Problem adequate (weak) variational
$H_{\alpha}^{\#}(0,1)-$**characterization**
for non-linear parabolic PDE
with singular order coefficient

Hilbert scale with norm

$$\|u\|_{\beta}^{2}:=\sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_{\nu}|^{2} \quad .$$

Non-linear parabolic PDE problems
with singular order coefficient
related to the one-dimensional Stefan problem

$$u_t-u_{xx}+t^{-1/4}u_x=0 \quad .$$

1- periodic solutions

Enabling quasi-optimal BEM approximation estimates
of the related variation auxiliary equation
$(u=Av\,,\,u_{xx}=Av_{xx}=Hv_x=Sv\,,\,v=Su=Hu_x)$

$$v\in H_{1/2}^{*}(0,1):\quad (\dot{v},w)_{-1/2}+(v',w')_{-1/2}+t^{-1/4}(v,w)_0=0\quad ,\ \forall w\in H_0^{\#}(0,1)\,.$$

STATUS QUO: Not-quasi-optimal FEM approximation estimates
For the Stefan problem ([NiJ2]) and for the non-stationary Navier-Stokes equations ([HeJ]) due to the fact that the energy norm $\|v\|_1(t)$ does not ensure finite energy for $t\to 0$ ([HeJ1], [NiJ2]). The proposition is that a variation formulation of the PDE system within respect to a problem adequate energy inner product ensures a well posed system, whereby the existence and uniqueness of the solution is approximated by BEM approximation solutions.

IDEA: functional-analytical approach applying fractional powers of the Stokes operator $A^{\alpha}:D(A^{\alpha})\to L_{\sigma}^{2}(\Omega)$ to define the time-weighted energy norm

$$\|u\|_{\alpha.\beta}^{2}(t):=t^{\beta}\|A^{-\alpha/2}u\|_0^2(t)+\int_0^t\tau^{\beta}\|A^{-(\alpha-1)/2}u\|_0^2(\tau)d\tau$$

This norm enables proper duality arguments and adequate energy norm estimates, while avoiding the usage of the lemma von Gronwall. It leverages on the counter example of Heywood & Walsh for standard energy norm [HeJ1].

Calderon-Zygmund and Riesz operators

The Calderon-Zygmund operator with symbol $|\nu|$ ([EsG] (3.17), (3.35)) is defined by

$$(\Lambda u)(x)=(\sum_{k=1}^{n}R_kD_ku)(x)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\sum_{k=1}^{n}p.v.\int_{-\infty}^{\infty}\sum_{k=1}^{n}\frac{x_k-y_k}{|x-y|^{n+1}}\frac{\partial u(y)}{\partial y_k}dy=-\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{\Delta_y u(y)}{|x-y|^{n-1}}dy=-(\Delta\Lambda^{-1})u(x)$$

whereby R_k denotes the Riesz operators ([AbH] p. 19, 106, [PeB] example 9.9)

$$R_ku=-i\pi^{-(n+1)/2}\Gamma(\frac{n+1}{2})p.v.\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}u(y)dy\,.$$

For $n\geq 2$ it holds ([EsG] (3.15))

$$\Lambda^{-1}u=\frac{1}{2}\pi^{-(n+1)/2}\Gamma(\frac{n-1}{2})\int_{-\infty}^{\infty}\frac{u(y)dy}{|x-y|^{n-1}}\quad .$$

The Riesz operators fulfill certain properties with respect to commutation with translations homothesis and rotation ([PeB], [StE]). Let $SO(n)$ denote the rotation group. If $j\neq j$ then R_jR_k is a singular convolution operator. On the other hand it holds $R_j^2=-(1/n)I+A_j$ where A_j is a convolution operator. The following identities are valid

$$\|R_j\|=1\;\;,\;\;R_j^*=-R_j\;\;,\;\;\sum R_j^2=-I\;\;,\;\;\sum\|R_ju\|^2=\|u\|^2\;\;,u\in L_2\;.$$

Let

$$m:=m(x):=(m_1(x),...m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho=\rho_{ik}\in SO(n)$, then

$$m(\rho(x))=\rho(m(x))\,,$$

whereby

$$m_j(\rho(x))=\sum\rho_{jk}m_k(x)$$

and

$$m(\rho(x))=c_n\int_{S^{n-1}}(\frac{\pi i}{2}sign(x\rho^{-1}(y))+\log\left|\frac{1}{x\rho^{-1}(y)}\right|)\frac{y}{|y|}d\sigma(y)$$

$$=c_n\int_{S^{n-1}}(\frac{\pi i}{2}sign(xy)+\log\left|\frac{1}{xy}\right|)\frac{y}{|y|}d\sigma(y)\;.$$

One-dimensional Calderon-Zygmund and Riesz Operators

In case of n=1 the Calderon-Zygmund and the Riesz operators are given by **(**[LiI] (1.2.31)-(1.2.33), [LiI1]):

$$Su(x):=\oint \frac{1}{4\sin^2\frac{x-y}{2}}u(y)dy$$

$$Hu(x):=\oint \frac{1}{2}\cot\frac{x-y}{2}u(y)dy\ .$$

For

$$Au(x):=-\oint \log 2\sin\frac{x-y}{2}u(y)dy$$

we note the relationship

$$A[u_x](x)=-H[u](x)$$

resp.

$$(HA)[u_x](x)=u(x)\ .$$

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Shift theorems for singular lower order coefficient, non-linear parabolic equation (1-periodic Cauchy problem)

$$u_t - u_{xx} + t^{-1/4} u_x = t^{-1/4} f$$

$$u(x,0) = \varphi(x)$$

A substitution of the variable (see Höllig paper)

$$y := x + \frac{4}{3} t^{3/4}$$

and putting

$$w(y,t) := u(x,t) \quad , \; g(y,t) := f(x,t)$$

gives

$$w_t - w_{xx} = t^{-1/4} g$$

$$w(y,0) = \varphi(y) \; .$$

This means that it becomes a linear heat equation, whereby there is a singular behavior of $t^{-1/4} g(\cdot,t)$ to be taken into account. We put

$$\Gamma(y,t) := \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}$$

and

$$G(y,t) := \frac{1}{\sqrt{4\pi t}} \sum_{-\infty}^{\infty} e^{-\frac{(y-\nu)^2}{4t}} = \frac{1}{\sqrt{4\pi t}} \sum_{-\infty}^{\infty} \Gamma(y-\nu,t) = 1 + 2\sum_{1}^{\infty} \cos(2\pi\nu y) e^{-4\pi^2\nu^2 t}$$

and

$$\varphi(y) =: \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos(2\pi\nu y) + b_\nu \sin(2\pi\nu y) \; .$$

Then it holds (Courant-Hilbert II, p. 155 ff):

$$w(y,t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos(2\pi\nu y) + b_\nu \sin(2\pi\nu y) e^{-4\pi^2\nu^2 t} = \int_0^1 \varphi(\xi) G(y-\xi,t) dt \cdot$$

From this it follows (see below):

in case of $g = 0$: $\quad \|w(t)\|_k^2 \le c t^{-(k-l)} \|\varphi\|_l^2 \cdot t^{-1/2}$?

in case of $\varphi = 0$: $\quad \int_0^T \|w\|_{k+2}^2 dt \le c \int_0^T t^{-1/2} \|g\|_k^2 dt$

Linear parabolic shift theorems

We consider the two parabolic equations

$\dot{w} - w'' = f$	$\dot{z} - z'' = 0$	in $(0,1) \times [0,T]$
$w(0,t) = w(1,t) = 0$	$z(0,t) = z(1,t) = 0$	for $t \in (0,T]$
$w(x,o) = 0$	$z(x,o) = g(x)$	for $x \in (0,1)$.

The following compatibility relations for the initial value function have to be fulfilled in order to ensure corresponding regularity of the solution z :

$$g(1) = 0 \ , \ g'(0) = 0 \ , \ g''(1) = g'^2(1) \ , \text{etc.}$$

Let $w_i := (w, \varphi_i)$ resp. $f_i := (f, \varphi_i)$ being the generalized Fourier coefficient related to the eigen pairs $-v_i'' = \lambda_i v_i$. Then it holds

$$\dot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \quad \text{and} \quad w_i(0) = 0 \ .$$

with the solution

$$w_i(t) = \int_0^t e^{-\lambda_i (t-\tau)} f_i(\tau) d\tau \ .$$

The following shift theorem holds true:

Lemma:

i) $\int_0^T t^{-1/2} \|w\|_{k+2}^2 dt \le c \int_0^T t^{-1/2} \|f\|_k^2 dt$

ii) $\|z(t)\|_k^2 \le ct^{-(k-l)} \|g\|_l^2$

iii) $\int_0^T t^{-1/2} \|z\|_{k+1/2}^2 dt \le \|z\|_k^2$ especially $\int_0^T t^{-1/2} \|z'\|_{-1/2}^2 dt = \int_0^T t^{-1/2} \|z\|_{1/2}^2 dt \le c \|g\|$

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Proof: i) It holds for $\tau \leq t$

$$\int_0^T t^{-1/2}\|w\|_{k+2}^2 dt = \sum \lambda_i^{k+2}\int_0^T t^{-1/2} w_i^2(t)dt \leq \sum \lambda_i^{k+2}\int_0^T\left[\int_0^t e^{-\lambda_i(t-\tau)}d\tau\right]\left[\int_0^t \tau^{-1/2}e^{-\lambda_i(t-\tau)} f_i^{\,2}(\tau)d\tau\right]dt$$

$$\leq \sum \lambda_i^{k+2}\int_0^T \lambda_i^{-1}\left[\int_0^t \tau^{-1/2}e^{-\lambda_i(t-\tau)} f_i^2(\tau)d\tau\right]dt \ .$$

Exchanging the order of integration gives

$$\int_0^T\int_0^t \tau^{-1/2}e^{-\lambda_i(t-\tau)}F_i^2(\tau)d\tau dt = \int_0^T\int_t^T \tau^{-1/2}e^{-\lambda_i(t-\tau)}F_i^{\,2}(\tau)dtd\tau = \int_0^T F_i^{\,2}(\tau)dt\left[\int_t^T e^{-\lambda_i(t-\tau)}d\tau\right] \leq \lambda_i^{-1}\int_0^T \tau^{-1/2}F_i^{\,2}(\tau)dt$$

and therefore

$$\int_0^T t^{-1/2}\|w\|_{k+2}^2 dt \leq c\int_0^T t^{-1/2}\|F\|_k^2 dt \ .$$

ii) From

$$z(x,t) = \sum z_\nu(t)\varphi_\nu(x)$$

it follows

$$\dot z - z'' = \sum(\dot z_\nu(t) + \lambda_\nu z_\nu(t))\varphi_\nu(x) = 0 \ .$$

Therefore

$$z_\nu(t) = z_\nu(0)e^{-\lambda_\nu t} \text{ and } z_\nu(0) = g_\nu = (g,\varphi_\nu) \ .$$

Putting $$C_{k.l}(t) := \sup_{\lambda_\nu \geq m > 0} \lambda_\nu^{k-l}e^{-2\lambda_\nu t}$$

it follows

$$\|z(t)\|_k^2 = \sum \lambda_\nu^k z_\nu^2(t) = \sum \lambda_\nu^k e^{-2\lambda_\nu t} g_\nu \leq C_{k.l}(t)\sum \lambda_\nu^l e^{-2\lambda_\nu t}$$

The conditions

$$(k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t} + \lambda^{k-l}(-2t)e^{-2\lambda_\nu t} = 0$$

resp.

$$(k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t} = 2t\lambda^{k-l}e^{-2\lambda_\nu t}$$

leads to (for the critical case $k > l$) $\lambda \approx t^{-1}$.

iii) $$\int_0^T t^{-1/2}\|z\|_{k+1/2}^2 dt = \sum \lambda_\nu^{k+1} g_\nu \int_0^T \lambda_\nu^{-1/2} t^{-1/2} e^{-2\lambda_\nu t} dt \leq c\sum \lambda_\nu^{k+1} g_\nu \lambda_\nu^{-1} \leq \|z\|_k^2$$

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