Available online at www.sciencedirect.com
ScienceDirect
J. Math. Pures Appl. 86 (2006) 529–540
JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
www.elsevier.com/locate/matpur

Integral inequalities for the Hilbert transform applied to a nonlocal transport equation

Antonio Córdoba [a,*,1], Diego Córdoba [b,2], Marco A. Fontelos [a,2]

[a] *Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain*
[b] *Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, C/ Serrano 123, Madrid 28006, Spain*

Received 20 April 2006
Available online 20 September 2006

Abstract

We prove several weighted inequalities involving the Hilbert transform of a function $f(x)$ and its derivative. One of those inequalities,

$$-\int \frac{f_x(x)[Hf(x)-Hf(0)]}{|x|^\alpha}\,dx \geqslant C_\alpha \int \frac{(f(x)-f(0))^2}{|x|^{1+\alpha}}\,dx,$$

is used to show finite time blow-up for a transport equation with nonlocal velocity.

Résumé

Dans cet article nous présentons la démonstration de plusieurs inegalités satisfaites par la transformation de Hilbert d'une fonction $f(x)$ et sa derivée $f'(x)$. Nous avons l'estimation suivante :

$$-\int \frac{f_x(x)[Hf(x)-Hf(0)]}{|x|^\alpha}\,dx \geqslant C_\alpha \int \frac{(f(x)-f(0))^2}{|x|^{1+\alpha}}\,dx,$$

où la constante C_α est strictement positive. Nous avons aussi utilisé cette inegalité pour démontrer l'explosion en un temps fini des solutions d'une équation de transport avec une vitesse nonlocale.

MSC: 35Q35; 76B03; 42A38; 35L67

Keywords: Integral inequalities; Hilbert transform; Transport equations

* Corresponding author.
E-mail addresses: antonio.cordoba@uam.es (A. Córdoba), dcg@imaff.cfmac.csic.es (D. Córdoba), marco.fontelos@uam.es (M.A. Fontelos).
[1] Partially supported by BFM2002-02269 grant.
[2] Partially supported by BFM2002-02042 grant.

0021-7824/$ – see front matter
doi:10.1016/j.matpur.2006.08.002

1. Introduction

In this paper we show the existence of finite-time singularities for a Burgers type equation with nonlocal velocity in one space variable,

$$f_t - (Hf)f_x = 0, \tag{1}$$

where $H(\cdot)$ denotes the Hilbert transform (see (2) below), improving the results of our previous paper [4] in the sense that we can obtain here blow-up for a much wider class of initial data. One motivation for the study of that equation is its analogy with both, the 2D quasi-geostrophic equation and the 3D Euler in vorticity form (see [2,3,5,6,8–10] and [11] for a variety of 1D models involving nonlocal operators, the majority of them have their origin in the seminal paper of Constantin, Lax and Majda [3]). For instance the 2D quasi-geostrophic equation belongs to the class of transport equations with nonlocal velocities:

$$\theta_t - R_2(\theta)\frac{\partial\theta}{\partial x_1} + R_1(\theta)\frac{\partial\theta}{\partial x_2} = 0,$$

where the velocity is $u = (-R_2(\theta), R_1(\theta))$ and $R(\theta) = (R_1(\theta), R_2(\theta))$ are the Riesz transforms of θ. This is a well-known model for the dynamic of mixtures of cold and hot air.

Another motivation for (1) is its similarity in structure with the Birkhoff–Rott equation for the evolution of vortex sheets (see [1,4] and [7]).

In this paper we present examples showing the existence of such singularities, our proofs below will take advantage of certain weighted norm inequalities satisfied by the Hilbert transform. There is a vast literature about such topic, but, in our case, we need very precise estimates, separately for both even and odd functions, which do not follow directly from the general theorems. Since the weights involved are powers of the independent variable, the Mellin transform is a very adequate instrument to produce those sharp estimates.

In the following we shall consider the Hilbert transform defined by the formulas:

$$Hf(x) = \frac{1}{\pi}\,\mathrm{PV}\int \frac{f(y)}{x-y}\,\mathrm{d}y, \tag{2}$$

or $\widehat{Hf}(\xi) = -\mathrm{i}\,\mathrm{sign}(\xi)\hat{f}(\xi)$, where $\hat{f}(\xi) = \int \mathrm{e}^{-2\pi\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x$ denotes the Fourier transform.

We shall make use also of the identity,

$$\int_0^\infty t^{-\beta}\frac{\mathrm{d}t}{t-1} = \pi\cot(\pi\beta), \tag{3}$$

valid for $0 < \mathrm{Re}(\beta) < 1$ and the explicit formula:

$$H\big(|x|^\alpha\big) = -\frac{1}{\pi}\tan\left(\frac{\alpha\pi}{2}\right)\mathrm{sign}(x)|x|^\alpha,$$

with $-3 < \alpha < 1$ and $\alpha \neq 0, -1, -2$. The distribution $1/|x|^\beta$ is defined by,

$$\frac{1}{|x|^\beta}(\varphi) = \mathrm{PV}\int \frac{\varphi(x) - \varphi(0)}{|x|^\beta}\,\mathrm{d}x \quad \text{if } 1 < \beta < 2,$$

and by,

$$\frac{1}{|x|^\beta}(\varphi) = \mathrm{PV}\int \frac{\varphi(x) - \varphi(0) - x\varphi_x(0)}{|x|^\beta}\,\mathrm{d}x \quad \text{if } 2 < \beta < 3.$$

To present our results we find it convenient to introduce the following functional spaces: For $0 < \alpha < 2$, $H^1_\alpha(\mathbb{R})$ is the closure of $\mathcal{C}^1_0(\mathbb{R})$ under the norm:

$$\|f\|^2 = \|f\|^2_{L^\infty} + \int \frac{(f(x) - f(0))^2}{|x|^{1+\alpha}}\,\mathrm{d}x + \int \frac{(f_x(x))^2}{|x|^{\alpha-1}}\,\mathrm{d}x,$$

and for $\alpha \leqslant 0$ we define $H^1_\alpha(\mathbb{R})$ as before taking out the correction term $f(0)$.

We will prove the following inequalities involving Hilbert transforms:

Theorem 1.1. (I) *For every α, $0 < \alpha < 2$, there exists a strictly positive constant C_α so that the inequality,*

$$-\int \frac{f_x(x)Hf(x)}{|x|^\alpha}\,dx \geqslant C_\alpha \int \frac{(f(x)-f(0))^2}{|x|^{1+\alpha}}\,dx, \tag{4}$$

holds for every even function in $\mathcal{C}^1(\mathbb{R}) \cap H^1_\alpha(\mathbb{R})$.

(II) *Let $f \in \mathcal{C}^1(\mathbb{R}) \cap H^1_\alpha(\mathbb{R})$ be a nonnegative (or nonpositive) function. Then for every α, $1 < \alpha < 2$, there exists a strictly positive constant C_α so that*

$$-\int \frac{f_x(x)[Hf(x)-Hf(0)]}{|x|^\alpha}\,dx \geqslant C_\alpha \int \frac{(f(x)-f(0))^2}{|x|^{1+\alpha}}\,dx. \tag{5}$$

(III) *The inequality,*

$$-\int f_x(x)Hf(x)|x|^\alpha\,dx \geqslant C_\alpha \int f^2(x)|x|^{\alpha-1}\,dx, \tag{6}$$

with $C_\alpha > 0$, is satisfied when $0 < \alpha < 2$ and $f \in \mathcal{C}^1(\mathbb{R}) \cap H^1_{-\alpha}(\mathbb{R})$ is an even function or when $1 < \alpha < 2$ and $f \in \mathcal{C}^1(\mathbb{R}) \cap H^1_{-\alpha}(\mathbb{R})$ is either nonnegative or nonpositive.

As it was mentioned before, we will use these inequalities to prove the formation of finite time singularities for the Burgers-type equation (1) with nonlocal velocity field.

The Cauchy problem is well posed for Eq. (1) giving us classical solutions $f(x,t)$ for a short period of time $0 \leqslant t < T$, for smooth enough initial data. Concerning blow-up solutions of Eq. (1) we have the following theorem:

Theorem 1.2. *Let $f(x,0) \in C^{1+\delta}_0(\mathbb{R})$ be positive and compactly supported. Then there is no global in time, locally bounded (in space) solution to Eq. (1).*

The analysis involved in Theorem 1.1 leads to other interesting inequalities about the Hilbert transform, namely:

Theorem 1.3. *Let $f \in H^1_\alpha(\mathbb{R})$. Then*

(I) *For $0 < \alpha < 2$ we have:*

$$\int_{-\infty}^{\infty} \frac{|(Hf)(x)-(Hf)(0)|^2}{|x|^{\alpha+1}}\,dx \geqslant \min\left\{\tan^2\left(\frac{1}{4}\pi\alpha\right), \cot^2\left(\frac{1}{4}\pi\alpha\right)\right\} \int_{-\infty}^{\infty} \frac{|(f(x)-f(0))|^2}{|x|^{\alpha+1}}\,dx. \tag{7}$$

(II) *For $-2 < \alpha < 0$ we have:*

$$\int_{-\infty}^{\infty} \frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,dx \geqslant \min\left\{\tan^2\left(\frac{1}{4}\pi\alpha\right), \cot^2\left(\frac{1}{4}\pi\alpha\right)\right\} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}}\,dx.$$

Theorem 1.4. (I) *Let f be an even function. Then for every α, $0 < \alpha < 2$, we have:*

$$\text{(A)}\quad \int_{-\infty}^{\infty} \frac{[f(x)-f(0)]Hf(x)}{|x|^\alpha x}\,dx \leqslant 0 \quad \textit{for } f \in H^1_\alpha(\mathbb{R}), \tag{8}$$

$$\text{(B)}\quad \int_{-\infty}^{\infty} \frac{f(x)Hf(x)}{|x|^{-\alpha} x}\,dx \geqslant 0 \quad \textit{for } f \in H^1_{-\alpha}(\mathbb{R}). \tag{9}$$

(II) *Let f be an odd function. Then for every α, $0 < \alpha < 2$, we have*:

$$\text{(A)} \quad \int_{-\infty}^{\infty} \frac{f(x)[Hf(x) - Hf(0)]}{|x|^{\alpha} x} \, \mathrm{d}x \geqslant 0 \quad \textit{for } f \in H^1_{\alpha}(\mathbb{R}), \tag{10}$$

$$\text{(B)} \quad \int_{-\infty}^{\infty} \frac{f(x)Hf(x)}{|x|^{-\alpha} x} \, \mathrm{d}x \leqslant 0 \quad \textit{for } f \in H^1_{-\alpha}(\mathbb{R}). \tag{11}$$

In Section 2 we present the proof of Theorem 1.1. Section 3 will be devoted to prove blow-up for Eq. (1) (Theorem 1.2). Finally Section 4 contains the proof of the inequalities for the Hilbert transform presented in Theorems 1.3 and 1.4.

2. Proof of Theorem 1.1

An important ingredient in the proofs of the above inequalities will be the use of Mellin transforms of functions $f(x)$ in $\mathbb{R}^+$. Such a transform is defined by the formula:

$$Mf(\lambda) = \int_0^{\infty} x^{\mathrm{i}\lambda} f(x) \frac{\mathrm{d}x}{x}.$$

It will be crucial in our analysis the fact that the restrictions of Hilbert transforms to the positive and negative real axis are Mellin multipliers as we will explicitly show below. A useful property of Mellin transforms is the following Plancherel's identity for functions supported in $\mathbb{R}^+$:

$$\int_0^{\infty} \overline{f(x)} g(x) \frac{\mathrm{d}x}{x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{Mf}(\lambda) Mg(\lambda) \, \mathrm{d}\lambda.$$

We can consider two different cases of $f(x)$, even and odd, and then discuss the general case.

Notice that $f_x(x)Hf(x) = (f(x) - f(0))_x H(f(x) - f(0)) = g_x(x)Hg(x)$ for $g(x) = f(x) - f(0)$ so we may, without loss of generality, assume that $f(0) = 0$ for $f(x)$ even. Then,

$$\int_{-\infty}^{\infty} \frac{f_x(x)Hf(x)}{|x|^{\alpha}} \, \mathrm{d}x = 2 \int_0^{\infty} \frac{f_x(x)Hf(x)}{x^{\alpha}} \, \mathrm{d}x, \tag{12}$$

and using Plancherel's identity it can be written in the form:

$$-\int_0^{\infty} \frac{f_x(x)Hf(x)}{x^{\alpha}} \, \mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F(\lambda)} m_s(\lambda) F(\lambda) \, \mathrm{d}\lambda, \tag{13}$$

where

$$F(\lambda) = M\left(\frac{f}{x^{\alpha/2}}\right)(\lambda) = \int_0^{\infty} x^{\mathrm{i}\lambda - \alpha/2 - 1} f(x) \, \mathrm{d}x,$$

and

$$m_s(\lambda) = -\overline{A(\lambda)} B(\lambda),$$

with $A(\lambda)$ and $B(\lambda)$ being Mellin multipliers that can be deduced from the Mellin transform of $f_x(x)$ and $Hf(x)$ respectively. In order to find them, let us compute:

$$\int_0^{\infty} x^{\mathrm{i}\lambda - \alpha/2} f_x(x) \, \mathrm{d}x = -\left(\mathrm{i}\lambda - \frac{\alpha}{2}\right) F(\lambda) \equiv A(\lambda) F(\lambda),$$

$$\int_0^\infty x^{i\lambda-\alpha/2-1} Hf(x)\,dx = \frac{1}{\pi}\int_0^\infty x^{i\lambda-\alpha/2-1}\left[\int_{-\infty}^\infty \frac{f(y)}{x-y}\,dy\right]dx$$
$$= \frac{1}{\pi}\int_0^\infty x^{i\lambda-\alpha/2-1}\left[\int_0^\infty \frac{f(y)}{x-y}\,dy + \int_0^\infty \frac{f(y)}{x+y}\,dy\right]dx$$
$$= \frac{2}{\pi}\int_0^\infty x^{i\lambda-\alpha/2}\left[\int_0^\infty \frac{f(y)}{x^2-y^2}\,dy\right]dx = \frac{2}{\pi}\int_0^\infty\left[\int_0^\infty \frac{x^{i\lambda-\alpha/2}}{x^2-y^2}\,dx\right]f(y)\,dy.$$

Then we evaluate the last integral in x performing the change of variables $x = yt$:

$$\int_0^\infty \frac{x^{i\lambda-\alpha/2}}{x^2-y^2}\,dx = \int_0^\infty \frac{x^{i\lambda-\alpha/2}}{x^2-y^2}\,dx = y^{i\lambda-\alpha/2-1}\int_0^\infty \frac{t^{i\lambda-\alpha/2}}{t^2-1}\,dt,$$

and the change $t^2 = s$ together with (3) to evaluate:

$$\int_0^\infty \frac{t^{i\lambda-\alpha/2}}{t^2-1}\,dt = \frac{1}{2}\int_0^\infty \frac{s^{(i\lambda-\alpha/2-1)/2}}{s-1}\,ds = \frac{\pi}{2}\cot\left(\frac{-i\lambda+\alpha/2+1}{2}\pi\right).$$

Hence

$$\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty \frac{x^{i\lambda-\alpha/2}}{x^2-y^2}\,dx\right]f(y)\,dy = \cot\left(\frac{-i\lambda+\alpha/2+1}{2}\pi\right)\int_0^\infty y^{i\lambda-\alpha/2-1}f(y)\,dy = -\tan\left(\frac{-i\lambda+\alpha/2}{2}\pi\right)F(\lambda).$$

Therefore

$$B(\lambda) = -\tan\left(\frac{-i\lambda+\alpha/2}{2}\pi\right),$$

and

$$m_s(\lambda) = \left(i\lambda+\frac{\alpha}{2}\right)\tan\left(\frac{-i\lambda+\alpha/2}{2}\pi\right),$$
$$\mathrm{Re}\big[m_s(\lambda)\big] = \frac{\lambda\sinh\pi\lambda + \frac{\alpha}{2}\sin\frac{1}{2}\pi\alpha}{\cosh\pi\lambda+\cos\frac{1}{2}\pi\alpha}. \tag{14}$$

If $-2 < \alpha < 2$, $\alpha \neq 0$, then

$$\frac{\lambda\sinh\pi\lambda + \frac{\alpha}{2}\sin\frac{1}{2}\pi\alpha}{\cosh\pi\lambda+\cos\frac{1}{2}\pi\alpha} > C_\alpha > 0,$$

and we may conclude that

$$-\int_0^\infty \frac{f_x(x)Hf(x)}{x^\alpha}\,dx \geqslant C_\alpha\int_{-\infty}^\infty \frac{|f(x)|^2}{|x|^{\alpha+1}}\,dx,$$

or in general, if $f(0) \neq 0$, we obtain inequality (4).

In the case of f being odd, we evaluate:

$$-\int_0^\infty \frac{f_x(x)(Hf(x)-Hf(0))}{x^\alpha}\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty d\lambda\,\overline{F(\lambda)}m_a(\lambda)F(\lambda),$$

where

$$m_a(\lambda) = -\overline{A(\lambda)}B(\lambda),$$

with $A(\lambda)$ and $B(\lambda)$ being the Mellin multipliers coming from the Mellin transform of $f_x(x)$ and $Hf(x) - Hf(0)$ respectively. To find them, let us compute:

$$\int_0^\infty x^{\mathrm{i}\lambda-\alpha/2} f_x(x)\,\mathrm{d}x = -(\mathrm{i}\lambda - \alpha/2)F(\lambda) \equiv A(\lambda)F(\lambda),$$

$$\begin{aligned}
&\int_0^\infty x^{\mathrm{i}\lambda-\alpha/2-1}\big((Hf)(x) - (Hf)(0)\big)\,\mathrm{d}x \\
&= \frac{1}{\pi}\int_0^\infty x^{\mathrm{i}\lambda-\alpha/2-1}\left[\int_0^\infty \frac{f(y)}{x-y}\,\mathrm{d}y + \int_0^\infty \frac{f(y)}{y}\,\mathrm{d}y - \int_0^\infty \frac{f(y)}{x+y}\,\mathrm{d}y + \int_0^\infty \frac{f(y)}{y}\,\mathrm{d}y\right]\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\infty x^{\mathrm{i}\lambda-\alpha/2-1} 2x^2\left[\int_0^\infty \frac{f(y)/y}{x^2-y^2}\,\mathrm{d}y\right]\mathrm{d}x = \frac{2}{\pi}\int_0^\infty \left[\int_0^\infty \frac{x^{\mathrm{i}\lambda-\alpha/2+1}}{x^2-y^2}\,\mathrm{d}x\right]\frac{f(y)}{y}\,\mathrm{d}y \\
&= \cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)\int_0^\infty y^{\mathrm{i}\lambda-\alpha/2-1} f(y)\,\mathrm{d}y = \cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)F(\lambda).
\end{aligned}$$

Hence

$$m_a(\lambda) = -\left(\mathrm{i}\lambda + \frac{\alpha}{2}\right)\cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right),$$

so that

$$\mathrm{Re}\left(-\left(\mathrm{i}\lambda + \frac{\alpha}{2}\right)\cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)\right) = \frac{-\frac{1}{2}\alpha\sin\frac{1}{2}\pi\alpha + \lambda\sinh\pi\lambda}{\cosh\pi\lambda - \cos\frac{1}{2}\pi\alpha},$$

and

$$-\int_0^\infty \frac{f_x(x)(Hf(x) - Hf(0))}{x^\alpha}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\lambda\,\frac{\lambda\sinh\pi\lambda - \frac{\alpha}{2}\sin\frac{1}{2}\pi\alpha}{\cosh\pi\lambda - \cos\frac{1}{2}\pi\alpha}\big|F(\lambda)\big|^2.$$

If f is an arbitrary function (without loss of generality we can again assume that $f(0) = 0$), then we can write:

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} \equiv f_s(x) + f_a(x),$$

where $f_s(x)$ and $f_a(x)$ are even and odd respectively.

Notice that

$$\begin{aligned}
-\int_{-\infty}^\infty \frac{f_x(x)(Hf(x) - Hf(0))}{|x|^\alpha}\,\mathrm{d}x &= -\int_{-\infty}^\infty \frac{[f_{s,x}(x) + f_{a,x}(x)](Hf_s(x) + Hf_a(x) - Hf_a(0))}{|x|^\alpha}\,\mathrm{d}x \\
&= -\int_{-\infty}^\infty \frac{f_{s,x}(x)Hf_s(x)}{|x|^\alpha}\,\mathrm{d}x - \int_{-\infty}^\infty \frac{f_{a,x}(x)(Hf_a(x) - Hf_a(0))}{|x|^\alpha}\,\mathrm{d}x \\
&= -2\int_0^\infty \frac{f_{s,x}(x)Hf_s(x)}{x^\alpha}\,\mathrm{d}x - 2\int_0^\infty \frac{f_{a,x}(x)(Hf_a(x) - Hf_a(0))}{x^\alpha}\,\mathrm{d}x.
\end{aligned}$$

Therefore:

$$
\begin{aligned}
-\int_{-\infty}^{\infty} \frac{f_x(x)(Hf(x)-Hf(0))}{|x|^\alpha}\,\mathrm{d}x &= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{Re}\big[m_s(\lambda)\big]\big|F_s(\lambda)\big|^2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \mathrm{Re}\big[m_a(\lambda)\big]\big|F_a(\lambda)\big|^2 \\
&\geqslant \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \inf_\lambda\big(\mathrm{Re}\big[m_s(\lambda)\big]\big)\big|F_s(\lambda)\big|^2 + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \inf_\lambda\big(\mathrm{Re}\big[m_a(\lambda)\big]\big)\big|F_a(\lambda)\big|^2 \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{\inf_\lambda(\mathrm{Re}[m_s(\lambda)]) + \inf_\lambda(\mathrm{Re}[m_a(\lambda)])}{2}\big(\big|F_s(\lambda)\big|^2 + \big|F_a(\lambda)\big|^2\big) \\
&\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \frac{\inf_\lambda(\mathrm{Re}[m_s(\lambda)]) - \inf_\lambda(\mathrm{Re}[m_a(\lambda)])}{2}\big(\big|F_s(\lambda)\big|^2 - \big|F_a(\lambda)\big|^2\big) \\
&= \inf_\lambda\big(\mathrm{Re}\big[m_s(\lambda)\big]\big) + \inf_\lambda\big(\mathrm{Re}\big[m_a(\lambda)\big]\big)\int_0^\infty \frac{|f_s(x)|^2 + |f_a(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x \\
&\quad + \inf_\lambda\big(\mathrm{Re}\big[m_s(\lambda)\big]\big) - \inf_\lambda\big(\mathrm{Re}\big[m_a(\lambda)\big]\big)\int_0^\infty \frac{|f_s(x)|^2 - |f_a(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x,
\end{aligned}
$$

where we have used the notation,

$$
F_a(\lambda) = M\left(\frac{f_a}{x^{\alpha/2}}\right), \qquad F_s(\lambda) = M\left(\frac{f_s}{x^{\alpha/2}}\right).
$$

It can be verified (see the lemma at the end of this section) that in the range $1 < \alpha < 2$, $\inf_\lambda(\mathrm{Re}[m_s(\lambda)]) > 0$, $\inf_\lambda(\mathrm{Re}[m_a(\lambda)]) < 0$ and $\inf_\lambda(\mathrm{Re}[m_s(\lambda)]) + \inf_\lambda(\mathrm{Re}[m_a(\lambda)]) > 0$. If $f(x)$ is everywhere nonnegative, then $|f_s(x)|^2 - |f_a(x)|^2 \geqslant 0$, and therefore:

$$
\begin{aligned}
&-\int_0^\infty \frac{f_x(x)(Hf(x)-Hf(0))}{x^\alpha}\,\mathrm{d}x \\
&\quad\geqslant \frac{\inf_\lambda(\mathrm{Re}[m_s(\lambda)]) + \inf_\lambda(\mathrm{Re}[m_a(\lambda)])}{2}\int_0^\infty \frac{|f_s(x)|^2 + |f_a(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x \\
&\quad= \frac{\inf_\lambda(\mathrm{Re}[m_s(\lambda)]) + \inf_\lambda(\mathrm{Re}[m_a(\lambda)])}{4}\int_0^\infty \frac{f^2(x) + f^2(-x)}{|x|^{\alpha+1}}\,\mathrm{d}x = C_\alpha \int_{-\infty}^{\infty} \frac{f^2(x)}{|x|^{\alpha+1}}\,\mathrm{d}x.
\end{aligned}
$$

Lemma 2.1. *For* $1 < \alpha < 2$*, we have*:

$$
\inf_\lambda\big(\mathrm{Re}\big[m_s(\lambda)\big]\big) + \inf_\lambda\big(\mathrm{Re}\big[m_a(\lambda)\big]\big) > 0.
$$

The proof of the lemma is based in two facts:
Fact 1:

$$
\min_\lambda\big(\pi\,\mathrm{Re}\big(m_a(\lambda)\big)\big) = \frac{-s}{1+t}.
$$

Fact 2: For every λ

$$
\frac{\lambda\sinh\lambda + s}{\cosh\lambda - t} > \frac{s}{1+t},
$$

where $s = \frac{\pi\alpha}{2}\sin\frac{\pi\alpha}{2}$ and $t = -\cos\frac{\pi\alpha}{2}$ $(1 < \alpha < 2)$. The first fact is trivial and the second is a consequence of the following observations:

$$\lambda\sinh\lambda > 2(\cosh\lambda - 1), \qquad \frac{s}{1+t} = 2\frac{\frac{\pi\alpha}{4}}{\tan\frac{\pi\alpha}{4}} < 2.$$

3. Applications to a nonlocal transport equation

This section contains the proof of the application described in the introduction dealing with a nonlocal partial differential equation, where the inequalities obtained in the previous section will be crucial in proving the formation of finite time singularities.

The equation is:

$$f_t - (Hf)f_x = 0.$$

In order to prove Theorem 1.2 we shall consider a general smooth initial data $f_0(x) = f(x,0)$, positive and compactly supported, and we will show that $f_x(x,t)$ blows up in finite time.

It is a straightforward exercise to obtain a priori inequality:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|f(\cdot,t)\|^2_{L^2}+\|f_{xx}(\cdot,t)\|^2_{L^2}\right) \leqslant C\left(\|f(\cdot,t)\|^2_{L^2}+\|f_{xx}(\cdot,t)\|^2_{L^2}\right)^{3/2},$$

which implies local (in time) existence for the Cauchy problem with initial data in Sobolev's space $H^2(\mathbb{R})$. We may thus assume a $\mathcal{C}^{1,\alpha}$ solution $f(x,t)$. We have a particle dynamics given by the ordinary differential equation $X'(t) = -Hf(X(t),t)$ and the equation implies that f is constant along trajectories. Let $\mathrm{supp}(f_0) \subset [a_0, b_0]$ then the positivity of f implies that $Hf(a_0) \leqslant 0$, $Hf(b_0) \geqslant 0$ which yields $\mathrm{supp}(f(x,t)) \subset [a_0, b_0]$ for $t > 0$.

If we change coordinates to a system of reference in which the maximum is stationary, i.e. we define $x_M(t)$ to be the trajectory where f reaches its maximum, and

$$x' = x - x_M(t), \qquad t' = t,$$

we obtain from (1) the equation:

$$\bar{f}_{t'} - x'_M(t)\bar{f}_{x'} - H\bar{f}(x',t')\bar{f}_{x'} = 0, \tag{15}$$

where $\bar{f}(x',t') = f(x' + x_M(t), t)$. In order to simplify notation let us omit from now on the explicit dependence of the function on the second variable $t = t'$.

But

$$\frac{\mathrm{d}x_M(t)}{\mathrm{d}t} = -Hf\big(x_M(t)\big),$$

$$Hf(x) = \frac{1}{\pi}\,\mathrm{PV}\int\frac{f(y)}{x-y}\,\mathrm{d}y = \frac{1}{\pi}\,\mathrm{PV}\int\frac{f(y)}{(x-x_M(t))-(y-x_M(t))}\,\mathrm{d}y = H\bar{f}(x'),$$

and

$$Hf\big(x = x_M(t)\big) = H\bar{f}(x' = 0),$$

so that (15) becomes

$$\bar{f}_{t'} - \big(H\bar{f}(x') - H\bar{f}(0)\big)\bar{f}_{x'} = 0,$$

also we have $\mathrm{supp}_{x'}(\bar{f}(\cdot,t)) \subset [-L, L]$ for a suitable finite constant L.

Hence, denoting $g = \bar{f}_{\max} - \bar{f}$,

$$g_{t'} = -\big(Hg(x') - Hg(0)\big)g_{x'}. \tag{16}$$

Dividing (16) by $|x|^\alpha$, $1 < \alpha < 2$, and using Theorem 1.1 we get:

$$\frac{\mathrm{d}}{\mathrm{d}t'}\int_{-L}^{L}\frac{g(x')}{|x'|^{\alpha}}\,\mathrm{d}x' = -\int_{-L}^{L}\frac{(Hg(x')-Hg(0))g_{x'}(x')}{|x'|^{\alpha}}\,\mathrm{d}x'$$
$$= -\int_{-\infty}^{\infty}\frac{(Hg(x')-Hg(0))g_{x'}(x')}{|x'|^{\alpha}}\,\mathrm{d}x' \geqslant C_{\alpha}\int_{-\infty}^{\infty}\frac{g^2(x')}{|x'|^{\alpha+1}}\,\mathrm{d}x'$$
$$\geqslant C_{\alpha}\int_{-L}^{L}\frac{g^2(x')}{|x'|^{\alpha+1}}\,\mathrm{d}x' \geqslant C_{L,\alpha}\left(\int_{-L}^{L}\frac{g(x')}{|x'|^{\alpha}}\,\mathrm{d}x'\right)^2,$$

where $C_{L,\alpha}$ is a positive constant. Therefore $\int_{-L}^{L}\frac{g(x')}{|x'|^{\alpha}}\,\mathrm{d}x'$ blows up in finite time and consequently f_x also blows up in finite time since

$$\int_{-L}^{L}\frac{g(x')}{|x'|^{\alpha}}\,\mathrm{d}x' \leqslant \sup_{x'}\frac{g(x')}{|x'|}\int_{-L}^{L}\frac{\mathrm{d}x'}{|x'|^{\alpha-1}} \leqslant \frac{2L^{2-\alpha}}{2-\alpha}\sup_{x}|g_x|.$$

4. Proof of Theorems 1.3 and 1.4

In the proof of Theorem 1.3 we will consider two different cases of $f(x)$, even and odd, and then discuss the general case.

If $f(x)$ is even, i.e. $f(x)=f(-x)$, then

$$\int_{-\infty}^{\infty}\frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{|H[f(x)-f(0)]|^2}{|x|^{\alpha+1}}\,\mathrm{d}x = 2\int_{0}^{\infty}\frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x, \tag{17}$$

since Hf is an odd function. In order to simplify the notation we can assume, without loss of generality, that $f(0)=0$.

To estimate the right-hand side of (17) we make use of the Mellin transform and write,

$$\int_{0}^{\infty}\frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\lambda\, m(\lambda)\left|F(\lambda)\right|^2, \tag{18}$$

where

$$F(\lambda) = \int_{0}^{\infty} x^{\mathrm{i}\lambda-\alpha/2-1} f(x)\,\mathrm{d}x,$$

and $m(\lambda)=|G(\lambda)|^2$ with $G(\lambda)$ being the Mellin multiplier for Hf with f an even function, that is:

$$\int_{0}^{\infty} x^{\mathrm{i}\lambda-\alpha/2-1} Hf(x)\,\mathrm{d}x = G(\lambda)F(\lambda),$$

where

$$G(\lambda) = -\tan\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right).$$

Then

$$m(\lambda) = \left|\tan\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)\right|^2 = \frac{\cos^2(\frac{1}{4}\pi\alpha)\sinh^2(\frac{1}{2}\pi\lambda)+\sin^2(\frac{1}{4}\pi\alpha)\cosh^2(\frac{1}{2}\pi\lambda)}{\cos^2(\frac{1}{4}\pi\alpha)\cosh^2(\frac{1}{2}\pi\lambda)+\sin^2(\frac{1}{4}\pi\alpha)\sinh^2(\frac{1}{2}\pi\lambda)} \geqslant \tan^2\left(\frac{1}{4}\pi\alpha\right),$$

as one can easily verify.

Therefore, inequality (7) follows immediately for the even case since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, m(\lambda)\big|F(\lambda)\big|^2 \geqslant \frac{1}{2\pi}\tan^2\left(\frac{1}{4}\pi\alpha\right)\int_{-\infty}^{\infty} \mathrm{d}\lambda\,\big|F(\lambda)\big|^2 = \tan^2\left(\frac{1}{4}\pi\alpha\right)\int_{-\infty}^{\infty}\frac{|f(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x,$$

in the range $0<\alpha<2$. In fact, one can show

$$\int_{-\infty}^{\infty}\frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x \geqslant \min\left\{\tan^2\left(\frac{1}{4}\pi\alpha\right),1\right\}\int_{-\infty}^{\infty}\frac{|f(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x$$

for $-2<\alpha<2$, $\alpha\neq 0$.

Next we consider $f(x)$ to be an odd function and $-2<\alpha<0$. Then (17) and (18) still hold, and we have to compute:

$$\int_0^{\infty} x^{\mathrm{i}\lambda-\alpha/2-1}Hf(x)\,\mathrm{d}x = \cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)\int_0^{\infty}\mathrm{y}^{\mathrm{i}\lambda-\alpha/2-1}f(y)\,\mathrm{d}y = \cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)F(\lambda).$$

Therefore,

$$\begin{aligned} m(\lambda) = \left|\cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)\right|^2 &= \frac{\cos^2(\frac{1}{4}\pi\alpha)\cosh^2(\frac{1}{2}\pi\lambda)+\sin^2(\frac{1}{4}\pi\alpha)\sinh^2(\frac{1}{2}\pi\lambda)}{\cos^2(\frac{1}{4}\pi\alpha)\sinh^2(\frac{1}{2}\pi\lambda)+\sin^2(\frac{1}{4}\pi\alpha)\cosh^2(\frac{1}{2}\pi\lambda)} \\ &\geqslant \min\left\{\cot^2\left(\frac{1}{4}\pi\alpha\right),1\right\}, \end{aligned}$$

which yields the inequality

$$\int_{-\infty}^{\infty}\frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x \geqslant \min\left\{\cot^2\left(\frac{1}{4}\pi\alpha\right),1\right\}\int_{-\infty}^{\infty}\frac{|f(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x,$$

for $-2<\alpha<0$. When $0<\alpha<2$, we estimate:

$$\int_{-\infty}^{\infty}\frac{|(Hf)(x)-(Hf)(0)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x.$$

As before let us now evaluate:

$$\int_0^{\infty} x^{\mathrm{i}\lambda-\alpha/2-1}\big((Hf)(x)-(Hf)(0)\big)\,\mathrm{d}x = \cot\left(\frac{-\mathrm{i}\lambda+\alpha/2}{2}\pi\right)\int_0^{\infty}\mathrm{y}^{\mathrm{i}\lambda-\alpha/2-1}f(y)\,\mathrm{d}y,$$

so that

$$\int_{-\infty}^{\infty}\frac{|(Hf)(x)-(Hf)(0)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x \geqslant \min\left\{\cot^2\left(\frac{1}{4}\pi\alpha\right),1\right\}\int_{-\infty}^{\infty}\frac{|f(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x.$$

Finally, given an arbitrary function f, we write:

$$f(x) = \frac{f(x)+f(-x)}{2}+\frac{f(x)-f(-x)}{2}\equiv f_s(x)+f_a(x),$$

where $f_s(x)$ and $f_a(x)$ are even and odd respectively. Therefore we have:

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{|(Hf)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x &= \int_{-\infty}^{\infty}\frac{|H[f_s(x)-f_s(0)]|^2}{|x|^{\alpha+1}}\,\mathrm{d}x+\int_{-\infty}^{\infty}\frac{|(Hf_a)(x)|^2}{|x|^{\alpha+1}}\,\mathrm{d}x \\ &\quad +2\int_{-\infty}^{\infty}\frac{H[f_s(x)-f_s(0)](Hf_a)(x)}{|x|^{\alpha+1}}\,\mathrm{d}x \end{aligned}$$

$$= \int_{-\infty}^{\infty} \frac{|H[f_s(x) - f_s(0)]|^2}{|x|^{\alpha+1}} \, dx + \int_{-\infty}^{\infty} \frac{|(Hf_a)(x)|^2}{|x|^{\alpha+1}} \, dx$$

$$\geqslant \min\left\{\tan^2\left(\frac{1}{4}\pi\alpha\right), 1\right\} \int_{-\infty}^{\infty} \frac{|f_s(x) - f_s(0)|^2}{|x|^{\alpha+1}} \, dx + \min\left\{\cot^2\left(\frac{1}{4}\pi\alpha\right), 1\right\} \int_{-\infty}^{\infty} \frac{|f_a(x)|^2}{|x|^{\alpha+1}} \, dx$$

$$\geqslant \min\left\{\tan^2\left(\frac{1}{4}\pi\alpha\right), \cot^2\left(\frac{1}{4}\pi\alpha\right)\right\} \int_{-\infty}^{\infty} \frac{|f(x) - f(0)|^2}{|x|^{\alpha+1}} \, dx,$$

for $-2 < \alpha < 0$.

Analogously, for $0 < \alpha < 2$ we get the estimate:

$$\int_{-\infty}^{\infty} \frac{|(Hf)(x) - (Hf)(0)|^2}{|x|^{\alpha+1}} \, dx \geqslant \min\left\{\tan^2\left(\frac{1}{4}\pi\alpha\right), \cot^2\left(\frac{1}{4}\pi\alpha\right)\right\} \int_{-\infty}^{\infty} \frac{|f(x)|^2}{|x|^{\alpha+1}} \, dx.$$

This completes the proof of Theorem 1.3.

Next, in order to prove Theorem 1.4 we will consider two different cases of $f(x)$, even and odd.

If $f(x)$ is even and $0 < \alpha < 2$, then

$$\int_{-\infty}^{\infty} \frac{[f(x) - f(0)]Hf(x)}{|x|^{\alpha} x} \, dx = 2 \int_{0}^{\infty} \frac{[f(x) - f(0)]Hf(x)}{x^{\alpha+1}} \, dx, \tag{19}$$

and again we can assume, without loss of generality, that $f(0) = 0$. As in the previous section we write:

$$\int_{0}^{\infty} \frac{f(x)Hf(x)}{x^{\alpha+1}} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, \overline{F(\lambda)} m(\lambda) F(\lambda), \tag{20}$$

where $m(\lambda)F(\lambda)$ is the Mellin transform of $Hf(x)$, that is,

$$m(\lambda) = -\tan\left(\frac{-i\lambda + \alpha/2}{2}\pi\right)$$

and therefore,

$$\int_{0}^{\infty} \frac{f(x)Hf(x)}{x^{\alpha+1}} \, dx = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, \frac{\sin\frac{1}{2}\pi\alpha}{\cosh \pi\lambda + \cos\frac{1}{2}\pi\alpha} \left|F(\lambda)\right|^2. \tag{21}$$

This last identity implies:

$$\int_{0}^{\infty} \frac{f(x)Hf(x)}{x^{\alpha+1}} \, dx < 0,$$

if $0 < \alpha < 2$, and

$$\int_{0}^{\infty} \frac{f(x)Hf(x)}{x^{\alpha+1}} \, dx > 0,$$

if $-2 < \alpha < 0$.

If $f(x)$ is odd then $Hf(x)$ is even and by the properties of the Hilbert transform we can write:

$$\int_{-\infty}^{\infty} \frac{f(x)Hf(x)}{|x|^{\alpha} x} \, dx = -\int_{-\infty}^{\infty} \frac{H(Hf(x))Hf(x)}{|x|^{\alpha} x} \, dx, \tag{22}$$

and applying previous inequalities to the even function $Hf(x)$ yields

$$\int_0^\infty \frac{f(x)Hf(x)}{x^{\alpha+1}}\,dx > 0,$$

if $0 < \alpha < 2$, and

$$\int_0^\infty \frac{f(x)Hf(x)}{x^{\alpha+1}}\,dx < 0,$$

if $-2 < \alpha < 0$.

References

[1] G.R. Baker, X. Li, A.C. Morlet, Analytic structure of 1D-transport equations with nonlocal fluxes, Physica D 91 (1996) 349–375.

[2] A.L. Bertozzi, A.J. Majda, Vorticity and the Mathematical Theory of Incompressible Fluid Flow, Cambridge Univ. Press, Cambridge, UK, 2002.

[3] P. Constantin, P. Lax, A. Majda, A simple one-dimensional model for the three-dimensional vorticity, Comm. Pure Appl. Math. 38 (1985) 715–724.

[4] A. Córdoba, D. Córdoba, M.A. Fontelos, Formation of singularities for a transport equation with nonlocal velocity, Ann. of Math. 162 (3) (2005) 1375–1387.

[5] D. Chae, A. Córdoba, D. Córdoba, M.A. Fontelos, Finite time singularities in a 1D model of the quasi-geostrophic equation, Adv. Math. 194 (2005) 203–223.

[6] S. De Gregorio, A partial differential equation arising in a 1D model for the 3D vorticity equation, Math. Methods Appl. Sci. 19 (1996) 1233–1255.

[7] D.W. Moore, The spontaneous appearance of a singularity in the shape of an evolving vortex sheet, Proc. R. Soc. London A 365 (1720) (1979) 105–119.

[8] T. Sakajo, On global solutions for the Constantin–Lax–Majda equation with a generalized viscosity term, Nonlinearity 16 (2003) 1319–1328.

[9] S. Schochet, Explicit solutions of the viscous model vorticity equation, Comm. Pure Appl. Math. 41 (1986) 531–537.

[10] Y. Yang, Behavior of solutions of model equations for incompressible fluid flow, J. Differential Equations 125 (1996) 133–153.

[11] M. Vasudeva, E. Wegert, Blow-up in a modified Constantin–Lax–Majda model for the vorticity equation, Z. Anal. Anwend. 18 (1999) 183–191.