Explicit Solutions of the Viscous Model Vorticity Equation

STEVEN SCHOCHET
Princeton University

Abstract

Explicit solutions are found for the viscous version of the model vorticity equation recently proposed by P. Constantin, P. D. Lax, and A. Majda:

$$w_t = H(w)w + \nu w_{xx},$$

where $H(w)$ is the Hilbert transform of w, and ν is a positive constant. Various properties of these solutions, including the fact that they blow up after a finite time, are discussed.

1. Introduction

The equation

$$w_t = H(w)w, \tag{1}$$

where $H(w)$ is the Hilbert transform of w, has recently been proposed by P. Constantin, P. D. Lax, and A. Majda [2] as a simple model for the vorticity equation of three-dimensional inviscid incompressible fluid flow, which can be written as (see [2])

$$\frac{D}{Dt}\omega = D(\omega)\omega, \tag{2}$$

where $D(\cdot)$ is a certain strongly singular integral operator and D/Dt is the convective derivative. See [2] for a discussion of the analogies between (1) and (2) and for the explicit solution of (1).

For viscous flow the vorticity equation (2) is modified to

$$\frac{D}{Dt}\omega = D(\omega)\omega + \nu\Delta\omega, \tag{3}$$

which suggests

$$w_t = H(w)w + \nu w_{xx} \tag{4}$$

as the viscous model vorticity equation. In this paper I present some explicit solutions of (4) that blow up in finite time. Various properties of these solutions,

Communications on Pure and Applied Mathematics, Vol. XXXIX 531–537 (1986)
 CCC 0010-3640/86/040531-07$04.00

some interesting in their own right and some relating to the usefulness of (4) as a model, will be discussed in Section 4.

2. Complexification

The Hilbert transform H is defined by

$$[H(f)](x) = \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{\infty} \frac{1}{x-y} f(y)\, dy. \tag{5}$$

One of the insights of [2], in slightly different notation than theirs, is that the fact that $H(f)$ is the unique function such that $Q = f + iH(f)$ is analytic in the upper half-plane and vanishes at infinity implies that (1) is the restriction to the real axis of the real part of

$$Q_t = -\tfrac{1}{2}iQ^2, \tag{6}$$

because $H(w) \cdot w = \mathcal{R}e[-\tfrac{1}{2}i(w + iH(w))^2]$. Since

$$w' + iH(w)' = [Q(z)_{\restriction \mathcal{I}m\, z=0}]' = [Q'(z)]_{\restriction \mathcal{I}m\, z=0} = w' + iH(w'),$$

(4) is the restriction of

$$Q_t = -\tfrac{1}{2}iQ^2 + \nu Q''; \tag{7}$$

that is, if Q satisfies (7) and is analytic in the upper half-plane and vanishes at infinity, then $\mathcal{R}e\, Q(z)_{\restriction \mathcal{I}m\, z=0}$ satisfies (4).

3. Explicit Solutions

It is easy to check that (7) has the stationary solutions

$$Q = -\frac{12\nu i}{(z-z_0)^2}; \tag{8}$$

if z_0 is chosen to lie in the lower half-plane, this yields a stationary solution of (4). Trying to determine how two such "polar solutions" interact (cf. [1], [4]) leads one to look for a solution of the form

$$Q(z,t) = \frac{A(t)}{z - z_1(t)} + \frac{B(t)}{[z - z_1(t)]^2} + \frac{C(t)}{z - z_2(t)} + \frac{D(t)}{[z - z_2(t)]^2}. \tag{9}$$

The poles of order one are included because $(z - z_1)^{-2} \cdot (z - z_2)^{-2}$ has such terms in its partial-fraction decomposition. Since it turns out that the solutions have

$$C(t) = -A(t), \qquad D(t) = B(t), \tag{10}$$

it will simplify the ensuing algebra to assume this from the outset.

Substituting (9) into (7), using partial fractions to express the result in terms of the form $f(t)/(z - z_j)^p$, equating coefficients of like terms on the two sides of the equation, and simplifying the results yields the following five equations for the four unknowns A, B, $z_1 + z_2$, $z_1 - z_2$:

$$B = -12\nu i, \tag{11a}$$

$$-\frac{1}{12}iA^2 + \frac{12\nu A}{z_1 - z_2} + \frac{144\nu^2 i}{(z_1 - z_2)^2} = 0, \tag{11b}$$

$$(z_1 - z_2)_t = -\tfrac{5}{6}iA, \tag{11c}$$

$$(z_1 + z_2)_t = 0, \tag{11d}$$

$$A_t = \tfrac{5}{6}iA^2/(z_1 - z_2). \tag{11e}$$

Since (11c, e) implies $[A(z_1 - z_2)]_t = 0$ in agreement with the solution

$$\frac{i}{\nu}A(z_1 - z_2) = 12(6 \pm \sqrt{6}) \equiv K_\pm \tag{12}$$

of (11b), the overdetermined set of equations (11a–e) is consistent, and so solutions of the form (9) do exist. Writing (12) as $iA = K_\pm \nu/(z_1 - z_2)$, substituting this into (11c), and solving the resulting ordinary differential equation one obtains

$$[z_1(t) - z_2(t)]^2 = [z_1(0) - z_2(0)]^2 - \tfrac{5}{3}K_\pm \nu t. \tag{13}$$

Taking the square root of (13) and combining it with the solution

$$z_1(t) + z_2(t) = z_1(0) + z_2(0) \tag{14}$$

of (11d) yields

$$z_1(t) = \tfrac{1}{2}\left[z_1(0) + z_2(0) + \left([z_1(0) - z_2(0)]^2 - \tfrac{5}{3}K_\pm \nu t\right)^{1/2}\right], \tag{15a}$$

$$z_2(t) = \tfrac{1}{2}\left[z_1(0) + z_2(0) - \left([z_1(0) - z_2(0)]^2 - \tfrac{5}{3}K_\pm \nu t\right)^{1/2}\right]. \tag{15b}$$

Finally, plugging (15) into (12) one obtains

$$A(t) = -K_{\pm}\nu i / \left([z_1(0) - z_2(0)]^2 - \tfrac{5}{3} K_{\pm} \nu t \right)^{1/2}. \tag{16}$$

Thus $z_1(0)$, $z_2(0)$, and the sign in $K_{\pm}$ can be chosen arbitrarily, and then (10), (11a), (15a, b), (16) determine the solution (9) to equation (7). If $z_1(0)$ and $z_2(0)$ both lie in the lower half-plane, then the real part of (9) on the real axis yields a solution to (4) for as long as $z_1(t)$ and $z_2(t)$ both remain in the lower half-plane. Denote this solution by $w(t, x, \nu, z_1(0), z_2(0), \pm)$.

4. Properties of the Solutions

LEMMA 1. *For all $z_1(0)$ and $z_2(0)$ in the lower half-plane and either choice of sign. the solution $w(t, x, \nu, z_1(0), z_2(0), \pm)$ of (4) blows up in finite time.*

Proof: Let $z_j(t) = x_j(t) + iy_j(t)$, $j = 1, 2$. By (14), $y_1(t) + y_2(t) =$ constant, and in view of the real part of (13), $|y_1(t) - y_2(t)| \to \infty$ as $t \to \infty$; thus one of the poles $z_j(t)$ must eventually cross the real axis, at which time the solution blows up.

Let $T^*(\nu, z_1(0), z_2(0), \pm)$ be the blow-up time of the solution $w(t, x, \nu, z_1(0), z_2(0), \pm)$, and define, as in [2], the "velocity" v corresponding to the "vorticity" w by

$$v(x) = \int_{-\infty}^{x} w(s)\, ds. \tag{17}$$

For the inviscid equation (1) the velocity typically remains bounded in L^p when the vorticity blows up (see [2]), but this is no longer true in the viscous case.

LEMMA 2. *For all solutions $w(t, x, \nu, z_1(0), z_2(0), \pm)$ the following holds:*
(i) *for all s, $v(t) \in W^{s,p}(R)$ for $t \in [0, T^*)$, $1 < p \leqq \infty$;*
(ii) $\lim_{t \nearrow T^*} \|v\|_{L^p} = \lim_{t \nearrow T^*} \|w\|_{L^p} = \infty$, $1 < p \leqq \infty$.

Proof: (i) Using (10), (11a), (12), one can write (9) as

$$w(x) + i[H(w)](x) = -\frac{K_{\pm}\nu i}{(x - z_1)(x - z_2)} - \frac{12\nu i}{(x - z_1)^2} - \frac{12\nu i}{(x - z_2)^2}. \tag{18}$$

It is now easy to verify that w and all of its derivatives are in L^p, $1 \leqq p \leqq \infty$, as long as z_1 and z_2 stay away from the real axis. Also, $\int_{-\infty}^{\infty} w = 0$ so that

$v(+\infty) = 0$ and $v = O(1/|x|)$ as $|x| \to \infty$, and hence $v \in L^p, 1 < p \leqq \infty$. Part (ii) follows from (18) on taking the limit $\mathscr{I}m\, z_j \to 0$, $j = 1$ or 2.

Equation (4) is thus a less successful qualitative model than (1), since if any solutions of the Navier-Stokes equations with initial data in H^2, say, do blow up, they do so in such a way that the velocity remains bounded in L^2.

On the other hand, equation (4), like (1) (see [2]), does have the appropriate scale invariance. That is, if $w(t, x)$ satisfies (4), then $\lambda^{1+\alpha}w(\lambda^{1+\alpha}t, \lambda^\alpha x)$ satisfies (4) with ν replaced by $\lambda^{\alpha-1}\nu$, and this is the same scaling law as for ω in (3). Our set of explicit solutions is also scale invariant, i.e.,

$$\lambda^{1+\alpha}w\left(\lambda^{1+\alpha}t, \lambda^\alpha x, \lambda^{\alpha-1}\nu, z_1(0), z_2(0), \pm\right) = w\left(t, x, \nu, \lambda^{-\alpha}z_1(0), \lambda^{-\alpha}z_2(0), \pm\right).$$

Taking $\alpha = -1$ and $\lambda = f(\nu)$ shows that

$$w\left(0, x, \nu, f(\nu)z_1(0), f(\nu)z_2(0), \pm\right) = w\left(0, x/f(\nu), \nu f(\nu)^{-2}, z_1(0), z_2(0), \pm\right)$$

$$= \nu f(\nu)^{-2} w_0\left(x/f(\nu)\right),$$

and the only function this can converge to as $\nu \to 0$ is zero, so the set of initial data for which the explicit solution is known is not large enough to examine this limit.

Solutions of (4) with fixed ν can, however, be compared with those of (1) with the same initial data. Let $\tau^*(\nu, z_1(0), z_2(0), \pm)$ be the blow-up time of the solution of (1) with initial data $w_0(x) = w(0, x, \nu, z_1(0), z_2(0), \pm)$.

LEMMA 3.

(i) $$T^* = \frac{12}{5K_\pm \nu} y_1(0) y_2(0)\left[1 + \frac{[x_1(0) - x_2(0)]^2}{[y_1(0) + y_2(0)]^2}\right];$$

(ii) *if* $c > .2042$, *then*

$$\tau^*\left(\nu, x_1(0) - ic\left|\tfrac{1}{2}(x_1(0) - x_2(0))\right|, x_2(0) - ic\left|\tfrac{1}{2}(x_1(0) - x_2(0))\right|, +\right)$$

$$= 2(1 + c^2)^2\left|\tfrac{1}{2}(x_1(0) - x_2(0))\right|^2 \Big/ \nu\left[K_+(1 + c^2) - 24(1 - c^2)\right];$$

(iii) *if* $c < .219$ *in* (ii), *then* $\tau^* > T^*$.

In other words, adding the diffusion sometimes makes the solution blow up sooner!

Proof: (i) At time T^*, y_1, say, is equal to 0. Solving the real and imaginary parts of (13) and (14) for the four unknowns x_1, x_2, y_2 and t yields, in particular, the formula for T^*.

(ii) As shown in [2], $\tau^* = 2/M$, where $M = \max\{[[H(w)](x)]^+ | w_0(x) = 0\}$. By translational invariance we may assume that $-x_2(0) = x_1(0) > 0$. A straightforward calculation from (18) shows that $w_0(x) = \nu y_1(0) x_1^4(0) EPx$, where P is positive and E is a quadratic expression in $(x/x_1(0))^2$ that is definite provided $c > .20414$ (rounded to 5 decimals). Then $M = [H(w_0)](0)$ and evaluating this from (18) yields the formula for τ^*.

(iii) In the case considered in (ii),

$$T^* = \frac{12}{5K_+\nu}(1 + c^2)\left|\tfrac{1}{2}(x_1(0) - x_2(0))\right|^2.$$

The condition $\tau^* > T^*$ reduces to $(1 - c^2)/(1 + c^2) > K_+/144$, which holds if $c < .21928$ (rounded to 5 decimals).

The phenomenon $\tau^* > T^*$ can be explained as follows:

Suppose $w_0(x_1) \approx 0$ but not equal to 0, $w_0(x_2) = 0$, and $[H(w_0)](x_1) > [H(w_0)](x_2) > 0$. The formula $w(t, x) = 4w_0(x)/[[2 - t[H(w_0)](x)]^2 + t^2 w_0^2(x)]$ from [2], shows that the solution of (1) becomes large at the point x_1 at time $t = 2/[H(w_0)](x_1)$, but remains bounded until it blows up, in a neighborhood of x_2, at time $t = 2/[H(w_0)](x_2)$. Although the solution of (4) in case (iii) also blows up at x_2 and not at x_1, the large value of w at x_1 will diffuse towards x_2 and hasten the blow-up of the solution to (4).

Two final remarks: First, equations (11c, e) for the motion of A and $z_1 - z_2$ can be written as a Hamiltonian system, with the Hamiltonian given up to a constant factor by the left side of (11b). In fact, equation (4) can be written as an infinite-dimensional Hamiltonian system; the details will be presented elsewhere.

Second, if instead of picking $z_1(0)$ and $z_2(0)$ in the lower half-plane we pick $z_2(0) = z_1(0)^*$, then $Q(t, z)$ is pure imaginary on the real axis, and $f = -\frac{1}{2}iQ(t, x)$ is a solution of $f_t = f^2 + \nu f_{xx}$ such that

(i) the solution exists for $t \in [0, \infty)$

and

(ii) $\min_{x \in R} f(t) < 0 < \max_{x \in R} f(t)$ for all t.

In [3] solutions with these two properties were constructed for a class of equations including $f_t = |f| f + \nu f_{xx}$.

Acknowledgments. I thank Andy Majda for presenting the results from [2] and proposing the problem of how solutions of (4) behave to his seminar on incompressible fluid dynamics at Princeton University. I also thank Mike Weinstein for several helpful suggestions.

The research for this paper was supported by the National Science Foundation postdoctoral fellowship #DMS84-14107.

Bibliography

[1] Chudnovsky, D. V., and Chudnovsky, G. V., *Pole expansions of nonlinear partial differential equations*, Il Nuovo Cimento, 40B, 1977, pp. 339–353.

[2] Constantin, P., Lax, P. D., and Majda, A., *A simple one-dimensional model for the three-dimensional vorticity equation*, Commun. Pure Appl. Math., 38, 1985, pp. 715–724.

[3] Haraux, A., and Weissler, F. B., *Nonuniqueness for a semilinear initial value problem*, Indiana U. Math. J. 31, 1982, pp. 167–189.

[4] Kruskal, M. D., *The KdV equation and evolution equations*, in *Nonlinear Wave Motion*, A. Newell ed., American Mathematical Society, Providence, 1974.

Received July, 1985.