

## Harmonic music “noise” signals between red (Brownian) and white noise

*Internet information*

Brownian noise or red noise is the kind of signal noise produced by Brownian motion. A Brownian motion (i.e. a Wiener process) is a continuous stationary stochastic process having independent increments, i.e.  $B(t) - B(0)$  is a normal random variable with mean  $\mu t$  and variance  $\sigma^2 t$ ,  $\mu, \sigma^2$  constant real numbers. The density function of a Brownian motion is given by

$$f_{B(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

The sample paths of Brownian motion are not differentiable, a mathematical fact explaining the highly irregular motions of small particles. The total variation of Brownian motion over a finite interval  $[0, T]$  is infinite. It holds

$$\text{Var}\left[\frac{B(t)}{t}\right] = \frac{\sigma^2}{t}.$$

If  $W(t)$  is a Wiener process on the interval  $[0, \infty)$ , then, as well the process

$$W^*(t) := \begin{cases} tW(1/t), & t > 0 \\ 0, & t = 0 \end{cases}.$$

White noise can be defined as the derivative of a Brownian motion (i.e. a Wiener process) in the framework of infinite dimensional distribution theory, as the derivative  $B'(t)$  of  $B(t)$ , does not exist in the ordinary sense. Not only  $B'(t)$ , but also all derivatives of Brownian motion are generalized functions on the same space. For each  $t$ , the white noise  $B'(t)$  is defined as a generalized function (distribution) on an infinite dimensional space. A Brownian motion is obtained as the integral of a white noise signal  $dB(t)$ , i.e.

$$B(t) = \int_0^t dB(\tau)$$

meaning that Brownian motion is the integral of the white noise  $dB(t)$  whose power spectral density is flat

$$S_0 = |\text{Fourier}[B'](\omega)|^2 = \text{const}.$$

This means, that the spectral density  $S_0$  for white noise is flat, i.e.  $S_0 / \omega^0 = c$  i.e. it is inversely proportional to  $\omega^0$ . It holds  $\text{Fourier}[B'](\omega) = i\omega \text{Fourier}[B](\omega)$ . Therefore the power spectrum of Brownian noise is given by

$$S(\omega) = |\text{Fourier}[B](\omega)|^2 = \frac{S_0}{\omega^2}.$$

This means, that the spectral density of Brownian (red) noise is  $S_0 / \omega^2$ , i.e. it is inversely proportional to  $\omega^2$ , meaning it has more energy at lower frequencies, even more so than pink noise. The above indicates a spectral density for harmonic music “noise” signals given by

$$S^*(\omega) = |\text{Fourier}[B^*](\omega)|^2 = \frac{S_0}{\omega}.$$