Deterministic and Statistical Models for Turbulence:

What Could Burgers Have Said to Kolmogorov?

Barbara Lee Keyfitz

Fields Institute and University of Houston

bkeyfitz@fields.utoronto.ca

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Introduction

Two Parts to talk

I. How statisticians are involved in programs at Fields

1. Thematic programs
2. DLSS
3. NPCDS

II. How mathematicians and statisticians look at the same problem with different tools: turbulence

1. Statistical (Kolmogorov)
2. Deterministic: modelling (Burgers)

Thematic Programs

Visitors Workshops Lecture Series Graduate Courses
Related Activities

- Probability and Its Applications August 1998 - June 1999 Organizing Committee D. Dawson (Fields), N. Madras (York), T. Salisbury (York), G. Slade (McMaster)
- Causal Interpretation and Identification of Conditional Independence Structures September to November 1999 Organizing Committee: Hélène Massam (University of Virginia), David Tritchler (University of Toronto)
- Fall 2005 Renormalization and Universality in Mathematics and Mathematical Physics
  Course: Percolation, Brownian motion, and SLE.
  Workshop, September 20-24, 2005: Percolation, SLE, and related topics. Organizing Committee: Ilia Binder (Toronto), Steffen Rohde (University of Washington)

DLSS

Distinguished Lecture Series in Statistical Science
(by nomination)
Week of November 7-11, 2005
Brad Efron, Stanford
http://www.fields.utoronto.ca/programs/scientific/statistical_lect
Previous Lectures:

- Sir David Cox
- Don Dawson
- Donald Fraser
- Peter G. Hall

NPCDS

National Program on Complex Data Structures

- Joint venture of SSC & 3 institutes
- National network in the statistical sciences

The broad goal is to foster nationally coordinated projects with substantial interactions with the large community of scientists involved in analysis of complex data sets, and to establish a framework for national networking of research activities in the statistical community.

- http://www.fields.utoronto.ca/programs/scientific/NPCDS/
- application of statistical methods for the analysis of data
- complex survey sample designs
- longitudinal biological, epidemiological, medical studies

NPCDS

Jamie Stafford (University of Toronto),
Director of the National Program
David Bellhouse (University of Western Ontario)
Richard Cook (University of Waterloo)
Paul Gustafson (UBC)
Mike Hidiroglou (Statistics Canada)
Nancy Reid (University of Toronto)
Randy Sitter (Simon Fraser University)
Ed Susko (Dalhousie University)
Louis-Paul Rivest (Université Laval).
Goal: establish 4-6 national projects

- form partnerships
- increase influence of statistical sciences on research agendas

NPCDS

Areas

- Data w complex structure
- Surveys w complex design
- Biology, Medicine, Industry, Environment, etc
- Data that is Longitudinal, Hierarchical, Correlated, Multi-level
- Emerging data types

NPCDS

Nine workshops run or planned; most recent –

- October 13-15, 2005
  Workshop on Current Issues in the Analysis of Incomplete Longitudinal Data; to be held at the Fields Institute
- August 17-19, 2005
  Workshop on Spatial/Temporal Modelling for Marine Ecological Systems; to be held at Dalhousie University
- May 24-28, 2005
  Workshop on Forest Fires and Point Processes; held at the Fields Institute
- May 4-6, 2005
  Workshop on Latent Variable Models and Survey Data for Social Sciences Research; held at Centre de recherches mathématiques, Montréal

NPCDS

Current projects underway

- Statistical Methods for Complex Survey Data (w MITACS) (Bellhouse, UWO)
- Statistical Genomics (Rafal Kustra, Toronto)
  proposal came from Canadian Consortium on Statistical Genomics
- Computer Experiments for Complex Systems (Bingham, SFU)
- Data Mining Methodology and Applications (Chipman, Acadia)

Operation: start with workshop, formulate project (60K/yr), with faculty, PDFs, partners (industry, gov't, medicine)

Turbulence

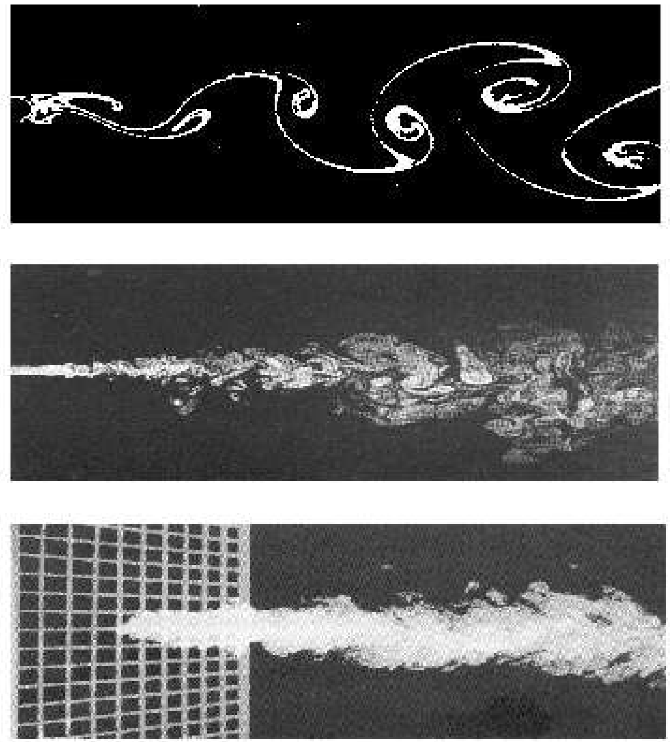
Navier-Stokes Equations

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} &= -\nabla p + \nu\Delta\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0 \end{aligned}$$

$\nu =$ viscosity, small
$Re = \frac{LV}{\nu} =$ Reynolds number, large

- Equations correct
- No existence theory
- No prediction of patterns

Problems:
– formation of eddies
– vortex stretching,
– typical scales
– energy cascade
(transfer of energy between scales)



Van Dyke, Lesieur

Background

- Routes to turbulence (instability of laminar flow)
- Hydrodynamic stability or fully developed turbulence
- Boundary layer or homogeneous
- Stationary, isotropic
- Practical models ($K - \epsilon$)

Start by defining mean flow and fluctuations

- mean flow uniform, steady, or even trivial
- Fluctuations modelled statistically or deterministically

Statistical Theory of Turbulence

Why a statistical theory?

- Experimental results are not reproducible in detail; statistical properties are reproducible
- Energy cascade, correlations etc, are statistical properties
- Modern theory: ‘chaotic’ behaviour can be deterministic
- Ergodic theorem: time averages = ensemble averages, so experiments (time traces or spatial correlations) are statements about random variable (initial condition) and stochastic process (time-dependent solution)
- Interesting questions about discrepancies, and theory of ‘large deviations’

Velocity $\mathbf{u}(t, \mathbf{x}, \omega)$ is stochastic process with $\mathbf{u}(0, \mathbf{x}, \omega) = \omega$

The Spectrum

Kinetic energy $E = \frac{1}{2}\int \mathbf{u}^2\,dx$

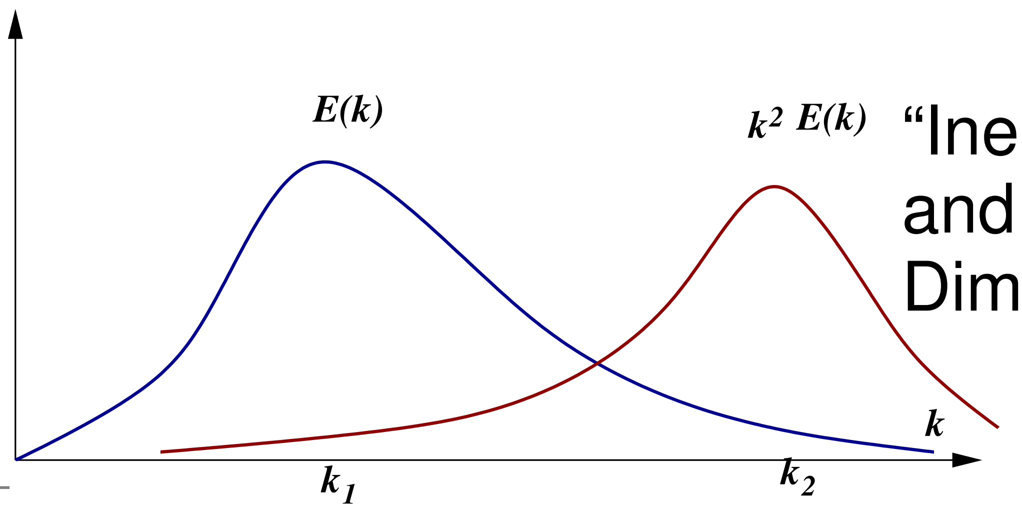
Energy dissipation $\frac{dE}{dt} = -\frac{1}{Re}\int (\nabla \mathbf{u})^2\,dx$ (from equation)

Fourier transform (or series): $\hat{\mathbf{u}}(\mathbf{k}, t)$; energy $E(k)$, $k = |\mathbf{k}|$;

$$E = \int E(k)\,dk \qquad k = \frac{1}{L} = \text{spatial scale of “eddy”}$$

Energy spectrum is $E(k)$

Dissipation spectrum (from $\dot{E}$) is $\frac{1}{Re}k^2 E(k)$



“Inertial range” between k_1 and k_2 (stationary behaviour)

Dimensional considerations:

$$E(k) = Ck^{-5/3}$$

Correlations

Velocity correlation

$$\langle u_i(t,\mathbf{x})u_j(t',\mathbf{x}')\rangle = \Gamma_{ij}(t-t',\mathbf{x}-\mathbf{x}')$$

for stationary, homogeneous turbulence
General properties:

$$E(k) = Ck^{-p}$$

implies

$$\langle |u_i(t,\mathbf{x}) - u_j(t,\mathbf{x}')|^2\rangle = C|\mathbf{x}-\mathbf{x}'|^{p-1}$$

Note:

- valid only in inertial range (convergence)
- experiments have $p-1 = 2/3$ in inertial range

Experimental Results

Two basic experimental results

1. Two-thirds law. $\langle |u_i(t,\mathbf{x}) - u_j(t,\mathbf{x}')|^2 \rangle = C\ell^{2/3}$ where $|\mathbf{x} - \mathbf{x}'| = \ell$.
2. Energy dissipation dE/dt as a function of Reynolds number approaches a positive, finite limit as $Re \to \infty$.

1941: Kolmogorov deduced these from the Navier-Stokes equation.

- Theory now used to develop further properties of turbulence, such as intermittency (current work of Chorin, Frisch, etc)

Kolmogorov’s 1941 Theory

Hypotheses:

1. Symmetries of NSE in statistical sense as $Re \to \infty$.
2. Flow is self-similar (small scales, large Re): $\exists!\ h$

$$\delta \mathbf{u}(\mathbf{x}, \lambda \mathbf{r}) = \lambda^h \delta \mathbf{u}(\mathbf{x}, \mathbf{r})$$

   where $\delta \mathbf{u}(\mathbf{x}, \mathbf{r}) = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})$
   (Called universality by Kolmogorov)

3. Finite, nonzero mean energy dissipation rate ε.

Theorem (4/5 law): In the limit $Re \to \infty$,

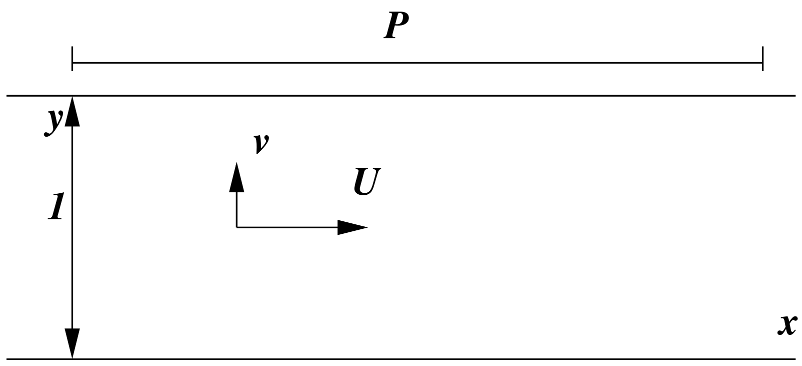
$$\langle (\delta \mathbf{u}(\mathbf{x}, \mathbf{r})^3 \rangle = -\frac{4}{5} \varepsilon |\mathbf{r}|$$

Proof: NSE $+$ (1), (2) and (3).
Thm $\Rightarrow h = 1/3$, and $h = 1/3 \Rightarrow p = 5/3$.

Burgers' Model

Infinite channel (x) width 1 (y):

$$\begin{aligned} \frac{dU}{dt} &= P - \nu U - \int_0^1 v^2\,dy \\ \frac{\partial v}{\partial t} &= Uv + \nu\frac{\partial^2 v}{\partial y^2} - 2v\frac{\partial v}{\partial y} \end{aligned}$$



$U(t) =$ laminar component $\quad P =$ pressure drop

$v(y,t) =$ turbulent fluctuation $\quad \nu =$ viscosity

- Uv and $\int v^2$ represent transfer of energy from laminar to turbulent modes & balance each other
- $-2v\frac{\partial v}{\partial y}$ is internal transfer among turbulent modes
- $E(t) = \frac{1}{2}U^2 + \frac{1}{2}\int_0^1 v^2\,dy$
  $\frac{dE}{dt} = PU - \nu\left(U^2 + \int (\frac{\partial v}{\partial y})^2 dy\right)$

Properties of Model

$$\dot{U} = P - \nu U - \int v^2, \; \dot{v} = Uv + \nu v_{yy} - (v^2)_y$$

1. Stability dependence of laminar flow on ν or Re
2. Energy transfer between modes of turbulent component
3. Coherent structures

Purpose of Model

- show three properties may result from balance of viscous damping and quadratic nonlinearities
- study dependence on parameters ν and P

Leaves out

- 3-D (even 2-D) effects
- vortices and vortex-stretching
- transition, main sequence, chaos

Loss of Stability

$$\dot U = P - \nu U - \int v^2, \qquad \dot v = Uv + \nu v_{yy} - (v^2)_y$$

Laminar solution $U = \frac{P}{\nu}$, $v \equiv 0$

Bifurcation to stationary soln, $\dot U = 0 = \dot v$; $v = \nu\varphi$, $\lambda = P/\nu^2$

$$\varphi'' - 2\varphi\varphi' + \varphi[\lambda - \int \varphi^2] = 0, \qquad \varphi(0) = 0 = \varphi(1)$$

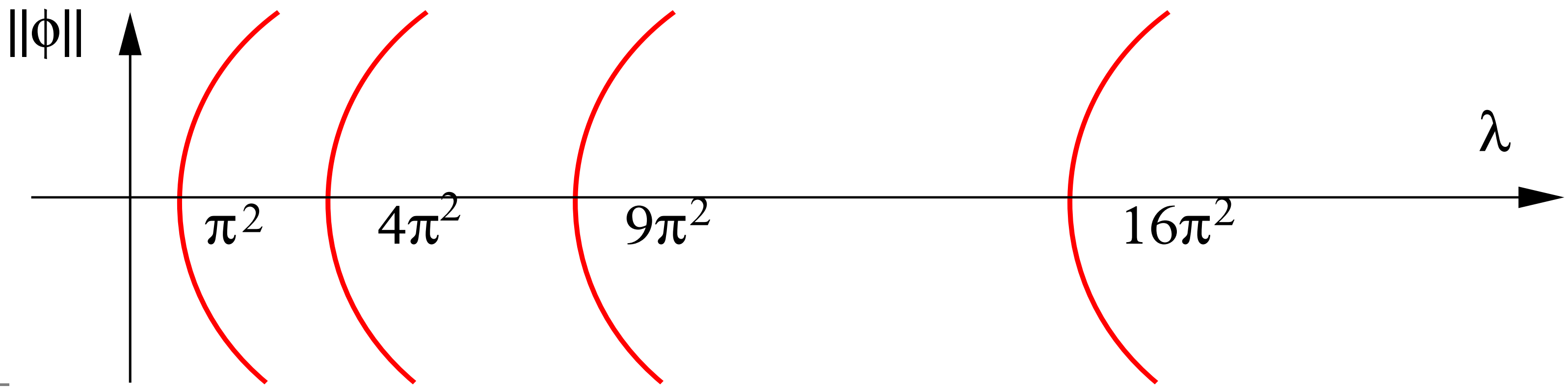
Linearize equation $\varphi'' + \lambda\varphi = 0$, $\lambda_n = n^2\pi^2$, $n = 1, 2, \ldots$

Wirtinger's inequality $\Rightarrow [\int \varphi^2][(\pi^2 - \lambda) + \int \varphi^2] \le 0$

Liapunov function $\mathcal{E}(t) = \frac{1}{2}(U - \frac{P}{\nu})^2 - \frac{1}{2}\int v^2$

$$\dot{\mathcal{E}}(t) \le -\nu(U - \frac{P}{\nu})^2 - \nu(\pi^2 - \lambda)\int v^2 < 0$$

Stable if $P/\nu^2 < \pi$, supercritical bifurcation at $n\pi$



Partition of Energy

Analysis of Fourier series: modes, transfer and decay

$$v = \sum_1^\infty \xi_n \sin n\pi y, \qquad \dot{U} = P - \nu U - \frac{1}{2}\sum_1^\infty \xi_n^2$$

$$\dot{\xi}_n = (U - \nu n^2\pi^2)\xi_n + n\pi\{\text{quadratic terms}\}$$

Decay if $\pi^2 n^2 > \frac{P}{\nu^2}$

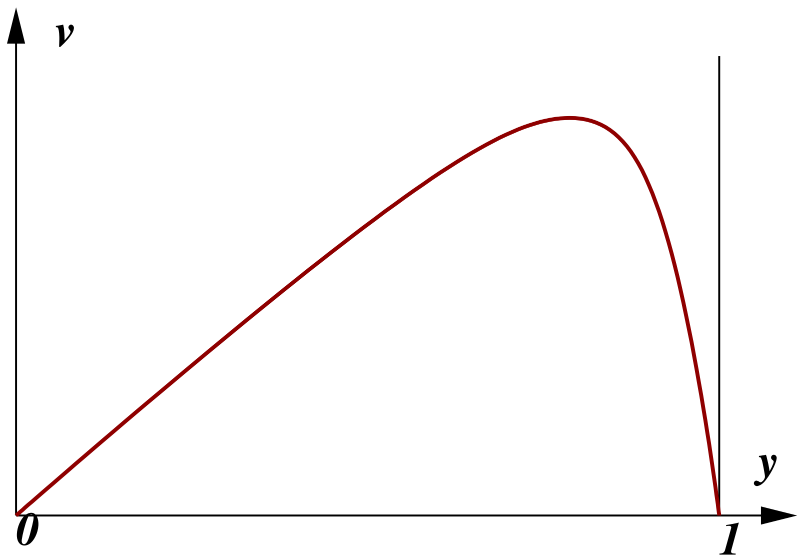
Exact solns for U const,

$v = \frac{U}{2}\left(y - 1 + \tanh\frac{U(1-y)}{2\nu}\right)$

$\xi_n = U/(\pi n)$ for $n \ll Re$,

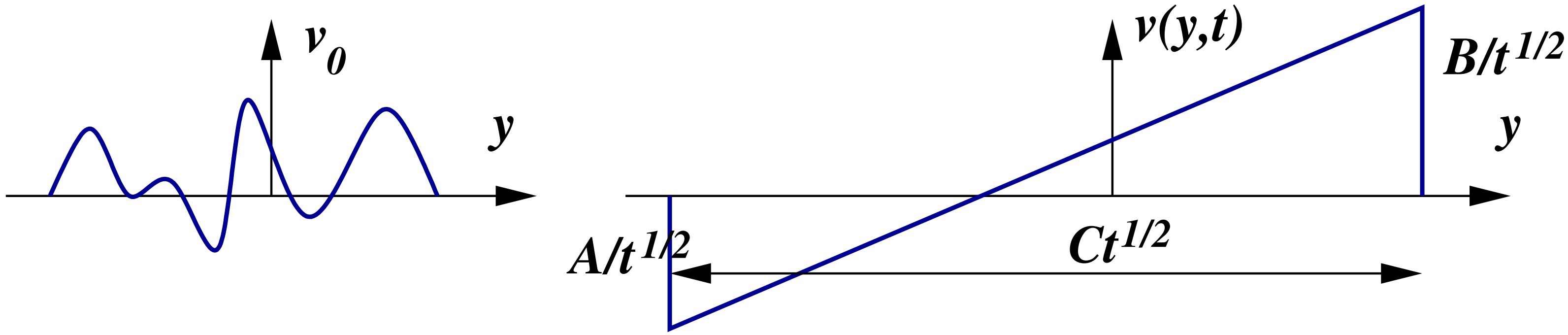
$\xi_n = 2\pi\nu e^{-\pi^2 n\nu/U}$ for $n > Re$,

- equipartition of energy dissipation in finite number of modes



Burgers' Equation

Infinite (in y) domain, $U = 0$, $v_t + 2vv_y = \nu v_{yy}$
Decay to N-waves:



$$A = \min_y \int_{-\infty}^{y} v_0(\eta)\, d\eta, \qquad B = \max_y \int_{-\infty}^{y} v_0(\eta)\, d\eta$$

Momentum $M = \int_{-\infty}^{\infty} v\, dy =$ const, Energy $= \mathcal{O}\left(\frac{M^{3/2}}{\sqrt{t}}\right)$

Correlation: $J(r) = \lim_{T\to\infty} \int_{-T}^{T} v(y)v(y+r)\, dy$

Find $R(r) = \dfrac{J(r)}{J(0)} = 1 - Cr^{2/3}$ and $E(k) = Ck^{-5/3}$

The Mathematics of Burgers' Equation

$$v_t + 2vv_y = \nu v_{yy}, \quad \text{small} \quad \nu$$

$\nu = 0$: conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

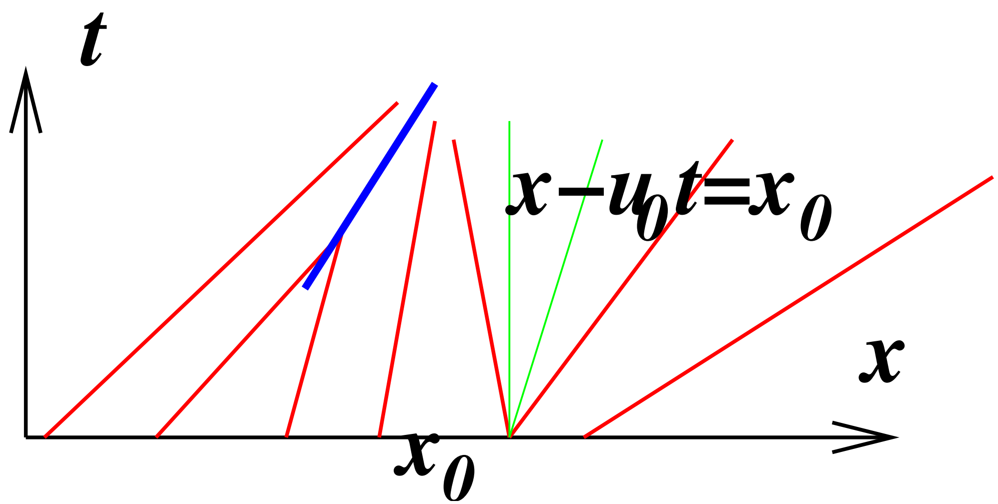
IC $u(x,0) = u_0(x)$

Method of characteristics:

on

$$\frac{dx}{dt} = u$$

$$\frac{d}{dt}u(x(t),t) = uu_x + u_t = 0$$



Thm: decay to N-waves

Results for $\nu > 0$

Conclusions

- Simple model shows energy relations between modes
- Formation of structures with particular size (determined from IC)
- Current theories are statistical
- Theory of Burgers' equation has developed
- Conjecture on transition: successive loss of stable modes through repeated branching (bifurcation cascade) replaced in 1970's when Ruelle and Takens produced example of a (finite-dimensional) system that became chaotic after 2 bifurcations (no cascade)

References

1. Uriel Frisch, *Turbulence: The Legacy of A. N Kolmogorov*, Cambridge, 1995.
2. J. M. Burgers, *A Mathematical Model Illustrating the Theory of Turbulence*, Adv. Appl. Math **1** (1948)
3. Alexandre J. Chorin, *Vorticity and Turbulence*, Springer, 1994.
4. C. R. Shalizi, http://cscs.umich.edu/∼crshalizi/notebooks/turbulence.html
5. Eugene Balkovsky and Boris I. Shraiman, *Olfactory Search at High Reynolds Number*, http://arxiv.org/abs/nlin/0109019