The Hilbert Transform of Wavelets are Wavelets.

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Abstract

Recently, we have discovered that the Hilbert transformation of a compactly-supported wavelet is, in a well defined sense, also a wavelet. That is, the HT (Hilbert Transform) wavelets are orthogonal to their translates, form a basis for $L^2(R)$ and define a multiresolution analysis. However, the HT scaling and wavelet functions do not have compact support. Their support is all of R . The scaling functions decays (vanishes at ∞) as $1/|x|$ and the wavelet function decays as $1/|x|^{p+1}$ where p is the number of vanishing moments.

1 Introduction

Recently, we have discovered that the Hilbert transformation of a compactly-supported wavelet is, in a well defined sense, also a wavelet. That is, the HT (Hilbert Transform) wavelets are orthogonal to their translates, form a basis for $L^2(R)$ and define a multiresolution analysis. However, the HT scaling and wavelet functions do not have compact support. Their support is all of R . The scaling functions decays (vanishes at ∞) as $1/|x|$ and the wavelet function decays as $1/|x|^{p+1}$ where p is the number of vanishing moments.

These various properties of the HT wavelets suggest that they might be useful for solving exterior boundary problems with a prescribed behavior at the point at ∞. For instance, acoustic radiation from a compact object is described by a solution of the wave equation that satisfies the Sommerfeld radiation condition at ∞. We examine a Galerkin method with an HT wavelet basis that can accurately resolve the near field while automatically preserving the correct far field rate of decay. This approach could allow the direct numerical simulation of scattering and radiation phenomena while avoiding the limitations of boundary element methods (nonuniqueness) and the constraints of artificial, nonreflective boundary conditions. The boundary element method is based on the potentials of a single and double layer. It is a consequence of the formulation, and not the physics, that the exterior Neumann problem does not have a unique solution for a frequency that is an eigenstate of the corresponding (adjoint) interior Dirichlet problem [20]. (And vice versa.) On the other hand, direct methods for the exterior problem avoid this problem. However, to use direct methods in a finite computational region requires an artificial boundary that allows energy to freely pass into and out of the computational domain without reflection at this boundary. In general this is difficult to achieve and can lower the resulting accuracy, if not the validity, of the calculation.

To a significant extent, our development of an HT Wavelet-Galerkin method can build upon, almost paraphrase, our previous development of the (compactly-supported) Wavelet-Galerkin method. A careful analysis of the problems caused by the singular Hilbert Transform and noncompact support of the basis functions will be required to maintain the accuracy and efficiency of the resulting algorithm. However, the difficulties in dealing with this basis may be far outweighed by the potential advantages of economy of representation and correct asymptotic behavior.

2 Wavelets and their Numerical Applications.

2.1 Compactly supported wavelets

Ingrid Daubechies defined the class of compactly supported wavelets [5]. Briefly, let ϕ be a solution of the *scaling* relation

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x - k).$$

The a_k are a collection of coefficients that categorize the specific wavelet basis. The expression ϕ is called the scaling function.

The normalization $\int \phi dx = 1$ of the scaling function obtains the condition

$$\sum a_k = 2.$$

The translates of ϕ are required to be orthonormal

$$\int \phi(x-k)\phi(x-m) = \delta_{k,m}.$$

From the scaling relation this implies the condition

$$\sum_{k=0}^{N} a_k a_{k-2m} = \delta_{0m}.$$

For coefficients verifying the above two conditions, the functions consisting of translates and dilations of the scaling function, $\phi(2^j x - k)$, form a complete, orthogonal basis for square integrable functions on the real line, $L^2(R)$.

If only a finite number of the a_k are nonzero then ϕ will have compact support.

Smooth scaling functions arise as a consequence of the degree of approximation of the translates. The conditions that the polynomials $1, x, \cdots, x^{p-1}$ be expressed as linear combinations of the translates of $\phi(x-k)$ is implied by the condition

$$\sum (-1)^k k^m a_k = 0$$

for $m = 0, 1, \cdots, p-1$.

The compactly supported wavelet is defined by the equation

$$\psi(x) = \sum (-1)^k a_{1-k} \phi(2x - k)$$

The translates of the scaling function and wavelet define orthogonal subspaces. i.e.

$$\int \phi(x)\psi(x-m)dx = \sum (-1)^k a_{1-k} a_{k-2m} = 0.$$

The orthogonal subspaces

$$V_j = \{2^{j/2}\phi(2^j x - m); m = \cdots, -1, 0, 1, \cdots\}$$

$$W_j = \{2^{j/2}\psi(2^j x - m); m = \cdots, -1, 0, 1, \cdots\}$$

are related by the condition

$$V_{j+1} = V_j \oplus W_j.$$

This is the basis of Mallat, or Wavelet, transform

$$V_0 \subset V_1 \subset \cdots \subset V_{j+1}$$

$$V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_j.$$

The following are equivalent results [7, 18].

- $\{1, x, \cdots, x^{p-1}\}$ are linear combinations of $\phi(x-k)$.
- $\|f - \sum c_k \phi(2^j x - k)\| \leq C 2^{-jp} \|f^{(p)}\|$.
- $\int x^m \psi(x) dx = 0$ for $m = 0, 1, \cdots, p-1$.
- $\int f(x)\psi(2^j x) dx \leq c 2^{-jp}$
- L_N where $L_{i,j} = a_{2i-j}$ has eigenvalues $1, \frac{1}{2}, \cdots, (\frac{1}{2})^{p-1}$.

2.2 Numerical Applications

The Wavelet-Galerkin Method.

For a PDE of the form

$$F(U, U_t, \cdots, U_x, U_{xx}, \cdots) = 0$$

define the wavelet expansion

$$U = \sum U_k \phi(x-k).$$

An approximation to the solution is defined by

$$\hat{U} = \sum_{k=-M}^{N} \hat{U}_k \phi(x-k).$$

In effect, the solution is projected onto the subspace spanned by

$$\Phi(M,N) = \{\phi(x-k) : k = -M, \cdots, N\}.$$

Herein and in what follows, we assume, for simplicity and without loss of generality, that the dilation factor 2^j has been normalized to 1 by a scale transformation $y = 2^j x$. In effect, the integers are the finest scale. To determine the coefficients of this expansion we substitute into the equation and again project the resulting expression onto the subspace $\Phi(M,N)$. This uniquely determines the coefficients U_k.

The projection requires $\hat{U}_k$ to verify the equations

$$\int_{-\infty}^{\infty} \phi(x-k) F(\hat{U}, \hat{U}_t, \hat{U}_x, \cdots) dx = 0$$

for $k = -M, \cdots, N$. To evaluate this expression we must know the *connection coefficients* of the form

$$\int \phi(x)\phi_x(x-k_1)\cdots\phi_{xx}(x-k_2)\cdots dx.$$

We have found *exact* methods for evaluating the functionals required in the Wavelet-Galerkin method [10]. A typical functional (three term connection coefficient) would be

$$\Omega(k,j) = \int \phi_{xx}(x)\phi_x(x-k)\phi(x-j)dx.$$

Since the scaling function used to define compact wavelets has a limited number of derivatives, the numerical evaluation of these expressions is often unstable or inaccurate.

The exact method is based on use of the *scaling relation*

$$\phi(x) = \sum_{k=0}^{N} a_k \phi(2x-k).$$

By the obvious manipulations a system of equations is found for the $\Omega(k, j)$. The system of equations is generally rank deficient (singular). The rank deficiency is cured and a unique solution is obtained by the inclusion of an additional set of linear equations that are obtained from the *moment* equations. The resulting system is non-singular and non-homogeneous and has a unique solution that is easily found by standard techniques. This technique is derived in the recent paper by Latto, Resnikoff and Tenenbaum [10].

The Wavelet-Capacitance Matrix Method

To solve boundary value problems we have developed an extension of the classical Capacitance Matrix method.

We will describe the method, as developed by Qian and Weiss [15], for the Harmonic Helmholtz equation

$$(-\Delta + \alpha)\, U = F$$

in a domain D with boundary conditions $U = g$ on the boundary of D. One version of the direct method is equivalent to a numerical implementation of the single layer potential [13]. A method based on the double layer potential is also a possibility [14]. The algorithm is based on the calculation of a numerical *partial Green's Function* [13].

The outline of our method is as follows. Regard the domain D as contained (embedded) in a periodic cell, S. We extend F from D to S in a smooth way. The extension $\hat{F}$ is periodic on S. We also define a periodic function $\hat{\rho}$ where $\hat{\rho}$ is zero except on the support of $\partial D \in S$. We determine $\hat{\rho}$ so that the periodic solution in S

$$(-\Delta + \alpha)\, U = \hat{F} + \hat{\rho}$$

will verify the boundary conditions $U = g$ on ∂D. By construction the equation $(-\Delta + \alpha)\, U = F$ is satisfied in D.

We have extended the method by allowing the support of $\hat{\rho}$ to be separate from the boundary of D, ∂D. When the equations are discretized by the Wavelet-Galerkin method, this extension eliminates the boundary residuals and defines a spectrally accurate method for non-separable domains. To our knowledge this algorithm is the first implementation of its type. We will present an extensive series of numerical calculations that support our conclusions about accuracy and convergence.

The numerical implementation is straightforward. In effect, we expand the solution in periodic, wavelet-Galerkin basis

$$U = \sum \sum U_{i,j}\, \phi(x - i) \phi(y - j)$$

where ϕ is a scaling function. To calculate the Green's Function we resolve the delta function in the space of translates of scaling function

$$\lambda_{x_0, y_0}(x, y) = \sum \sum \phi(x_0 - i)\phi(x - i)\phi(y_0 - j)\phi(y - j).$$

Since the translates of the dilated scaling function are orthogonal and complete in L^2, the above expression implies that for a square integrable function f

$$f(x_0, y_0) = \int\int dxdy \lambda_{x_0,y_0}(x, y) f(x, y),$$

which is the definition of the delta function. Here we can remove the dilation factor by an affine change of variable.

Therefore, we solve, by the wavelet-Galerkin method [15], the equation

$$(-\Delta + \alpha)\, G(x, x_0; y, y_0) = \lambda_{x_0,y_0}(x, y)$$

for the partial Green's Function, G. To find the usual Capacitance Matrix, C, we discretize the boundary into a series of points $\hat{x}_j$ and form the matrix whose (i, j) component is $G(\hat{x}_i, \hat{x}_j)$. The evaluation of G requires only one solution of the periodic, fast, wavelet-Galerkin solver [14].

In our formulation of the algorithm, we discretize the boundary by the points $\hat{x}_j$ and the support of $\hat{\rho}$ in S by the points $\hat{y}_j$. The definition of the capacitance matrix is then

$$C_{i,j} = G(\hat{x}_i, \hat{y}_j).$$

Depending on the cardinality of the sets $\hat{x}$ and $\hat{y}$, the system of equations for the discrete potential $\hat{\rho}$ are determined, overdetermined or underdetermined. We have examined these possibilities and present the results in ref. [15]. In general, if $\hat{y}$ is exterior to $\hat{x}$, we obtain excellent numerical results that depend stably on the choice of $\hat{y}$.

In terms of the (extended) Capacitance Matrix, the discrete potential of a single layer is a solution of the system

$$\hat{g} = C\hat{\rho}.$$

For non-determined systems we use a singular value decomposition of C to find the least square or minimal norm solution [9].

The Capacitance Matrix is a fast and general method for solving boundary value problems in nonseparable domains. It uses fast periodic solvers based on the FFT to drive direct or iterative (Conjugate Gradient) algorithms. The geometry at the boundary is enforced by potentials with singular support on the boundary. The use of functions with singular support effectively restricts the Capacitance Matrix method to low order solvers, requiring a high level of discretization to produce accurate results. Due to *boundary residuals*, the introduction of higher order solvers can cause the rate of convergence to become worse. For problems with complicated geometries this fact limits the applicability of the method.

By combining a reformulation of the Capacitance Matrix method with a wavelet discretization, we have defined a Wavelet-Capacitance Matrix method. This allows the use of higher order approximations with rapid (even spectral) convergence and produces highly accurate solutions for low to moderate levels of discretization. In effect, we cure the Capacitance Matrix method of its' most serious limitation, while retaining the method's advantages.

The method applies equally to equations with three space dimensions and problems with a time dependence. For instance, we have already applied the method to the long time integration of Euler and Navier-Stokes flows, with excellent results [23]. Figure 1 shows the evolution of Navier-Stokes flow in an L-shape region.

3 The Hilbert Transform of Wavelets.

3.1 The Hilbert Transform of Wavelets

The basic scaling function satisfies a scaling relation of the form

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x-k).$$

It is also true that the Hilbert transform of ϕ

$$H(\phi)(y) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{\phi(x)}{x-y}$$

is a solution of the same scaling relation. Although ϕ may have compact support, the Hilbert transform has support on the real line and decays as y^{-1}. The Hilbert transform of the related wavelet, ψ, is also noncompact and decays like y^{-p-1} where

$$\int x^m \psi(x) dx = 0$$

for $m = 0, 1, \cdots, p-1$. We expect to use these properties to obtain wavelet-Galerkin solutions with the proper behavior at ∞.

The definition of Hilbert transform is [16]

$$\tilde{\phi} = H(\phi)(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\phi(t)}{t-x}\, dt.$$

It is true that $H : L^2(R) \to L^2(R)$ is an isometry map, and the Hilbert transform of wavelets is a complete, orthogonal basis set on $L^2(R)$. The Hilbert transform preserves orthogonality of translates.

$$\int \tilde{\phi}(x-n)\tilde{\phi}(x-m) dx = \delta_{nm}.$$

It also preserves the local (Lipshutz) continuity.

$\tilde{\phi}$ is a scaling function.

The basic compactly-supported scaling function satisfies a scaling relation of the form

$$\phi(x) = \sum_{k=0}^{N-1} a_k \phi(2x-k).$$

Apply the Hilbert Transform to the above scaling relation and use the identity

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\phi(2t-k)}{t-x}\,dt = \tilde{\phi}(2x-k),$$

which is a consequence of the change of variable $t' = 2t-k$. Therefore, $\tilde{\phi}(x)$ verifies the scaling relation

$$\tilde{\phi}(x) = \sum_{k=0}^{N-1} a_k \tilde{\phi}(2x-k).$$

We note that the Hilbert transform of $\phi(x)$ exists as a well defined function since $\phi(x)$ is integrable, square-integrable (has compact support). In fact, $\phi(x)$ is generally Lipshutz continuous, implying that $\tilde{\phi}(x)$ is Lipshutz continuous with that same modulus [16].

Orthogonality of trasnslates of $\tilde{\phi}(x)$.

The orthogonality of translates of $\tilde{\phi}(x)$ is a direct consequence of the (distributional) identity on $L^1(R)$ [2, 16]

$$\frac{1}{\pi^2}\int_{-\infty}^{+\infty} \frac{1}{(t-x)(s-x)}\,dx = \delta(t-s),$$

where, as usual, the integral is evaluated in the principal value sense [16]. Therefore,

$$\int \tilde{\phi}(x-n)\tilde{\phi}(x-m)dx = \int \phi(x-n)\phi(x-m)dx = \delta_{nm}.$$

Asymptotic behavior of $\tilde{\phi}(x)$, $\tilde{\psi}(x)$.

The moment equation

$$\int x^m \psi(x)\,dx = 0,$$

where $(m = 0, 1, \cdots, p-1)$, implies the asymptotic behavior. As $x \to \infty$

$$\tilde{\phi}(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\phi(t)}{t-x}\,dt \to -\frac{M_0}{\pi x} - \frac{M_1}{\pi x^2} - \cdots - \frac{M_{p-1}}{\pi x^p},$$

where $M_j = \int t^j \phi(t)dt$ for $j = 0, 1, \ldots, p-1$ are the *moments* of $\phi(t)$ and are known explicitly. Since the moments of $\psi(t)$ vanish

$$\tilde{\psi}(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\psi(t)}{t-x}\, dt$$

$$= -\frac{1}{\pi x}\int_{-\infty}^{+\infty} (1 + \frac{t}{x} + \frac{t^2}{x^2} + \cdots)\psi(t)\, dt$$

$$\rightarrow -\frac{C}{\pi x^{p+1}},$$

where $C = \int t^p \psi(t)dt$.

It is straightforward to show that there are finite linear combinations of translates of $\tilde{\phi}$(x) with coefficients depending on the moments that decay as x^{-j} for $j = 1, 2, \ldots, p$.

A comment on the applicability of HT wavelets.

The scaling and wavelet functions of compact support can be defined on the real line, R, or on the circle (is periodic). The HT scaling and wavelet functions do not have compact support and are defined on the real line, R. The periodic Hilbert Transform is defined by a $\cot(x)$ kernel [6]. It is easy to see that the periodic Hilbert transform of a periodic, compact-support scaling or wavelet function is not a periodic solution of the scaling relation. The reason for this result is explained in the Section 4.

3.2 Evaluation of the HT scaling function, $\tilde{\phi}(x)$.

The Hilbert Transform, $\tilde{\phi}$, verifies the same scaling relation as ϕ.

$$\tilde{\phi}(x) = \sum_{k=0}^{N-1} a_k \tilde{\phi}(2x-k).$$

If values of $\tilde{\phi}$ are known at the integers then the values of $\tilde{\phi}$ are known at the dyadic rationals $x = m/2^j$ by recursion

$$\tilde{\phi}(m/2^j) = \sum_{k=0}^{N-1} a_k \tilde{\phi}(m/2^{j-1} - k).$$

$(\tilde{\phi}(m) \; |m < 0, m > N-1)$ can be computed directly since the integrals are nonsingular. The scaling relation can be used to infer certain values of $\tilde{\phi}(m)$ without direct calculation.

The values $(\tilde{\phi}(m) \; |0 < m < N-1)$ are defined by the *principal values* of singular integrals [16]. To avoid direct evaluation of these integrals we form for $\tilde{\phi}(m)$, for

$m \in [0, N-1]$, the system of equations

$$\begin{array}{ll} \tilde{\phi}(0) & = a_0\tilde{\phi}(0) + a_1\tilde{\phi}(-1) + a_2\tilde{\phi}(-2) + \cdots + a_{N-1}\tilde{\phi}(-N+1) \\ \tilde{\phi}(1) & = a_0\tilde{\phi}(2) + a_1\tilde{\phi}(1) + a_2\tilde{\phi}(0) + \cdots + a_{N-1}\tilde{\phi}(-N+3) \\ \vdots & \vdots \\ \tilde{\phi}(N-1) & = a_0\tilde{\phi}(2N-2) + a_1\tilde{\phi}(2N-3) + \cdots + a_{N-1}\tilde{\phi}(N-1) \end{array}$$

The recursion matrix L_N that determines the vector

$$\vec{\phi} = [\tilde{\phi}(0), \tilde{\phi}(1), \cdots, \tilde{\phi}(N-1)]^t$$

$$\vec{\phi} = L_N\vec{\phi} + \vec{f}$$

is

$$L_{i,j} = a_{2i-j-1},$$

where

$$\vec{f_i} = \sum_{j<1, j>N} a_{2i-j-1}\tilde{\phi}(j-1).$$

In the compact support case $\vec{f} \equiv 0$ and the above system is an eigenvalue problem.

The direct evaluation of $\vec{\phi}$ involves a singular integral. The direct evaluation of $\vec{f}$ involves nonsingular integrals. Therefore, we calculate $\vec{f}$ and use the recursion matrix to find $\vec{\phi}$. However, $\vec{\phi}$ is not uniquely determined by the recursion matrix equation. There is the constraint on $\tilde{\phi}$,

$$\sum_{m=-\infty}^{+\infty} \tilde{\phi}(m) = 0.$$

We fix the normalization of $\vec{\phi}$ by this constraint.

To numerically evaluate the normalization of $\vec{\phi}$ we use MacLaurin's Formula

$$\sum_{i=0}^{n} f(x_i) = \int_{x_0}^{x_n} f(x)\,dx + \frac{1}{2}[f(x_0) + f(x_n)]$$

$$+ \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}[f^{(2k-1)}(x_n) - f^{(2k-1)}(x_0)] + R_{2m}.$$

The constants B's are the Bernoulli number, R_{2m} is the remainder. The integral in the formula is not singular and the result allows the unique determination of $\tilde{\phi}$. Figure 2 shows the $D6$ scaling function and its' Hilbert Transform as evaluated by this method. We will continue the development and analysis of methods for the fast evaluation of the HT wavelets. Specifically, we will evaluate the stability and accuracy of the algorithms and find a more direct inclusion of the asymptotic behavior in the evaluation of the HT wavelets.

3.3 Expansion of function in the HT wavelet basis.

Figure 2 also compares the truncation errors for $\frac{\sin(x)}{x}$ using the HT wavelet and compactly-supported wavelet (CS wavelets). The expansion of this function in the HT basis has a lower truncation error per mode.

The HT above wavelet expansion use the fact that the Galerkin coefficients of a function in the HT basis are equal to the Galerkin coefficients of the Hilbert transform of the function in the CS wavelet basis. The HT scaling function is evaluated by the previously described algorithm.

3.4 Representation of differential operators in the HT wavelet basis.

The Galerkin coefficients (Connection Coefficients) for differential operators in the HT wavelet basis can be shown, in certain cases, to be identical to the Galerkin coefficients (Connection Coefficients) in the CS wavelet basis. This important result uses the properties of the Hilbert transform and assumes a certain smoothness of the basic wavelet function. In effect, we require that the Hilbert operator and differentiation commute.

For instance, the two term connection coefficient required for the Galerkin approximation of the first derivative are of the form

$$\Omega_k = \int \phi(x-k)\phi_x(x)\,dx.$$

The corresponding HT connection coefficient is

$$\Phi_k = \int \tilde{\phi}(x-k)\tilde{\phi}_x(x)\,dx.$$

By definition and integration by parts

$$\Phi_k = \frac{1}{\pi^2}\int \phi(t-k)\,dt \int \phi_x(s)\,ds \int \frac{dx}{(t-x)(s-x)}$$

and by the principal value for the Hilbert kernel

$$\Omega_k = \Phi_k.$$

By the same argument, the general two term connection coefficients required for the Galerkin representation of linear differential operators are also identical in the CS and HT bases.

Furthermore, the general n-term connection coefficients required for the Galerkin representation of nonlinear expressions are also identical in the CS and HT bases. To show this requires an analysis of the Cauchy type integral

$$\Theta(t_1, t_2, \ldots, t_n) = \frac{1}{\pi^n}\int_{-\infty}^{\infty} \frac{dx}{(t_1-x)(t_2-x)\cdots(t_n-x)},$$

for $\{t_1, t_2, \ldots, t_n\}$ on the real line. It can be shown that, as a distribution on $L^1(R^n)$ [2, 16],

$$\Theta(t_1, t_2, \ldots, t_n) = \delta(t_1 - t_2)\delta(t_2 - t_3)\cdots\delta(t_n - t_1).$$

This immediately implies the identity of n-term connection coefficients in the CS and HT bases.

See Bremermann [2] and Roos [16] for further discussion of distributions and the Hilbert transformation.

3.5 Applications of HT Wavelets.

Exterior Boundary Value Problems

We can apply the HT Wavelet-Galerkin method to boundary value and scattering problems in R and R^2.

The Schrodinger equation in R^n for $n = 1, 2, 3$ can be written

$$(\Delta + U(\hat{x}))\, V = \lambda V.$$

The Schrodinger equation has, of course, many applications. In general, it is assumed that $U(\hat{x})$ decays sufficiently fast at ∞ so that for some μ depending on n [1]

$$\int (1 + |\hat{x}|^{\mu})|U(\hat{x})|dx < \infty.$$

We can approximate the potential U in the HT wavelet basis and apply the Galerkin method with HT wavelet basis to discretize the Schrodinger Equation. This leads to a (discrete) eigenvalue problem for $\lambda = -k_j^2$ with solutions $V_j(\hat{x}) \in L^2(R^n)$. In one dimension, there is a well known class of reflectionless potentials that have closed form solutions. In this case comparison with the exact solutions over R is possible.

In R^2 there are various scattering problems with exact closed form solutions that decay at ∞. These are associated with the Kadomtsev-Petviashvili and Davey-Stewartson equations as so-called lump solutions [1]. We can apply the Galerkin method with a HT wavelet basis to solve these problems and compare to the exact solutions in specific cases.

Also of interest is the problem of the acoustic radiation from an compact object, D, into the domain exterior to D containing the point at infinity in R^2 or R^3. This requires finding solutions of the reduced wave equation

$$\left(\Delta + k^2\right) P = 0$$

with prescribed normal derivative $\frac{\partial P}{\partial \hat{n}}$ on the boundary of D and that verify the Sommerfeld radiation condition

$$\lim_{|\hat{x}|\to\infty} |\hat{x}|^{\frac{n-1}{2}} [\imath k P(\hat{x}) - \nabla_{\hat{x}} P(\hat{x})] = 0$$

in R^n.

The Sommerfeld radiation essentially requires that for $|\hat{x}| \to \infty$, $P \to |\hat{x}|^{\frac{-n+1}{2}} e^{ik|x|}$.

We can discretize the system by the Galerkin method with HT wavelet basis and impose the Neumann boundary conditions on ∂D by a variant of the Capacitance Matrix method.

4 Summary

The Hilbert Transform acts as an involution on the space of square-integrable solutions of a scaling relation. The Hilbert transform preserves the orthogonality, local smoothness, and connection (coefficients) of a scaling function (wavelet) basis. In other words, the Hilbert transform of a wavelet is a wavelet.

The Hilbert transform also acts as an involution on the space of solutions of a more general class of linear functional-differential equations. Rvachev [17] has made an extensive study of differential dilation equations of the form

$$Ly(x) = \lambda \sum_{k=1}^{M} c_k y(ax - b_k).$$

where $L = D^n + a_1 D^{n-1} + \cdots + a_n$, $D = \frac{d}{dx}$, and $|a| > 1$. These systems can have C^∞ solutions with compact support. In general, the translates of a compact solution can define a basis for Sobolev spaces. However, the translates are not orthogonal [17]. It is simple to verify that the Hilbert transform of a solution of the above differential-dilation equation is also a solution of the equation.

In addition, by a result of Stein [19], the Hilbert transform is the unique, bounded linear operator on $L^2(R)$ that commutes with translation, dilation and reflection. Therefore, the Hilbert transform is the unique operator that preserves the $L^2(R)$ solution space of the one-dimensional, linear, translation-dilation equations. The higher dimensional scaling relations involve translations, dilations and rotations of the argument vector. The appropriate bounded linear operators on $L^2(R^n)$ that map solutions into solutions would be the Riesz operators described in reference [19].

We suggest that the Hilbert wavelets may have several useful numerical applications. These include exterior boundary value problems and the inversion of the Radon transform. The inverse Radon transform requires the evaluation of derivatives and Hilbert transforms. A Galerkin approximation could be a natural application of the Hilbert wavelets [24].

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