.2) with

$$\boldsymbol{\varphi}(x,t) \equiv \boldsymbol{\varphi}_N = \sum_{k=1}^N \gamma_k(t) \boldsymbol{\psi}_l(x), \tag{2.14}$$

where $\gamma_l \in C_0^1([0,T))$. By the linearity of (2.8) in $\boldsymbol{\varphi}$, it is enough to show this statement for $N = 1$. Now, (2.2) with $\boldsymbol{\varphi}(x,t) = \gamma(t)\boldsymbol{\psi}(x)$ and (2.8) can be written in the following forms

$$\int_0^T \gamma'(t) g(t) dt = -\int_0^T \gamma(t) G(t) dt - \gamma(0) g(0) \tag{2.2'}$$

[9] $\mathbb{N}$ denotes the set of all positive integers.

and

$$g(t) = \int_0^t G(s)ds + g(0), \quad t \in [0, T) \tag{2.8'}$$

respectively, where

$$g(t) = (\boldsymbol{v}(t), \boldsymbol{\psi})$$
$$G(t) = \{-\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{\psi}) - (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\psi}) + (\boldsymbol{f}, \boldsymbol{\psi})\} \in L^1(0, T)$$

From Lemma 2.2 we already know that $(2.2')$ implies $(2.8')$. Conversely, from classical results on Lebesgue integration (see, $e.g.$, Titchmarsh, 1964, §11), one shows that $(2.8')$ implies $(2.2')$. [10] To complete the equivalence of the two formulations, it remains to show that every $\boldsymbol{\varphi} \in \mathcal{D}_T$ together with their first and second spatial derivatives and first time derivatives, can be approximated in Ω_T by functions of the type (2.14). This is the objective of the following lemma.

Lemma 2.3 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, $n \geq 2$, and let $T > 0$. Then, there exists a sequence of functions $\{\boldsymbol{\psi}_r\} \subset \mathcal{D}(\Omega)$ with the following properties. Given $\boldsymbol{\varphi} \in \mathcal{D}_T$ and $\varepsilon > 0$ there are $N = N(\boldsymbol{\varphi}, \varepsilon) \in \mathbb{N}$ functions $\gamma_k \in C_0^1([0, T))$, $k = 1, \ldots, N$, such that*

$$\max_{t\in[0,T]} \|\boldsymbol{\varphi}_N(t) - \boldsymbol{\varphi}(t)\|_{C^2(\Omega)} + \max_{t\in[0,T]} \left\| \frac{\partial \boldsymbol{\varphi}_N(t)}{\partial t} - \frac{\partial \boldsymbol{\varphi}(t)}{\partial t} \right\|_{C^0(\Omega)} < \varepsilon,$$

with $\boldsymbol{\varphi}_N$ given in (2.14). Moreover, $\{\boldsymbol{\psi}_k\}$ can be chosen to be an orthonormal basis in $H(\Omega)$.

Proof. Let $H^m = H^m(\Omega)$ be the completion of $\mathcal{D}(\Omega)$ in the norm $\|\cdot\|_m$ of the Sobolev space $\boldsymbol{W}^{m,2}(\Omega)$ and let $\{\boldsymbol{\Phi}_r\}$ be a basis of H^m constituted by elements of $\mathcal{D}(\Omega)$. [11] For arbitrary $\eta > 0$, let $0 \equiv t_0 < t_1 < \ldots < t_n \equiv T$ be a partition of $[0, T]$ such that

$$\|\boldsymbol{\varphi}(t') - \boldsymbol{\varphi}(t'')\|_m < \eta, \quad t', t'' \in [t_{k-1}, t_k]. \tag{2.15}$$

Denoting by $(\cdot,\cdot)_m$ the scalar product in H^m and setting

$$\boldsymbol{\varphi}_l(x, t) = \sum_{r=1}^{l} (\boldsymbol{\varphi}, \boldsymbol{\Phi}_r)_m \boldsymbol{\Phi}_r(x),$$

[10] If $G \in C([0, T))$, equation $(2.2')$ is obtained from $(2.8')$ after multiplying this latter by $\gamma'(t)$ and integrating by parts.

[11] This is always possible, owing to the separability of $\boldsymbol{W}^{m,2}$.

we have

$$\lim_{l\to\infty} \|\boldsymbol{\varphi}_l(t) - \boldsymbol{\varphi}(t)\|_m = 0, \quad \text{for all} \quad t \in [0, T], \tag{2.16}$$

and so, by the Schwarz inequality, we find for all $t \in [t_{k-1}, t_k]$

$$\|\boldsymbol{\varphi}_l(t) - \boldsymbol{\varphi}_l(t_k)\|_m \leq \|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(t_k)\|_m < \eta. \tag{2.17}$$

Thus, from (2.15)-(2.17), for $t \in [t_{k-1}, t_k]$ and sufficiently large l we find

$$\begin{aligned} \|\boldsymbol{\varphi}_l(t) - \boldsymbol{\varphi}(t)\|_m &\leq \|\boldsymbol{\varphi}_l(t) - \boldsymbol{\varphi}_l(t_k)\|_m + \|\boldsymbol{\varphi}_l(t_k) - \boldsymbol{\varphi}(t_k)\|_m \\ &+ \|\boldsymbol{\varphi}(t) - \boldsymbol{\varphi}(t_k)\|_m < 3\eta. \end{aligned}$$

Choosing $m > n/2$, by the Sobolev embedding theorem we conclude

$$\max_{t\in[0,T]} \|\boldsymbol{\varphi}_l(t) - \boldsymbol{\varphi}(t)\|_{C^2(\Omega)} < C\eta$$

with $C = C(\Omega, m, n)$. Moreover, for all $t \in [0, T]$, it is

$$\lim_{l\to\infty} \left\| \frac{\partial \boldsymbol{\varphi}_l(t)}{\partial t} - \frac{\partial \boldsymbol{\varphi}(t)}{\partial t} \right\|_m = 0,$$

and so, by the same kind of argument used before, we show

$$\lim_{l\to\infty} \max_{t\in[0,T]} \left\| \frac{\partial \boldsymbol{\varphi}_l(t)}{\partial t} - \frac{\partial \boldsymbol{\varphi}(t)}{\partial t} \right\|_{C^0(\Omega)} = 0.$$

To the set $\{\boldsymbol{\Phi}_r\}$ we can apply the Schmidt orthogonalization procedure in L^2, thus obtaining another system $\{\boldsymbol{\psi}_r\}$, whose generic element is a linear combination of $\boldsymbol{\Phi}_1, \ldots, \boldsymbol{\Phi}_\ell$, $\ell = \ell(r)$. Since H^m is dense in H, it is easy to show that $\{\boldsymbol{\psi}_r\}$ satisfies all requirements stated in the lemma which, consequently, is proved.

From what we have shown, we deduce the following result.

Lemma 2.4 *A measurable function $\boldsymbol{v} : \Omega_T \to \mathbb{R}^n$ is a weak solution of the problem (0.1)-(0.4) in Ω_T if and only if*

a) $\boldsymbol{v} \in V_T$*;*

b) $\boldsymbol{v}$ *verifies (2.8), for all $t \in [0, T)$ and all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$.*

Remark 2.7 Differentiating (2.8) with respect to t and recalling that $\boldsymbol{v} \in V_T$, we find

$$\frac{d}{dt}(\boldsymbol{v}(t), \boldsymbol{\psi}) = -\nu(\nabla \boldsymbol{v}(t), \nabla \boldsymbol{\psi}) - (\boldsymbol{v}(t) \cdot \nabla \boldsymbol{v}(t), \boldsymbol{\psi}) + (\boldsymbol{f}, \boldsymbol{\psi}) \tag{2.18}$$

for a.a. $t \in [0, T)$ and all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$. It is easily seen that the right-hand side of (2.18) defines a linear, bounded functional in $\boldsymbol{\psi} \in H^1(\Omega)$. In fact, denoting by $\mathcal{F}$ such a functional, by the Schwarz inequality and by (2.12) we have,

$$\begin{aligned} |\mathcal{F}(\boldsymbol{\psi})| &\leq (\nu\|\nabla \boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_4^2 + \|\boldsymbol{f}\|_2)\, \|\boldsymbol{\psi}\|_{1,2} \\ &\leq c\left(\|\nabla \boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_2^{\alpha}\|\nabla \boldsymbol{v}\|_2^{\beta}\right) \|\boldsymbol{\psi}\|_{1,2} \end{aligned} \tag{2.19}$$

where

$$\alpha = \beta = 1, \quad \text{if } n = 2$$

$$\alpha = 1/2, \ \beta = 3/2, \quad \text{if } n = 3.$$

Thus, denoting by $H^{-1}(\Omega)$ the dual space of $H^1(\Omega)$, for almost all $t \in [0, T)$, there exists $\boldsymbol{v}_t \in H^{-1}(\Omega)$ such that

$$\frac{d}{dt}(\boldsymbol{v}(t), \boldsymbol{\psi}) = \langle \boldsymbol{v}_t, \boldsymbol{\psi} \rangle, \quad \boldsymbol{\psi} \in H^1(\Omega),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H^{-1} and H^1. Notice that $\boldsymbol{v}_t$ is in H^{-1} but *not* necessarily in $W^{-1,2}$. Moreover, by (2.18), (2.19) and the assumption $\boldsymbol{v} \in V_T$, we find

$$\langle \boldsymbol{v}_t, \boldsymbol{\psi} \rangle = -\nu(\nabla \boldsymbol{v}(t), \nabla \boldsymbol{\psi}) - (\boldsymbol{v}(t) \cdot \nabla \boldsymbol{v}(t), \boldsymbol{\psi}) + (\boldsymbol{f}, \boldsymbol{\psi}), \quad \boldsymbol{v}_t \in L^{\sigma}(0, T; H^{-1}(\Omega)),$$

where $\sigma = 2$ if $n = 2$ and $\sigma = 4/3$ if $n = 3$.

Remark 2.8 The method of proof used for Lemma 2.3 enables us to give a density result which will be used several times in the next sections. To this end, we recall standard facts concerning the theory of approximation of functions. Let $w \in L^q(0, T; X)$, $1 \leq q < \infty$. For $T > h > 0$, the mollifier w_h (in the sense of Friederichs) of w is defined by

$$w_h(t) = \int_0^T j_h(t - s) w(s) ds \tag{2.20}$$

where $j_h(s)$ is an even, positive, infinitely differentiable function with support in $(-h, h)$, and $\int_{-\infty}^{\infty} j_h(s) ds = 1$. We have (see, $e.g.$, Hille and Phillips, 1957)

Lemma 2.5 Let $w \in L^q(0,T;X)$, $1 \le q < \infty$. Then $w_h \in C^k([0,T];X)$, for all $k \ge 0$. Moreover

$$\lim_{h\to 0} \|w_h - w\|_{L^q(0,T;X)} = 0.$$

Finally, if $\{w_k\} \subset L^q(0,T;X)$ converges to w in the norm of $L^q(0,T;X)$ then

$$\lim_{k\to\infty} \|(w_k)_h - w_h\|_{L^q(0,T;X)} = 0.$$

We also have.

Lemma 2.6 $\mathcal{D}_T$ *is dense in* $L^2(0,T;H^1(\Omega))$.

Proof. Let $\{\boldsymbol{\Phi}_r\} \subset \mathcal{D}(\Omega)$ be a basis of H^1 and let $\boldsymbol{w} \in L^2(0,T;X)$. Denoting by $(\cdot,\cdot)_1$ the scalar product in H^1, and setting

$$\boldsymbol{w}_{l,h}(x,t) = \sum_{r=1}^{l} (\boldsymbol{w}_h, \boldsymbol{\Phi}_r)_1 \boldsymbol{\Phi}_r(x),$$

we have

$$\lim_{l\to\infty} \|\boldsymbol{w}_{l,h}(t) - \boldsymbol{w}_h(t)\|_1 = 0, \quad \text{for all} \quad t \in [0,T] \text{ and } h < T. \tag{2.21}$$

Clearly, $\boldsymbol{w}_{l,h} \in \mathcal{D}_T$. By Lemma 2.5, for a given $\varepsilon > 0$, there is $h > 0$

$$\int_0^T \|\boldsymbol{w}_h(t) - \boldsymbol{w}(t)\|_1^2 < \varepsilon.$$

On the other hand, from (2.21) and the Lebesgue dominated convergence theorem, we have for all fixed h

$$\lim_{l\to\infty} \int_0^T \|\boldsymbol{w}_{l,h}(t) - \boldsymbol{w}_h(t)\|_1^2 = 0.$$

Thus the result follows from the last two displayed relations and the triangle inequality.

3 Existence of Weak Solutions.

The aim of this section is to prove the following existence theorem of weak solutions.

Theorem 3.1 *Let Ω be any domain in $\mathbb{R}^n$ and let $T > 0$. Then, for any given*

$$\boldsymbol{v}_0 \in H(\Omega), \quad \boldsymbol{f} \in \boldsymbol{L}^2(\Omega_T),$$

there exists at least one weak solution to (0.1)-(0.3) in Ω_T. This solution verifies, in addition, the following properties

i) The energy inequality:

$$\|\boldsymbol{v}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 d\tau \le 2 \int_0^t (\boldsymbol{v}(\tau), \boldsymbol{f}(\tau)) d\tau + \|\boldsymbol{v}_0\|_2^2, \quad t \in [0, T]. \tag{EI}$$

ii) $\lim_{t \to 0} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 = 0.$

Proof. We shall use the so called "Faedo-Galerkin" method. Let $\{\boldsymbol{\psi}_r\} \subset \mathcal{D}(\Omega)$ be the basis of $H(\Omega)$ given in Lemma 2.3. We shall look for approximating solutions $\boldsymbol{v}_k$ of the form

$$\boldsymbol{v}_k(x,t) = \sum_{r=1}^{k} c_{kr}(t) \boldsymbol{\psi}_r(x), \quad k \in \mathbb{N}, \tag{3.1}$$

where the coefficients c_{kr} are required to satisfy the following system of ordinary differential equations

$$\frac{dc_{kr}}{dt} + \sum_{i=1}^{k} a_{ir} c_{ki} + \sum_{i,s=1}^{k} a_{isr} c_{ki} c_{ks} = f_r, \quad r = 1, \dots, k, \tag{3.2}$$

with the initial condition

$$c_{kr}(0) = C_{0r} \quad r = 1, \dots, k, \tag{3.3}$$

where

$$a_{ir} = \nu(\nabla \boldsymbol{\psi}_i, \nabla \boldsymbol{\psi}_r), \quad a_{isr} = (\boldsymbol{\psi}_i \cdot \nabla \boldsymbol{\psi}_s, \nabla \boldsymbol{\psi}_r),$$

$$f_r = (\boldsymbol{f}, \boldsymbol{\psi}_r), \quad C_{0r} = (\boldsymbol{v}_0, \boldsymbol{\psi}_r).$$

Since $f_r \in L^2(0,T)$ for all $r = 1, \dots, k$, from the elementary theory of ordinary differential equations, we know that problem (3.1)–(3.3) admits a unique solution $c_{kr} \in W^{1,2}(0, T_k)$, $r = 1, \dots, k$, where $T_k \le T$. Multiplying (3.2) by c_{kr}, summing over r and employing the orthonormality conditions on $\{\boldsymbol{\psi}_r\}$ along with the identity

$$(\boldsymbol{\psi} \cdot \nabla \boldsymbol{\psi}, \boldsymbol{\psi}) = 0, \quad \text{for all } \boldsymbol{\psi} \in \mathcal{D}(\Omega),$$

we obtain for all $t \in [0,T)$

$$\|\boldsymbol{v}_k(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{v}_k(\tau)\|_2^2 d\tau = 2\int_0^t (\boldsymbol{v}_k(\tau), \boldsymbol{f}(\tau))d\tau + \|\boldsymbol{v}_{0k}\|_2^2 \tag{3.4}$$

with $\boldsymbol{v}_{0k} = \boldsymbol{v}_k(0)$. Since $\|\boldsymbol{v}_{0k}\|_2 \leq \|\boldsymbol{v}_0\|_2$, Using in (3.4) the Schwarz inequality along with Gronwall's lemma, we easily deduce the following bound

$$\|\boldsymbol{v}_k(t)\|_2^2 + \int_0^t \|\nabla \boldsymbol{v}_k(\tau)\|_2^2 d\tau \leq M, \quad \text{for all } t \in [0,T) \tag{3.5}$$

with M independent of t and k. From this inequality it follows, in particular, that $|c_{kr}(t)| \leq M^{1/2}$ for all $r = 1, \ldots, k$ which in turn, by standard results on ordinary differential equations, implies $T_k = T$, for all $k \in \mathbb{N}$. We shall now investigate the properties of convergence of the sequence $\{\boldsymbol{v}_k\}$ when $k \to \infty$. To this end, we begin to show that, for any fixed $r \in \mathbb{N}$, the sequence of functions

$$G_k^{(r)}(t) \equiv (\boldsymbol{v}_k(t), \boldsymbol{\psi}_r)$$

is uniformly bounded and uniformly continuous in $t \in [0,T]$. The uniform boundedness follows at once from (3.5). To show the uniform continuity, we observe that from (3.2), (3.5), with the help of the Schwarz inequality it easily follows that

$$\begin{aligned} |G_k^{(r)}(t) - G_k^{(r)}(s)| \leq & S_1 \int_s^t \left(\|\nabla \boldsymbol{v}_k(\tau)\|_2 + \|\boldsymbol{f}(\tau)\|_2\right) d\tau \\ & + S_2 M^{1/2} \int_s^t \|\nabla \boldsymbol{v}_k(\tau)\|_2 d\tau, \end{aligned} \tag{3.6}$$

where

$$S_1 \equiv \|\boldsymbol{\psi}_r\|_2, \quad S_2 = \max_{x \in \Omega} |\boldsymbol{\psi}_r(x)|.$$

Thus, using the Schwarz inequality into (3.6) and recalling (3.5), we readily show the equicontinuity of $G_k^{(r)}(t)$. By the Ascoli-Arzelà theorem, from the sequence $\{G_k^{(r)}(t)\}_{k\in\mathbb{N}}$ we may then select a subsequence –which we continue to denote by $\{G_k^{(r)}(t)\}_{k\in\mathbb{N}}$– uniformly converging to a continuous function $G^{(r)}(t)$. The selected sequence $\{G_k^{(r)}(t)\}_{k\in\mathbb{N}}$ may depend on r. However, using the classical Cantor diagonalization method, we end up with a sequence –again denoted by $\{G_k^{(r)}(t)\}_{k\in\mathbb{N}}$– converging to $G^{(r)}$, for all $r \in \mathbb{N}$, uniformly in t$\in [0,T)$. This information, together with (3.5) and the weak compactness of the space H, allows us to infer the existence of $\boldsymbol{v}(t) \in H(\Omega)$ such that

$$\lim_{k\to\infty} (\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{\psi}_r) = 0 \quad \text{uniformly in } t \in [0,T] \text{ and for all } r \in \mathbb{N}. \tag{3.7}$$

Let us now prove that $\boldsymbol{v}_k(t)$ converges weakly in $\boldsymbol{L}^2$ to $\boldsymbol{v}(t)$, uniformly in $t \in [0,T]$, that is,

$$\lim_{k\to\infty} (\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{u}) = 0, \quad \text{uniformly in } t \in [0,T] \text{ and for all } \boldsymbol{u} \in \boldsymbol{L}^2(\Omega). \tag{3.8}$$

By the Helmholtz-Weyl orthogonal decomposition (2.1), it is enough to show (3.8) for $\boldsymbol{u} \in H(\Omega)$. To this end, writing

$$\boldsymbol{u} = \sum_{r=1}^{\infty} u_r \boldsymbol{\psi}_r \equiv \sum_{r=1}^{N} u_r \boldsymbol{\psi}_r + \boldsymbol{u}^{(N)}$$

and using the Schwarz inequality together with (3.5), we find

$$\begin{aligned} |(\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{u})| &\le \sum_{r=1}^{N} |(\boldsymbol{v}_k(t) - \boldsymbol{v}(t), u_r \boldsymbol{\psi}_r)| + |(\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{u}^{(N)})| \\ &\le \sum_{r=1}^{N} \|\boldsymbol{u}\|_2 |(\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{\psi}_r)| + 2M^{1/2} \|\boldsymbol{u}^{(N)}\|_2. \end{aligned}$$

For $\varepsilon > 0$, we choose N so large that

$$\|\boldsymbol{u}^{(N)}\|_2 < \varepsilon.$$

Further, by (3.6) we can pick $k = k(\boldsymbol{u}, \varepsilon)$ so that

$$\sum_{r=1}^{N} \|\boldsymbol{u}\|_2 |(\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{\psi}_r)| < \varepsilon,$$

and (3.8) follows from (3.7) and the last two displayed inequalities. In view of (3.5) we clearly have $\boldsymbol{v} \in L^\infty(0,T; H(\Omega))$. Again from (3.5), by the weak compactness of the space $L^2(\Omega_T)$ it follows the existence of $\tilde{\boldsymbol{v}} \in L^2(0,T; H^1(\Omega))$ such that for $m = 1, \dots, n$ (with $\partial_m = \partial/\partial x_m$)

$$\lim_{k\to\infty} \int_0^T (\boldsymbol{v}_k - \tilde{\boldsymbol{v}}, \boldsymbol{w}) ds = \lim_{k\to\infty} \int_0^T (\partial_m(\boldsymbol{v}_k - \tilde{\boldsymbol{v}}), \boldsymbol{w}) ds = 0, \text{ for all } \boldsymbol{w} \in \boldsymbol{L}^2(\Omega_T).$$

Choosing in this inequality $\boldsymbol{w} \in \mathcal{D}_T$ and using (3.8), it is easy to show that $\boldsymbol{v} = \widetilde{\boldsymbol{v}}$. Thus, in particular, we find

$$\lim_{k\to\infty}\int_0^T (\partial_m(\boldsymbol{v}_k - \boldsymbol{v}), \boldsymbol{w})ds = 0, \quad \text{for all } \boldsymbol{w} \in \boldsymbol{L}^2(\Omega_T), \quad m = 1, \ldots, n. \tag{3.9}$$

We wish now to prove that (3.8) and (3.9) imply the *strong* convergence of $\{\boldsymbol{v}_k\}$ to $\boldsymbol{v}$ in $\boldsymbol{L}^2(\omega \times [0,T])$, for all $\omega \subset\subset \Omega$. To show this, we need the following *Friederichs inequality*, see, $e.g.$, Galdi (1994, Lemma II.4.2): *Let C be a cube in $\mathbb{R}^n$, then for any $\eta > 0$, there exist $K(\eta, C) \in \mathbb{N}$ functions $\boldsymbol{\omega}_i \in L^\infty(C)$, $i = 1, \ldots, K$ such that*

$$\int_0^T \|\boldsymbol{w}(t)\|_{2,C}^2 dt \le \sum_{i=1}^K \int_0^T (\boldsymbol{w}(t), \boldsymbol{\omega}_i)_C^2 dt + \eta \int_0^T \|\nabla \boldsymbol{w}(t)\|_{2,C}^2 dt.$$

If we apply this inequality with $\boldsymbol{w} \equiv \boldsymbol{v}_k - \boldsymbol{v}$ and use (3.5), (3.7) we find

$$\lim_{k\to\infty}\int_0^T \|\boldsymbol{v}_k(t) - \boldsymbol{v}(t)\|_{2,C}^2 dt = 0 \tag{3.10}$$

With the help of (3.8)-(3.10), we shall now show that $\boldsymbol{v}$ is a weak solution to (0.1)-(0.3). Since we already proved that $\boldsymbol{v} \in V_T$, by Lemma 2.4, it remains to show that $\boldsymbol{v}$ satisfies (2.8). Integrating (3.1) from 0 to $t \le T$ we find

$$\begin{aligned}\int_0^t \{-\nu(\nabla \boldsymbol{v}_k, \nabla \boldsymbol{\psi}_r) - (\boldsymbol{v}_k \cdot \nabla \boldsymbol{v}_k, \boldsymbol{\psi}_r)\}\, ds \\ = -\int_0^t (\boldsymbol{f}, \boldsymbol{\psi}_r)ds + (\boldsymbol{v}_k(t), \boldsymbol{\psi}_r) - (\boldsymbol{v}_0, \boldsymbol{\psi}_r).\end{aligned} \tag{3.11}$$

From (3.8), (3.9) we at once obtain

$$\lim_{k\to\infty}(\boldsymbol{v}_k(t) - \boldsymbol{v}(t), \boldsymbol{\psi}_r) = 0, \quad \lim_{k\to\infty}\int_0^t (\nabla \boldsymbol{v}_k(s) - \nabla \boldsymbol{v}(s), \boldsymbol{\psi}_r)ds = 0. \tag{3.12}$$

Furthermore, denoting by C a cube containing the support of $\boldsymbol{\psi}_r$, we have

$$\begin{aligned}&\left|\int_0^t (\boldsymbol{v}_k \cdot \nabla \boldsymbol{v}_k, \boldsymbol{\psi}_r) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{\psi}_r)ds\right| \\ &\quad \le \left|\int_0^t ((\boldsymbol{v}_k - \boldsymbol{v}) \cdot \nabla \boldsymbol{v}_k, \boldsymbol{\psi}_r)_C ds\right| + \left|\int_0^t (\boldsymbol{v} \cdot \nabla(\boldsymbol{v}_k - \boldsymbol{v}), \boldsymbol{\psi}_r)_C ds\right|.\end{aligned} \tag{3.13}$$

Setting $S = \max_{x\in C}|\boldsymbol{\psi}(x)|$, by (3.5) we also have

$$\left|\int_0^t ((\boldsymbol{v}_k - \boldsymbol{v})\cdot\nabla\boldsymbol{v}_k, \boldsymbol{\psi}_r)_C\right| \leq S\left(\int_0^t \|\boldsymbol{v}_k - \boldsymbol{v}\|_{2,C}^2 dt\right)^{1/2}\left(\int_0^t \|\nabla\boldsymbol{v}_k\|_2^2 dt\right)^{1/2}$$
$$\leq SM^{1/2}\left(\int_0^t \|\boldsymbol{v}_k - \boldsymbol{v}\|_{2,C}^2 dt\right)^{1/2}$$

and so, using (3.10), we find

$$\lim_{k\to\infty}\left|\int_0^t ((\boldsymbol{v}_k - \boldsymbol{v})\cdot\nabla\boldsymbol{v}_k, \boldsymbol{\psi}_r)ds\right| = 0. \tag{3.14}$$

Furthermore, we have

$$\left|\int_0^t (\boldsymbol{v}\cdot\nabla(\boldsymbol{v}_k - \boldsymbol{v}), \boldsymbol{\psi}_r)_C ds\right| \leq \sum_{m=1}^n \left|\int_0^t (\partial_m(\boldsymbol{v}_k - \boldsymbol{v}), v_m\boldsymbol{\psi}_r)_C ds\right|$$

and since $v_m\boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega_T)$, from (3.9) we deduce

$$\lim_{k\to\infty}\left|\int_0^t (\boldsymbol{v}\cdot\nabla(\boldsymbol{v}_k - \boldsymbol{v}), \boldsymbol{\psi}_r)_C ds\right| = 0. \tag{3.15}$$

Therefore, passing into the limit $k \to \infty$ in (3.11), from (3.12)-(3.15) we conclude

$$\begin{aligned}\int_0^t \{-\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{\psi}_r) - (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{\psi}_r)\}\, ds \\ = -\int_0^t (\boldsymbol{f}, \boldsymbol{\psi}_r)ds + (\boldsymbol{v}(t), \boldsymbol{\psi}_r) - (\boldsymbol{v}_0, \boldsymbol{\psi}_r).\end{aligned} \tag{3.16}$$

However, from Lemma 2.3 we know that every function $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$ can be uniformly approximated in $C^2(\overline{\Omega})$ by functions of the form

$$\boldsymbol{\psi}_N(x) = \sum_{r=1}^N \gamma_r\boldsymbol{\psi}_r(x), \quad N \in \mathbb{N},\ \gamma_r \in \mathbb{R}.$$

So, writing (3.16) with $\boldsymbol{\psi}_N$ in place of $\boldsymbol{\psi}$, we may pass to the limit $N \to \infty$ in this new relation and use the fact that $\boldsymbol{v} \in V_T$ to show the validity of (2.8) for all $\boldsymbol{\psi} \in \mathcal{D}(\Omega)$. We shall now prove the energy inequality (EI). To this end, we shall take the $\liminf$ as $k \to \infty$ of both sides of (3.4). By the definition of $\boldsymbol{v}_{0k}$, the properties of $\boldsymbol{f}$, and (3.8) we deduce

$$\lim_{k\to\infty}\left\{\int_0^t(\boldsymbol{v}_k(\tau),\boldsymbol{f}(\tau))d\tau+\|\boldsymbol{v}_{0k}\|_2^2\right\}=\int_0^t(\boldsymbol{v}(\tau),\boldsymbol{f}(\tau))d\tau+\|\boldsymbol{v}_0\|_2^2.$$

Moreover, by (3.8), (3.9) and a classical property of weak limits, we find that

$$\liminf_{k\to\infty}\left\{\|\boldsymbol{v}_k(t)\|_2^2+2\nu\int_0^t\|\nabla\boldsymbol{v}_k(\tau)\|_2^2d\tau\right\}\geq\|\boldsymbol{v}(t)\|_2^2+2\nu\int_0^t\|\nabla\boldsymbol{v}(\tau)\|_2^2d\tau, \tag{3.17}$$

and (EI) follows from (3.4) and the last two displayed relations. From (EI) we deduce at once

$$\limsup_{t\to 0}\|\boldsymbol{v}(t)\|_2^2\leq\|\boldsymbol{v}_0\|_2^2.$$

On the other hand, $\boldsymbol{v}(t)$ is weakly continuous in $\boldsymbol{L}^2$ (see Lemma 2.2), and so we have

$$\liminf_{t\to 0}\|\boldsymbol{v}(t)\|_2^2\geq\|\boldsymbol{v}_0\|_2^2,$$

which implies

$$\lim_{t\to 0}\|\boldsymbol{v}(t)\|_2^2=\|\boldsymbol{v}_0\|_2^2.$$

This relation together with the $\boldsymbol{L}^2$ weak continuity of $\boldsymbol{v}$ allows us to conclude

$$\lim_{t\to 0}\|\boldsymbol{v}(t)-\boldsymbol{v}_0\|_2=0,$$

and the theorem is thus proved.

Remark 3.1 In the literature, one may find many different definitions of weak solution (see, *e.g.*, Lions, 1969; Masuda, 1984; von Wahl (1985)). The one chosen here is due to Leray and Hopf. Likewise, there are many different constructive procedures of weak solutions (see Leray 1934a, 1934b; Kiselev and Ladyzhenskaya, 1957; Shinbrot, 1973). Since, as we shall see in the next section, there is no uniqueness guaranteed for weak solutions in dimension 3 (or higher), these procedures may conceivably lead to different solutions.

4 The Energy Equality and Uniqueness of Weak Solutions.

An interesting feature of weak solutions that should be emphasized, is that they obey only an energy *inequality* rather than the energy *equality* (that is (EI) with the equality sign), as should be expected from the physical point of view. To analyze this fact in more detail, let us take, for simplicity, $\boldsymbol{f} \equiv 0$. Then, any "physically reasonable" solution should be such that the associated kinetic energy $E(t)$ at a certain time t ($= \frac{1}{2}\|\boldsymbol{v}(t)\|_2^2$) is equal to $E(\sigma)$ ($\sigma < t$) minus the amount of energy dissipated by viscosity in the time interval $[\sigma, t]$ ($= \nu \int_\sigma^t \|\nabla \boldsymbol{v}(\tau)\|^2 d\tau$). According to (EI), however, a weak solution not only does not satisfy *a priori* this property but, in fact, its kinetic energy could even *increase* in certain time intervals. Therefore, a first question to ask is if it is possible to construct weak solutions for which the corresponding kinetic energy is a decreasing function of time. To this end, it would be enough that weak solutions would satisfy the following relation

$$\|\boldsymbol{v}(t)\|_2^2 + 2\nu \int_\sigma^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 d\tau \le \|\boldsymbol{v}(\sigma)\|_2^2, \qquad \text{for almost all } \sigma \ge 0, \text{ and all } t \in [\sigma, T). \tag{4.1}$$

Inequality (4.1) is usually called the *strong energy inequality* (SEI).

It is easy to see that if Ω is bounded, then the solutions constructed in Theorem 3.1 obey the (SEI). In fact, from (3.10), by taking $C \supset \Omega$, it follows that

$$\lim_{k\to\infty} \|\boldsymbol{v}_k(\sigma) - \boldsymbol{v}(\sigma)\|_2 = 0, \quad \text{for almost all } \sigma \in [0, T). \tag{4.2}$$

On the other hand, from (3.4) (with $\boldsymbol{f} \equiv 0$) we have

$$\|\boldsymbol{v}_k(t)\|_2^2 + 2\nu \int_\sigma^t \|\nabla \boldsymbol{v}_k(\tau)\|_2^2 d\tau = \|\boldsymbol{v}_k(\sigma)\|_2^2, \qquad \text{for all } \sigma \in [0, T) \text{ and } t \in [\sigma, T).$$

and so, passing to the limit $k \to \infty$ in this relation and using (3.17) (with 0 replaced by σ) and (4.2), we recover (SEI). With much more effort, one can show existence of weak solutions obeying (SEI) when Ω is either the whole $\mathbb{R}^n$ (Leray 1934b), or an exterior domain (Galdi and Maremonti, 1986; Sohr, von Wahl and Wiegner, 1986; Miyakawa and Sohr, 1988), or a half space (Borchers and Miyakawa, 1988). It is interesting to observe that all proofs given by these authors rely on a certain estimate for the pressure field, which implies, in particular, the following property for p:

$$p \in L^r(0, T; L^q(\Omega)), \quad \text{for suitable exponents } r, q.$$

This is much more than the regularity property proved in Theorem 2.1. On the other hand, one knows how to prove this estimate only for a certain type of domains and, therefore, *it is not known if* (SEI) *holds for an arbitrary* Ω *(no matter how smooth).*

The strong energy inequality, even though more reasonable than the energy inequality, still presents an undesired feature, in all time intervals I (*if any*) where it holds as a *strict* inequality. Actually, in any of such intervals, the kinetic energy *is* decreased by a certain amount, say M_I, which is *not* due to the dissipation. It seems therefore interesting to furnish sufficient conditions on a weak solution in order that it verifies an *energy equality* and to compare them with those ensuring its *uniqueness*. As we shall see, the former (see Theorem 4.1) are weaker than the latter (see Theorem 4.2), and they are both verified by a weak solution in dimension 2, but not *a priori* in dimension 3. Thus, the question of the existence of a three dimensional weak solution which 1) satisfies the energy equality and 2) is unique, remains open.

In this section we provide conditions on a weak solution under which 1) and 2) above are met. The following theorem holds.

Theorem 4.1 *Let* $\boldsymbol{v}$ *be a weak solution in* Ω_T. *Assume*

$$\boldsymbol{v} \in L^4(0,T;\boldsymbol{L}^4(\Omega)). \tag{4.3}$$

Then $\boldsymbol{v}$ *verifies the energy equality*

$$\|\boldsymbol{v}(t)\|_2^2 + 2\nu\int_0^t \|\nabla\boldsymbol{v}(\tau)\|_2^2 d\tau = 2\int_0^t (\boldsymbol{v}(\tau),\boldsymbol{f}(\tau))d\tau + \|\boldsymbol{v}_0\|_2^2, \tag{4.4}$$

for all $t \in [0,T)$.

Proof. Let $\{\boldsymbol{v}_k\} \subset \mathcal{D}_T$ be a sequence converging to $\boldsymbol{v}$ in $L^2(0,T;H^1(\Omega))$, see Lemma 2.6. We choose in (2.4) (with $s=0$) $\boldsymbol{\varphi} = (\boldsymbol{v}_k)_h \equiv \boldsymbol{v}_{h,k}$, where $(\,\cdot\,)_h$ is the mollification operator defined in (2.20), see Lemma 2.5. Observing that

$$\begin{aligned}\left|\int_0^T \{(\boldsymbol{v}\cdot\,\nabla\boldsymbol{v}_{k,h},\boldsymbol{v}) - (\boldsymbol{v}\cdot\nabla\boldsymbol{v}_h,\boldsymbol{v})\}\,dt\right| & \\ \le \int_0^T \|\boldsymbol{v}\|_4^2\|\nabla(\boldsymbol{v}_{k,h}-\boldsymbol{v}_h)\|_2 & \\ \le \|\boldsymbol{v}_{k,h}-\boldsymbol{v}_h\|_{L^2(0,T;H^1)}\left(\int_0^T \|\boldsymbol{v}(t)\|_4^4 dt\right)^{1/2}, &\end{aligned} \tag{4.5}$$

by a standard procedure which makes use of Lemma 2.5, we find in the limit $k \to \infty$

$$\int_0^t \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{v}_h}{\partial t} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{v}_h) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}_h, \boldsymbol{v}) \right\} ds$$

$$= -\int_0^t (\boldsymbol{f}, \boldsymbol{v}_h) ds + (\boldsymbol{v}(t), \boldsymbol{v}_h(t)) - (\boldsymbol{v}_0, (\boldsymbol{v}_0)_h). \tag{4.6}$$

Since the kernel $j_h(s)$ in (2.20) is even in $(-h, h)$, we obtain

$$\int_0^t \left(\boldsymbol{v}, \frac{\partial \boldsymbol{v}_h}{\partial t} \right) = \int_0^t \int_0^t \frac{dj_h(t-t')}{dt} \left(\boldsymbol{v}(t), \boldsymbol{v}(t') \right) dt dt' = 0.$$

Moreover, by Lemma 2.5 and (4.5) with $\boldsymbol{v}_h$ in place of $\boldsymbol{v}_{k,h}$ and $\boldsymbol{v}$ in place of $\boldsymbol{v}_h$, respectively, we obtain

$$\lim_{h \to 0} \int_0^t (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}_h) ds = \int_0^t (\nabla \boldsymbol{v}, \nabla \boldsymbol{v}) ds$$

$$\lim_{h \to 0} \int_0^t (\boldsymbol{f}, \boldsymbol{v}_h) ds = \int_0^t (\boldsymbol{f}, \boldsymbol{v}) ds$$

$$\lim_{h \to 0} \int_0^T (\boldsymbol{v} \cdot \nabla \boldsymbol{v}_h, \boldsymbol{v}) ds = \int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}) ds.$$

Now, $\boldsymbol{v}(t) \in H^1(\Omega)$, for a.a. $t \in [0, T)$ and so, for any such fixed t, denoting by $\{\boldsymbol{\psi}_k\}$ a sequence from $\mathcal{D}(\Omega)$ converging to $\boldsymbol{v}$ in H^1 we have

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}_k, \boldsymbol{\psi}_k)| \le |(\boldsymbol{v} \cdot \nabla \boldsymbol{v}, (\boldsymbol{v} - \boldsymbol{\psi}_k))| + |(\boldsymbol{v} \cdot \nabla(\boldsymbol{v} - \boldsymbol{\psi}_k), \boldsymbol{v})|$$

$$\le +\|\boldsymbol{v}\|_4 \|\nabla \boldsymbol{v}\|_2 \|\boldsymbol{v} - \boldsymbol{\psi}_k\|_4 + \|\boldsymbol{v}\|_4^2 \|\nabla(\boldsymbol{v} - \boldsymbol{\psi}_k)\|_2.$$

By the Sobolev embedding theorem it follows that [12]

$$\|u\|_4 \le c \left(\|u\|_2 + \|\nabla u\|_2 \right), \quad u \in W^{1,2}(\Omega)$$

and so we deduce

$$\lim_{k \to \infty} (\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}_k, \boldsymbol{\psi}_k) = (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}).$$

However, since $\boldsymbol{v}(t) \in H(\Omega)$ for a.a. t, we get

$$(\boldsymbol{v} \cdot \nabla \boldsymbol{\psi}_k, \boldsymbol{\psi}_k) = \tfrac{1}{2}(\boldsymbol{v}, \nabla(\boldsymbol{\psi}_k)^2) = 0, \quad \text{for all } k \in \mathbb{N},$$

[12] Recall that the space dimension is 2 or 3.

which furnishes
$$\int_0^t (\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{v})ds = 0.$$
Finally, by the weak L^2 continuity, and recalling that $\int_0^h j_h(s)ds = 1/2$, we have
$$\begin{aligned}(\boldsymbol{v}(t),\boldsymbol{v}_h(t)) &= \int_0^h j_h(s)\,(\boldsymbol{v}(t),\boldsymbol{v}(t+s))\,ds\\ &= \int_0^h j_h(s)\,(\|\boldsymbol{v}(t)\|_2^2 + (\boldsymbol{v}(t),(\boldsymbol{v}(t+s)-\boldsymbol{v}(t)))\,ds\\ &= \tfrac{1}{2}\|\boldsymbol{v}(t)\|_2^2 + O(h).\end{aligned}$$
Likewise,
$$(\boldsymbol{v}_0,(\boldsymbol{v}_0)_h) = \tfrac{1}{2}\|\boldsymbol{v}_0\|_2^2 + O(h).$$
Therefore, the theorem follows by letting $h\to 0$ in (4.6).

Remark 4.1 From $(2.12)_1$, for a weak solution $\boldsymbol{v}$ we have
$$\int_0^T \|\boldsymbol{v}(t)\|_4^4 dt \le c\int_0^T \|\boldsymbol{v}(t)\|_2^2\|\nabla\boldsymbol{v}(t)\|_2^2 dt < \infty, \quad n=2,$$
and so every weak solution, in dimension 2, obeys the energy equality. However, by $(2.12)_2$, we have only
$$\boldsymbol{v}\in L^{8/3}(0,T;\boldsymbol{L}^4(\Omega)), \quad n=3$$
and the question of whether a weak solution obeys the energy equality remains open.

Remark 4.2 Recalling that every weak solution is L^2 weakly continuous in time, all weak solutions satisfying (4.4) –and so, all weak solutions in dimension 2– belong to $C^0([0,T);\boldsymbol{L}^2(\Omega))$.

Remark 4.3 Serrin (1963, Theorem 5) proves (4.4) for $n=3$ under the assumption
$$\boldsymbol{v}\in L^r(0,T;\boldsymbol{L}^s(\Omega)), \quad \frac{3}{s}+\frac{2}{r}=1, \ s\in[3,\infty]. \qquad (*)$$
This condition, however, is stronger than (4.3) for any choice of r and s in their ranges. Actually, for $s=4$, it furnishes $\boldsymbol{v}\in L^8(0,T;\boldsymbol{L}^4(\Omega))$ which implies (4.3). For $s>4$, by the convexity inequality we find
$$\|\boldsymbol{v}\|_4^{2(r+4)/3} \le \|\boldsymbol{v}\|_s^r\|\boldsymbol{v}\|_2^{(8-r)/3},$$

and so $(*)$ implies (4.3), since $r \geq 2$ and $\boldsymbol{v} \in V_T$. If $s < 4$, by the Sobolev embedding theorem we have [13]

$$\|\boldsymbol{v}\|_4 \leq c\|\boldsymbol{v}\|_s^{s/2(6-s)}\|\nabla\boldsymbol{v}\|_2^{3(4-s)/2(6-s)},$$

which, by the Hölder inequality, gives

$$\int_0^T \|\boldsymbol{v}\|_4^{4(6-s)/(9-2s)}dt \leq c\left(\int_0^T \|\boldsymbol{v}\|_s^r dt\right)^{(s-3)/(9-2s)}\left(\int_0^T \|\nabla\boldsymbol{v}\|_2^2 dt\right)^{3(4-s)/(9-2s)}.$$

Since $s \geq 3$, also in this case $(*)$ implies (4.3).

Remark 4.4 The result proved in Theorem 4.1 is due to Lions (1960) and is a particular case of that stated in Shinbrot (1974, Theorem 4.4), where assumption (4.3) is replaced by the following one:

$$\boldsymbol{v} \in L^r(0,T;\boldsymbol{L}^q(\Omega)), \quad \frac{1}{r}+\frac{1}{q}=\frac{1}{2}, \quad q \geq 4.$$

However, unlike Theorem 4.1, the proof given by Shinbrot requires certain restrictions on the domain Ω (such as boundedness of its boundary) which are not explicitly formulated by the author. For related questions, we also refer to Taniuchi (1997).

Our next objective is to give sufficient conditions under which a weak solution is unique in the class of weak solutions. The basic idea is due to Leray (1934b, pp.242-244), who gave this result for the Cauchy problem ($\Omega \equiv \mathbb{R}^n$). The generalization to an arbitrary domain is due to Serrin (1963, Theorem 6). The procedure to prove uniqueness is essentially the same as that we have just used for proving the energy equality and, here as there, one approximates the solutions by a suitable sequence of functions from $\mathcal{D}_T$. The main difficulty is to show the convergence of the nonlinear terms along these sequences. Apparently, the condition $\boldsymbol{v} \in V_T$ satisfied by a weak solution does not guarantee this convergence in dimension 3, while it does in dimension 2. The following lemmas play a fundamental role in estimating the nonlinear term. The first one is a simple consequence of the Hölder and Sobolev inequalities (see Serrin 1963, Lemma 1; Masuda 1984, Lemma 2.4); the second one is a clever application of Dini's theorem on the uniform convergence of sequences of monotonically decreasing functions (Masuda, 1984, Lemma 2.7).

[13] Recall that the space dimension is 2 or 3.

Lemma 4.1 *Let r, s satisfy*

$$\frac{n}{s}+\frac{2}{r}=1,\ s\in[n,\infty].$$

and let $\boldsymbol{v},\boldsymbol{w}\in V_T$, $\boldsymbol{u}\in L^r(0,T;\boldsymbol{L}^s(\Omega))$. Then,

$$\left|\int_0^T(\boldsymbol{v}\cdot\nabla\boldsymbol{w},\boldsymbol{u})dt\right|\leq c\left(\int_0^T\|\nabla\boldsymbol{w}\|_2^2dt\right)^{1/2}\left(\int_0^T\|\nabla\boldsymbol{v}\|_2^2dt\right)^{n/2s}\left(\int_0^T\|\boldsymbol{u}\|_s^r\|\boldsymbol{v}\|_2^2dt\right)^{1/r}.$$

with the exception of the single case $s=n=2$.

Lemma 4.2 *Let $\boldsymbol{w}\in L^2(\tau,T;H^1(\Omega))$, $\boldsymbol{v}\in L^\infty(\tau,T;\boldsymbol{L}^n(\Omega))$. Assume that*

$$\int_\tau^t\|\nabla\boldsymbol{w}\|_2^2ds>0,\quad \textit{for all } t\in(\tau,T)$$

and that $\boldsymbol{v}$ is right continuous at $t=\tau$ in the L^n-norm. Then, for any $\varepsilon>0$ there exists $M=M(\boldsymbol{w},\boldsymbol{v},\varepsilon)>0$ such that

$$\left|\int_\tau^t(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{v})ds\right|\leq\varepsilon\int_\tau^t\|\nabla\boldsymbol{w}\|_2^2ds+M\int_\tau^t\|\boldsymbol{w}\|_2^2ds,\quad \textit{for all } t\in(\tau,T).$$

We also have

Lemma 4.3 *Let $\boldsymbol{v}\in V_T$. Then, there exists a sequence of functions $\{\boldsymbol{v}_k\}\subset L^2(0,T;H^1(\Omega))$ such that*

(i) $\boldsymbol{v}_k$ tends to $\boldsymbol{v}$ in $L^2(0,T;H^1(\Omega))$

(ii) $\boldsymbol{v}_k(t)\in\mathcal{D}(\Omega)$ for a.a. $t\in[0,T)$

Moreover, their mollifiers $(\boldsymbol{v}_k)_h\equiv\boldsymbol{v}_{h,k}$ ($\in\mathcal{D}_T$), see (2.20), satisfy the following properties

$$\lim_{k\to\infty}\int_0^t(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}_{h,k})ds=\int_0^t(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v}_h)ds,$$

for all $\boldsymbol{u}\in V_T$.

Proof. Let $(\ \cdot,\cdot\ )_1$ denote the scalar product in H^1. Let $\{\boldsymbol{\Phi}_r\}$ be an orthonormal basis in $H^1(\Omega)$ constituted by elements of $\mathcal{D}(\Omega)$, and set

$$\boldsymbol{v}_k(t)=\sum_{r=1}^k(\boldsymbol{v}(t),\boldsymbol{\Phi}_r)_1\boldsymbol{\Phi}_r.$$

Clearly, $\boldsymbol{v}_k$ satisfies (i), by the Lebesgue dominated convergence theorem, and (ii). Now, we have

$$\boldsymbol{v}_{h,k}(t) = \sum_{r=1}^{k} (\boldsymbol{v}_h(t), \boldsymbol{\Phi}_r)_1 \boldsymbol{\Phi}_r,$$

and

$$\lim_{k\to\infty} \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_{1,2} = 0, \quad \text{for all } t \in [0,T).$$

By the Sobolev embedding theorem and by the property of mollifiers, we also have

$$\begin{aligned} \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_4 &\le c\|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_{1,2} \le c \max_{t\in[0,T]} \|\boldsymbol{v}_h(t)\|_{1,2} \quad n = 2, \\ \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_3 &\le c\|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_{1,2} \le c \max_{t\in[0,T]} \|\boldsymbol{v}_h(t)\|_{1,2} \quad n = 3, \end{aligned} \tag{4.7}$$

from which we deduce, in particular, for all $t \in [0,T)$

$$\begin{aligned} &\lim_{k\to\infty} \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_4 = 0 \quad \text{for } n = 2 \\ &\lim_{k\to\infty} \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_3 = 0 \quad \text{for } n = 3. \end{aligned} \tag{4.8}$$

Let us first consider the case $n = 2$. We know from Remark 4.1 that $\|\boldsymbol{u}\|_4^2 \le C\|\nabla \boldsymbol{u}\|_2$ and so, by the Hölder inequality,

$$\int_0^t |(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}_{h,k} - \boldsymbol{v})| ds \le C \int_0^t \|\nabla\boldsymbol{u}\|_2^2 \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_4^2. \tag{4.9}$$

The result then follows from (4.9), (4.7), $(4.8)_1$ and the Lebesgue dominated convergence theorem. In the case $n = 3$, by the Sobolev theorem, we have $\|\boldsymbol{u}\|_6 \le c\|\nabla\boldsymbol{u}\|_2$ and, in place of (4.9), we find

$$\int_0^t |(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}_{h,k} - \boldsymbol{v})| ds \le C \int_0^t \|\nabla\boldsymbol{u}\|_2^2 \|\boldsymbol{v}_{h,k}(t) - \boldsymbol{v}_h(t)\|_3^2,$$

and the result follows as in the case $n = 2$.

We are now in a position to show the following uniqueness theorem.

Theorem 4.2 *Let $\boldsymbol{v}$, $\boldsymbol{u}$ be two weak solutions in Ω_T corresponding to the same data $\boldsymbol{v}_0$ and $\boldsymbol{f}$. Assume that $\boldsymbol{u}$ satisfies the energy inequality* (EI) *and that $\boldsymbol{v}$ satisfies at least one of the next two conditions:*

(i) $\boldsymbol{v} \in L^r(0,T;\boldsymbol{L}^s(\Omega))$, *for some* r, s *such that* $\dfrac{n}{s} + \dfrac{2}{r} = 1$, $s \in (n, \infty]$;

(ii) $\boldsymbol{v} \in L^\infty(0,T;\boldsymbol{L}^n(\Omega))$, *and* $\boldsymbol{v}(t)$ *is right continuous for* $t \in [0,T)$ *in the* L^n*-norm.*

Then $\boldsymbol{v} = \boldsymbol{u}$ *a.e. in* Ω_T*.*

Proof. Let $\{\boldsymbol{u}_{h,k}\}$ be a sequence of functions of the type introduced in the previous theorem, and let $\{\boldsymbol{v}_{h,k}\}$ be the sequence of Lemma 4.3 . We choose $\boldsymbol{\varphi} = \boldsymbol{u}_{h,k}$ in (2.4), with $s = 0$, and $\boldsymbol{\varphi} = \boldsymbol{v}_{h,k}$ in (2.4), with $s = 0$ and with $\boldsymbol{u}$ in place of $\boldsymbol{v}$. We thus obtain

$$\int_0^t \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{u}_{h,k}}{\partial \tau} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}_{h,k}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_{h,k}) \right\} d\tau$$
$$= -\int_0^t (\boldsymbol{f}, \boldsymbol{u}_{h,k}) d\tau + (\boldsymbol{v}(t), \boldsymbol{u}_{h,k}(t)) - (\boldsymbol{v}_0, (\boldsymbol{v}_0)_{h,k}), \tag{4.10}$$

$$\int_0^t \left\{ \left(\boldsymbol{u}, \frac{\partial \boldsymbol{v}_{h,k}}{\partial \tau} \right) - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_{h,k}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_{h,k}) \right\} d\tau$$
$$= -\int_0^t (\boldsymbol{f}, \boldsymbol{v}_{h,k}) d\tau + (\boldsymbol{u}(t), \boldsymbol{v}_{h,k}(t)) - (\boldsymbol{v}_0, (\boldsymbol{v}_0)_{h,k}). \tag{4.11}$$

We wish to let $k \to \infty$ in these relations. The only terms which need a little care are the nonlinear ones. From Lemma 4.1 and the assumptions made on $\boldsymbol{v}, \boldsymbol{u}$ it follows that

$$\left| \int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_{h,k} - \boldsymbol{u}_h) d\tau \right| = \left| \int_0^t (\boldsymbol{v} \cdot \nabla (\boldsymbol{u}_{h,k} - \boldsymbol{u}_h), \boldsymbol{v}) d\tau \right|$$
$$\leq C \left(\int_0^t \|\nabla (\boldsymbol{u}_{h,k} - \boldsymbol{u}_h)\|_2^2 d\tau \right)^{\frac{1}{2}}$$

where C depends on $\boldsymbol{v}$. Therefore, from Lemma 2.5, we find

$$\lim_{k \to \infty} \int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_{h,k}) d\tau = -\int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}) d\tau. \tag{4.12}$$

Moreover, from Lemma 4.3, we have

$$\lim_{k \to \infty} \int_0^t (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_{h,k}) d\tau = \int_0^t (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_h) d\tau. \tag{4.13}$$

Thus, letting $k \to \infty$ in (4.10), (4.11) and using (4.12), (4.13) and Lemma 2.5, we find

$$\int_0^t \left\{ \left(\boldsymbol{v}, \frac{\partial \boldsymbol{u}_h}{\partial \tau} \right) - \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}_h) - (\boldsymbol{v} \cdot \nabla \boldsymbol{v}, \boldsymbol{u}_h) \right\} d\tau$$
$$= - \int_0^t (\boldsymbol{f}, \boldsymbol{u}_h) d\tau + (\boldsymbol{v}(t), \boldsymbol{u}_h(t)) - (\boldsymbol{v}_0, (\boldsymbol{v}_0)_h), \tag{4.14}$$

$$\int_0^t \left\{ \left(\boldsymbol{u}, \frac{\partial \boldsymbol{v}_h}{\partial \tau} \right) - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h) - (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_h) \right\} d\tau$$
$$= - \int_0^t (\boldsymbol{f}, \boldsymbol{v}_h) d\tau + (\boldsymbol{u}(t), \boldsymbol{v}_h(t)) - (\boldsymbol{v}_0, (\boldsymbol{v}_0)_h). \tag{4.15}$$

By Fubini's theorem and the properties of the mollifier, we show

$$\int_0^t \left(\boldsymbol{v}, \frac{\partial \boldsymbol{u}_h}{\partial \tau} \right) d\tau = - \int_0^t \left(\boldsymbol{u}, \frac{\partial \boldsymbol{v}_h}{\partial \tau} \right) d\tau,$$

and so, adding (4.14) and (4.15) furnishes

$$- \int_0^t \{ \nu(\nabla \boldsymbol{v}, \nabla \boldsymbol{u}_h) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}_h) - (\boldsymbol{v} \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}_h) \} d\tau$$
$$= - \int_0^t (\boldsymbol{f}, \boldsymbol{u}_h + \boldsymbol{v}_h) d\tau + (\boldsymbol{v}(t), \boldsymbol{u}_h(t)) + (\boldsymbol{u}(t), \boldsymbol{v}_h(t))$$
$$- (\boldsymbol{v}_0, (\boldsymbol{v}_0)_h) - (\boldsymbol{v}_0, (\boldsymbol{v}_0)_h). \tag{4.16}$$

We now want to let $h \to 0$ in this relation. Again, the main difficulty is given by the nonlinear terms, the other terms being easily treated by means of Lemma 2.5. By the same reasoning leading to (4.12) we find

$$\lim_{h \to 0} \int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{u}_h, \boldsymbol{v}) d\tau = \int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) d\tau. \tag{4.17}$$

Concerning the other nonlinear term, we shall distinguish the three cases:

a) $s > n$;

b) $s = n$;

c) $s = \infty$.

In case a), since $\boldsymbol{u} \in V_T$, from Lemma 4.1 we obtain

$$\int_0^t |(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v})| d\tau \leq C\|\boldsymbol{v}_h - \boldsymbol{v}\|_{L^r(0,T;L^s(\Omega))}$$

with $C = C(\boldsymbol{u})$, and so, by Lemma 2.5 we find

$$\lim_{h\to 0}\int_0^t (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}_h) d\tau = \int_0^t (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}) d\tau. \tag{4.18}$$

In case b), we shall consider only the case $n = 3$, the case $n = 2$ being treated in a similar way. We thus observe that by the Hölder and Sobolev inequalities, and recalling that $\boldsymbol{u} \in V_T$, it follows that

$$\int_0^t \|\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{3/2} d\tau \leq \int_0^t \|\boldsymbol{u}\|_6 \|\nabla\boldsymbol{u}\|_2 d\tau \leq c\int_0^t \|\nabla\boldsymbol{u}\|_2^2 d\tau \leq C. \tag{4.19}$$

Therefore, setting $\boldsymbol{w} \equiv \boldsymbol{u}\cdot\nabla\boldsymbol{u}$, by the property of the mollifier, we obtain

$$\begin{aligned}\left|\int_0^t (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v}_h - \boldsymbol{v}) d\tau\right| &\equiv \left|\int_0^t (\boldsymbol{w}, \boldsymbol{v}_h - \boldsymbol{v}) d\tau\right| = \left|\int_0^t (\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{v}) d\tau\right| \\ &\leq \operatorname{ess}\sup_{\mathrm{t}\in[0,\mathrm{T}]} \|\boldsymbol{v}(t)\|_3 \int_0^t \|\boldsymbol{w} - \boldsymbol{w}_h\|_{3/2} d\tau.\end{aligned}$$

By (4.19), we have $\boldsymbol{w} \in L^1(0, T; \boldsymbol{L}^{3/2}(\Omega))$ and so, by Lemma 2.5, we conclude the validity of (4.18). Finally, in case c), from the Schwarz inequality and the fact that $\boldsymbol{u} \in V_T$, we easily establish that $\boldsymbol{w} \in L^2(0, T; \boldsymbol{L}^1(\Omega))$ and so, using the following relation

$$\begin{aligned}\left|\int_0^t (\boldsymbol{w}, \boldsymbol{v}_h - \boldsymbol{v}) d\tau\right| &= \left|\int_0^t (\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{v}) d\tau\right| \\ &\leq \left(\int_0^t \|\boldsymbol{w} - \boldsymbol{w}_h\|_1^2\right)^{1/2} \left(\int_0^t \|\boldsymbol{v}\|_\infty^2\right)^{1/2},\end{aligned}$$

we again arrive at (4.18). Letting $h \to 0$ in (4.16), and using (4.17), (4.18), we obtain

$$\begin{aligned}&-\int_0^t \{\ 2\nu(\nabla\boldsymbol{v}, \nabla\boldsymbol{u}) + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})\} \, d\tau \\ &\quad = -\int_0^t (\boldsymbol{f}, \boldsymbol{u} + \boldsymbol{v}) d\tau + 2[(\boldsymbol{v}(t), \boldsymbol{u}(t)) - (\boldsymbol{v}_0, \boldsymbol{v}_0)],\end{aligned} \tag{4.20}$$

with $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$. By Remark 4.3, $\boldsymbol{v}$ obeys the energy equality

$$\|\boldsymbol{v}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{v}\|_2^2 d\tau = 2\int_0^t (\boldsymbol{v}, \boldsymbol{f}) d\tau + \|\boldsymbol{v}_0\|_2^2, \tag{4.21}$$

while, by assumption, $\boldsymbol{u}$ obeys the energy inequality

$$\|\boldsymbol{u}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{u}\|_2^2 d\tau \le 2\int_0^t (\boldsymbol{u}, \boldsymbol{f}) d\tau + \|\boldsymbol{v}_0\|_2^2. \tag{4.22}$$

Adding $2 \times (4.20)$, (4.21) and (4.22), and observing that

$$\int_0^t (\boldsymbol{w} \cdot \nabla \boldsymbol{v}, \boldsymbol{v}) d\tau = 0$$

we obtain

$$\|\boldsymbol{w}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{w}\|_2^2 d\tau \le 2\int_0^t (\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}) d\tau. \tag{4.23}$$

If $s > n$, we employ Lemma 4.1 on the term on the right-hand side of (4.23) together with the Young inequality to deduce

$$\begin{aligned}\int_0^t (\boldsymbol{w} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}) d\tau &\le c \left(\int_0^t \|\nabla \boldsymbol{w}\|^2 d\tau\right)^{1-1/r} \left(\int_0^t \|\boldsymbol{v}\|_s^r \|\boldsymbol{w}\|_2^2 d\tau\right)^{1/r} \\ &\le \nu \int_0^t \|\nabla \boldsymbol{w}\|^2 d\tau + c_1 \int_0^t \|\boldsymbol{v}\|_s^r \|\boldsymbol{w}\|_2^2 d\tau.\end{aligned}$$

Replacing this inequality into (4.23), we find

$$\|\boldsymbol{w}(t)\|_2^2 \le c_1 \int_0^t \|\boldsymbol{v}\|_s^r \|\boldsymbol{w}\|_2^2 d\tau,$$

which, with the help of Gronwall's lemma, allows us to conclude $\boldsymbol{v} = \boldsymbol{u}$ a.e. in Ω_T. If $s = n$, we set

$$\mathcal{T} = \{\tau \in [0, T] : \ \|\boldsymbol{w}(s)\|_2 = 0, \quad \text{for all } s \in [0, \tau]\}.$$

Clearly, $\mathcal{T}$ is not empty and, in virtue of the L^2 weak continuity of $\boldsymbol{w}$, it is also closed. Let us denote by τ_0 its maximum. If $\tau_0 = T$, there is nothing to prove. Therefore, assuming $\tau_0 < T$, we have

$$\int_{\tau_0}^t \|\nabla \boldsymbol{w}\|_2^2 ds > 0, \quad \text{for all } t \in [\tau_0, T).$$

By Lemma 4.2, it then follows

$$\left|\int_{\tau_0}^{t}(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{v})d\tau\right| \le \varepsilon\int_{\tau_0}^{t}\|\nabla\boldsymbol{w}\|_2^2 ds + M\int_{\tau_0}^{t}\|\boldsymbol{w}\|_2^2 ds, \quad \text{for all } t\in(\tau_0,T).$$

Replacing this inequality into (4.23), and recalling that $\boldsymbol{w}(s)=0$ for all $s\le\tau_0$, we find

$$\|\boldsymbol{w}(t)\|_2^2 \le M\int_{\tau_0}^{t}\|\boldsymbol{w}\|_2^2 ds,$$

which, with the help of Gronwall's lemma, again implies $\boldsymbol{v}=\boldsymbol{u}$ a.e. in Ω_T. The theorem is thus proved.

Remark 4.5 If Ω is a bounded or an exterior domain with a sufficiently smooth boundary, or a half space, one can furnish an important generalization of the uniqueness result given in the previous theorem. Such a generalization, instead of hypothesis (ii), requires only

$$\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{L}^n(\Omega)). \tag{4.24}$$

This result, due to Kozono and Sohr (1996a) (see also Sohr and von Wahl (1984), under more restrictive assumptions on $\boldsymbol{v}$, and the review article of Kozono (1998)) will be proved in Section 7, Theorem 7.2, in the case $\Omega=\mathbb{R}^n$.

Remark 4.6 Since in dimension 2 every weak solution belongs to the class $C^0([0,T);\boldsymbol{L}^2(\Omega))$, see Remark 4.2, by Theorem 4.2 it follows that every such weak solution is unique in the class of weak solutions assuming the same data, a fact discovered for the first time by Lions and Prodi (1959). In dimension 3, by the Sobolev inequality, we have

$$\|\boldsymbol{v}\|_s \le c\|\boldsymbol{v}\|_2^{(6-s)/2s}\|\nabla\boldsymbol{v}\|_2^{3(s-2)/2s}, \quad s\in[2,6]$$

and so, for $\boldsymbol{v}\in V_T$, we find

$$\boldsymbol{v}\in L^r(0,T;\boldsymbol{L}^s(\Omega)), \quad \frac{3}{s}+\frac{2}{r}=\frac{3}{2},$$

and the condition in Theorem 4.2 is *not* satisfied. The problem of whether a three dimensional weak solution obeying the energy inequality is unique in its class is an outstanding open question. In this respect, we wish to mention the contribution of Ladyzhenskaya (1969), in her effort to *disprove* uniqueness. Specifically, using a method introduced by Golovkin (1964) in a different context, she constructs two distinct three dimensional solutions $\boldsymbol{v}_i$, $i=1,2$, with rotational symmetry, corresponding to the same data, in a *non-cylindrical* domain Q_T of the space-time. This latter is defined as

$$Q_T = \{(r,z,t):\ t\in[0,T],\ r\in[\eta\sqrt{t},\ell\sqrt{t}],\ z\in[-\ell\sqrt{t},\ell\sqrt{t}],\ \eta << \ell\},$$

where (r,z) denote cylindrical coordinates. Both solutions belong to the Leray-Hopf class in the sense that

$$\max_{t\in[0,T]}\int_{\Omega(t)}|\boldsymbol{v}_i|^2+\int_{Q_T}|\nabla\boldsymbol{v}_i|^2<\infty$$

where

$$\Omega(t)\equiv\{r\in[\eta\sqrt{t},\ell\sqrt{t}],\ z\in[-\ell\sqrt{t},\ell\sqrt{t}]\}.$$

Moreover, they match the (vanishing) initial data in the following sense

$$\lim_{t\to 0}\int_{\Omega(t)}|\boldsymbol{v}_i|^2=0$$

and obey "stress-free" boundary conditions. [14] Finally, they satisfy the following condition

$$\int_0^T\left(\int_{\Omega(t)}|\boldsymbol{v}_i|^s\right)^{r/q}dt<M(\varepsilon)<\infty,\quad i=1,2,$$

with exponents s,r such that

$$\frac{3}{s}+\frac{2}{r}=1+\varepsilon,\quad \varepsilon>0,$$

$(M(\varepsilon)\to\infty$ as $\varepsilon\to 0)$. However, this result can not be considered completely satisfactory, in that the space-time domain Q_T where the solutions $\boldsymbol{v}_i$ exist is *not* cylindrical (that is, of the type $\Omega\times I$ with Ω a fixed spatial domain and I a time interval). Rather, it expands when time increases and reduces to a single point when time goes to zero. In the same paper, Ladyzhenskaya furnishes another counter example to uniqueness in a class of solutions slightly weaker than the Leray-Hopf one, in that the spatial derivatives are summable with an exponent *strictly* less than 2. This time the boundary conditions are the usual adherence conditions, but the space-time domain is still non-cylindrical.

[14] That is, the normal component of $\boldsymbol{v}$ is prescribed, together with the tangential component of the vorticity field.

Remark 4.7 For later purposes, we wish to notice that the condition

$$\boldsymbol{v} \in L^{r_1}(0,T;\boldsymbol{L}^{s_1}(\Omega)), \quad \text{for some } s_1 \in (n,\infty],\ r_1 \in [2,\infty), \ \text{with } \ \frac{n}{s_1}+\frac{2}{r_1}=1 \tag{A}$$

is weaker than

$$\boldsymbol{v} \in L^{r_2}(0,T;\boldsymbol{L}^{s_2}(\Omega)), \quad \text{for some } s_2 \in (n,\infty],\ r_2 \in (2,\infty), \text{ with } \ \frac{n}{s_2}+\frac{2}{r_2}<1 \tag{B}$$

in the sense that if $\boldsymbol{v}$ satisfies (B), then, by the Hölder inequality, $\boldsymbol{v}$ satisfies (A) with $s_1 = s_2$ and $r_1 = 2s_2/(s_2 - n)$ ($< r_2$).

Remark 4.8 In the literature, one may find many other uniqueness theorems for weak solutions, see, among others, Prodi (1959), Lions and Prodi (1959), Ladyzhenskaya (1967). However, in all these papers one compares two weak solutions *each of which* possesses more regularity than that established in the existence Theorem 3.1. It is therefore worth emphasizing that Theorem 4.2 compares two weak solutions of which *only one* possesses extra regularity. For uniqueness results related to Theorem 4.2, in a class of "very weak" solutions, see Foiaş (1961), Fabes, Jones and Riviére (1972), H. Kato (1993), Chemin (1999), Monniaux (1999), Amann (1999).

5 Regularity of Weak Solutions.

The regularity theory for weak solutions to the Navier-Stokes equations presents different features, according to whether one looks for *interior regularity* or *regularity for the initial-boundary value problem*. In the first case, denoting by $R = \omega \times (t_1, t_2)$ a bounded domain strictly contained in Ω_T, one considers a field $\boldsymbol{v}$ that satisfies the identity (2.2) for all solenoidal test functions $\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(R)$, (hereafter denoted by $(2.2)_0$), which is divergence free in R and, further, verifies the following condition

$$\boldsymbol{v} \in L^2(t_1,t_2;W^{1,2}(\omega)) \cap L^\infty(t_1,t_2;L^2(\omega)). \tag{5.1}$$

In the second case, one requires that $\boldsymbol{v}$ is a weak solution to the initial-boundary value problem, in the sense of Definition 2.1. Now, let us consider the field $\overline{\boldsymbol{v}}$ defined in (2.3). As already observed in Section 2, $\overline{\boldsymbol{v}}$ satisfies $(2.2)_0$ with $\boldsymbol{f} \equiv 0$. However, this field –though infinitely differentiable in the space variables–

need have no time derivative at all and, in fact, it may even have (integrable) singularities in the time interval $[0, T)$. This example, due to Serrin (1962), leads us to the following considerations. First, for interior regularity, one can *not* expect to prove a result where the amount of regularity in time is more than that assumed at the outset. Second, the existence of such "bad" solutions is due to the fact that the possible singularities are absorbed by the pressure term. For instance, the field (2.3) with a "bad" behavior in time could also be a solution to the quasi-linear (vector) heat equation $(0.1'')$, on condition that, however, the force $\boldsymbol{f}$ is chosen to have an equally "bad" behavior. On the other hand, $\overline{\boldsymbol{v}}$ does not meet the boundary conditions (0.3) hidden in requirement a) of Definition 2.1, unless it is identically zero, and so there is hope that one can "gain" regularity in time by dealing with solutions of the initial-boundary value problem.

The aim of this section is to furnish sufficient conditions for regularity of weak solutions. As we shall see, these conditions do not overlap completely with those ensuring uniqueness, and there is an interesting question which is still left open. Moreover, as in the case of uniqueness, one shows that every weak solution in dimension 2 is regular, provided the data are regular enough. In dimension 3, the regularity of weak solutions is an outstanding open problem. We shall report, without proof, the interior regularity results, due essentially to Ohyama (1960), Serrin (1962) and Struwe (1988), see Theorem 5.1. Successively, in Theorem 5.2, we shall give a result concerning the regularity of weak solutions of the initial-boundary value problem (in the sense of Definition 2.1). In doing this, we shall follow the method of Galdi and Maremonti (1988). For further regularity results, see H. Kato (1977/78,1986,1989,1993), Tanaka (1987).

Theorem 5.1 *Let $\boldsymbol{v}$ be a solenoidal field in $\omega \times (t_1, t_2)$, satisfying $(2.2)_0$ with $\boldsymbol{f} = 0$,* [15] *and (5.1). Assume, in addition, that $\boldsymbol{v}$ verifies at least one of the following two conditions:*

(i) $\boldsymbol{v} \in L^r(t_1, t_2; \boldsymbol{L}^s(\omega))$, *for some r, s such that* $\dfrac{n}{s} + \dfrac{2}{r} = 1, \;\; s \in (n, \infty)$;

(ii) $\boldsymbol{v} \in L^\infty(t_1, t_2; \boldsymbol{L}^n(\omega))$, *and, given $\epsilon > 0$ there is $\rho > 0$ such that*

$$\int_{B_\rho \cap \omega} |\boldsymbol{v}(x,t)|^n dx < \epsilon, \quad \text{for all } t \in (t_1, t_2)$$

where B_ρ is a ball of radius ρ.

[15] For the general case $\boldsymbol{f} \not\equiv 0$, we refer the reader to the papers of Serrin and Struwe.

Then, $\boldsymbol{v}$ is of class $C^\infty(\omega)$, and each space derivative is bounded in compact subregions of $\omega \times (t_1, t_2)$. If, in addition,

$$\frac{\partial \boldsymbol{v}}{\partial t} \in L^2(t_1, t_2; \boldsymbol{L}^q(\omega)), \quad \textit{for some } q \geq 1,$$

then, the space derivatives of $\boldsymbol{v}$ are absolutely continuous functions of time.

Remark 5.1 For $n = 2$, a possible choice of exponents is $s = r = 4$. Therefore, from Remark 4.1, we conclude that every two dimensional weak solution is regular in the sense specified in Theorem 5.1. On the other hand, three dimensional weak solutions do not satisfy either of assumption (i), (ii), see Remark 4.6, and nothing can be said about their regularity. An interesting variant of Theorem 5.1(i) has been given by Takahashi (1990, 1992), who replaces the Lebesgue space L^r with the Lorentz space $L^{(r)}$ ("L^r*-weak*"), requiring, however, that the corresponding "norm" be sufficiently small. In particular, denoting by $B_R(x_0)$ a ball of radius R centered at x_0, he shows that a sufficient condition for a weak solution $\boldsymbol{v}$ to be of class L^∞ in $B_R(x_0) \times (-R^2 + t_1, t_1)$ [16] is that it satisfies an estimate of the type

$$\|\boldsymbol{v}(t)\|_{\sigma, B_R(x_0)} \leq \frac{\varepsilon}{(t_1 - t)^{(\sigma - n)/2\sigma}}, \quad t \in (-R^2 + t_1, t_1), \ \sigma \in (n, \infty]$$

with a "small" ε. As we shall see in Theorem 7.3, a necessary condition for $\boldsymbol{v}$ to become irregular at a time t_1 [17] is that

$$\|\boldsymbol{v}(t)\|_\sigma \geq \frac{C}{(t_1 - t)^{(\sigma - n)/2\sigma}}, \quad t < t_1,$$

with $C = C(n, \sigma, \nu) > 0$; Takahashi also extends Theorem 5.1(i) to the case $s = \infty$.

We shall now be concerned with the regularity of weak solutions to the initial-boundary value problem, in the sense of Definition 2.1. For simplicity, we shall assume that $\boldsymbol{f} \equiv 0$. Before going into details, we wish to outline the main idea underlying the proof. To this end, let $\boldsymbol{v}$ be a weak solution in Ω_T and let $\boldsymbol{u}$ be

[16] And hence regular, in the sense of Theorem 5.1.

[17] See Definition 6.1.

a weak solution in Ω_T to the following initial-boundary value problem

$$\begin{aligned}
&\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} = \nu \Delta \boldsymbol{u} + \nabla \pi \\
&\operatorname{div} \boldsymbol{u} = 0 \\
&\boldsymbol{u}(x,0) = \boldsymbol{v}_0, \quad x \in \Omega \\
&\boldsymbol{u}(y,t) = 0, \quad y \in \partial\Omega, \ t > 0.
\end{aligned} \tag{5.2}$$

By this we mean that $\boldsymbol{u} \in V_T$ and that it satisfies the following relation

$$\int_0^\infty \left\{ \left(\boldsymbol{u}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) \right\} dt = -(\boldsymbol{v}_0, \boldsymbol{\varphi}(0)), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_T. \tag{5.3}$$

Thus, $\boldsymbol{v}$ becomes the coefficient of a "linearized" Navier-Stokes equation. Notice, also, that $\boldsymbol{v}$ and $\boldsymbol{u}$ are both weak solutions to the *same* Navier-Stokes problem with the *same* data $\boldsymbol{v}_0$. The question is now to determine the weakest conditions on $\boldsymbol{v}$ in order that:

a) $\boldsymbol{v} = \boldsymbol{u}$, a.a. in Ω_T.

b) $\boldsymbol{u}$ has more regularity than that originally assumed for $\boldsymbol{v}$.

If b) is met, then, by a), $\boldsymbol{v}$ becomes more regular and then $\boldsymbol{u}$ becomes more regular too and so, by a boot-strap argument, we can conclude that $\boldsymbol{v}$ becomes as much regular as allowed by the data. In this latter respect, we wish to emphasize that this method only requires $\boldsymbol{v}_0 \in H(\Omega)$, since regularity is established in the semi-open interval $(0, T]$. On the other hand, we shall prove regularity up to the boundary of Ω which, therefore, will be assumed suitably smooth.

Remark 5.2 Instead of the linearized problem (5.2), we could consider the following one:

$$\begin{aligned}
&\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{v} = \nu \Delta \boldsymbol{u} + \nabla \pi \\
&\operatorname{div} \boldsymbol{u} = 0 \\
&\boldsymbol{u}(x,0) = \boldsymbol{v}_0, \quad x \in \Omega \\
&\boldsymbol{u}(y,t) = 0, \quad y \in \partial\Omega, \ t > 0.
\end{aligned}$$

With such a choice, one could find conditions on $\nabla \boldsymbol{v}$ (instead of $\boldsymbol{v}$) under which the weak solution $\boldsymbol{v}$ becomes regular. This can be done exactly along the same lines we shall follow hereafter for problem (5.2). We shall limit ourselves to state the corresponding results, without proof, in Remarks 5.3 and 5.6.

Let us first consider condition a). Since the system (5.2) is *linear* in $\boldsymbol{u}$, we expect that the conditions on $\boldsymbol{v}$ which ensure a), should be weaker than those ensuring the uniqueness of a weak solution to the full nonlinear Navier-Stokes problem. Actually, we have

Lemma 5.1 *Let $\boldsymbol{v} \in V_T$ and let $\boldsymbol{u}$ be a weak solution to (5.2) in Ω_T. Then, if*

$$\boldsymbol{v} \in L^4(0,T;\boldsymbol{L}^4(\Omega)), \tag{5.4}$$

we have $\boldsymbol{v} = \boldsymbol{u}$, a.a. in Ω_T.

Proof. Reasoning exactly as in the proof of Lemma 2.1, we show that $\boldsymbol{u}$ satisfies the following relation

$$\int_0^t \left\{ \left(\boldsymbol{u}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{\varphi}) \right\} ds = (\boldsymbol{u}(t), \boldsymbol{\varphi}(t)) - (\boldsymbol{v}_0, \boldsymbol{\varphi}(0)),$$

$$\text{for all } t \in [0,T) \text{ and all } \boldsymbol{\varphi} \in \mathcal{D}_T.$$

Subtracting the integral equation in the previous relation from that in (2.4) with $\boldsymbol{f} \equiv 0$, and setting $\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v}$ we find

$$\int_0^t \left\{ \left(\boldsymbol{w}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{\varphi}) - (\boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{\varphi}) \right\} ds = (\boldsymbol{w}(t), \boldsymbol{\varphi}(t)). \tag{5.5}$$

From now on, the proof is the same as that of Theorem 4.1. Specifically, we denote by $\{\boldsymbol{w}_k\} \subset \mathcal{D}_T$ a sequence converging to $\boldsymbol{v}$ in $L^2(0,T;H^1(\Omega))$. We then choose in (5.5) $\boldsymbol{\varphi} = \boldsymbol{w}_{h,k}$, and pass to the limits $k \to \infty$ and $h \to 0$. Reasoning as in Theorem 4.1, we show

$$\tfrac{1}{2}\|\boldsymbol{w}(t)\|_2^2 = \int_0^t \left\{ (\boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{w}) - \nu \|\nabla \boldsymbol{w}\|_2^2 \right\} ds.$$

However, since $\boldsymbol{v}(t) \in H(\Omega)$ for a.a. t, we get

$$\int_0^t (\boldsymbol{v} \cdot \nabla \boldsymbol{w}, \boldsymbol{w}) ds = 0,$$

and the lemma follows.

The major assumption on the weak solution $\boldsymbol{v}$ comes into the proof of point b). To show this, however, we need some preliminary considerations. The first one concerns well known results for the steady *Stokes system*, obtained as a suitable linearization of the full steady-state Navier-Stokes system (0.4). Specifically, we have (see, $e.g.$, Galdi, 1994, Theorem IV.6.1).

Lemma 5.2 *Let Ω be a bounded domain of $\mathbb{R}^n$, of class C^{m+2}, $m \geq 0$. For any $\boldsymbol{F} \in \boldsymbol{W}^{m,q}(\Omega)$, $1 < q < \infty$, there exists one and only one solution $\boldsymbol{u}, \phi$ [18] to the following Stokes problem*

$$
\begin{aligned}
&-\nu \Delta \boldsymbol{u} = \nabla \phi + \boldsymbol{F} \\
&\operatorname{div} \boldsymbol{u} = 0 \\
&\boldsymbol{u}(y) = 0, \quad y \in \partial\Omega,
\end{aligned}
$$

such that

$$\boldsymbol{u} \in \boldsymbol{W}^{m+2,q}(\Omega), \quad \phi \in \boldsymbol{W}^{m+1,q}(\Omega).$$

This solution satisfies the estimate:

$$\|\boldsymbol{u}\|_{m+2,q} + \|\phi\|_{m+1,q} \leq c\|\boldsymbol{F}\|_{m,q}.$$

Moreover, the problem

$$
\begin{aligned}
&-\nu \Delta \boldsymbol{a} = \nabla \phi + \lambda \boldsymbol{a} \\
&\operatorname{div} \boldsymbol{a} = 0 \\
&\boldsymbol{a}(y) = 0, \quad y \in \partial\Omega,
\end{aligned}
$$

admits a denumerable number of positive eigenvalues $\{\lambda_r\}$ clustering at infinity, and the corresponding eigenfunctions $\{\boldsymbol{a}_r\}$ form an orthonormal basis in H.

Our second preliminary result concerns an estimate for the nonlinear term, which strengthens that given in Lemma 4.2.

Lemma 5.3 *Let*

$$\boldsymbol{v} \in C^0([0,T]; \boldsymbol{L}^n(\Omega)), \quad \boldsymbol{u} \in \boldsymbol{W}^{2,2}(\Omega), \quad \boldsymbol{a} \in L^2(\Omega).$$

Then, given $\eta > 0$ there exists $M = M(\boldsymbol{v}, \eta) > 0$ such that

$$|(\boldsymbol{v} \cdot \nabla \boldsymbol{u}, \boldsymbol{a})| \leq \varepsilon \left(\|P\Delta \boldsymbol{u}\|_2^2 + \|\boldsymbol{a}\|_2^2\right) + M\|\nabla \boldsymbol{u}\|_2^2,$$

where P is the orthogonal projection operator from $\boldsymbol{L}^2$ to H (see Section 2).

[18] With the normalization condition $\int_\Omega \phi = 0$.

Proof. We extend $\boldsymbol{v}$ to zero outside Ω, and let $\boldsymbol{v}_\eta$ be the spatial mollifier of $\boldsymbol{v}$, that is,

$$\boldsymbol{v}_\eta(x,t) = \int_{\mathbb{R}^3} J_\eta(x-\xi)\boldsymbol{v}(\xi,t)d\xi,$$

with $J_\eta(\sigma)$ an infinitely differentiable function vanishing for $|\sigma| > \eta$ and normalized to 1. It is well known that

$$\sup_{x\in\Omega} |\boldsymbol{v}_\eta(x,t)| \le c(\eta)\|\boldsymbol{v}(t)\|_3$$

and that

$$\lim_{\eta\to 0} \|\boldsymbol{v}_\eta(t) - \boldsymbol{v}(t)\|_3 = 0, \quad \text{for all } t \in [0,T].$$

Using the continuity assumption on $\boldsymbol{v}$, by an argument completely analogous to that adopted in the proof of Lemma 2.3, we show that this limit is taken uniformly in $t \in [0,T]$. In view of this, by Sobolev's theorem and Lemma 5.2, we thus have

$$\begin{aligned} |(\boldsymbol{v}\cdot\nabla\boldsymbol{u},\boldsymbol{a})| &\le |((\boldsymbol{v}-\boldsymbol{v}_\eta)\cdot\nabla\boldsymbol{u},\boldsymbol{a})| + |(\boldsymbol{v}_\eta\cdot\nabla\boldsymbol{u},\boldsymbol{a})| \\ &\le \|\boldsymbol{v}-\boldsymbol{v}_\eta\|_3\|\nabla\boldsymbol{u}\|_6\|\boldsymbol{a}\|_2 + \sup_{x\in\Omega}|\boldsymbol{v}_\eta(x,t)|\|\nabla\boldsymbol{u}\|_2\|\boldsymbol{a}\|_2 \\ &\le \varepsilon\|P\Delta\boldsymbol{u}\|_2\|\boldsymbol{a}\|_2 + M\|\nabla\boldsymbol{u}\|_2\|\boldsymbol{a}\|_2, \end{aligned}$$

and the result follows after using Cauchy's inequality on the last line of this inequality.

Using these lemmas we can now show the first regularity result for $\boldsymbol{v}$.

Lemma 5.4 *Let Ω be an arbitrary domain in $\mathbb{R}^n$, uniformly of class C^2.* [19] *Assume that $\boldsymbol{v} \in V_T$ and that it satisfies at least one of the following two conditions:*

(i) $\boldsymbol{v} \in L^r(0,T;\boldsymbol{L}^s(\Omega))$, *for some r, s such that* $\dfrac{n}{s} + \dfrac{2}{r} = 1$, $s \in (n,\infty]$;

(ii) $\boldsymbol{v} \in C^0([0,T];\boldsymbol{L}^n(\Omega))$.

[19] Ω is said *uniformly of class C^m*, $m \ge 0$, if Ω lies on one part of its boundary $\partial\Omega$ and, for each $x_0 \in \partial\Omega$, there exists a ball B centered at x_0 and of radius independent of x_0, such that $\partial\Omega \cap B$ admits a Cartesian representation of the form $x_n = \gamma(x_1,\dots,x_{n-1})$, where γ is a function of class C^m in its domain, with its derivatives up to order m inclusive uniformly bounded by the same constant, independently of x_0. If Ω is uniformly of class C^m, for all $m \ge 0$, we shall say that Ω is uniformly of class C^∞.

Then, for any $\boldsymbol{v}_0 \in H(\Omega)$*, there exists one weak solution* $\boldsymbol{u}$ *to (5.2) in* Ω_T *such that*

$$\boldsymbol{u} \in C^0((\varepsilon, T]; H^1(\Omega)) \cap L^2(\varepsilon, T; \boldsymbol{W}^{2,2}(\Omega))$$

$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(\varepsilon, T; H(\Omega)),$$

where ε *is an arbitrary positive number. Moreover, by* Lemma 5.1 *and* Remarks 4.1, 4.3, $\boldsymbol{v} = \boldsymbol{u}$ *a.e. in* Ω_T*.* [20]

Proof. To avoid unessential technical difficulties, we limit ourselves to give the proof in the case Ω bounded and $n = 3$, referring the reader to Galdi and Maremonti (1988) for the proof in the general case. We shall use the Faedo-Galerkin method of Theorem 3.1, with the basis $\{\boldsymbol{a}_r\}$ of H constituted by the eigenvectors of the Stokes problem (Lemma 5.3). Thus, we shall look for approximating solutions $\boldsymbol{u}_k$ of the form

$$\boldsymbol{u}_k(x,t) = \sum_{r=1}^{k} c_{kr}(t)\boldsymbol{a}_r(x), \quad k \in \mathbb{N},$$

where the coefficients c_{kr} are required to satisfy the following system of ordinary differential equations

$$\frac{d}{dt}(\boldsymbol{u}_k, \boldsymbol{a}_r) + \nu(\nabla \boldsymbol{u}_k, \nabla \boldsymbol{a}_r) + (\boldsymbol{v} \cdot \nabla \boldsymbol{u}_k, \boldsymbol{a}_r) = 0 \quad r = 1, \ldots, k, \tag{5.6}$$

with the initial condition

$$c_{kr}(0) = (\boldsymbol{v}_0, \boldsymbol{a}_r) \quad r = 1, \ldots, k.$$

As in Theorem 3.1, we show that this system of ordinary differential equations admits a (unique) solution in the time interval $[0, T]$, as a consequence of the following relation, which is obtained by multiplying (5.6) by c_{kr} and summing over the index r:

$$\|\boldsymbol{u}(t)\|_2^2 + 2\nu \int_0^t \|\nabla \boldsymbol{u}(\tau)\|_2^2 d\tau = \|\boldsymbol{v}_0\|_2^2. \tag{5.7}$$

For simplicity, here as in the following relations, we shall omit the subscript "k". We next multiply (5.6) by $\lambda_k c_{kr}$ and by dc_{kr}/dt, respectively, sum over r, and employ the properties of the eigenfunctions $\boldsymbol{a}_k$ to obtain

$$\tfrac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|_2^2 + \nu \|P\Delta \boldsymbol{u}\|_2^2 = (\boldsymbol{v} \cdot \nabla \boldsymbol{u}, P\Delta \boldsymbol{u}), \tag{5.8}$$

[20] Of course, in case (ii), we have $\boldsymbol{v}_0 \in H(\Omega) \cap \boldsymbol{L}^n(\Omega)$.

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}(t)\|_2^2 + \nu\|D_t\boldsymbol{u}\|_2^2 = -(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, D_t\boldsymbol{u}), \tag{5.9}$$

where P is the orthogonal projection operator from $\boldsymbol{L}^2$ to H (see Section 2) and D_t denotes differentiation with respect to t. We wish now to increase the trilinear form $(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{a})$. Let us first consider the case (i), $i.e.$, $\infty \geq s > 3$ (=n). By the Hölder inequality we have

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{a})| \leq \|\boldsymbol{v}\|_s\|\nabla\boldsymbol{u}\|_{2s/(s-2)}\|\boldsymbol{a}\|_2;$$

Furthermore, since $2s/(s-2) \in [2,6)$, by the Sobolev theorem and Lemma 5.3 we obtain

$$\|\nabla\boldsymbol{u}\|_{2s/(s-2)} \leq c\|\boldsymbol{u}\|_{2,2}^{3/s}\|\nabla\boldsymbol{u}\|_2^{(s-3)/s} \leq c_1\|P\Delta\boldsymbol{u}\|_2^{3/s}\|\nabla\boldsymbol{u}\|_2^{(s-3)/s}$$

and so, it follows that

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{a})| \leq c\|\boldsymbol{v}\|_s\|\nabla\boldsymbol{u}\|_2^{(s-3)/s}\|P\Delta\boldsymbol{u}\|_2^{3/s}\|\boldsymbol{a}\|_2.$$

Employing Young's inequality, with exponents $2s/(s-3)$, $2s/3$ and $1/2$ we thus conclude

$$|(\boldsymbol{v}\cdot\nabla\boldsymbol{u}, \boldsymbol{a})| \leq c\|\boldsymbol{v}\|_s^{2(s-3)/s}\|\nabla\boldsymbol{u}\|_2^2 + \eta\|P\Delta\boldsymbol{u}\|_2^2 + \eta\|\boldsymbol{a}\|_2^2 \tag{5.10}$$

with arbitrary $\eta > 0$ and $c = c(\Omega, s, \eta)$. Summing (5.8) and (5.9), and using (5.10) with $\boldsymbol{a} = P\Delta\boldsymbol{u}$ and $\boldsymbol{a} = D_t\boldsymbol{u}$, respectively, for sufficiently small η we find

$$\frac{d}{dt}\|\nabla\boldsymbol{u}\|_2^2 + c_1(\|P\Delta\boldsymbol{u}\|_2^2 + \|D_t\boldsymbol{u}\|_2^2) \leq c_2\|\boldsymbol{v}\|_s^{2(s-3)/s}\|\nabla\boldsymbol{u}\|_2^2.$$

Integrating this relation furnishes

$$\|\nabla\boldsymbol{u}(t)\|_2^2 + c_3\int_\tau^t(\|P\Delta\boldsymbol{u}\|_2^2 + \|D_\tau\boldsymbol{u}\|_2^2)d\tau \leq \|\nabla\boldsymbol{u}(\tau)\|_2^2\exp\left[c_2\int_0^T\|\boldsymbol{v}(\tau)\|_s^r d\tau\right]$$

for all $t \in [s,T]$, $s \geq \varepsilon$.

If we integrate this inequality on $\tau \in [\varepsilon, t]$ and use (5.7), we obtain the following limitations

$$\|\boldsymbol{u}(t)\|_{1,2} + \int_\varepsilon^t(\|P\Delta\boldsymbol{u}\|_2^2 + \|\boldsymbol{u}_\tau\|_2^2) \leq M, \quad \text{for all } t \in [\varepsilon, T], \tag{5.11}$$

where M depends on $\|\boldsymbol{v}_0\|_2$, ε, Ω, and s. Using these *a priori* estimates on the approximating solutions $\{\boldsymbol{u}_k\}$ and proceeding as in the proof of Theorem 3.1, we easily show that from the sequence $\{\boldsymbol{u}_k\}$ we can select a subsequence which converges to a weak solution $\boldsymbol{u}$ of the problem (5.2) and which, in addition, satisfies

$$\boldsymbol{u} \in L^\infty(\varepsilon, T; H^1(\Omega)) \cap L^2(\varepsilon, T; \boldsymbol{W}^{2,2}(\Omega))$$
$$\frac{\partial \boldsymbol{u}}{\partial t} \in L^2(\varepsilon, T; H(\Omega)).$$

From these properties and the identity:

$$\|\boldsymbol{u}(t+h) - \boldsymbol{u}(t)\|_{1,2}^2 = \int_0^h \left(\frac{d}{ds} \|\boldsymbol{u}(s+h) - \boldsymbol{u}(t)\|_2^2 \right.$$
$$\left. -(\Delta(\boldsymbol{u}(s+h) - \boldsymbol{u}(t)), \frac{\partial}{\partial s}(\boldsymbol{u}(s+h) - \boldsymbol{u}(t))) \right) ds,$$

we deduce

$$\boldsymbol{u} \in C^0((\varepsilon, T]; H^1(\Omega)),$$

and so the result follows under the assumption (i). In case (ii), instead of (5.10), we use the estimate showed in Lemma 5.3 and proceed exactly as in case (i). The lemma is thus proved.

Remark 5.3 The same conclusion of Lemma 5.4 can be obtained under the following alternative assumptions, see Remark 5.2,

(i)′ $\nabla \boldsymbol{v} \in L^{r'}(0, T; \boldsymbol{L}^{s'}(\Omega))$, $\frac{2}{r'} + \frac{n}{s'} = 2$, $s' \in (n, \infty]$,

(i)″ $\nabla \boldsymbol{v} \in C^0([0, T]; \boldsymbol{L}^{n/2}(\Omega))$.

A similar result, for the case $\Omega \equiv \mathbb{R}^n$, was first obtained by Beirão da Veiga (1995a, 1995b).

Remark 5.4 Once we have established that $\boldsymbol{v}$ has the “minimum” regularity ensured by Lemma 5.4, we shall prove, in the next two lemmas that, in fact, $\boldsymbol{v}$ must be of class C^∞ in $\overline{\Omega} \times (\varepsilon, T]$, if Ω is uniformly of class C^∞. Now, while the assumption (i) coincides with that made for uniqueness when $s > n$, the assumption (ii) for $s = n$ is stronger. Actually, if we compare it with the analogous assumption for uniqueness in a domain with a compact boundary (see Remark 4.5), we see that regularity requires *continuity* in time, while uniqueness only requires *essential boundedness*. Though it may be very likely that this latter weaker condition also ensures regularity, no proof is so far available. To add more weight to this conjecture, there is the recent remarkable contribution of Necas, Ruzicka and Sverák (1996) who rule out a possible counter example to regularity proposed by Leray (1934b, pp. 225, 245) (see Remark 7.4). This weak solution (whose existence has been disproved by the previous authors) would satisfy *neither* conditions (i), (ii) of Lemma 5.4 but *only* the weaker assumption of being in the class $L^\infty(0,T;\boldsymbol{L}^n(\Omega))$. We shall return on the importance of this condition in Section 7.

In the next lemma we show that a weak solution satisfying either (i) or (ii) of Lemma 5.4, possesses time derivative of arbitrary order. The method of proof is borrowed from Heywood (1980).

Lemma 5.5 *Let* Ω *and* $\boldsymbol{v}$ *satisfy the assumption of* Lemma 5.4. *Then,*

$$D_t^\ell \boldsymbol{v} \in L^2(\varepsilon,T;\boldsymbol{W}^{2,2}(\Omega)), \quad \textit{for all } \ell \geq 0. \tag{5.12}$$

Proof. By Lemmas 5.1 and 5.4, it is enough to prove (5.12) for the solution $\boldsymbol{u}$ to (5.2). For $\ell = 0$ the result has already proved in Lemma 5.4. We then construct a solution $\boldsymbol{u}$ to (5.2) satisfying (5.12) for $\ell = 1$. By uniqueness, it will coincide with $\boldsymbol{v}$ which will then verify (5.12) wit $\ell = 1$. With this information on the coefficient, we shall then construct a solution $\boldsymbol{u}$ to (5.2) which satisfies (5.12) with $\ell = 2$. By uniqueness, it will coincide with $\boldsymbol{v}$ and so, by induction, we can prove (5.12) for arbitrary $k \in \mathbb{N}$. Here, for simplicity, we shall prove (5.12) for $\ell = 1$, referring the reader to the paper of Galdi and Maremonti (1988) for a proof in the general case. To construct the solution $\boldsymbol{u}$ we shall use the Faedo-Galerkin method. So, in addition to the estimates on the approximating solution that we have already obtained in the proof of Lemma 5.4, we obtain the following ones. We differentiate (5.6) with respect to time, multiply by dc_{kr}/dt, and sum over r from 1 to k to obtain

$$\tfrac{1}{2}\frac{d}{dt}\|D_t\boldsymbol{u}\|_2^2 + \nu\|\nabla D_t\boldsymbol{u}\|_2^2 = -(D_t\boldsymbol{v}\cdot\nabla\boldsymbol{u}, D_t\boldsymbol{u}), \tag{5.13}$$

where, as before, we have omitted the subscript "k". From the Hölder inequality, the Sobolev theorem, and Lemma 5.2 we find

$$\begin{aligned} |(D_t\boldsymbol{v}\cdot\nabla\boldsymbol{u}, D_t\boldsymbol{u})| &\leq \|D_t\boldsymbol{v}\|_2\|\nabla\boldsymbol{u}\|_3\|D_t\boldsymbol{u}\|_6 \leq c\|D_t\boldsymbol{v}\|_2\|\boldsymbol{u}\|_{2,2}\|\nabla D_t\boldsymbol{u}\|_2 \\ &\leq \|c\|D_t\boldsymbol{v}\|_2^2\|P\Delta\boldsymbol{u}\|_2^2 + \eta\|\nabla D_t\boldsymbol{u}\|_2^2, \end{aligned} \tag{5.14}$$

where η is a small positive number. Multiplying (5.6) by $\lambda_r c_{kr}$, summing over r and recalling the second part of Lemma 5.2, we find

$$(P\Delta\boldsymbol{u}, D_t\boldsymbol{u}) = \|P\Delta\boldsymbol{u}\|_2^2 + (\boldsymbol{v}\cdot\nabla\boldsymbol{u}, P\Delta\boldsymbol{u}).$$

From Lemma 5.5 and by the Sobolev theorem, we know that $\boldsymbol{v} \in C^0(\varepsilon, T; \boldsymbol{L}^3(\Omega))$ and so, we may use Lemma 5.3 in the preceding relation to obtain

$$\|P\Delta\boldsymbol{u}\|_2 \le c(\|D_t\boldsymbol{u}\|_2 + \|\nabla\boldsymbol{u}\|_2), \tag{5.15}$$

with a constant c independent of $t \in [\varepsilon, T]$. Replacing this inequality into (5.14), and recalling that $\|\nabla\boldsymbol{u}\|_2 \le C$ with C independent of t, we deduce

$$|(D_t\boldsymbol{v}\cdot\nabla\boldsymbol{u}, D_t\boldsymbol{u})| \le c\|D_t\boldsymbol{v}\|_2^2\|D_t\boldsymbol{u}\|_2^2 + \eta\|\nabla D_t\boldsymbol{u}\|_2^2.$$

With this estimate, equation (5.13) furnishes

$$\frac{d}{dt}\|D_t\boldsymbol{u}\|_2^2 + c_1\|\nabla D_t\boldsymbol{u}\|_2^2 \le c_2\|D_t\boldsymbol{v}\|_2^2\|D_t\boldsymbol{u}\|_2^2.$$

Integrating this inequality from τ to t and then on τ from ε to t, and recalling Lemma 5.4, we obtain that the weak solution $\boldsymbol{u}$ satisfies

$$D_t\boldsymbol{u} \in L^\infty(\varepsilon, T; \boldsymbol{L}^2(\Omega)) \cap L^2(\varepsilon, T; H^1(\Omega)). \tag{5.16}$$

By uniqueness, the same properties hold for $\boldsymbol{v}$. Notice that, by virtue of Lemmas 5.2 and 5.4, and (5.16) it also follows that

$$\boldsymbol{v}, \boldsymbol{u} \in L^\infty(\varepsilon, T; \boldsymbol{W}^{2,2}(\Omega)). \tag{5.17}$$

We now differentiate (5.6) with respect to t, multiply by $\lambda_r dc_{kr}/dt$ and sum over r. We get

$$\tfrac{1}{2}\frac{d}{dt}\|\nabla D_t\boldsymbol{u}\|_2^2 + \nu\|P\Delta D_t\boldsymbol{u}\|_2^2 = (D_t\boldsymbol{v}\cdot\nabla\boldsymbol{u}, P\Delta D_t\boldsymbol{u}) + (\boldsymbol{v}\cdot\nabla D_t\boldsymbol{u}, P\Delta D_t\boldsymbol{u}). \tag{5.18}$$

By the Hölder inequality, the Sobolev theorem, (5.17), and Lemma 5.2 it easily follows that

$$\begin{aligned}|(D_t\boldsymbol{v}\cdot\nabla\boldsymbol{u}, P\Delta D_t\boldsymbol{u})| &\le \|D_t\boldsymbol{v}\|_3\|\nabla\boldsymbol{u}\|_6\|P\Delta D_t\boldsymbol{u}\|_2 \le c\|D_t\boldsymbol{v}\|_{1,2}\|\boldsymbol{u}\|_{2,2}\|P\Delta D_t\boldsymbol{u}\|_2\\ &\le c_1\|D_t\boldsymbol{v}\|_{1,2}\|P\Delta D_t\boldsymbol{u}\|_2 \le c_2\|D_t\boldsymbol{v}\|_{1,2}^2 + \eta\|P\Delta D_t\boldsymbol{u}\|_2^2\\ |(\boldsymbol{v}\cdot\nabla D_t\boldsymbol{u}, P\Delta D_t\boldsymbol{u})| &\le \|\boldsymbol{v}\|_\infty\|\nabla D_t\boldsymbol{u}\|_2\|P\Delta D_t\boldsymbol{u}\|_2\\ &\le c\|D_t\boldsymbol{u}\|_{1,2}\|P\Delta D_t\boldsymbol{u}\|_2 \le c_2\|D_t\boldsymbol{u}\|_{1,2}^2 + \eta\|P\Delta D_t\boldsymbol{u}\|_2^2.\end{aligned}$$

We now replace this inequalities into (5.18) and integrate with respect to time twice, as we already did many times previously. If we then use (5.16), we arrive at (5.12) with $\ell = 1$. As we noticed, the general case is treated by an elementary induction procedure.

The next lemma provides regularity in space and time for a weak solution, for sufficiently smooth Ω.

Lemma 5.6 *Let $\boldsymbol{v}$ be a weak solution satisfying the assumption of Lemma 5.4. Assume, further, Ω uniformly of class C^m, $m \geq 2$. Then*

$$D_t^\ell \boldsymbol{v} \in L^2(\varepsilon, T; \boldsymbol{W}^{k,2}(\Omega)), \quad \textit{for all } \ell \geq 0 \textit{ and all } k = 2, \ldots, m. \tag{5.19}$$

Proof. The main idea is to write (5.2) as a Stokes system of the following type

$$\begin{aligned} &-\nu\Delta\boldsymbol{u} = -\frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{v}\cdot\nabla\boldsymbol{u} - \nabla\pi \equiv \boldsymbol{F} - \nabla\pi \\ &\operatorname{div}\boldsymbol{u} = 0 \\ &\boldsymbol{u}(y,t) = 0, \quad y \in \partial\Omega,\ t > 0. \end{aligned} \tag{5.20}$$

Then, as in the previous lemma, the proof is again based on an inductive argument and the "interplay" between $\boldsymbol{v}$ and $\boldsymbol{u}$. Specifically, knowing that

$$D_t^\ell \boldsymbol{F} \in L^2(\varepsilon, T; \boldsymbol{W}^{k,2}(\Omega)),$$

by Lemma 5.2 we deduce

$$D_t^\ell \boldsymbol{u} \in L^2(\varepsilon, T; \boldsymbol{W}^{k+2,2}(\Omega)),$$

and so, by uniqueness,

$$D_t^\ell \boldsymbol{v} \in L^2(\varepsilon, T; \boldsymbol{W}^{k+2,2}(\Omega)).$$

If we plug this information back into (5.20), we obtain that $\boldsymbol{F}$ has more spatial regularity than that assumed at the outset and, by induction, we obtain the proof. Referring the reader to the paper of Galdi and Maremonti (1988) for full details, we wish here to give a proof of the lemma for the case $m = 3$, $\ell = 1$. By what we said, it is enough to show that

$$(D_t^2\boldsymbol{u} + D_t\boldsymbol{v}\cdot\nabla\boldsymbol{u} + \boldsymbol{v}\cdot\nabla D_t\boldsymbol{u}) \in L^2(\varepsilon, T; \boldsymbol{W}^{1,2}(\Omega)).$$

From Lemma 5.5 we know already [21]

$$D_t^2 \boldsymbol{u} \in L^2(\varepsilon, T; \boldsymbol{W}^{1,2}(\Omega)).$$

By the Hölder inequality and Sobolev theorem, we have ($\partial_i = \partial/\partial x_i$)

$$\begin{aligned}
\|D_t \boldsymbol{v} \cdot \nabla \boldsymbol{u}\|_2 &\le \|D_t \boldsymbol{v}\|_4 \|\nabla \boldsymbol{u}\|_4 \le \|D_t \boldsymbol{v}\|_{1,2} \|\boldsymbol{u}\|_{2,2} \\
\|D_t \partial_i \boldsymbol{v} \cdot \nabla \boldsymbol{u}\|_2 &\le \|D_t \boldsymbol{v}\|_{2,2} \|\boldsymbol{u}\|_{2,2} \\
\|D_t \boldsymbol{v} \cdot \partial_i \nabla \boldsymbol{u}\|_2 &\le \|D_t \boldsymbol{v}\|_{1,2} \|\boldsymbol{u}\|_{2,2} \\
\|\boldsymbol{v} \cdot \nabla D_t \boldsymbol{u}\|_2 &\le \|\boldsymbol{v}\|_\infty \|\nabla D_t \boldsymbol{u}\|_2 \le \|\boldsymbol{v}\|_{2,2} \|D_t \boldsymbol{u}\|_{1,2} \\
\|\partial_i \boldsymbol{v} \cdot \nabla D_t \boldsymbol{u}\|_2 &\le \|\partial_i \boldsymbol{v}\|_4 \|\nabla D_t \boldsymbol{u}\|_4 \le \|\boldsymbol{v}\|_{2,2} \|D_t \boldsymbol{u}\|_{2,2} \\
\|\boldsymbol{v} \cdot \partial_i \nabla D_t \boldsymbol{u}\|_2 &\le \|\boldsymbol{v}\|_\infty \|\partial_i \nabla D_t \boldsymbol{u}\|_2 \le \|\boldsymbol{v}\|_{2,2} \|D_t \boldsymbol{u}\|_{2,2},
\end{aligned}$$

and the result follows from these inequalities and Lemma 5.5.

From the preceding lemma and the Sobolev theorem we at once deduce the following result

Theorem 5.2 *Let $\boldsymbol{v}$ be a weak solution in Ω_T, corresponding to $\boldsymbol{f} \equiv 0$ and to $\boldsymbol{v}_0 \in H(\Omega)$. Assume that $\boldsymbol{v}$ satisfies at least one of the following two conditions:*

(i) $\boldsymbol{v} \in L^r(0, T; \boldsymbol{L}^s(\Omega))$, *for some r, s such that* $\dfrac{n}{s} + \dfrac{2}{r} = 1, \ \ s \in (n, \infty]$;

(ii) $\boldsymbol{v} \in C^0([0, T]; \boldsymbol{L}^n(\Omega))$.

Then, if Ω is uniformly of class C^∞, we have

$$\boldsymbol{v} \in \boldsymbol{C}^\infty(\overline{\Omega} \times (0, T]).$$

Remark 5.5 Intermediate regularity results, with Ω only of class C^m, $m \ge 2$, can be directly obtained from Lemma 5.6, and the Sobolev theorem. We leave it to the reader as an exercise.

Remark 5.6 The same conclusion of Theorem 5.2 can be obtained under the following alternative assumptions, see Remarks 5.2, 5.3

[21] Recall that, by uniqueness, $\boldsymbol{v} = \boldsymbol{u}$.

(i)$'$ $\nabla \boldsymbol{v} \in L^{r'}(0,T;\boldsymbol{L}^{s'}(\Omega))$, $\frac{2}{r'}+\frac{n}{s'}=2$, $s' \in (n/2,\infty]$,

(i)$''$ $\nabla \boldsymbol{v} \in C^0([0,T);\boldsymbol{L}^{n/2}(\Omega))$.

Remark 5.7 Every weak solution in dimension 2 is L^2 strongly continuous, and, thus, by Theorem 5.2, it is regular in space and time. Regularity of weak solutions in dimension two was first obtained by Leray (1934a), Ladyzhenskaya (1958)

Remark 5.8 Theorem 5.2(i), for $\Omega = \mathbb{R}^3$ was proved for the first time by Leray (1934b, pp. 224-227), while for $\Omega = \mathbb{R}^n$, $n \geq 2$, and $s < \infty$ it is due to Fabes, Jones and Riviere (1972); see also Fabes, Lewis and Riviere (1977a, 1977b). Sohr (1983) proved Theorem 5.2(i) with $s < \infty$, for domains with a bounded boundary. An attempt to prove Sohr's result was already made by Kaniel and Shinbrot (1967). However, their proof contains an oversight which leads to the Corollary at p. 323 of their paper, where it is stated that, if Ω is of class C^∞, then any weak solution corresponding to initial data in $C^\infty(\overline{\Omega})$ and satisfying condition (i) is in $C^\infty(\overline{\Omega}\times[0,T])$. This result can not hold as stated, due to the fact that if a solution is regular *up to the time* $t=0$ *included*, then certain compatibility conditions have to be met, see Solonnikov (1964, p. 97 of the english translation), Heywood (1980, Remark at p. 677). The same oversight is contained in the book of Temam (1977, pp. 303, 307). The question of "how much smooth" a solution can be up to the time $t=0$, *without* compatibility conditions is studied by Rautmann (1983), von Wahl (1983) and Temam (1980). That condition (ii) implies regularity was first discovered by von Wahl (1986), in the case of a bounded domain. This latter result was extended to domains with a bounded boundary by Giga (1986). The case $n=3$, $s'=2$, $r'=4$ of Remark 5.6 for $\Omega = \mathbb{R}^3$, is due to Leray (1934b, p. 227); see also Section 6.

Remark 5.9 Regularity results involving assumptions on the pressure, rather than the velocity, have been given by Kaniel (1969) and, more recently, by Berselli (1999).

6 More Regular Solutions and the "Théorème de Structure".

The aim of this section is two-fold. On one hand, we would like to show that regular solutions do exist in three dimension if either we restrict ourselves to a "short" time interval, or if we choose initial data "small" compared to viscosity. [22] On the other hand, we wish to give more information about the possible formation of singularities for a weak solution, along the lines of the so-called "théorème de structure", Leray (1934b, pp. 244-245).

We have the following result due to Heywood (1980) [23].

Theorem 6.1 *Let* $\Omega \subset \mathbb{R}^3$ *be uniformly of class* C^2. *Then, for any* $\boldsymbol{v}_0 \in H^1(\Omega)$, *there exists* $T > 0$ *and at least one weak solution in* Ω_T *such that*

$$\boldsymbol{v} \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;\boldsymbol{W}^{2,2}(\Omega)).$$

The number T *is bounded from below by a constant depending only on* $\|\nabla \boldsymbol{v}_0\|_2$, ν *and the* C^2*-regularity of* Ω. *In the case when* Ω *is bounded or* $\Omega = \mathbb{R}^n$ *we have*

$$T \geq \nu^3 C/\|\nabla \boldsymbol{v}_0\|_2^4,$$

where C *depends only on* Ω. *Moreover, there is a decreasing function* $G = G(\lambda)$, $\lambda > 0$, *such that if*

$$\|\boldsymbol{v}_0\|_2 \leq G(\|\nabla \boldsymbol{v}_0\|_2),$$

T *can be taken as an arbitrary positive number. In the case when* Ω *is bounded or* $\Omega = \mathbb{R}^n$ *we have* $G = C\nu^2/\|\nabla \boldsymbol{v}_0\|_2$, *with* C *depending only on* Ω.

Proof. We shall show the result for the case Ω bounded, referring the reader to the paper of J. Heywood for the general case. To show the existence of such a solution, we then use the Faedo-Galerkin method of Theorem 3.1, with the basis of the eigenfunctions of the Stokes problem, see Lemma 5.3. In addition to the estimate (3.4) with $\boldsymbol{f} \equiv 0$, we obtain the following one. We multiply (3.2) (with $\boldsymbol{f} \equiv 0$) by $\lambda_r c_{kr}$ and sum over r, to get (as usual, we omit the subscript "k")

$$\tfrac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}(t)\|_2^2 + \nu\|P\Delta \boldsymbol{v}\|_2^2 = (\boldsymbol{v}\cdot\nabla\boldsymbol{v}, P\Delta\boldsymbol{v}). \tag{6.1}$$

Using the Hölder inequality, the Sobolev theorem, and Lemma 5.2 we have the following two different ways of increasing the term N (say) on the right-hand side of this equation, namely,

a) $N \leq \|\boldsymbol{v}\|_6\|\nabla\boldsymbol{v}\|_3\|P\Delta\boldsymbol{v}\|_2 \leq c\|\nabla\boldsymbol{v}\|_2^{3/2}\|P\Delta\boldsymbol{v}\|_2^{3/2} \leq c\nu^{-3}\|\nabla\boldsymbol{v}\|_2^6 + \frac{1}{2}\nu\|P\Delta\boldsymbol{v}\|_2^2$

[22] We assume hereafter, for simplicity, that $\boldsymbol{f} \equiv 0$. We also notice that existence of regular and global solutions in dimension 2 has been established in Theorem 5.2, see Remark 5.7.

[23] Actually, Heywood requires more regularity on the boundary than that requested in Theorem 6.1.

b) $N \le \|\boldsymbol{v}\|_3 \|\nabla \boldsymbol{v}\|_6 \|P\Delta \boldsymbol{v}\|_2 \le c\|\boldsymbol{v}\|_2^{1/2} \|\nabla \boldsymbol{v}\|_2^{1/2} \|P\Delta \boldsymbol{v}\|_2^2$.

Replacing a) in (6.1) and setting $y(t) = \|\nabla \boldsymbol{v}(t)\|_2^2$ we find

$$\frac{dy}{dt} \le \nu^{-3} c y^3, \tag{6.2}$$

which, by Gronwall's lemma, and (6.1) and a), in turns gives

$$\|\nabla \boldsymbol{v}(t)\|_2 + \int_0^t \|P\Delta \boldsymbol{v}(s)\|_2^2 ds \le M, \quad \text{for all } t \in [0, T) \tag{6.3}$$

where $[0, T)$ is the maximal interval of existence of the differential inequality (6.2). By classical comparison theorems for differential inequalities, we have $T \ge \nu^3 / 2c\|\nabla \boldsymbol{v}_0\|_2^4$. In case b) we find

$$\frac{d}{dt}\|\nabla \boldsymbol{v}(t)\|_2^2 + (\nu - c\|\boldsymbol{v}\|_2^{1/2} \|\nabla \boldsymbol{v}\|_2^{1/2}) \|P\Delta \boldsymbol{v}\|_2^2 \le 0,$$

which, once integrated, furnishes (6.3) for arbitrary $T > 0$, provided

$$\nu > c\|\boldsymbol{v}_0\|_2^{1/2} \|\nabla \boldsymbol{v}_0\|_2^{1/2}.$$

Using the estimate (6.3) along the approximating solutions, together with the procedure employed in Theorem 3.1, we then show the result.

From this theorem, Theorem 5.2, and $(2.12)_2$ we then obtain the following result.

Theorem 6.2 *Let $\Omega \subset \mathbb{R}^3$ be uniformly of class C^∞.* [24] *Then, for any $\boldsymbol{v}_0 \in H^1(\Omega)$ there exist $T > 0$ and a unique solution to (0.1)-(0.3) with $\boldsymbol{f} \equiv 0$, which assumes the data $\boldsymbol{v}_0$ and which is of class $\boldsymbol{C}^\infty(\overline{\Omega} \times (0, T))$. Moreover, there exists a positive constant $C(\Omega)$ such that, if*

$$\|\boldsymbol{v}_0\|_2 \le G(\|\nabla \boldsymbol{v}_0\|_2),$$

with G defined in Theorem 6.1, *we can take T arbitrarily large.*

We shall now derive some other consequences of Theorem 6.1. Following Leray, we are able to specify better the set of times where a weak solution can be irregular. This can be done for all those Ω for which a *strong energy*

[24] See Remark 5.5.

inequality holds (see (4.1)). Specifically, we have the following result of “partial regularity”.

Theorem 6.3 (Théorème de Structure) *Let* Ω *satisfy the assumption of* Theorem 6.2. [25] *Assume* $\boldsymbol{v}$ *is a weak solution in* Ω_T*, for all* $T > 0$*, corresponding to* $\boldsymbol{f} \equiv 0$ *and satisfying the strong energy inequality* (4.1). *Then, there exists a union* $\mathcal{T}$ *of disjoint open time intervals such that:*

(i) The Lebesgue measure of $(0,\infty) - \mathcal{T}$ *is zero;*

(ii) $\boldsymbol{v}$ *is of class* C^∞ *in* $\overline{\Omega} \times \mathcal{T}$*,*

(iii) There exists $T^* \in (0,\infty)$ [26] *such that* $\mathcal{T} \supset (T^*,\infty)$*;*

(iv) If $\boldsymbol{v}_0 \in H^1(\Omega)$ *then* $\mathcal{T} \supset (0,T_1)$ *for some* $T_1 > 0$.

Proof. Since

$$\|\boldsymbol{v}(t)\|_2^2 + \int_0^\infty \|\nabla \boldsymbol{v}(\tau)\|_2^2 d\tau \le \|\boldsymbol{v}_0\|_2^2 \quad \text{for all } T > 0,$$

and since $\boldsymbol{v}$ verifies (4.1) for almost all $s > 0$, we can find T^* with the following properties:

a) $\|\boldsymbol{v}(T^*)\|_2 \le G(\|\nabla \boldsymbol{v}(T^*)\|_2)$,

b) The strong energy inequality (4.1) holds with $s = T^*$,

where G is the function introduced in Theorem 6.1. Let us denote by $\widetilde{\boldsymbol{v}}$ the solution of Theorem 6.1 corresponding to the data $\boldsymbol{v}(T^*)$. By a), $\widetilde{\boldsymbol{v}}$ exists for all times $t \ge T^*$ and, by Theorem 6.2, it is of class C^∞ in $\Omega \times (T^*,\infty)$. By the uniqueness Theorem 4.2 we must have $\boldsymbol{v} = \widetilde{\boldsymbol{v}}$ in $\Omega \times (T^*,\infty)$, and part (iii) is proved. Next, denote by I the subset of $(0,T^*)$ where the following conditions are met:

a) $\|\boldsymbol{v}(t)\|_{1,2} < \infty$, for $t \in I$,

b) The strong energy inequality (4.1) holds with $s \in I$.

[25] See Remark 5.5.

[26] T^* can be estimated from above by a quantity depending only on $\|\boldsymbol{v}_0\|_2$ and Ω, see Heywood (1980, Theorem 8 (ii)). See also Remark 6.3.

Clearly, $(0,T^*) - I$ is of zero Lebesgue measure. Moreover, for every $t_0 \in I$ we can construct in the time interval $(t_0, t_0 + T(t_0))$ a solution $\widetilde{\boldsymbol{v}}$ assuming at t_0 the initial data $\boldsymbol{v}(t_0)$ $(\in H^1(\Omega))$. From Theorems 6.1 and 4.1, we know that $\widetilde{\boldsymbol{v}}$ is of class C^∞ in $\Omega \times (t_0, t_0 + T(t_0))$ and that it coincides with $\boldsymbol{v}$, since this latter satisfies the energy inequality with $s = t_0$. It is obvious that the set $\bigcup_{t_0 \in I} (t_0, t_0 + T(t_0)) - I$ has zero Lebesgue measure. Finally, if $\boldsymbol{v}_0 \in H^1(\Omega)$, by Theorems 6.1 and 6.2, there exists $T_1 > 0$ such that $\boldsymbol{v}$ is of class C^∞ in $\Omega \times (0, T_1)$. The theorem thus follows with $\mathcal{T} \equiv \bigcup_{t_0 \in I} (t_0, t_0 + T(t_0)) \cup (T^*, \infty)$.

Remark 6.1 It is likely that Theorem 6.3 holds for any (sufficiently smooth) domain. However, no proof is so far available, since one can prove the strong energy inequality only for certain domains (see Section 4). On the other hand, Heywood (1988) has shown that for any Ω, uniformly of class C^2, and any $\boldsymbol{v}_0 \in H(\Omega)$ there exists at least one corresponding weak solution $\boldsymbol{v}$ satisfying the following condition: There exists an open set $R \subseteq [0, \infty)$ such that

a) $[0, \infty) - R$ has zero Lebesgue measure;

b) For every compact interval $[\alpha, \beta] \subset R$ there holds

$$\sup_{t \in [\alpha, \beta]} \|\boldsymbol{v}(t)\|_{1,2}^2 + \int_\alpha^\beta \left(\|\boldsymbol{v}(\tau)\|_{2,2}^2 + \|D_\tau \boldsymbol{v}(\tau)\|_2^2 \right) ds < \infty.$$

Since it is not known if weak solutions in dimension 3 are unique in their class, we can not conclude from this result that *any* weak solution satisfies a) and b). Notice that, by Theorem 5.2, every weak solution satisfying b) is of class $\boldsymbol{C}^\infty(\overline{\Omega} \times (\alpha, \beta])$, if Ω is uniformly of class C^∞.

Our next objective is to investigate when and in which way a weak solution $\boldsymbol{v}$ can become irregular, and to give a more precise estimate of the set of the possible irregular times. From Theorem 6.3, we know that this set is the complement to $(0, \infty)$ of a union $\mathcal{T}$ of intervals, and that, under suitable assumptions on the smoothness of Ω, $\boldsymbol{v} \in C^\infty(\overline{\Omega} \times \mathcal{T})$.

For simplicity, *in the remaining part of this section, we shall assume that the domain* $\Omega \subset \mathbb{R}^3$ *is either bounded and uniformly of class* C^∞*, or* $\Omega = \mathbb{R}^3$*, and that* $\boldsymbol{f} \equiv 0$.

Following Leray (1934b, p. 224) we give the following

Definition 6.1 We shall say that a solution $\boldsymbol{v}$, *becomes irregular at the time* t_1 if and only if

a) t_1 is finite;

b) $\boldsymbol{v} \in C^\infty(\overline{\Omega} \times (t_0, t_1))$, for some $t_0 < t_1$;

c) It is not possible to extend $\boldsymbol{v}$ to a regular solution in (t_0, t') with $t' > t_1$.

The number t_1 will be called *epoch of irregularity* (“époque de irrégularité”, Leray, *loc. cit.*).

We shall denote by $\mathcal{I} = \mathcal{I}(\boldsymbol{v})$ the set of all possible epochs of irregularity. As we know from Theorem 6.3, the one-dimensional Lebesgue measure of $\mathcal{I}$ is zero.

We have the following result which is essentially due to Leray (1934b, pp. 245-246) and Scheffer (1976a).

Theorem 6.4 *Let $\boldsymbol{v}$ be a weak solution in Ω_T, for all $T > 0$, corresponding to the initial data $\boldsymbol{v}_0 \in H(\Omega)$, and satisfying the strong energy inequality (4.1). Let t_1 be an epoch of irregularity for $\boldsymbol{v}$. Then, the following properties hold:*

(i) $\|\nabla \boldsymbol{v}(t)\|_2$ diverges as $t \to t_1^-$ in such a way that

$$\|\nabla \boldsymbol{v}(t)\|_2 \geq \frac{C\nu^{3/4}}{(t_1 - t)^{1/4}}, \quad t < t_1,$$

with $C = C(\Omega) > 0$;

(ii) There exists a constant $C > 0$, depending only on Ω, such that

$$t_1 \leq C\nu^{-5}\|\boldsymbol{v}_0\|_2^4.$$

(iii) The one-half dimensional Hausdorff measure of $\mathcal{I}(\boldsymbol{v})$ is equal to zero. [27]

[27] Let S be a subset of $\mathbb{R}^n$. The m-dimensional (spherical) Hausdorff measure $\mathcal{H}^m$ of S is given by

$$\mathcal{H}^m(S) = \lim_{\delta \to 0} \mathcal{H}^m_\delta(S),$$

where

$$\mathcal{H}^m_\delta(S) = \inf \sum_i (2^{-1} \operatorname{diam} B_i)^m,$$

the infimum being taken over all at most countable coverings $\{B_i\}$ of S constituted by closed balls B_i with

$$\operatorname{diam} B_i < \delta, \quad \text{for all } i,$$

see, *e.g.*, Simon (1983).

Proof. Let t_1 be an epoch of irregularity. Then,

$$\lim_{t\to t_1^-} \|\nabla \boldsymbol{v}(t)\|_2 = \infty. \tag{6.4}$$

Actually, assuming that (6.4) does not hold, there would exist a sequence $\{\tau_k\}$ tending to t_1, $\tau_k < t_1$ for all $k \in \mathbb{N}$, and a number $M > 0$ such that

$$\|\nabla \boldsymbol{v}(\tau_k)\|_2 \leq M.$$

Since $\boldsymbol{v}(\tau_k) \in H^1(\Omega)$, by Theorem 6.1 we may construct a solution $\overline{\boldsymbol{v}}$ with initial data $\boldsymbol{v}(\tau_k)$, in a time interval $(\tau_k, \tau_k + T_1)$ where

$$T_1 \geq A/\|\nabla \boldsymbol{v}(\tau_k)\|_2^4 \geq AM \equiv T_0,$$

and A depends only on Ω and ν. The solution $\overline{\boldsymbol{v}}$ belongs to $L^\infty(\tau_k, \tau_k + T_0; H^1(\Omega))$ and so, by the Sobolev theorem, it satisfies Theorem 5.2(i) with $s = 6$ and $r = 4$ (for instance). Therefore, $\overline{\boldsymbol{v}} \in C^\infty(\overline{\Omega} \times (\tau_k, \tau_k + T_0])$. Moreover, by the uniqueness Theorem 4.2, $\boldsymbol{v} = \overline{\boldsymbol{v}}$ in $[\tau_k, \tau_k + T_0]$. We may now select τ_k such that $\tau_k + T_0 > t_1$, contradicting the assumption that t_1 is an epoch of irregularity, and (6.4) follows. We next operate as in the proof of Theorem 6.1, to show that $y(t) \equiv \|\nabla \boldsymbol{v}(t)\|_2^2$ satisfies (6.2) in the time interval (t_0, t_1). Integrating (6.2) we then find

$$\frac{1}{\|\nabla \boldsymbol{v}(t)\|_2^4} - \frac{1}{\|\nabla \boldsymbol{v}(\tau)\|_2^4} \leq \nu^{-3} c(\tau - t), \quad t_0 < t < \tau < t_1.$$

Letting $\tau \to t_1$ and recalling (6.4), we prove (i). Property (ii) is simply obtained, by integrating the inequality in (i) from 0 to t_1, and then using the energy inequality (EI) in Theorem 3.1. To show (iii) we observe that the set $\mathcal{T}$ introduced in Theorem 6.3, can be decomposed as follows

$$\mathcal{T} = \left(\bigcup_{i\in I} (\tau_i, s_i) \right) \cup (T^*, \infty), \quad \tau_i < s_i,$$

where $T^* < \infty$, each s_i is an epoch of irregularity, and

$$\begin{aligned} &(\tau_i, s_i) \subset [0, T^*], \quad \text{for all } i \in I; \\ &(\tau_i, s_i) \cap (\tau_j, s_j) = \emptyset, \quad i \neq j. \end{aligned} \tag{6.5}$$

From (i) and the energy inequality (EI) we at once deduce that

$$\sum_{i\in I}(\tau_i - s_i)^{1/2} \le C\sum_{i\in I}\int_{\tau_i}^{s_i} \|\nabla \boldsymbol{v}(\tau)\|_2^2 dt \le C_1\|\boldsymbol{v}_0\|_2^2.$$

Thus, for every $\delta > 0$ we can find a finite part I_δ of I such that

$$\sum_{i\notin I_\delta}(\tau_i - s_i) < \delta, \qquad \sum_{i\notin I_\delta}(\tau_i - s_i)^{1/2} < \delta. \tag{6.6}$$

By $(6.5)_1$, $\cup_{i\in I}(\tau_i, s_i) \subset [0, T^*]$ and so the set

$$[0, T^*] - \cup_{i\in I_\delta}(\tau_i, s_i)$$

consists of a finite number of disjoint closed intervals B_j, $j = 1, \dots, N$. Clearly,

$$\bigcup_{j=1}^{N} B_j \supset \mathcal{I}(\boldsymbol{v}). \tag{6.7}$$

By $(6.5)_2$, we have that each interval (τ_i, s_i), $i \notin I_\delta$, is included in one and only one B_j. Denote by I_j the set of all indeces i satisfying $B_j \supset (\tau_i, s_i)$. We thus have

$$\begin{aligned} I &= I_\delta \cup \left(\bigcup_{j=1}^{N} I_j\right) \\ B_j &= \left(\bigcup_{i\in I_j}(\tau_i, s_i)\right) \cup \left(B_j \cap \mathcal{I}(\boldsymbol{v})\right). \end{aligned} \tag{6.8}$$

By Theorem 6.3, the set $\mathcal{I}$ has zero Lebesgue measure and so, from $(6.8)_2$ we have

$$\operatorname{diam} B_j = \sum_{i\in I_j}(\tau_i - s_i).$$

Thus, by (6.6),

$$\operatorname{diam} B_j \le \sum_{i\notin I_\delta}(\tau_i - s_i) < \delta \tag{6.9}$$

and, again by (6.6) and $(6.8)_1$,

$$\sum_{j=1}^{N}(\operatorname{diam} B_j)^{1/2} \le \sum_{j=1}^{N}\left(\sum_{i\in I_j}(\tau_i - s_i)\right)^{1/2} \le \sum_{i\notin I_\delta}(\tau_i - s_i)^{1/2} < \delta. \tag{6.10}$$

Therefore, property (iii) follows from (6.7), (6.9) and (6.10).

Remark 6.2 From Theorem 6.4(i) it follows that a sufficient condition for the absence of epochs of irregularity is that

$$\nabla \boldsymbol{v} \in L^4(0,T;\boldsymbol{L}^2(\Omega)),$$

a fact discovered for the first time by Leray (1934b, p. 227) when $\Omega = \mathbb{R}^3$. As we already noticed, this is a particular case of the more general conditions furnished in Remark 5.6.

Remark 6.3 From Theorem 6.4(ii) it follows that the number T^* introduced in Theorem 6.3 is bounded above by $\nu^{-5}C\|\boldsymbol{v}_0\|_2^4$, with $C = C(\Omega)$. Moreover, assume $\boldsymbol{v}_0 \in H^1(\Omega)$. By Theorem 6.1 we then know that any epoch of irregularity t_1 satisfies the following estimate

$$t_1 \geq \nu^3 C/\|\nabla \boldsymbol{v}_0\|_2^4,$$

with $C = C(\Omega)$. Thus, from this inequality and Theorem 6.4(ii), it follows that there exists $A = A(\Omega) > 0$ such that if

$$\|\boldsymbol{v}_0\|_2\|\nabla \boldsymbol{v}_0\|_2 \leq A\nu^2,$$

the set $\mathcal{I}(\boldsymbol{v})$ is empty, and we reobtain the second part of Theorem 6.2.

Remark 6.4 There is a wide range of results concerning "partial regularity" of "suitable" weak solutions, that we will not treat here. In this regard, we refer the reader to the work of Scheffer (1976a, 1976b, 1977, 1978, 1980, 1982, 1985), Foiaş and Temam (1979), Caffarelli, Kohn and Nirenberg (1982), Maremonti (1987), Wu (1991), Lin (1998), and Ladyzhenskaya and Seregin (1999).

7 Existence in the Class $L^r(0,T;\boldsymbol{L}^s(\Omega))$, $2/r+n/s = 1$, and Further Regularity Properties.

Theorem 5.1 has revealed that the functional class

$$L^{s,r}(\Omega_T) \equiv L^r(0,T;\boldsymbol{L}^s(\Omega)), \quad 2/r + n/s = 1, \tag{7.1}$$

plays a crucial role in the study of regularity of weak solutions. However, as we have seen in Remark 4.6, unless $n = 2$, it is not known whether a *generic* weak solution belongs to this class, for a suitable choice of r and s. It seems, therefore, of the utmost importance to investigate under which assumptions on the initial data $\boldsymbol{v}_0$ [28] one can *construct* a weak solution which, in addition, belongs to such a class. For example, in Theorem 6.1 we have shown that this happens if $\boldsymbol{v}_0 \in H^1(\Omega)$. Our main objective in this section is to show existence of weak solutions in the class (7.1), [29] under mild assumption on $\boldsymbol{v}_0$, namely, that it belongs to Lebesgue spaces $\boldsymbol{L}^\sigma(\Omega)$. Though obvious, it is worth noticing that, in order to show regularity of weak solutions, it would *not* really matter if existence in the class (7.1) is proved for a *short* time T (say) only, on condition that one could take σ and T suitably. For instance, regularity would trivially follow if we could take $\sigma = 2$ and T a decreasing function of $\|\boldsymbol{v}_0\|_2$. However, the existence theory known so far, with data in L^q, requires $\sigma \geq n$. [30, 31]

In order to avoid technical difficulties, in what follows we shall assume that $\Omega = \mathbb{R}^n$, referring to Giga (1986, Theorem 4) for the more general case when Ω has a (non-empty) compact boundary. [32] The results we shall prove will be then an immediate consequence of suitable estimates for solutions to the heat equation and of the classical successive approximation method applied to the *linearized Stokes problem* (see (7.3) below). In fact, using a decomposition lemma of the Helmholtz-Weyl type, we shall see that the assumption $\Omega = \mathbb{R}^n$ allows us to treat this latter problem as a (vector) heat equation.

We have the following.

Lemma 7.1 *Let $\boldsymbol{F} = \{F_{ij}\}$ be a second order tensor field with*

$$F_{ij} \in L^r(\mathbb{R}^n),\ \partial_i F_{ij} \in L^s(\mathbb{R}^n), \quad j = 1, \dots, n\,, \quad 1 < r, s < \infty.$$

Then, there exists a second order tensor field $\boldsymbol{G} = \{G_{ij}\}$ with $\partial_j \partial_i G_{ij} = 0$, [33]

[28] Throughout this section, for the sake of simplicity, we shall assume $\mathbf{f} \equiv 0$.

[29] See Remark 4.7.

[30] This is another way of obtaining regularity of weak solution for $n = 2$.

[31] Weak solutions with data in L^σ, $2 < \sigma < n$, have been constructed by Calderon (1990a). For existence of strong solutions with data in suitable Besov spaces, larger than L^n, see Cannone (1997), Kozono (1998), Kozono and Yamazaki (1998), Amann (1999), and the extensive literature cited therein.

[32] For the Cauchy problem, see also T. Kato (1984).

[33] We shall use the Einstein summation convention over repeated indeces. This condition on G_{ij} has to be understood in the distributional sense.

and a scalar field p such that, for all $i, j = 1, \dots, n$,

$$
\begin{gathered}
G_{ij} \in L^r(\mathbb{R}^n), \quad \partial_i G_{ij} \in L^s(\mathbb{R}^n), \quad p \in L^r(\mathbb{R}^n), \quad \partial_i p \in L^s(\mathbb{R}^n) \\
\partial_i F_{ij} = \partial_i G_{ij} + \partial_j p \\
\|G_{ij}\|_r \leq c(n,r)\|F_{ij}\|_r \\
\|\partial_i G_{ij}\|_s \leq c(n,s)\|\partial_i F_{ij}\|_s.
\end{gathered}
$$

Proof. Without loss of generality, we may assume that F_{ij} are smooth functions with compact support in $\mathbb{R}^n$, see Galdi (1994, Lemma VII.4.3). We set

$$
\begin{aligned}
p(x) &= \int_{\mathbb{R}^n} \mathcal{E}(x-y)\partial_i\partial_j F_{ij}(y)dy \\
G_{ij} &= \delta_{ij}p - F_{ij},
\end{aligned}
$$

where $\mathcal{E}(\xi)$ is the fundamental solution of Laplace's equation. It is clear that

$$
\begin{aligned}
\partial_i F_{ij} &= \partial_i G_{ij} + \partial_j p \\
\partial_j \partial_i G_{ij} &= 0.
\end{aligned}
$$

Moreover, from the Calderon-Zygmund theorem on singular integrals we find that

$$
\begin{aligned}
\|\nabla p\|_s &\leq c(n,s)\|\partial_i F_{ij}\|_s \\
\|p\|_r &\leq c(n,r)\|F_{ij}\|_r,
\end{aligned}
\tag{7.2}
$$

and the lemma is proved.

Our next objective is to prove some existence results for weak solutions $\boldsymbol{u}$ to the following Cauchy problem for the linearized *Stokes system*:

$$
\left.
\begin{aligned}
&\frac{\partial \boldsymbol{u}}{\partial t} = \nu \Delta \boldsymbol{u} + \nabla p + \operatorname{div} \boldsymbol{F} \\
&\operatorname{div} \boldsymbol{u} = 0
\end{aligned}
\right\} \text{ in } \mathbb{R}^n_T
\\
\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x)
\tag{7.3}
$$

where $\boldsymbol{F}$ is a given second-order tensor field, and $\{\operatorname{div} \boldsymbol{F}\}_j = \partial_i F_{ij}$. As usual, we shall say that $\boldsymbol{u}$ is a weak solution to (7.3) if $\boldsymbol{u} \in V_T$ (see Definition 2.1), and it satisfies the following relation

$$\int_0^\infty \left\{ \left(\boldsymbol{u}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{\varphi}) \right\} dt = \int_0^\infty (\boldsymbol{F}, \nabla \boldsymbol{\varphi}) dt - (\boldsymbol{u}_0, \boldsymbol{\varphi}(0)), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_T. \tag{7.4}$$

Before proving our results, however, we wish to recall some well-known properties concerning the heat equation and classical inequalities. Denote by $W(x,t)$, $(x,t) \in \mathbb{R}^n_T$, the *Weierstrass function*, that is,

$$W(x,t) = \frac{1}{(4\pi\nu t)^{n/2}} \exp\left\{ -\frac{x^2}{4\nu t} \right\}.$$

By a direct computation, we show that

$$\begin{aligned} |W(x,t)| &\le \frac{c}{(x^2+t)^{n/2}} \\ |\partial_k W(x,t)| &\le \frac{c}{(x^2+t)^{(n+1)/2}}, \end{aligned} \tag{7.5}$$

where $c = c(n,\nu)$. For $u_0 \in L^\sigma(\mathbb{R}^n)$ and $f \in L^{s,r}(\mathbb{R}^n_T)$ the convolutions

$$U(x,t) = \int_{\mathbb{R}^n} W(x-y,t) u_0(y) dy \equiv W * u_0$$

and

$$U_1(x,t) = \int_0^t \left(\int_{\mathbb{R}^n} W(x-y,t-\tau) f(y,\tau) dy \right) d\tau$$

are called the *volume potential* and *volume heat potential*, respectively. It is well known, see, *e.g.* Ladyzhenskaya, Ural´ceva and Solonnikov (1968, Chapter IV, §1), that the volume potentials solve the following Cauchy problems for the *heat equation*

$$\frac{\partial U}{\partial t} = \nu \Delta U \quad \text{in } \mathbb{R}^n \times \{t > 0\},$$
$$\lim_{t\to 0} \|U(t) - u_0\|_2 = 0$$

and

$$\frac{\partial U_1}{\partial t} = \nu \Delta U_1 + f \quad \text{in } \mathbb{R}^n \times \{t > 0\},$$
$$\lim_{t \to 0} U_1(x,t) = 0, \quad x \in \mathbb{R}^n.$$

For $f \in L^{q_1}(\mathbb{R}^n)$, $1 < q_1 < \infty$, $n \geq 1$, we set

$$Tf(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}} dy, \quad 0 < \lambda < n.$$

Then, the following *Hardy-Littlewood-Sobolev inequality* holds (see, $e.g.$ Stein (1970))

$$\|Tf\|_p \leq c\|f\|_{q_1}, \quad \frac{1}{q_1} = \frac{1}{p} + \frac{\lambda}{n}. \tag{7.6}$$

Finally, if $t^\alpha u \in BC([0,T); L^q(\mathbb{R}^n))$, and $w \in L^{r,s}(\mathbb{R}^n_T)$, we set

$$\langle\!\langle u \rangle\!\rangle_{q,\alpha,T} = \sup_{t \in [0,T)} t^\alpha \|u(t)\|_q$$

$$\|w\|_{r,s,T} \equiv \|w\|_{L^{r,s}(\mathbb{R}^n_T)}.$$

We are now in a position to prove the following result.

Lemma 7.2 *Let* $n < s < \infty$, $1 < q_1 < \infty$, $1/s_1 = 1/s + 1/2$, $1/q_2 = 1 - n/2s$, $\alpha = (1 - n/s)/2$. *Assume*

$$\boldsymbol{F} \in L^{s/2,q_1}(\mathbb{R}^n_T)$$

$$\operatorname{div} \boldsymbol{F} \in L^{s_1,q_2}(\mathbb{R}^n_T)$$

$$t^{2\alpha}\boldsymbol{F} \in BC([0,T); \boldsymbol{L}^{s/2}(\mathbb{R}^n))$$

$$t^{\alpha}\boldsymbol{F} \in BC([0,T); \boldsymbol{L}^{s_1}(\mathbb{R}^n)).$$

Assume also that $\boldsymbol{u}_0 \in H(\mathbb{R}^n)$ *and that*

$$W * \boldsymbol{u}_0 \in L^{s,r}(\mathbb{R}^n_T)$$

$$t^\alpha W * \boldsymbol{u}_0 \in BC([0,T); \boldsymbol{L}^s(\mathbb{R}^n))$$

where W *is the Weiestrass function and*

$$\frac{1}{r} = \frac{1}{q_1} - \frac{1}{2}\left(1 - \frac{n}{s}\right).$$

Then, there exists a unique weak solution $\boldsymbol{u}$ *to (7.3) such that*

$$\boldsymbol{u} \in L^{s,r}(\mathbb{R}^n_T)$$

$$t^\alpha \boldsymbol{u} \in BC([0,T); \boldsymbol{L}^s(\mathbb{R}^n)).$$

This solution satisfies the following estimates

$$\begin{aligned} \|\boldsymbol{u}\|_{s,r,T} &\le \|W*\boldsymbol{u}_0\|_{s,r,T} + C\|\boldsymbol{F}\|_{s/2,q_1,T} \\ \langle\!\langle \boldsymbol{u}\rangle\!\rangle_{s,\alpha,T} &\le \langle\!\langle W*\boldsymbol{u}_0\rangle\!\rangle_{s,\alpha,T} + C\langle\!\langle \boldsymbol{F}\rangle\!\rangle_{s/2,2\alpha,T} \\ \|\boldsymbol{u}\|_{2,\infty,T} &\le \|\boldsymbol{u}_0\|_2 + C\langle\!\langle \boldsymbol{F}\rangle\!\rangle_{s_1,\alpha,T} \\ \|\nabla\boldsymbol{u}\|_{2,2,T} &\le C(\|\boldsymbol{u}_0\|_2 + \|\mathrm{div}\,\boldsymbol{F}\|_{s_1,q_2}), \end{aligned} \tag{7.7}$$

with $C=C(\nu,n,s,q_1)$.

Proof. Uniqueness of the solution in the class V_T is easy to show along the same lines of the proof of Theorem 4.2, and we leave it to the reader. In view of the Helmholtz-Weyl decomposition result given in Lemma 7.1, it is enough to give the proof of existence for the following non-homogeneous heat equation problem

$$\begin{aligned} &\frac{\partial u}{\partial t} = \nu\Delta u + \mathrm{div}\,\boldsymbol{F} \quad \text{in } \mathbb{R}^n\times(0,T) \\ &u(x,0)=u_0(x), \end{aligned}$$

where u is the j-th component of the velocity field, and $\boldsymbol{F}=(G_{j1},\ldots,G_{jn})$, with $\boldsymbol{G}$ given in Lemma 7.1. A solution to this problem may be written as the sum of the volume potential corresponding to u_0 and to the heat volume potential corresponding to $\mathrm{div}\,\boldsymbol{F}$, namely,

$$u(t) = W(t)*u_0 + \int_0^t W(t-\tau)*\mathrm{div}\,\boldsymbol{F}(\tau)d\tau. \tag{7.8}$$

Integrating by parts in the space variables in the last integral, we have

$$u(t) = W(t)*u_0 - \int_0^t \partial_k W(t-\tau)*F_k(\tau)d\tau. \tag{7.9}$$

From (7.9), we find

$$\|u(t)\|_\sigma \le \|W(t)*u_0\|_\sigma + \int_0^t \|\partial_k W(t-\tau)*F_k(\tau)\|_\sigma\, d\tau. \tag{7.10}$$

Using $(7.5)_2$ it follows that

$$|\partial_k W(t-\tau)*F_k(\tau)| \le C\int_{\mathbb{R}^n} \frac{|\boldsymbol{F}(y,\tau)|}{\big(|x-y|^2+(t-\tau)\big)^{(n+1)/2}}dy.$$

Since

$$\frac{1}{(|x|^2+t)^{(n+1)/2}} \le \frac{c(\beta)}{|x|^{\beta(n+1)}t^{(n+1)(1-\beta)/2}}, \quad \beta \in (0,1),$$

we obtain

$$|\partial_k W(t-\tau) * F_k(\tau)| \le \frac{C}{(t-\tau)^{(n+1)(1-\beta)/2}} \int_{\mathbb{R}^n} \frac{|\boldsymbol{F}(y,\tau)|}{|x-y|^{\beta(n+1)}} dy.$$

Thus, choosing $\beta < n/(n+1)$, and using (7.6) we deduce

$$\|\partial_k W(t-\tau) * F_k(\tau)\|_\sigma \le \frac{\|\boldsymbol{F}\|_{\sigma_1}}{(t-\tau)^{\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma}\right)+\frac{1}{2}}}, \quad \frac{1}{\sigma_1} = \frac{1}{\sigma}+1-\frac{(n+1)\beta}{n}. \tag{7.11}$$

Since β is arbitrary in $(0, n/(n+1))$, we have that this last relation holds for all $\sigma_1 < \sigma$. Thus, inserting the inequality in (7.11) into (7.10), we conclude

$$\|u(t)\|_\sigma \le \|W(t)*u_0\|_\sigma + \int_0^t \frac{\|\boldsymbol{F}(\tau)\|_{\sigma_1}}{(t-\tau)^{\frac{n}{2}\left(\frac{1}{\sigma_1}-\frac{1}{\sigma}\right)+\frac{1}{2}}} d\tau, \quad 1 < \sigma_1 < \sigma < \infty. \tag{7.12}$$

We next differentiate (7.8) with respect to x_k and take the L^2-norm of both sides of the resulting equation, to get

$$\|\partial_k u(t)\|_2 \le \|\partial_k (W(t) * u_0)\|_2 + \int_0^t \|\partial_k W(t-\tau) * \operatorname{div} \boldsymbol{F}(\tau)\|_2 d\tau. \tag{7.13}$$

Proceeding as before, one shows that

$$\|\partial_k W(t-\tau) * \operatorname{div} \boldsymbol{F}(\tau)\|_2 \le C \frac{\|\operatorname{div} \boldsymbol{F}\|_{s_1}}{(t-\tau)^{\frac{1}{2}\left(1+\frac{n}{s}\right)}}$$

where $1/s_1 = 1/s + 1/2$. Replacing this estimate into (7.13), we deduce

$$\|\nabla u(t)\|_2 \le \|\nabla (W(t) * u_0)\|_2 + \int_0^t \frac{\|\operatorname{div} \boldsymbol{F}\|_{s_1}}{(t-\tau)^{\frac{1}{2}\left(1+\frac{n}{s}\right)}} d\tau, \quad \frac{1}{s_1} = \frac{1}{s} + \frac{1}{2}. \tag{7.14}$$

We now choose in (7.12) $\sigma = s$, $\sigma_1 = s/2$ to obtain

$$\|u(t)\|_s \le \|W(t) * u_0\|_s + \int_0^t \frac{\|\boldsymbol{F}(\tau)\|_{s/2}}{(t-\tau)^{\frac{1}{2}\left(1+\frac{n}{s}\right)}} d\tau. \tag{7.15}$$

We take the L^r-norm in time of both sides of this relation. If $s>n$, we may apply inequality (7.6) with $n=1$, $p=r$ and $\lambda=(1-n/s)/2$ to the integral in (7.15) to show the validity of (7.7_1). To show (7.7_2), we multiply both sides of (7.16) by t^α, $\alpha=(1-n/s)/2$, and notice that

$$t^\gamma\alpha\int_0^t \frac{d\tau}{(t-\tau)^{\frac{1}{2}\left(1+\frac{n}{s}\right)}\tau^{(1+\gamma)\alpha}}=B_\gamma=\text{const},\quad \gamma=0,1. \tag{7.16}$$

To show (7.7_3), we take in (7.12) $\sigma=2$, $\sigma_1=s_1$ and notice that, by (7.17),

$$\int_0^t \frac{\|\boldsymbol{F}(\tau)\|_{s_1}}{(t-\tau)^{\frac{1}{2}\left(1+\frac{n}{s}\right)}}d\tau\leq\langle\!\langle\boldsymbol{F}\rangle\!\rangle_{s_1,\alpha,T}\int_0^t\frac{d\tau}{(t-\tau)^{\frac{1}{2}\left(1+\frac{n}{s}\right)}\tau^\alpha}=B_0\langle\!\langle\boldsymbol{F}\rangle\!\rangle_{s_1,\alpha,T}.$$

Finally, to prove of $(7.7)_4$, we take the L^2-norm in time of (7.14), apply (7.6) with $n=1$, $p=2$, $\lambda=(1-n/2s)/2$, and notice that, for the solution $W(t)*u_0$ of the Cauchy problem for the heat equation it is

$$\|\nabla(W(t)*u_0)\|_2^2\leq\frac{1}{2\nu}\|u_0\|_2^2.$$

The lemma is thus proved.

Before proving the main result of this section, we need a further preliminary lemma. The first part is a simple consequence of the Young inequality for convolutions while the second is due to Giga (1986, Lemma p. 196). [34]

Lemma 7.3 *Let* $1<\sigma,s\leq\infty$,

$$\frac{1}{r}=\frac{n}{2}\left(\frac{1}{\sigma}-\frac{1}{s}\right),$$

and let $a\in L^\sigma(\mathbb{R}^n)$. *Then, there exists* $C=C(\nu,n,s,\sigma)$ *such that the following properties hold, for all* $t\in(0,T]$ *and all* $T>0$*:*

(i) $\|W*u_0\|_s\leq C\,t^{-1/r}\|u_0\|_\sigma,\quad \sigma\leq s$

(ii) $\|W*u_0\|_{s,r,T}\leq C\,\|u_0\|_\sigma,\quad \sigma<s.$

We shall now prove the main result of this section.

[34] Actually, Giga's lemma applies to more general situations than the Cauchy problem for the heat equation described in Lemma 7.3.

Theorem 7.1 *Let $n \le \sigma < s < \infty$, and let*

$$\frac{2}{r} + \frac{n}{s} = \frac{n}{\sigma}.$$

Then, for any $\boldsymbol{v}_0 \in H(\mathbb{R}^n) \cap \boldsymbol{L}^\sigma(\mathbb{R}^n)$, there exists $T > 0$ and a unique weak solution $\boldsymbol{v}$ to the Navier-Stokes equations in Ω_T such that $\boldsymbol{v} \in L^{s,r}(\mathbb{R}^n_T)$. Moreover, denoting by f_η the (spatial) mollifier of the function f, we have that the number T is estimated as follows:

(i) If $\sigma > n$:

$$T \ge \frac{C}{\|\boldsymbol{v}_0\|_\sigma^{1/\beta_1}}, \quad \beta_1 = \frac{1}{2}\left(1 - \frac{n}{\sigma}\right);$$

(ii) If $\sigma = n$:

$$T \ge \left(\frac{C - \|\boldsymbol{v}_0 - \boldsymbol{v}_{0\eta}\|_n}{\|\boldsymbol{v}_{0\eta}\|_q}\right)^{1/\beta_2}, \quad \beta_2 = \frac{1}{2}\left(1 - \frac{n}{q}\right),$$

where $C = C(\nu, n, s, \sigma) > 0$, q is arbitrary in (n, s), and η is taken as small as to satisfy the condition $\|\boldsymbol{v}_0 - \boldsymbol{v}_{0\eta}\|_n < C$.

Proof. We use the method of successive approximations. We set

$$\boldsymbol{v}_1(x,t) \equiv W(t) * \boldsymbol{v}_0,$$

and, for $k = 1, 2, \dots,$ $\boldsymbol{v}_{k+1}$ solves the following Stokes-like problem

$$\int_0^\infty \left\{ \left(\boldsymbol{v}_{k+1}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \right) - \nu(\nabla \boldsymbol{v}_{k+1}, \nabla \boldsymbol{\varphi}) \right\} dt = \int_0^\infty (\boldsymbol{v}_k \otimes \boldsymbol{v}_k, \nabla \boldsymbol{\varphi}) dt - (\boldsymbol{v}_0, \boldsymbol{\varphi}(0)), \quad \text{for all } \boldsymbol{\varphi} \in \mathcal{D}_T. \tag{7.17}$$

Using (7.7) and the Hӧlder inequality, we find that [35]

$$\begin{aligned}
\|\boldsymbol{v}_{k+1}\|_{s,r,T} &\le \|\boldsymbol{v}_1\|_{s,r,T} + C\|\boldsymbol{v}_k\|^2_{s,q_1,T} \le \|\boldsymbol{v}_1\|_{s,r,T} + CT^{\beta_1}\|\boldsymbol{v}_k\|^2_{s,r,T} \\
\langle\!\langle \boldsymbol{v}_{k+1} \rangle\!\rangle_{s,\alpha,T} &\le \langle\!\langle \boldsymbol{v}_1 \rangle\!\rangle_{s,\alpha,T} + C\langle\!\langle \boldsymbol{v}_k \rangle\!\rangle^2_{s,\alpha,T} \\
\|\boldsymbol{v}_{k+1}\|_{2,\infty,T} &\le \|\boldsymbol{v}_0\|_2 + C\|\boldsymbol{v}_k\|_{2,\infty,T}\langle\!\langle \boldsymbol{v}_k \rangle\!\rangle_{s,\alpha,T} \\
\|\nabla \boldsymbol{v}_{k+1}\|_{2,2,T} &\le C(\|\boldsymbol{v}_0\|_2 + \|\boldsymbol{v}_k\|_{s,\frac{2s}{(s-n)},T}\|\nabla \boldsymbol{v}_k\|_{2,2,T}) \\
&\le C(\|\boldsymbol{v}_0\|_2 + T^{\beta_1}\|\boldsymbol{v}_k\|_{s,r,T}\|\nabla \boldsymbol{v}_k\|_{2,2,T}).
\end{aligned} \tag{7.18}$$

[35] Throughout the proof of this theorem, we denote by C a generic constant which depends, at most, on $n, \sigma, s, \nu,$.

Denote by $K_0^{(1)}=K_0^{(1)}(T)$ and $K_0^{(2)}=K_0^{(2)}(T)$ two majorants for $\|\boldsymbol{v}_1\|_{s,r,T}$ and $\langle\langle \boldsymbol{v}_1\rangle\rangle_{s,\alpha,T}$, respectively. We want to show that there exist $T>0$ and $K_0^{(i)}$, $i=1,2$, such that

$$\begin{aligned}\|\boldsymbol{v}_{k+1}\|_{s,r,T}&\le 2K_0^{(1)}\\ \langle\langle \boldsymbol{v}_{k+1}\rangle\rangle_{s,\alpha,T}&\le 2K_0^{(2)}\end{aligned}\qquad\text{for all } k=1,2,\dots . \tag{7.19}$$

We proceed by induction. From $(7.18)_{1,2}$ we obtain

$$\begin{aligned}\|\boldsymbol{v}_{k+1}\|_{s,r,T}&\le K_0^{(1)}\left(1+CT^{\beta_1}K_0^{(1)}\right)\\ \langle\langle \boldsymbol{v}_{k+1}\rangle\rangle_{s,\alpha,T}&\le K_0^{(2)}\left(1+CK_0^{(2)}\right).\end{aligned}$$

Thus, (7.19) follows whenever the following conditions are met

$$\begin{aligned}CT^{\beta_1}K_0^{(1)}&<1\\ CK_0^{(2)}&<1.\end{aligned}\tag{7.20}$$

Let us first consider the case $\sigma>n$. From Lemma 7.3 we find

$$\|\boldsymbol{v}_1\|_{s,r,T}\le C\|\boldsymbol{v}_0\|_\sigma.$$

Thus, we choose

$$K_0^{(1)}=C\|\boldsymbol{v}_0\|_\sigma$$

and condition $(7.19)_1$ is certainly satisfied for those T such that

$$T^{\beta_1}\|\boldsymbol{v}_0\|_\sigma<C. \tag{7.21}$$

Moreover, again from Lemma 7.3, we find for $t\in[0,T]$

$$t^{\frac12\left(1-\frac ns\right)}\|\boldsymbol{v}_1(t)\|_s\le Ct^{-\frac1r+\frac12\left(1-\frac ns\right)}\|\boldsymbol{v}_0\|_\sigma\le CT^{\beta_1}\|\boldsymbol{v}_0\|_\sigma$$

and so, choosing

$$K_0^{(2)}=CT^{\beta_1}\|\boldsymbol{v}_0\|_\sigma,$$

condition (7.19_2) is satisfied again for those T verifying (7.21). In the case $\sigma=n$, observing that $\boldsymbol{v}_{0\eta}\in\boldsymbol{L}^q(\mathbb{R}^n)$, for all $q\in(1,\infty]$, from Lemma 7.3 we deduce for any $q\in(n,s)$

$$\|\boldsymbol{v}_0\|_{s,r,T}\le\|\boldsymbol{v}_0-\boldsymbol{v}_{0\eta}\|_n+CT^{\beta_2}\|\boldsymbol{v}_{0\eta}\|_q.$$

Thus, choosing

$$K_0^{(1)} = \|\boldsymbol{v}_0 - \boldsymbol{v}_{0\eta}\|_n + CT^{\beta_2}\|\boldsymbol{v}_{0\eta}\|_q,$$

we see that $(7.19)_1$ is satisfied if we select η sufficiently small and T such that

$$T^{\beta_2}\|\boldsymbol{v}_{0\eta}\|_q \le C - \|\boldsymbol{v}_0 - \boldsymbol{v}_{0\eta}\|_n. \tag{7.22}$$

Likewise, we show that if we take $K_0^{(2)}$ of the same form as $K_0^{(1)}$, condition $(7.19)_2$ is satisfied for a choice of T of the type (7.22). Using (7.19) and (7.20), into $(7.18)_{3,4}$ we also find that

$$\begin{aligned} \|\boldsymbol{v}_{k+1}\|_{2,\infty,T} &\le \|\boldsymbol{v}_0\|_2 \\ \|\nabla \boldsymbol{v}_{k+1}\|_{2,2,T} &\le C\|\boldsymbol{v}_0\|_2. \end{aligned} \tag{7.23}$$

Let us now show that the sequence $\{\boldsymbol{v}_k\}$ is converging to a weak solution belonging to the space $L^{s,r}$. To this end, we write (7.17) for $\boldsymbol{v}_{k+1}$ and for $\boldsymbol{v}_k$, then subtract the two resulting equations and apply the estimates of Lemma 7.2 to find ($k \ge 1$, $\boldsymbol{v}_0 \equiv 0$)

$$\|\boldsymbol{v}_{k+1} - \boldsymbol{v}_k\|_{s,r,T} \le CT^{\beta_1}\left(\|\boldsymbol{v}_k\|_{s,r,T} + \|\boldsymbol{v}_{k-1}\|_{s,r,T}\right)\|\boldsymbol{v}_k - \boldsymbol{v}_{k-1}\|_{s,r,T}.$$

If we employ $(7.19)_1$ and $(7.20)_1$ into this inequality, we end up with an estimate of the following form

$$\|\boldsymbol{v}_{k+1} - \boldsymbol{v}_k\|_{s,r,T} \le \alpha\|\boldsymbol{v}_k - \boldsymbol{v}_{k-1}\|_{s,r,T}, \tag{7.24}$$

where α is a constant strictly less than one and independent of k. From (7.24) it easy to show that $\{\boldsymbol{v}_k\}$ is a Cauchy sequence in the space $L^{s,r}(\mathbb{R}^n_T)$. In fact, (7.24) implies

$$\|\boldsymbol{v}_{k+1} - \boldsymbol{v}_k\|_{s,r,T} \le K_0^{(1)}\alpha^k,$$

and so, for all $k' = k + l$, $l > 0$,

$$\|\boldsymbol{v}_k - \boldsymbol{v}_{k'}\|_{s,r,T} \le \sum_{i=1}^{l}\|\boldsymbol{v}_{k+i} - \boldsymbol{v}_{k+i-1}\|_{r,s,T} \le \alpha^k\sum_{i=1}^{l}\alpha^i \le \frac{\alpha^{k+1}}{1-\alpha} \to 0 \text{ as } k \to \infty.$$

Denoting by $\boldsymbol{v}$ the limit field, from (7.23) we also deduce that $\boldsymbol{v} \in V_T$ and, by a simple calculation which uses (7.17) and the convergence properties of $\{\boldsymbol{v}_k\}$, that $\boldsymbol{v}$ satisfies (2.8) (with $\boldsymbol{f} \equiv 0$). The existence proof is thus completed. Since uniqueness is a consequence of Theorem 4.2 and Remark 4.7, the theorem is completely proved.

We shall now analyze some consequences of Theorem 7.1. We begin with the following result which improves Theorem 4.2(ii), see also Remark 4.5

Theorem 7.2. *Let $\boldsymbol{v}$, $\boldsymbol{u}$ be two weak solutions in Ω_T corresponding to the same data $\boldsymbol{v}_0$. Assume that $\boldsymbol{u}$ satisfies the energy inequality* (EI) *and that $\boldsymbol{v}\in L^\infty(0,T;\boldsymbol{L}^n(\mathbb{R}^n))$. Then $\boldsymbol{v}=\boldsymbol{u}$ a.e. in $\mathbb{R}^n_T$.*

Proof. As in the proof of Theorem 4.2, we establish (4.23). Let $\mathcal{T}$ and τ_0 be defined as in that proof, and assume $\tau_0<T$. Thus, (4.23) implies

$$\|\boldsymbol{w}(t)\|_2^2+2\nu\int_{\tau_0}^t\|\nabla\boldsymbol{w}\|_2^2d\tau\le 2\int_{\tau_0}^t(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{v})d\tau,\quad t\in(\tau_0,T). \tag{7.25}$$

We shall show that

$$\boldsymbol{v}(t)\in\boldsymbol{L}^n(\mathbb{R}^n),\quad \text{for all } t\in[0,T], \tag{7.26}$$

and so, in particular, that $\boldsymbol{v}(\tau_0)\in\boldsymbol{L}^n(\mathbb{R}^n)$. In fact, denote by $E\subset[0,T]$ the set where possibly (7.26) does not hold. Clearly, E is of zero Lebesgue measure. Let $t_*\in E$ and let $\{t_k\}\subset[0,T]-E$ be a sequence converging to t_*. By assumption, it follows that there exists $\boldsymbol{U}\in\boldsymbol{L}^n(\Omega)$ such that

$$\lim_{k\to\infty}(\boldsymbol{v}(t_k),\boldsymbol{\psi})=(\boldsymbol{U},\boldsymbol{\psi}),\quad \text{for all } \boldsymbol{\psi}\in\boldsymbol{C}_0^\infty(\Omega).$$

On the other hand, by the weak L^2 continuity, we have

$$\lim_{k\to\infty}(\boldsymbol{v}(t_k),\boldsymbol{\psi})=(\boldsymbol{v}(t_*),\boldsymbol{\psi}),\quad \text{for all } \boldsymbol{\psi}\in\boldsymbol{C}_0^\infty(\Omega),$$

and (7.26) follows. Now, by Theorem 7.1, we infer that there exists a weak solution $\widetilde{\boldsymbol{v}}$, say, assuming the initial data $\boldsymbol{v}(\tau_0)$ and belonging to the space $L^r(\tau_0,\tau_0+T(\tau_0);\boldsymbol{L}^s(\mathbb{R}^n))$, $2/r+n/s=1$, $s>n$. In view of Theorem 4.1 and Remark 4.3, $\boldsymbol{v}$ satisfies the energy equality in $[\tau_0,\tau_0+T(\tau_0))$ and so, from Theorem 4.2(i), we conclude $\boldsymbol{v}=\widetilde{\boldsymbol{v}}$ in $[\tau_0,\tau_0+T(\tau_0))$. We then use (7.25), and reason as in the proof of Theorem 4.2(i) to show $\boldsymbol{v}=\boldsymbol{u}$ in $[\tau_0,\tau_0+T(\tau_0))$, contradicting the fact that τ_0 is a maximum.

Another consequence of Theorem 7.1 is contained in the following one, which extends the results of Theorem 6.4(i) to the case $\Omega=\mathbb{R}^n$. [36]

Theorem 7.3 *Let $\boldsymbol{v}$ be a weak solution in $\mathbb{R}^n_T$, for all $T>0$, corresponding to the initial data $\boldsymbol{v}_0\in H(\mathbb{R}^n)$, and satisfying the strong energy inequality (4.1)*

[36] We refer to Giga (1986), for the more general case when Ω has a compact boundary.

and let t_1 *be an epoch of irregularity for* $\boldsymbol{v}$*. Then,* $\|\boldsymbol{v}(t)\|_\sigma$ *diverges as* $t \to t_1^-$*, for all* $n < \sigma < \infty$*, in such a way that*

$$\|\boldsymbol{v}(t)\|_\sigma \geq \frac{C}{(t_1 - t)^{(\sigma-n)/2\sigma}}, \quad t < t_1,$$

with $C = C(n, \sigma, \nu) > 0$*;*

Proof. Reasoning as in the proof of Theorem 6.4(i), we show that there can not exist a sequence $\{\tau_k\}$, say, tending to t_1, along which $\|\boldsymbol{v}(\tau_k)\|_\sigma$ stays bounded. [37] In fact, otherwise, in view of Theorem 7.2, we could construct a solution $\widetilde{\boldsymbol{v}}$, having $\boldsymbol{v}(\tau_k)$ as initial data and belonging to $L^r(\tau_k, \tau_k + T_k; \boldsymbol{L}^s(\mathbb{R}^n))$, for some $r = 2s/(s-n)$, $s > n$, [38] and $\tau_k + T_k > t_1$. By Theorem 5.2(i), $\boldsymbol{v} \in C^\infty(\overline{\mathbb{R}_n} \times (\tau_k, \tau_k + T_k))$ and by the uniqueness Theorem 4.2(i), $\boldsymbol{v} = \widetilde{\boldsymbol{v}}$ on $(\tau_k, \tau_k + T_k)$, contradicting the assumption that t_1 is an epoch of irregularity. From Theorem 7.2(i), we then have

$$(t_1 - t) \geq C / \|\boldsymbol{v}(t)\|_\sigma^{2\sigma/(\sigma-n)}, \quad t < t_1,$$

and the result is proved.

Remark 7.1 From Theorem 7.3, we reobtain the sufficient condition for the absence of epochs of irregularity given in Theorem 5.2. The estimate of Theorem 7.3 was first obtained for $n = 3$ by Leray (1934b, pp. 227). Actually, following the work of Leray, *loc. cit.* pp. 222-224, we could show that this estimate also holds in the case $\sigma = \infty$.

As we have noticed in Remark 5.4, one important point which is left out in Theorem 5.4 is to show that a weak solution $\boldsymbol{v}$ which in addition satisfies

$$\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{L}^n(\Omega)) \tag{7.27}$$

is in fact regular. So far, it is not known whether this property is true or not. The last part of this section will be devoted to investigate the kind of regularity achieved by weak solution satisfying (7.27). This will be obtained by means of Theorem 7.1.

We begin to show the following result.

[37] Recall that, from the Definition 6.1 of epoch of irregularity, it follows that $\boldsymbol{v}(t) \in L^q(\mathbb{R}^n)$ for all $q \geq 2$.

[38] See Remark 4.7.

Lemma 7.4 *Let $\boldsymbol{v}$ be a weak solution in $\mathbb{R}^n_T$, verifying (7.27). Then, for any $t_0 \in [0,T)$, there exists $\delta(t_0)>0$ such that $\boldsymbol{v} \in C([t_0,t_0+\delta(t_0));\boldsymbol{L}^n(\mathbb{R}^n))$. In particular, $\boldsymbol{v}(t)$ is right continuous in the $\boldsymbol{L}^n$-norm, at each $t\in[0,T)$.*

Proof. We already know that $\boldsymbol{v}(t) \in \boldsymbol{L}^n(\mathbb{R}^n)$, for all $t\in[0,T)$, see (7.26). Therefore, for any fixed $t_0 \in [0,T)$, by Theorem 7.1 we know that there exists $\delta(t_0)>0$ such that

$$\boldsymbol{v} \in L^r(t_0,t_0+\delta(t_0);\boldsymbol{L}^n(\mathbb{R}^n)), \quad \text{for all } s>n \text{ and } r=2s/(s-n), \tag{7.28}$$

and thus $\boldsymbol{v}$ is regular in $I=(t_0,t_0+\delta(t_0))$. We may then multiply the Navier-Stokes equations (0.1) –written in $\mathbb{R}^n\times I$, with $\boldsymbol{f}\equiv 0$– by $|\boldsymbol{v}|^{n-2}\boldsymbol{v}$, and integrate by parts over $\mathbb{R}^n$, to obtain [39]

$$\frac{1}{n}\frac{d}{dt}\|\boldsymbol{v}\|_n^n+\nu D_1(\boldsymbol{v})+4\nu\frac{n-2}{n^2}D_2(\boldsymbol{v}) = -(n-2)\int_{\mathbb{R}^n} p|\boldsymbol{v}|^{n-4}\boldsymbol{v}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{v}dx, \tag{7.29}$$

where

$$D_1(\boldsymbol{v}) = \int_{\mathbb{R}^n}|\boldsymbol{v}|^{n-2}|\nabla\boldsymbol{v}|^2dx$$

$$D_2(\boldsymbol{v}) = \int_{\mathbb{R}^n}|\nabla|\boldsymbol{v}|^{n/2}|^2dx.$$

We now apply the Cauchy-Schwarz inequality in the integral at the right-hand side of (7.29) to deduce

$$\frac{1}{n}\frac{d}{dt}\|\boldsymbol{v}\|_n^n+\tfrac{1}{2}\nu D_1(\boldsymbol{v})+4\nu\frac{n-2}{n^2}D_2(\boldsymbol{v}) \le C\int_{\mathbb{R}^n}p^2|\boldsymbol{v}|^{n-2}dx. \tag{7.30}$$

Since

$$\Delta p = \partial_i\partial_j(v_iv_j),$$

from the Calderon-Zygmund theorem on singular integrals we obtain

$$\|p\|_{(n+2)/2} \le C\|\boldsymbol{v}\|_{n+2}^2.$$

Using this inequality at the right-hand side of (7.30), we conclude

$$\frac{1}{n}\frac{d}{dt}\|\boldsymbol{v}\|_n^n+\tfrac{1}{2}\nu D_1(\boldsymbol{v})+4\nu\frac{n-2}{n^2}D_2(\boldsymbol{v}) \le C\|\boldsymbol{v}\|_{n+2}^{n+2}. \tag{7.31}$$

By the same procedure, one also shows

$$\left|\frac{d}{dt}\|\boldsymbol{v}\|_n^n\right| \le C\left(D_1(\boldsymbol{v})+D_2(\boldsymbol{v})+\|\boldsymbol{v}\|_{n+2}^{n+2}\right). \tag{7.32}$$

[39] For this type of technique, see Rionero and Galdi (1979), and Beirão da Veiga (1987).

From (7.28) we know that $\boldsymbol{v} \in L^{n+2}(t_0, t_0 + \delta(t_0); \boldsymbol{L}^{n+2}(\mathbb{R}^n))$ and, by assumption, that $\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{L}^n(\mathbb{R}^n))$. Therefore, (7.31) gives

$$\int_{t_0}^{t_0+\delta(t_0)} [D_1(\boldsymbol{v}) + D_2(\boldsymbol{v})]\, dt \leq M,$$

which, in turn, once replaced into (7.32), allows us to infer

$$\int_{t_0}^{t_0+\delta(t_0)} \left| \frac{d}{dt} \|\boldsymbol{v}\|_n^n \right| \leq M.$$

Thus, $\|\boldsymbol{v}(t)\|_n$ is continuous in $[t_0, t_0 + \delta(t_0))$. On the other hand, the weak continuity of $\boldsymbol{v}$ in $\boldsymbol{L}^2$, along with the uniform boundedness in $\boldsymbol{L}^n$, implies that $\boldsymbol{v}$ is weakly continuous in $\boldsymbol{L}^n$ and we conclude the continuity of $\boldsymbol{v}$ in $\boldsymbol{L}^n(\mathbb{R}^n)$.

We are now able to prove the following partial regularity result (Sohr and von Wahl, 1984, Theorem III.4).

Theorem 7.4 *Let $\boldsymbol{v}$ be a weak solution in $\mathbb{R}^n_T$ verifying the condition $\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{L}^n(\mathbb{R}^n))$. Then, there exists a set $E \subset [0, T]$ with the following properties*

(i) $\boldsymbol{v} \in C^\infty\left(\overline{\mathbb{R}_n} \times E\right)$;

(ii) The set $S \equiv [0, T] - E$ is at most countable;

(iii) For every epoch of irregularity t_1 $(\in S)$ we have

$$\limsup_{t \to t_1^-} \|\boldsymbol{v}(t)\|_n > \lim_{t \to t_1^+} \|\boldsymbol{v}(t)\|_n.$$

Proof. Point (i) is already known from Theorem 6.3. [40] For $t_0 \in S$, by Theorem 7.1, we may construct a regular solution in $(t_0, t_0 + \delta(t_0))$. We can take a rational number in $(t_0, t_0 + \delta(t_0))$ to show that S is countable. Let now t_1 be an epoch of irregularity. Then, by Theorem 7.3, $\boldsymbol{v}$ is right continuous at t_1 in the $\boldsymbol{L}^n$-norm, that is,

$$\lim_{t \to t_1^+} \|\boldsymbol{v}(t)\|_n = \|\boldsymbol{v}(t_1)\|_n.$$

[40] We recall that, by assumption and by Theorem 4.1, $\boldsymbol{v}$ satisfies the energy *equality* in $[0, T)$.

Since $\boldsymbol{v}$ is weakly continuous at t_1, we also have that

$$\limsup_{t\to t_1^-}\|\boldsymbol{v}(t)\|_n \text{ exists.}$$

If

$$\limsup_{t\to t_1^-}\|\boldsymbol{v}(t)\|_n \le \|\boldsymbol{v}(t_1)\|_n = \lim_{t\to t_1^+}\|\boldsymbol{v}(t)\|_n,$$

we would then have that $\boldsymbol{v}(t)$ is strongly continuous in $\boldsymbol{L}^n$ at t_1. From Theorem 5.2(ii) it then follows that

$$\boldsymbol{v}\in C\left([t_1-\eta_1,t_1];\boldsymbol{L}^n(\mathbb{R}^n)\right), \quad \eta_1>0,$$

On the other hand, by Lemma 7.1, we also have that

$$\boldsymbol{v}\in C\left([t_1,t_1+\eta_2];\boldsymbol{L}^n(\mathbb{R}^n)\right), \quad \eta_2>0,$$

and so,

$$\boldsymbol{v}\in C\left([t_1-\eta,t_1+\eta];\boldsymbol{L}^n(\mathbb{R}^n)\right), \quad \eta>0,$$

and, by Theorem 5.2(ii), t_1 can not be an epoch of irregularity.

Remark 7.2 Condition (ii) in Theorem 7.4 can be refined in the following way, see Kozono and Sohr (1996b), Beirão da Veiga (1996). Let the assumption of that theorem be satisfied and let t_1 be any instant of time. Set

$$a\equiv\limsup_{t\to t_1^-}\|\boldsymbol{v}(t)\|_n-\lim_{t\to t_1^+}\|\boldsymbol{v}(t)\|_n.$$

Then, there exists a constant C independent of the particular solution $\boldsymbol{v}$ such that, if $a<C$, then necessarily $a=0$, that is, $\boldsymbol{v}$ is strongly continuous in $\boldsymbol{L}^n$ at t_1 and, therefore, smooth at t_1. Further investigation on the structure of the possible irregular points of a solution satisfying the assumption of Theorem 7.4, has been more recently carried out by J. Neustupa (1999).

Remark 7.3 The estimate from below for the time T of existence of a solution with data in $\boldsymbol{L}^n$ may play a crucial role in the theory of regularity. Though it is very unlikely that we can give for T a bound of the type

$$T\ge C\|\boldsymbol{v}_0\|_n^{-\beta}, \quad \beta>0,$$

we may still conjecture the following estimate

$$T\ge f(\|\boldsymbol{v}_0\|_n) \tag{7.33}$$

where $f(\lambda)$ is a positive, strictly decreasing function of λ. The following two possibilities may then arise

i) $\lim_{\lambda\to\infty} f(\lambda) = f_0 > 0;$

ii) $\lim_{\lambda\to\infty} f(\lambda) = 0.$

In the case i), no epoch of irregularity can exist. In fact, we have $T \geq f_0$. Let t_1 be an epoch of irregularity. Then, we could choose t_0 such that $t_1 - t_0 < f_0/2$ (say), and we would conclude, by Theorem 7.1, that $\boldsymbol{v}$ is regular in $(t_0, t_0 + f_0)$, contradicting the fact that t_1 is an epoch of irregularity. In case ii), we distinguish again the following two possibilities:

ii)$'$ $\limsup_{t\to t_1^-} \|\boldsymbol{v}(t)\|_n = \infty;$

ii)$''$ $\boldsymbol{v} \in L^\infty(0, t_1; \boldsymbol{L}^n(\Omega)).$

In case ii)$'$, for t very close to t_1, we would have

$$\|\boldsymbol{v}(t)\|_n \geq f^{-1}(t_1 - t), \quad t < t_1,$$

and, therefore, since

$$\lim_{\lambda\to 0} f^{-1}(\lambda) = \infty,$$

a condition even weaker than (7.27) –depending on f– would imply regularity. In case ii)$''$, setting

$$M = \text{ess} \sup_{t\in[0,t_1]} \|\boldsymbol{v}(t)\|_n.$$

we would have $T \geq f(M)$, and so, reasoning as in case i), we would deduce that t_1 can not be an epoch of irregularity. From all the above, we then conclude that, if an estimate of the type (7.33) holds for T, then a condition weaker than (7.27) and depending on f, would suffice to ensure regularity of a weak solution. However, we only have for T the estimate of Theorem 7.1(ii).

Remark 7.4 In view of Theorem 6.4(i) and Theorem 7.3, we deduce that a weak solution $\boldsymbol{v}$ in dimension 3 will never go through an epoch of irregularity t_1, provided that the condition $\boldsymbol{v} \in V_T$ is incompatible with the following ones:

$$\left.\begin{aligned} \|\nabla\boldsymbol{v}(t)\|_2 &\geq \frac{C}{(t_1 - t)^{1/4}} \\ \|\boldsymbol{v}(t)\|_\sigma &\geq \frac{C}{(t_1 - t)^{(\sigma-n)/2\sigma}} \end{aligned}\right. \quad t < t_1, \;\; \sigma > n. \tag{7.34}$$

With this in mind, J.Leray (1934b, p.225) proposed a *possible* counter example to the existence of a global regular solution. This counter example *would* lead to a weak solution possessing just one epoch of irregularity. Even though the existence of such a solution has been recently ruled out by Necas, Ruzicka and Sverák (1996) (see also Tsai (1998)), we deem it interesting to reproduce and discuss it here. This solution is constructed as follows. *Assume* that the following system of equations

$$\left.\begin{array}{c} \nu\Delta \boldsymbol{U}(y) - \alpha\,[\boldsymbol{U}(y) + \boldsymbol{y}\cdot\nabla\boldsymbol{U}(y)] + \nabla P(y) = \boldsymbol{U}(y)\cdot\nabla\boldsymbol{U}(y) \\ \mathrm{div}\,\boldsymbol{U}(y) = 0 \end{array}\right\} \quad y \in \mathbb{R}^n,$$

admits a non-zero solution $\boldsymbol{U} \in \boldsymbol{W}^{1,2}(\mathbb{R}^n)$, for some $\alpha > 0$, and set

$$\lambda(t) = (2\alpha(t_1 - t))^{-1/2}, \quad t < t_1.$$

Then, the function

$$\boldsymbol{u}(x,t) = \begin{cases} \lambda(t)\boldsymbol{U}(\lambda(t)x) & \text{if } t < t_1 \\ 0 & \text{if } t \geq t_1 \end{cases} \tag{7.35}$$

is a weak solution to the Navier-Stokes problem in $\mathbb{R}^n_T$. By a simple calculation which uses (7.35) we show that

$$\begin{aligned} \|\boldsymbol{u}(t)\|_s &= C\,(\lambda(t))^{1-n/s}, \quad s \in [2,\infty), \quad t < t_1 \\ \|\nabla\boldsymbol{u}(t)\|_2 &= C(\lambda(t))^{1/2}, \quad t < t_1. \end{aligned} \tag{7.36}$$

From (7.36) it is clear that $\boldsymbol{u}$ satisfies all requirements of a weak solution and that, in fact, it possesses even more regularity, such as strong L^2-continuity in time. However, $\boldsymbol{u}$ blows up at t_1 exactly in the way prescribed by (7.34), so that t_1 is the only epoch of irregularity. Moreover, $\|\boldsymbol{u}(t)\|_n \leq C$, uniformly in t, and $\|\boldsymbol{u}(t)\|_n$ becomes irregular at t_1 just in the way predicted by Theorem 7.4(iii). As we mentioned, such a solution does not exist, since Necas, Ruzicka and Sverák, *loc. cit.*, have shown that $\boldsymbol{U} \equiv 0$. This result gives more weight to the conjecture that the class $L^{n,\infty}(\mathbb{R}^n_T)$ is a regularity class.

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