A Kummer function based Zeta function theory to prove the Riemann Hypothesis

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January 28, 2017

Summary

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function $f(x)$ it holds

$$M[f](s)=\frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \quad , \quad M[-xf'(x)](s)=\frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2})=\frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2}) .$$

The corresponding entire Zeta function is given by ([EdH] 1.8)

$$\xi(s):=\frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s)=(1-s)\cdot\zeta(s)M[-xf'(x)](s)=\xi(1-s) .$$

The central idea is to replace

$$M[xf'(x)](s)\rightarrow M[f_H(x)](s)$$

with $f_H(x):=H[f](x)$, $\hat{f}_H(0)=0$ and

$$M[f_H(x)](s)=2\pi\cdot M\left[x\cdot{}_1F_1(1,\frac{3}{2},-\pi x^2)\right](s)=\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s) .$$

This enables the definition of an alternative *entire* Zeta function in the form (§2)

$$\xi^*(s):=(1-s)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)\cdot\zeta(s) .$$

with same zeros as $\xi(s)$. It enables a modified formula for $J(x)$ ([EdH] 1.13 ff.).

The fractional part function

$$\rho(x):=\{x\}:=x-[x]=\frac{1}{2}-\sum_1^\infty\frac{\sin 2\pi\nu x}{\pi\nu}\in L_2^\#(0,1)$$

is linked to the Zeta function by ([TiE] (2.1.5), lemma 2.1)

$$\zeta(1-s)=(s-1)M[\rho](s-1)=M[-x\rho'(x)](s-1) .$$

The Hilbert transform of the fractional part function is given by

$$\rho_H(x)=\sum_1^\infty\frac{\cos 2\pi\nu x}{\pi\nu}=-\frac{1}{\pi}\log 2\sin(\pi x)\in L_2^\#(0,1) \quad , \quad \hat{\rho}_H(0)=0 \quad , \quad \rho'_H\in H_{-1}^\# .$$

Applying the idea of above leads to the replacement (§3)

$$M[-x\rho'(x)](s-1)=\zeta(1-s) \qquad \rightarrow \qquad M[-\rho_H(x)](s-1)$$

$$M[-\rho_H(x)](s-1)=M\left[\sum_1^\infty\frac{\cos 2\pi\nu x}{\pi\nu})\right](s-1)=2\pi\Gamma(\frac{1-s}{2})\Gamma^{-1}(\frac{s}{2})\cdot\chi(s)\cdot\frac{\zeta(s)}{s}$$

with same zeros as $\zeta(1-s)$.

The integral function representations of the Zeta functions above based on the Hilbert transforms of the Gaussian and the fractional part functions enable all "convolution" related Polya-RH criteria ([CaD]), e.g. the Hilbert-Polya conjecture, Polya polynomial criteria ([EdH] 12.5), as the Hilbert transform is defined by a singular (convolution) integral operator.

The $H_{-1}^\#$ Hilbert space is the same as applied in [BaB] to reformulate the Beurling-Nyman criterion. The non-vanishing constant Fourier term of the series causes same "self-adjoint integral operator" building issue than in case of the Gaussian function.

The related entire Riemann function enables the definition of correspondingly defined alternative Keiper-Li coefficients ([LaG]). It is enabled by the zeros of the concerned Kummer functions and the related zeros of the Hilbert transformed Hermite polynomials. The challenging part to verify the RH (prime number counting error function) criterion

$$\pi(x)-li(x)=O(\sqrt{x}\log x)=O(x^{\frac{1}{2}+\varepsilon})$$

is the asymptotical behavior of the exponential (integral) function ([EdH] 1.14 ff.)

$$Ei(x):=-\int_x^\infty e^{-y}d\log y=-\int_x^\infty e^{-y}\frac{dy}{y}=\int_{-\infty}^x\frac{e^t}{t}dt$$

given by Ramanujan's asymptotic power series ([BeB] IV)

$$Ei(x^2)\approx\frac{e^{-x^2}}{x}\sum_{k=0}^\infty(-1)^{k+1}\frac{k!}{x^{2k+1}}.$$

The non-normalized (exponential) error function is given by ([AbM] 13.6)

$$erf(x):=\int_0^x e^{-t^2}dt=\sum_{k=0}^\infty\frac{(-1)^k}{k!}\frac{x^{2k+1}}{2k+1}.$$

Its relationship to the Kummer functions is given by ([AbM] 7.15)

$$erf(x)=xF_1(\frac{1}{2},\frac{3}{2},-x^2)=xe^{-x^2}{}_1F_1(1,\frac{3}{2},x^2).$$

The asymptotics of the corresponding non-normalized $erfc(x)-$ function is given by (lemma D4, [OlF] chapter 3, 1.1; chapter 12 1.1, ([AbM] 7.1.23)

$$erfc(x)=1-erf(x)\approx e^{-x^2}\sum_{k=0}^\infty(-1)^k\frac{1\cdot3\cdot....(2k-1)}{2^{k+1}x^{2k+1}}.$$

It further holds ([LeN] 9.13)

$$erfc(x)=\frac{1}{2}e^{-x^2}\Psi(1,\frac{3}{2},x^2)=\frac{1}{2}e^{-x^2}\Psi(\frac{1}{2},\frac{1}{2},x^2)=\frac{1}{2}e^{-x^2}\left[\sqrt{\pi}e^{x^2}-2\sqrt{x}{}_1F_1(1,\frac{3}{2},x^2)\right]$$

i.e.

$$erfc(x)=\frac{\sqrt{\pi}}{2}-\sqrt{x}e^{-x^2}F_1(1,\frac{3}{2},x^2)=\frac{\sqrt{\pi}}{2}-\sqrt{x}{}_1F_1(\frac{1}{2},\frac{3}{2},-x^2).$$

The above relationships provide the linkage of the concerned Kummer functions with the RH ($li(x)-$ function) convergence criterion.

We further note the following properties:

i) the function represented by

$$\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)\cdot x^{2k+1}},\quad |x|>1.$$

has the value $\pi/2$ as $x\to0$, $x>0$ ([BeB] IV, (10.2))

ii) $H\left[e^{-x^2}\right]=2\sqrt{\pi}F(x)$, $H^+\left[\frac{e^{-x}}{\sqrt{\pi x}}\right]=2\frac{F(\sqrt{x})}{\sqrt{x}}\approx\frac{1}{x}\left[1+\sum_{k=1}^\infty\frac{1\cdot3\cdot...(2k-1)}{(2x)^k}\right]$, ([GaW])

iii) $$\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)^s}=\prod_p(1-\frac{\sin(\frac{\pi}{2}s)}{p^s})^{-1}$$ for $\mathrm{Re}(s)>1$ ([BeB] 5, Corollary 4)

implying the convergence of the series

$$\sum_p p^{-1}\sin(\frac{\pi}{2}p).$$

We also note the following properties of the concerned hypergeometric functions ([AbM] p.507, [OlF] p. 44/67).

$$ {}_1F_1(\frac{1}{2};\frac{3}{2},-x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{x^n}{n!} \quad \text{and} \quad {}_1F_1(1;\frac{3}{2},-x) . $$

They are related to the error function and the Dawson function by

$$ erf(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{x^{2n+1}}{n!} \quad \text{and} \quad F(x)=e^{-x^2}\int_0^x e^{t^2}dt=\frac{\sqrt{\pi}}{2i}e^{-x^2}erf(ix) . $$

The corresponding Mellin transforms (valid in the critical stripe) are given by

$$ \int_0^{\infty} x^{s/2}\,{}_1F_1(\frac{1}{2};\frac{3}{2},-x)\frac{dx}{x}=\frac{1}{1-s}\Gamma(\frac{s}{2}) \quad , $$

$$ \frac{1}{2}\int_0^{\infty} x^{s/2}\,{}_1F_1(1;\frac{3}{2},-x)\frac{dx}{x}=\Gamma(s)\sin(\frac{\pi}{2}s) . $$

The function

$$ {}_1F_1(\frac{1}{2};\frac{3}{2},z) $$

has only imaginary zeros ρ_n fulfilling ([SeA], [SeA1], Note O5/38/39)

$$ n-1<a_{2n-1}:=\frac{\left|\operatorname{Im}(\rho_n)\right|-\pi}{2\pi}<n-\frac{1}{2}<a_{2n}:=\frac{\left|\operatorname{Im}(\rho_n)\right|}{2\pi}<n . $$

The analog approach based on the fractional part function

There is an analog approach to the Gaussian function above with respect to the fractional part function $\rho(x)$ and its relationship to the $\zeta-$function by the equality

$$M[-\rho'(x)](1-s)=\zeta(s)=\chi(s)\zeta(1-s).$$

It corresponds to the isomorphism of

$$\dot{H}_\beta(-\infty,\infty)\approx l_2^\beta \quad .$$

We note that the function $\rho_H(x)=H[\rho](x)$ has mean value zero, i.e the norm below is defined and the prerequisites of the theorems in Note S36 are fulfilled.

The function $\rho'_H(x)$ has a convergent Fourier series representation in a weak $H_{-1}^{\#}(0,1)-$ sense, which is equivalent to the $\cot(\pi x)-$function, i.e.

$$\cot(\pi x)=2\sum_1^\infty \sin(2\pi\nu x)=-\rho'_H(x)\in H_{-1}^{\#}(0,1).$$

This generalized Fourier series representation of $\cot(\pi x)$ is Cesàro summable (mean of order one) ([ZyA] VI-3, VII-1). It leads to a $\zeta-$function representation in the form

$$M[-\rho'_H(x)](1-s)=M[\cot(\pi x)](1-s)=(2\pi)^{s-1}\zeta(1-s)\Gamma(1-s)\cos(\frac{\pi}{2}s)=\chi(s)\cot(\frac{\pi}{2}s)\zeta(s).$$

We note (see also Notes S32/33) that

$$S(u,v)=\int_{S^1} u\cdot dg=-i\sum_{-\infty}^{\infty} n u_n v_n$$

defines an inner product on $l_2^{1/2}=(l_2^{-1/2})^*$ ([NaS]), i.e. the generalized Fourier coefficients $\left\{\sqrt{|n|}u_n\right\}$ are square summable.

For functions with vanishing constant Fourier term (i.e. with zero mean, where the Hilbert transform defines a unitary mapping, see also Note S36) the norm of the corresponding dual space is given by

$$\|u\|_{-1/2}^2=\sum_1^\infty \frac{1}{n}|u_n|^2 .$$

The Hilbert space $l_2^{-1/2}$ is part of the Hilbert scale l_2^β whereby it holds

$$(u,v)_\beta\leq\|u\|_{\beta-1/2}\cdot\|v\|_{\beta+1/2}.$$

Especially one gets for $u\in l_2^{-1}$, $v\in l_2^0=l_2$

$$(u,v)_{-1/2}\leq\|u\|_{-1}\cdot\|v\|_0 \quad .$$

The distributional Hilbert spaces $l_2^{-\beta}$ ($\beta=0,1/2,1$) play a key role

- in [BrK3] in order to define an alternative new ground state energy for the harmonic quantum oscillator (see also Note O52 for the Weyl-berry conjecture)
- in [BrK1] providing a global, unique solution of the non-stationary, non-linear 3D-Navier-Stokes equations
- in [BaB] (see also [BrK2]) where a functional analysis reformulation of the Nyman criterion is provided (see below).

The Dirichlet series (see also Notes S44/45/47) on the critical line

$$f(s):=\sum_{1}^{\infty} a_n e^{-s\log n} \qquad g(s):=\sum_{1}^{\infty} b_n e^{-s\log n}$$

are linked to the Hilbert space $H^{\#}_{-1/2} \cong l_2^{-1/2}$ by ([LaE] §227, Satz 40):

$$((f,g))_{-1/2} := \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(1/2+it)g(1/2-it)dt = \sum_{1}^{\infty}\frac{1}{n}a_n b_n .$$

As it holds ([EdH] 9.8)

$$\frac{1}{2\omega}\int_{-\omega}^{\omega}|\Xi(t)|^2 dt \approx \log\omega$$

one gets (see also Notes S32/33)

$$\|\Xi\|^2_{-1/2} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\Xi(t)|^2 dt = \sum_{n=1}^{\infty}\frac{1}{n} = \zeta(1) = \infty \quad , \quad \|\Xi\|^2_{-1} = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} = \int_0^1 \frac{\log x}{x-1}dx = \left[\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\right]^{-1} .$$

From Remark 2.8 below we recall the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\Omega(t)|^2 dt := \frac{1}{2\pi}\int_{-\infty}^{\infty}|\Gamma(1/2+it)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\pi dt}{\cosh(\pi t)} = \frac{1}{2} \text{ , i.e. } \Omega \in l_2^{-1/2} .$$

Theorem (Bagchi-Nyman criterion, [BaB]): Let

$$\gamma_k := \left\{\rho\left(\frac{n}{k}\right) \middle| n = 1,2,3,....\right\} \quad \text{for} \quad k = 1,2,3,...$$

and Γ_k be the closed linear span of γ_k. Then the Nyman criterion states that the following statements are equivalent:

The Riemann Hypothesis is true $\Leftrightarrow$ $\gamma \in \overline{\Gamma}_k$.

Alternatively to the double infinite matrix γ_k above we propose the analog defined double infinite matrix

$$\gamma_k^H := \{\rho_H(n/k) | n = 1,2,3,....\} \quad \text{for} \quad k = 1,2,3,... .$$

For any $u \in l_2^{-1}$, $v \in l_2^0 = l_2$, as it holds

$$(u,v)_{-1/2} \le \|u\|_{-1}\cdot\|v\|_0 \ ,$$

the inner product $(u,v)_{-1/2}$ is defined.
Putting

$$\tilde{u} := \Xi \cong \gamma \in l_2^{-1} ,$$

$$\tilde{v} := -\log(2\sin(\pi\circ)) = d\Phi \in L_2$$

this leads to the representation

$$(\gamma,\gamma_k)_{-1/2} \cong (\Xi,\rho_H)_{-1/2} = (\Xi,d\Phi)_{-1/2} \cong (\Xi,\rho'_H)_0 = (\Xi,S[\rho_H]) = (S[\Xi],\rho_H) \quad .$$

As $l_2^{-1/2}$ is dense in l_2^{-1} with respect to the $l_2^{-1}-$norm, γ belongs to the closed linear span of $\{\gamma_k^H\}_{k\in N}$, i.e.

$$\gamma \in l_2^{-1} = \overline{l_2^{-1/2}}^{\|\circ\|_{-1}}$$

which fulfills the Bagchi criterion.

The analog approach related to the von Mangoldt density function $\psi(x)$

We propose the Landau density function $\vartheta(x)$ alternatively to the Riemann density function $J(x)$ and the von Mangoldt function $\psi(x)$. They are related by

$$d[x\vartheta'] = d\psi = \log x dJ \ .$$

With respect to the scope of [KoJ], [ViJ] we note the asymptotics

$$\lim_{\lambda\to\infty} \vartheta'(\lambda x) = \frac{d}{dx}\left[\sum_{n=1}^{\infty}\Lambda(n)\log(\frac{\lambda x}{n})\right] = \lim_{\lambda\to\infty}\frac{\psi(\lambda x)}{\lambda x} = 1 \ .$$

The alternative approach leads to th alternative asymptotics in the form

$$\lim_{x\to\infty}\frac{\psi(x)}{x} = 1 \qquad \to \qquad \lim_{\to\infty}\frac{\vartheta(x)}{\log x} = \log(2\pi)$$

The Riemann and von Mangoldt densities are related to the Zeta function by (see also notes S19/30/41, O19/20/21)

$$\log\zeta(s) = s\int_0^\infty x^{-s-1}J(x)dx = \int_0^\infty x^{-s}dJ(x)$$

$$-\log'\zeta(s) = s\int_0^\infty x^{-s-1}\psi(x)dx = s\int_0^\infty x^{-s}\left[\frac{\psi(x)}{x}\right]dx = \int_0^\infty x^{-s}d\psi(x) = \int_0^\infty x^{-s}(\log x)dJ(x) = s\int_0^\infty x^{-s}d\vartheta(x) \ .$$

It holds ([LaE] §50, [ScW] IV, ([PrK] III §3, Note S39)

$$\vartheta(x) = \sum_{n=1}^{\infty}\Lambda(n)\log(\frac{x}{n}) = \psi(x)\log x - \sum_{1}^{\infty}\Lambda(n)\log(n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left[-\frac{\zeta'(s)}{s\zeta(s)}\right]x^s\frac{ds}{s} \propto x$$

resp. with ([PrK] VII §4, Note S50)

$$c := -\frac{\zeta'(0)}{\zeta(0)} = \log(2\pi) \quad , \quad \sum_{n=1}^{\infty}\frac{1}{2n}x^{-2n} = -\frac{1}{2}\log(1-\frac{1}{x^2}) = -\frac{1}{2}\log\frac{x^2-1}{x^2} \quad , \quad x^2 > 1$$

$$\vartheta(x) = \int_0^x\frac{\psi(t)}{t}dt = x - c\cdot\log x - \sum_\rho\frac{x^\rho}{\rho^2} - \sum_n\frac{x^{-2n}}{(2n)^2} - (1+\gamma)$$

$$\vartheta(x) = x - c\log x - \sum_\rho\frac{x^\rho}{\rho^2} + \frac{1}{2}\log(1-\frac{1}{x^2}) + \sum_n\frac{2n-1}{(2n)^2}x^{-2n} - (1+\gamma)$$

$$\psi_0(x) := \frac{1}{2}\psi(x+0) - \psi(x-0) = x + \lg(2\pi) - \sum_\rho\frac{x^\rho}{\rho} - \frac{1}{2}\log(1-\frac{1}{x^2}) \ .$$

The Landau density function $\vartheta(x)$ is linked to the Zeta function by ([OsH] Bd. 1, 8, [KoJ], Note O51)

(*) $$-\frac{\log'\zeta(s)}{s} = -\frac{\zeta'(s)}{s\zeta(s)} = s\int_0^\infty x^{-s-1}\vartheta(x)dx = \int_0^\infty x^{-s}d\vartheta(x) =: \int_0^\infty e^{-sx}dT(x) = \int_0^\infty x^{-s}dT(\log x) \quad , \quad s = \sigma + it \quad , \quad \sigma > 0 \ .$$

The proof of (*) applies the fundamental identity ([LaE] §48, [ScW] IV, ([PrK] III §6)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^s}{s}\frac{ds}{s} = \begin{cases}\log y & 1 \le y \\ 0 & 0 < y \le 1\end{cases} \ .$$

With respect to corresponding convergent Dirchlet series representation we refer to [LaE] Bd. 2, theorem 51, Note S47.

Remark: We note the "regularity" relationship between the above three density functions (in a weak "Hilbert scale" framework) given by

$$H_{\beta+1/2} \to H_\beta \to H_{\beta+1/2}$$

$$J(x) \to \psi(x) \to \vartheta(x) \ .$$

Remark ([OsH] Bd. I, §8, Note O51): As $T(x)=\vartheta(e^x)$ is monotone increasing, and

$$f(s):=-\frac{\log'\zeta(s)}{s}=\int_0^\infty e^{-sx}dT(x)=\int_0^\infty x^{-s}dT(\log x) \quad , \; s=\sigma+it \; , \; \sigma>0$$

convergent. Then, as

$$\lim_{s\to 0^+}[sf(s)]=-\lim_{s\to 0^+}\frac{\zeta'(s)}{\zeta(s)}=-\frac{\zeta'(0)}{\zeta(0)}=\log(2\pi)$$

exists, this holds also for

$$\lim_{x\to\infty}\frac{T(x)}{x}$$

and both limits are identical, i.e.

$$\lim_{s\to 0^+}[sf(s)]=\lim_{x\to\infty}\frac{T(x)}{x}=\lim_{x\to\infty}\frac{T(\log x)}{\log x}=\lim_{x\to\infty}\frac{\vartheta(x)}{\log x}=\log(2\pi) \quad .$$

Remark: Von Mangoldt proved the Euler conjecture, i.e. that ([PrK] III, §5)

$$\sum_{n=1}^\infty\frac{\mu(n)}{n}=1-\frac{1}{2}-\frac{1}{3}-\frac{1}{5}+\frac{1}{6}-...=0 \quad .$$

The convergence of this series is a consequence of the PNT.

The convergence of ([PrK] III, §5)

$$(**) \quad \sum_{n=1}^\infty\frac{\mu(n)\log n}{n}=-1 \quad \text{i.e.} \quad \sum_{n=1}^\infty\frac{\mu(n)}{n}\log(\frac{1}{n})=1$$

was proven by E. Landau ([LaE] §150). It cannot be derived from the PNT. In this context we recall the corresponding comment from E. Landau concerning his proof ([LaE] §159)

"... (it) goes deeper than the prime number theorem ..." .

The Landau theorem (**) can be represented in the following form

$$1=\sum_{n=1}^\infty\frac{1}{n}\mu(n)\log(\frac{1}{n})=\sum_1^\infty\frac{1}{n}a_nb_n=:((u,v))_{-1/2}:=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega}u(1/2+it)v(1/2-it)dt$$

i.e. the $H_{-1/2}-$ inner product of the related functions exists,
i.e.

$$u(\frac{1}{2}+it):=\sum_{n=1}^\infty\frac{\mu(n)}{n^s}\in H_{-1/2} \quad , \quad v(\frac{1}{2}-it):=\sum_{n=1}^\infty\frac{\log(1/n)}{n^s}\in H_{-1/2} \, .$$

From [PrK] III, §5, we recall for $\sigma>1$

$$\frac{1}{\zeta(s)}=\sum_{n=1}^\infty\frac{\mu(n)}{n^s} \quad , \quad -\frac{\zeta'(s)}{s}=\sum_{n=1}^\infty\frac{1}{n}\frac{1}{n^s}$$

i.e.

$$-\frac{\zeta'(s)}{s\zeta(s)}=\left[\sum_{n=1}^\infty\frac{\mu(n)}{n^s}\right]\cdot\left[\sum_{n=1}^\infty\frac{1}{n}\frac{1}{n^s}\right] \, .$$

Remark: We note that for

$$\tilde{T}(\log x):=\sum_{n\le x}\frac{1}{n}\log(\frac{x}{n}) \quad , \quad x\ge 1$$

the inverse mapping is given by ([ScW] (3.8))

$$\sigma(x):=\tilde{T}^{-1}(\log x)=\sum_{n\le x}\frac{\mu(n)}{n}\log(\frac{x}{n}) \, .$$

With

$$\log(\frac{xy}{n})+(\log\frac{1}{n})=\log(\frac{x}{n})+\log(\frac{y}{n})$$

it follows

$$\vartheta(xy)+\sum\Lambda(n)\log n=\vartheta(x)+\vartheta(y)$$

$$\sigma(xy)+1=\sigma(x)+\sigma(y)\,.$$

Remark: The asymptotics of the Riemann, the von Mangoldt and the Landau functions are given by

$$J(x)=\sum_{n\le x}\frac{\Lambda(n)}{\log n}\approx\frac{x}{\log x}\quad,\quad\psi(x)=\sum_{n\le x}\Lambda(n)\approx x\quad,\quad\vartheta(x)=\sum_{n\le x}\Lambda(n)\log(\frac{x}{n})\approx x\,.$$

The asymptotics $\psi(x)\approx x$ leads to the PNT, whereby the convergence of the summand

$$\sum_{\rho}\frac{x^{\rho}}{\rho}$$

requires special attention ([EdH] 4.1, [LaE] §89), i.e.

$$\psi(x)\approx x\quad\text{iff}\quad\lim_{x\to\infty}\sum_{\rho}\frac{x^{\rho-1}}{\rho}=0\,.$$

Remark: The relative error in $\pi(x)-Li(x)$ goes to zero faster than $x^{-1/2-\varepsilon}$ as $x\to\infty$ is equivalent to the RH ([EdH] 5.1).

Remark: In [ViJ] a quick distributional way to the Prime Number Theorem (PNT) is provided. In this context we note that the regularity of the applied Dirac function is given by

$$\delta,H[\delta]\in H_{-1/2-\varepsilon}$$

where

$$\pi H[\delta(x)]=\frac{1}{x}=\log'(\frac{x}{n})$$

and (in a distributional sense)

$$\psi'(x)=\sum_{n\le x}\Lambda(n)\delta(x-n)\quad,\quad\vartheta'(x)=\sum_{n\le x}\Lambda(n)H[\delta](x-n)\,.$$

For the relationship to the $\cot(\pi x)-$ function we refer to [EsR] example 78, appendix "Cardinal series").

Putting $c=-\zeta'(0)/\zeta(0)=\log(2\pi)$ one gets from the PNT

$$\frac{\psi(x)}{x}\approx 1\qquad\text{resp.}\qquad\frac{\psi(x)}{x-c}=\frac{\psi(x)/x}{1-c/x}\approx\frac{\vartheta'(x)}{1-c/x}\approx\frac{\vartheta(x)}{x-c\log x}$$

where

$$\vartheta(x)=x-c\log x-\sum_{\rho}\frac{x^{\rho}}{\rho^{2}}+\frac{1}{2}\log(1-\frac{1}{x^{2}})+\sum_{n}\frac{2n-1}{(2n)^{2}}x^{-2n}-(1+\gamma)\,.$$

Proposal: The above provides alternatives in the form

$$\pi(x)\frac{\log x}{x}\approx\frac{\psi(x)}{x}\approx\frac{\psi(x)}{x-c}\approx\frac{\vartheta(x)}{x-c\log x}\qquad\to\qquad\frac{\vartheta(x)}{x-c\log x+\frac{1}{2}\log(1-\frac{1}{x^{2}})}=\frac{\vartheta(x)}{x+\log\frac{\sqrt{x^{2}-1}}{x^{1+c}}}\,.$$

resp.

$$\pi(x)\approx\frac{\psi(x)}{\log x}\approx\frac{\psi(x)}{\log x}\frac{x}{x}\qquad\to$$

$$\pi(x\approx li^{*}(x)\approx\frac{\vartheta(x)}{\log x}\frac{1}{1+\frac{1}{x\log x}\cdot(\frac{\log\sqrt{x^{2}-1}}{1+\log(2\pi)})}\approx\frac{\vartheta(x)}{\log x}\cdot\frac{x}{1+x}$$

The Riemann Hypothesis states that

$$\zeta(s) \neq 0 \quad \text{for all } s=\sigma+it \text{ with } 1/2<\sigma<1,$$

i.e.

$$\frac{1}{\zeta(s)}, \frac{\zeta'(s)}{s\zeta(s)}$$

has no poles in case of $1/2<\sigma<1$.

We note the following equivalent critera for the RH:

i) $\pi(x)=Li(x)+O(\sqrt{x}\log x)$

ii) $\pi(x)=Li(x)+O(x^{1/2+\varepsilon})$, $\varepsilon>0$, $((H^{*}_{-1/2-\varepsilon}=H_{1/2+\varepsilon})$

iii) $\psi(x)=x+O(\sqrt{x}\log^2 x)$

iv) The series $\sum_{n=1}^{\infty}\mu(n)n^{-s}$ is convergent for $\operatorname{Re}(s)>1/2$ and

$$\frac{1}{\zeta(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} .$$

Remarks: iv) states that

$$\frac{1}{\zeta(s)}$$

is holomorphic for $\operatorname{Re}(s)>1/2$; from ii) one can derive that

$$-\frac{\zeta'(s)}{\zeta(s)}-\frac{1}{1-s}$$

is holomorphic for $\operatorname{Re}(s)>1/2$; von Mangoldt explicit formula regarding $\psi(x)$ given by

$$\psi_0(x):=\begin{cases}\psi(x) & x\notin N\\ \psi(x-1\frac{1}{2})+\frac{1}{2}\Lambda(x) & x\in N\end{cases}=x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\frac{1}{2}\log(1-\frac{1}{x^2})+c \quad , \quad c:=-\frac{\zeta'(0)}{\zeta(0)}=\log(2\pi)\approx 1.84 .$$

The proof that iii) is valid in case the RH is true, is based on the estimate

$$\psi(x)=x-\sum_{\rho,|\rho|\le T}\frac{x^{\rho}}{\rho}+O(\frac{x\log^2 x}{T}) \quad \text{for } 2\le T\le x .$$

Putting $T=\sqrt{x}$ with $x\ge 4$ this leads to

$$\psi(x)=x+O(\sqrt{x}\sum_{\rho,|\rho|\le\sqrt{x}}\frac{1}{|\rho|})+O(\frac{x\log^2 x}{T}) .$$

Because of

$$\sum_{\rho,|\rho|\le\sqrt{x}}\frac{1}{|\rho|}=O(\sum_{\rho,|\rho|\le\sqrt{x}}\frac{\log n}{n})+O(1)=O(\log^2 x)$$

one gets

$$\psi(x)=x+O(\sqrt{x}\log^2 x) .$$

Proposal: Combining von Mangoldt's formula with the Landau function leads to

$$\psi_0(x)+\vartheta(x)=2x+c(1-\log x)-\sum_{\rho}\frac{(1+\rho)x^{\rho}}{\rho^2}+\sum_{n}\frac{2n-1}{(2n)^2}x^{-2n}-(1+\gamma)$$

$$\psi_0(x)-\vartheta(x)=c\log x+\sum_{\rho}\frac{(1+\rho)x^{\rho}}{\rho^2}-\sum_{n}\frac{2n-1}{(2n)^2}x^{-2n}+1+\gamma-\log(2\pi)$$

which indicates a replacement in the form

$$\pi(x)\approx\frac{\psi(x)}{\log x} \quad \to \quad \pi(x)\approx\frac{\psi_0(x)+\vartheta(x)}{\psi_0(x)-\vartheta(x)} .$$

We summaries the related $H_\beta -$norm estimates on the critical line $\sigma = 1/2$:

$$\|\zeta\|_{-1/2}^2 = \sum_{n=1}^{\infty} \frac{1}{n} = \zeta(1) = \infty \quad , \quad \|\zeta\|_{-1}^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \left[\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \right]^{-1} < \infty \, .$$

From

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad , \quad -\frac{\zeta'(s)}{s} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^s}$$

one gets

$$\left(\left(\frac{1}{\zeta(s)}, 1 \right) \right)_{-1/2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = \infty$$

$$\left(\left(\frac{1}{\zeta(s)}, v(\bar{s}) \right) \right)_{-1/2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log n < \infty$$

$$\left(\left(\frac{1}{\zeta(\bar{s})}, -\frac{\zeta'(s)}{s} \right) \right)_{-1/2} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} < \infty \, .$$

Remark: A related $l_2^{-1} -$ identity is given by ([ApT] 3.12)

$$\sum_{n \le x} \frac{\varphi(n)}{n^2} + \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^2} = \frac{6}{\pi^2} (\log x + \gamma) + +O(\frac{\log x}{x}) \, .$$

Remark: The following identities are valid (Note S41)

i) $\quad -\log \zeta(s) = \sum_p \log(1 - \frac{1}{p^s})^{-1} = \sum_p \sum_n \frac{1}{n} p^{-ns} = \sum_n \frac{\Lambda(n)}{\log n} \cdot \frac{1}{n^s}$ ([PrK] III §3)

ii) $\quad -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1} = \sum_p \sum_n (\log p) p^{-ns}$ ([PrK] III §3)

iii) $\quad \frac{\log' \zeta(s)}{s} = \frac{\zeta'(s)}{s\zeta(s)} = \frac{1}{s-1} - \sum_\rho \frac{1}{\rho(s-\rho)} + \sum_n \frac{1}{2n(s+2n)} - \frac{c}{s}$ ([PrK] VII §2, [EdH] 10.6)

iv) $\quad \log''(s) = \left(\frac{\zeta'}{\zeta} \right)'(s) = \sum_{n=2}^{\infty} \Lambda(n)(\log n) n^{-s} = \frac{1}{(s-1)^2} - \sum_\rho \frac{1}{(s-\rho)^2} - \sum_{n=1}^{\infty} \frac{1}{(s+2n)^2}$

and therefore

$$\lim_{s \to 0} \frac{\log' \zeta(s)}{s} = \lim_{s \to 0} \log'' \zeta(s) = 1 - \sum_\rho \frac{1}{\rho^2} - \sum_{n=1}^{\infty} \frac{1}{4n^2} = 1 - \frac{\pi^2}{24} - \sum_\rho \frac{1}{\rho^2} \, .$$

Remark:As $\log'(s)/s$ has no pole at $s = 0$ its poles are identical to the zeros of the Zeta function.

Remark [LuB]: $\log''(s) = O(\log t)$ for $\sigma \ge 1/2 + \delta$, $\delta > 0$

Remark:

i) $\Lambda(n)=\sum_{d|n}\mu(d)\log\frac{n}{d}$ ([PrK] III §6)

ii) $J(x)=\sum_{n<x}\frac{\Lambda(n)}{\log n}$, $J(x)=\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}$ ([PrK] VII §2)

iii) $\psi(x)=\sum_{n<x}\Lambda(n)$, $\psi(x)=\int_{a-i\infty}^{a+i\infty}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]x^s\frac{ds}{s}$

iv) $\psi(x)=x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\sum_{n}\frac{x^{-2n}}{2n}-\log(2\pi)$,

where $\sum_{\rho}\frac{1}{|\rho|}$ is divergent , $\sum_{\rho}\frac{1}{|\rho|^{1+\delta}}$ is convergent

v) $\vartheta(x)=\int_0^x\psi(t)\frac{dt}{t}=x-\sum_{\rho}\frac{x^{\rho}}{\rho^2}-\sum_{n}\frac{x^{-2n}}{(2n)^2}-\frac{\zeta'(0)}{\zeta(0)}\log x+constant$ ([EdH] 4.1)

vi) $\sum_{n<x}\Lambda(n)\log(\frac{x}{n})=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left[-\frac{\log'\zeta(s)}{s}\right]x^s\frac{ds}{s}=x+O(xe^{-\log^{\frac{1}{11}}x})$ ([LaE] Bd.1, XII, §51)

Remark: The formula (*) motivates the definition of correspondingly defined alternative Keiper-Li coefficients ([LaG]). From [VoA] we recall the sufficient and necessary condition for the Riemann Hypothesis (see also Riemann's estimate of $N(T)$, [EdH] 6.7, 9.8):

$$\frac{2\omega_N}{2\pi}\approx\frac{N}{2\pi}\left[\log\frac{N}{2\pi}-(1-\gamma)\right].$$

Additive number theory, Goldbach conjecture and the circle method

Additive number theory is the study of sums of h-fold hA of a set A of integers for $h \geq 2$. Instead of analyzing the arithmetic nature of corresponding sets/sequences of integers one considers metric structures of corresponding sums of sets of integers. The Schnirelmann-Goldbach theorem states that every integer greater than 1 can be represented as a sum of a finite number of primes (NaM), i.e. the set of primes builds a basis of finite order h of the set of integer numbers. The Schnirelman number is the number of primes which one needs maximal to build this representation.

The natural density of a set

$$A := \{\alpha_1 < \alpha_2 < ... < \alpha_n < ..., n \in N\}$$

is defined by

$$d(A) = \lim_{n\to\infty} \frac{n}{\alpha_n}$$

if the limit exits. Obviously the density of the set of integers is 1. As

$$\lim_{n\to\infty} \frac{n}{\log n} = 0$$

the "asymptotic density" of the set of prime numbers is 0. Any natural number $n > 1$ either is a prime number or a unique (up to permutation of factors) product

$$n = p_1^{n_1} p_2^{n_2} ... p_k^{n_k}$$

which is called the canonical representation of n. Thus the prime numbers form a multiplicative basis for the set of natural numbers. In this context we refer to the above densities

$$\vartheta(x) = \sum_{n=1}^{\infty} \Lambda(n) \log(\frac{x}{n}) \quad , \quad \sigma(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \log(\frac{x}{n})$$

and its related multiplicative" properties

$$\vartheta(xy) = \vartheta(x) + \vartheta(y) - \sum \Lambda(n) \log n \quad , \quad \sigma(xy) = \sigma(x) + \sigma(y) - 1 .$$

The binary Goldbach problem states that every even integer greater 2 can be represented as the sum of two primes. The tertiary Goldbach conjecture is about a Schnirelman number 3. The theorem from Ramaré gives a proof for a Schnirelman number 7.

The metric in a Hilbert space is defined by its norm. The negative result of [DiG] concerning asymptotic basis of second order in case of C^0- metric indicates an alternative metric in form of a $l_2^{\beta}-$norm with $\beta \leq 0$.

Let $A := \{n_1, n_2, ...n_k, ...\}$ denote a set of integers and x denote the variable of the generating function $F(x)$ of a number theoretical function $f(n)$. Then

i) $x = e^{-s}$ is a one-to-one mapping to (in case of $0 \notin A$, generalized) Dirichlet sums and therefore a one-to-one mapping to the Hilbert scale H_{β}

ii) $x = e^{2\pi i s}$ is a one-to-one mapping to Weyl sums and therefore a one-to-one mapping to the Hilbert scale $l_2^{\beta} \cong H_{\beta}$.

The circle method (defined on the open unit disc, [RaH] IV) is applied to additive number theory questions (e.g. [ErP1] [LaE] [LuB] [PrK]). The key conceptual element of the circle method is the definition of the partition number function based on prime number generation power series in combination with the Cauchy integral formula (e.g. ([PrK] VI, ([OsH] Bd. 1, 1.7). The challenge is that this results into "different " (problem depending) definitions of related partition number counting functions depending from even/odd and positive-pairwise/negative-pairwise different even/odd summands ([OsH] Bd. 1, 1.7). The advantage of the circle method (and the central concept why it has been established) is the fact that the convergence of all to be considered power series is always ensured, as the circle method operates in the open unit disk.

The circle method is about Fourier analysis over Z, which acts on the circle R/Z. The analyzed functions are complex-valued power series

$$f(x)=\sum_0^\infty a_n z^n \quad , \quad |z|<1 .$$

The fundamental principle is ([ViI] chapter I, lemma 4)

$$r^n a_n=\int_0^1 f(re^{2\pi it})e^{-2\pi\, \mathrm{int}}dt \quad , \quad 0<r<1 .$$

The circle method is applied to additive prime number problems. Hardy-Littlewood [HaG2] resp. Vinogradov [ViI] applied the Farey arcs resp. major and minor arcs ([HeH]) to derive estimates for corresponding Weyl sums ([WaA]) supporting attempts to prove the 2-primes resp. 3-primes Goldbach conjectures. All those attempts require estimates for purely trigonometric sums ([ViI]), as there is no information existing about the distribution of the primes, which jeopardizes all attempts to prove both conjectures.

We propose an alternative framework to leverage on the idea of the circle method to prove both Goldbach conjectures: th concept is about an replacement of the discrete Fourier transformation applied for power functions $f(x)$ by continuous Hilbert- (H), Riesz- (A) resp. Calderon-Zygmund-transformations (S) (which are Pseudo Differential Operators of order 0, -1 and 1) with distributional, periodical Hilbert space domains $H_\alpha^\#(0,1)$. The analogue fundamental principle is

$$-nf_n(x)=\frac{1}{2}\int_0^1\frac{f_n(y)}{\sin^2(\pi(x-y))}dy=:[Sf_n](x)=\left[A^{-1}f_n\right](x)$$

for

$$f_n(y):=a_n\cos 2\pi ny+b_n\sin 2\pi ny \quad .$$

The circle method is based on convergent power series with the open unit disk as domain. The Dirichlet series theory is an extension of the concept of power series replacing

$$\sum_1^\infty a_n e^{-xn}\to\sum_1^\infty a_n e^{-x\log n} .$$

The relationship between the Dirichlet series (see also Notes S44/45)

$$f(s):=\sum_1^\infty a_n e^{-s\log n} \qquad g(s):=\sum_1^\infty b_n e^{-s\log n}$$

and the Hilbert space $H_{-1/2}^\# \cong l_2^{-1/2}$ on the critical line is given by ([LaE] §227, Satz 40):

$$((f,g))_{-1/2}:=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega}f(1/2+it)g(1/2-it)dt=\sum_1^\infty\frac{1}{n}a_n b_n .$$

The cardinal series theory is an extension of the Dirichlet series theory.

The change leads to a generalized circle method on the circle in a $H^{\#}_{-\beta}-$ framework based on generalized Fourier series representations leveraging the method into two directions

- move from the open unit disk domain to the unit circle domain
- move from complex-value power series representations to generalized Fourier series representations with unit circle domain (resp. cardinal series with domain R) (e.g. [LiI]).

We propose to apply the properties of the zeros of the concerned Kummer functions for an alternative "prime number counting" process (Lemma 2.4, Notes O5/6/22/27):

all zeros z_n of the functions

$$F_1(\frac{1}{2},\frac{3}{2},z)$$

are complex-valued and lie in the horizontal stripe

$$2(n-1) < \omega_{2n-1} := \frac{|\mathrm{Im}(z_n)| - \pi}{\pi} < 2n-1 < \omega_{2n} := \frac{|\mathrm{Im}(z_n)|}{\pi} < 2n \quad .$$

As it holds

$$[\omega_{2n}] - [\omega_{2n-1}] = 1$$

resp.

$$2n - \frac{1}{2} < \frac{\omega_{2n-1} + \omega_{2n}}{2} + 1 < 2n + \frac{1}{2}$$

we propose a replacement in the form

$$2n = p + q \approx \frac{\omega_{2n-1} + \omega_{2n}}{2} + 1$$

to define an alternative definition of $H(x)$ which denote the number of prime pairs (p,q) for which it holds $p+q \le x$ given by ([LaE1], Note O30)

$$H(x) = \sum_{p \le x} \pi(x-p) \approx \int_2^{x-2} \pi(x-t)\frac{dt}{\log t} \approx \int_2^{x-2} \frac{x-t}{\log(x-t)}\frac{dt}{\log t}$$

For the relationships to the Hardamard gap condition, the Schnirelmann's density, the Littlewood-Paley function and corresponding Fourier series ([ZyA] XV) we refer to the Notes O5-7, O22-27,O33-35, S36-S38).

Our proposed enhanced circle method framework enables

- convergence and asymptotic analysis in a (distributional) Hilbert space framework with inner product on $l_2^{1/2}=(l_2^{-1/2})^*$ and appropriate linkage to the Fourier-Stieltjes integral concept ([NaS])

$$S(u,v)=\int_{S^1} u\cdot dg=-i\sum_{-\infty}^{\infty} nu_n v_n \quad , \quad ((u,v))_{-1/2}:=\lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega} u(1/2+it)v(1/2-it)dt=\sum_1^{\infty}\frac{1}{n}a_n b_n$$

- a generalized Schnirelmann's density concept in the form
-

$$\lim_{n\to\infty}\frac{A(n)}{n}\to\sum_1^{\infty}\frac{1}{n}|a_n|^2=\|\vec{a}\|_{-1/2}^2 \quad , \quad \vec{a}=(a(n))_{n\in N}\in l_2^{-1/2} .$$

It provides the linkage to

- the full power of spectral theory and of conformal mapping theory
- to probability theory([BiP]) and its linkage to Linnik's dispersion (variance) method ([LiJ])
- a convergent series representation of the (not fixed, not unique, non-measurable) ground state energy of the Hamiltonian operator of a free string ([BrK3])
- Hardy and BMO (bounded mean oscillation) spaces (→ dispersion method)
- an alternative "Dirac function" functionality with slightly (but critical) better regularity requirements than (see also Note O52)

$$\delta(x)=\frac{1}{2\pi}\int_0^{2\pi} e^{ikx}dk=\frac{1}{\pi}\int_0^{2}\cos(kx)dk\in l_2^{-1/2-\varepsilon}$$

- the Teichmüller theory ([NaS])
- Ramanujan's (main) master theorem ([BeB], lemma A10)
- the inverse formula of Stieltjes for BMO density functions (Note S33)
- the concept of of logarithmic capacity of sets and convergence of Fourier series to functions fulfilling ([ZyA] V-11)

$$\sum_1^{\infty} n\left[a_n^2+b_n^2\right]<\infty$$

- harmonic analysis by ([StE])

$$[\varphi]^2:=\frac{\pi}{2}\sum_1^{\infty}\nu(a_\nu^2+b_\nu^2)=\frac{1}{2}\iint|dh(z)|^2dxdy=\frac{1}{4\pi}\int_{\partial B}\int_{\partial B}\frac{|\varphi(w)-\varphi(\zeta)|^2}{|w-\zeta|^2}ds(w)d\zeta<\infty$$

  and the related energy of the harmonic continuation $h=\mathrm{E}(\varphi)$ to the boundary
- Jacobians of the Riemann surfaces ([BiI]), "mute" winding numbers ([BoJe]), topological degree (H. Brezis), electric field integral equation theory
- a global unique weak $H_{-1/2}-$solution of the generalized 3D Navier-Stokes initial value problem with not vanishing (generalized) non-linear energy term *www.navier-stokes-equations.com* (Note O55)

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2+\|u\|_{1/2}^2\le|(Bu,u)_{-1/2}|\le c\cdot\|u\|_{-1/2}\|u\|_1^2 .$$

The Goldbach problem

The binary Goldbach problem states that every even integer greater 2 can be represented as the sum of two primes. Every integer n can be represented in the form $n = n_1 + n_2$ in $n-1$ different ways. The relative frequency of the occurrence of primes is $\log^{-1} n$, i.e.

$$\pi(x) \approx \int_2^x \frac{dn}{\log n} \approx \frac{x}{\log x} \ .$$

Therefore, an even n has about

$$(n-1)\left[\frac{1}{\log n}\right]^2$$

representations as a sum of two primes.

The current state of verification of the Goldbach conjecture is, that it is true for nearly all even integers, i.e. ([LaE] V), let $h(n)$ denote the number of the first n even positive integers, which can not represented as a sum of two primes, then there exists a constant $\vartheta < 1$ that

$$\lim_{n\to\infty} \frac{h(n)}{n^{\vartheta}} = 0 \text{ , i.e } \lim_{n\to\infty} \frac{h(n)}{n} = 0$$

leading to Schnirelmann's "density" concept ([ScL]).

The result above states that for at most 0% of all even positive integers the Goldbach conjecture is not true.

The complementary set of all even integers which cannot be represented as a sum of two primes has the natural (Schnirelmann) density zero, i.e. ([OsH] Bd. 2, 21)

$$G(x) = O(\frac{x}{\log^{\alpha} x}) \quad \forall \alpha > 0 \ .$$

From [PrK] II, §4, we recall the theorem of Brun, i.e.

If p' goes through all twins prime pairs, then the following series is convergent

$$\sum_{p'} \frac{1}{p'}$$

We note that the binary Goldbach problem is inaccessible to the dispersion (variance) method as given in [LiJ] X.2. The main difficulty is the calculation of a term which is asymptotically equal to the number of solutions of the equation

$$\nu_1(n-p_1) = \nu_2(n-p_2) \ , \ \nu_1 \neq \nu_2 \text{, where } \nu_1, \nu_2, p_1, p_2 \text{ are primes.}$$

Remark: The dispersion method in binary additive problems is about the concepts of dispersion, covariance, and the Chebysev inequality ([LiJ]).The central concept is that of the independence of events relating to different primes. The dispersion method simply takes for use a finite field of elementary events. Its application to concrete binary additive problems involves a great deal of rather cumbersome computations (the calculation of the dispersion of the number of solutions). The construction of the fundamental inequality for the dispersion closely resembled Vinogradov's method for the estimation of double trigonometric sums. The latter one somehow corresponds to the double integral representation of the Hilbert-transformed Gaussian function above.

We propose to define generalized variances with respect to the appropriate l_2^{β} − distributional Hilbert space framework applying corresponding asymptotic analysis for the corresponding generalized (distributional) Fourier series representations ([EsR], [VlV]).

Let $H(x)$ denote the number of prime pairs (p,q) for which it holds $p+q \le x$ given by ([LaE1], Note O30)

$$H(x) = \sum_{p \le x} \pi(x-p) \approx \int_2^{x-2} \pi(x-t)\frac{dt}{\log t} \approx \int_2^{x-2} \frac{x-t}{\log(x-t)}\frac{dt}{\log t}$$

Stäckel's approximation formula is given by ([LaE1])

$$G(n) \approx \frac{n}{\log^2 n} \cdot \frac{n}{\varphi(n)} = \frac{n}{\log^2 n} \cdot \prod_p (1-\frac{1}{p})^{-1} = \frac{n}{\log^2 n} \cdot \prod_p \frac{p}{p-1} \cdot .$$

It provides the mean value for the corresponding variance (dispersion) calculation "to attack" the binary Goldbach problem. We propose an alternative definition of $H(x)$ based on the corresponding density function $\vartheta(x)$ and the related alternative li-function

$$li^*(x) \approx \frac{\vartheta(x)}{\log x} \frac{1}{1+\frac{1}{x\log x}\cdot(\frac{\log\sqrt{x^2-1}}{1+\log(2\pi)})} \approx \frac{\vartheta(x)}{\log x} \cdot \frac{x}{1+x} .$$

Remark: A Schnirelmann density corresponds to the probability to pick an element $n_k \in A$ out of the total numbers of integers. The concept builds on the simplest function of period 1 ([WeH])

$$e(nx) = e^{(2n)\pi i x} \text{ for all integers } n .$$

For any sequence $a(n) = a_n$ and any integer m it holds

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e(m \cdot a_k) = \int_0^1 e(mx)dx = 0 .$$

It also holds the following inverse:

If for any integer m it holds

$$\sum_{k=1}^{n} e(m \cdot a_k) = o(n)$$

then the numbers $a_n \bmod 1$ build a uniform dense distribution on the unit circle. Vinogradov's solution concept it built on the Weyl sums. The root cause of current handicaps to prove appropriate estimates in this framework are due to corresponding estimates of the Weyl sums and not due to Goldbach problem specific challenges.

We propose to apply an analog Weyl sums based concept replacing the exponential function by corresponding Kummer functions and its related zeros (see also Notes O13/16 resp. Notes O6/O7/O27).

Remark: Let

$$a_n := N(p_1, p_2 \in P | n = p_1 + p_2) .$$

Then for appropriate constants c_1, c_2 it holds ([PrK] V)

$$N\left(n \le x; | \frac{c_1 x}{\log^2 x} < a_n < \frac{c_2 x}{\log^2 x}\right) > c_3 \cdot x .$$

Our proposed replacements above

$$\pi(x) \approx \frac{\psi(x)}{\log x} \quad \rightarrow \quad \pi(x) \approx \frac{\psi_0(x) + \vartheta(x)}{\psi_0(x) - \vartheta(x)} .$$

are supposed to enable appropriate estimates to verify the binary Goldbach problem.