

## A NONLINEAR INTEGRAL EQUATION OCCURRING IN A SINGULAR FREE BOUNDARY PROBLEM<sup>1</sup>

BY

KLAUS HÖLLIG<sup>2</sup> AND JOHN A. NOHEL

ABSTRACT. We study the Cauchy problem

$$\begin{cases} u_t = \phi(u_x)_x, & (x, t) \in \mathbf{R} \times \mathbf{R}_+, \\ u(\cdot, 0) = f \end{cases}$$

with the piecewise linear constitutive function  $\phi(\xi) = \xi_+ = \max(0, \xi)$  and with smooth initial data  $f$  which satisfy  $xf'(x) \geq 0$ ,  $x \in \mathbf{R}$ , and  $f''(0) > 0$ . We prove that free boundary  $s$ , given by  $u_x(s(t)^+, t) = 0$ , is of the form

$$s(t) = -\kappa\sqrt{t} + o(\sqrt{t}), \quad t \rightarrow 0^+,$$

where the constant  $\kappa = 0.9034\dots$  is the (numerical) solution of a particular nonlinear equation. Moreover, we show that for any  $\alpha \in (0, 1/2)$ ,

$$\left| \frac{d^2}{dt^2} f(s(t)) \right| = O(t^{\alpha-1}), \quad t \rightarrow 0^+.$$

The proof involves the analysis of a nonlinear singular integral equation.

### 1. Introduction and result. We study the Cauchy problem

$$(1) \quad \begin{cases} u_t = \phi(u_x)_x, & (x, t) \in \mathbf{R} \times \mathbf{R}_+, \\ u(\cdot, 0) = f \end{cases}$$

with the piecewise linear constitutive function  $\phi: \mathbf{R} \rightarrow \mathbf{R}_+$  given by  $\phi(\xi) = \xi_+ = \max(\xi, 0)$ ; the initial data  $f: \mathbf{R} \rightarrow \mathbf{R}$  are assumed smooth, specifically  $f \in C^3(\mathbf{R})$  with bounded derivatives, and satisfy the conditions

$$(2) \quad \begin{cases} xf'(x) \geq 0, & x \in \mathbf{R}, \\ f''(0) > 0. \end{cases}$$

Our motivation for the study of the Cauchy problem (1), (2) is its similarity to the well-known one-phase Stefan problem (in one space dimension) [3, 4, 7, 8] in which one would assume  $f'(x) \equiv -1$  for  $x < 0$ , as well as  $f'(x) > 0$  for  $x > 0$ , so that  $f'$  has a jump discontinuity at  $x = 0$ . The assumption (2) yields a different behavior of

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the solution  $u$  and of the resulting free boundary. Indeed, here (cf. the Theorem below), the free boundary  $s$ , given by  $u_x(s(t)^+, t) = 0$ , is of the form

$$(3) \quad s(t) = -\kappa\sqrt{t} + O(t^{1/2+\alpha}), \quad t \rightarrow 0^+,$$

where  $\kappa$  is a positive constant and  $0 < \alpha < 1/2$ . Thus, the function  $s$  is not (infinitely) differentiable at  $t = 0$ , contrary to the situation for the Stefan problem [7].

The result (3) is established by solving a nonlinear integral equation (15) with kernels which depend on the unknown function  $s$  and which are also singular in the sense that the integral on  $(0, t)$  of the kernel does not approach zero as  $t \rightarrow 0^+$ . One consequence of this is that the integral operator defined by (15) is not compact in a suitable Hölder class.

The principal motivation for the study of the Cauchy problem (1), (2) is that it serves as a prototype of nonlinear parabolic problems which arise as monotone “convexifications” of nonlinear diffusion equations with nonmonotone constitutive functions  $\phi$  (see [5 and 6]); in [6, §4] the reader will also find the formulation and preliminary analysis of such a convexified problem, corresponding to a piecewise linear nonmonotone  $\phi$  (specifically,  $\phi'((-\infty, a) \cup (b, \infty)) > 0$ ,  $\phi'(a, b) < 0$ ,  $0 < a < b < \infty$ ). The analysis in [5] shows the existence the infinitely many solutions  $u$  of the nonmonotone problem, each having  $u_x$  bounded, and  $u_x$  omitting values in  $[a, b]$ ; thus each solution  $u$  exhibits phase changes. Numerical experiments further suggest the conjecture that the “physically correct” solution of the nonmonotone problem is the one which, as  $t \rightarrow \infty$ , approaches the unique solution of the appropriately related convexified monotone problem. However, for small  $t > 0$  the behavior of the solution of (1), (2) is qualitatively different (see (3)). The present study of (1), (2) is intended as a step towards the understanding of this intriguing phenomenon. The relation of the convexified problem in [6] to the Cauchy problem (1), (2) is clear (the particular boundary conditions in [6] do not play a role in the analysis of the free boundary curve).

It is simple to give a formal explanation for (3). We rewrite (1), (2) as the free boundary problem

$$(4a) \quad \begin{cases} u_t = u_{xx}, & s(t) < x < \infty, & t \in \mathbf{R}_+, \\ u_x(s(t), t) = 0, \\ u(\cdot, 0) = f. \end{cases}$$

From the constitutive function  $\phi$  one also has the equation

$$(4b) \quad \begin{cases} u_t(x, t) = 0, & -\infty < x < s(t), & t \in \mathbf{R}_+, \\ u(\cdot, 0) = f. \end{cases}$$

Therefore, assuming the continuity of  $u$  across the free boundary  $s(t)$  and assuming that  $s$  is monotone decreasing (cf. paragraph preceding the Theorem), we have

$$(5) \quad \begin{cases} u(s(t), t) = f(s(t)), & t \in \mathbf{R}_+, \\ s(0) = 0. \end{cases}$$

Differentiating (5) with respect to  $t$  and using  $u_x(s(t)^+, t) = 0$ , where “+” denotes the limit from the right, we obtain

$$(6) \quad f'(s(t))s'(t) = u_{xx}(s(t)^+, t).$$

Since by the assumption (2)

$$f'(x) = f''(0)x + O(|x|^2), \quad |x| \rightarrow 0^+,$$

a simple calculation formally yields (3) with  $\kappa = \sqrt{2}$  (provided one assumes continuity from the right of  $u_t$  and  $u_{xx}$  up to the free boundary  $s$ ).

The rigorous treatment of the problem consists of analyzing, in §3, the nonlinear integral equation (15) for the free boundary  $x = s(t)$ . Our analysis shows that (3) holds, but that the constant  $\kappa$  is the solution of the nonlinear equation (16); its numerical value is  $\kappa = 0.9034\dots$ , and not  $\kappa = \sqrt{2}$  which was predicted by the above formal calculation. It also follows that  $s(t)$  is smooth for  $t > 0$  thus justifying (5) and (6) for positive  $t$ ; in particular, one sees from (6) that  $s$  is as smooth as the initial function  $f$  is. We remark that for  $t \geq \epsilon > 0$  the problem (1), (2) can also be viewed as a one-phase Stefan problem; consequently, the results in Kinderlehrer and Nirenberg [7] yield the regularity of the free boundary for  $t > 0$ .

The existence of a unique generalized continuous solution for problem (1), and hence of a unique free boundary, follows from nonlinear semigroup theory for  $m$ -accretive operators [1, 2]. Approximating (1) by an implicit Euler scheme, one can also show the existence of the free boundary  $s$  which is Hölder continuous on  $[0, \infty)$  with exponent  $1/2$  and monotone decreasing. However, using such general methods, it is not possible to analyze the precise behavior of  $s$  at  $t = 0$ .

Our main result is

**THEOREM.** *Define*

$$(7) \quad r(t) = df(s(t))/dt.$$

*Then for any  $\alpha \in (0, 1/2)$  there exists  $T > 0$  such that  $r$  is continuous on  $[0, T]$  and satisfies*

$$(8) \quad t^{1-\alpha}|r'(t)| \leq c(f), \quad 0 < t \leq T,$$

*where  $c(f) > 0$  is a constant which depends on the data  $f$ . Moreover, (3) holds with*

$$\kappa := \left( 2 \frac{r(0)}{f''(0)} \right)^{1/2} = 0.9034\dots$$

*The constant  $\kappa$  is the (numerical) solution of equation (16) in §3.*

By the definition of  $\kappa$ , the result (3) follows from (7) and the assertion (8). To see this, we solve (7) for  $s$ . Let  $R(t) = \int_0^t r(\tau) d\tau$  and integrate (7), obtaining

$$R(t) = f(s(t)) - f(0).$$

Define the function  $g$  implicitly by

$$g(-\text{sign}(x)\sqrt{f(x) - f(0)}) = x.$$

Since we assume that

$$(9) \quad f(x) - f(0) = \beta^2 x^2 + O(|x|^3), \quad |x| \rightarrow 0$$

( $\beta^2 = f''(0)/2$ ),  $g$  is well defined for small  $|x|$  and

$$(10) \quad g(x) = -\beta^{-1}x + O(|x|^2), \quad |x| \rightarrow 0.$$

For a small interval  $[0, T]$ , the monotone decreasing solution of (7) is given by

$$(11) \quad s(t) = g(\sqrt{R(t)}), \quad 0 \leq t \leq T,$$

and (3) follows from (8) and (10).

The Theorem describes the regularity of the free boundary at  $t = 0$ . It is sharp in the sense that, unless  $f'''(0) = 0$ , the estimate (8) does not hold for  $\alpha > 1/2$  (cf. the Remark at the end of §3).

It should also be observed that the second derivatives of the solution  $u$  are not continuous at the point  $(x, t) = (0, 0)$ , because using (6), (7) and the definition of  $\kappa$  one has

$$\lim_{t \rightarrow 0^+} u_{xx}(s(t)^+, t) = r(0) = \frac{\kappa^2}{2} f''(0) \neq \lim_{x \rightarrow 0^+} u_{xx}(x, 0) = f''(0).$$

However, on the set  $\{t: f'(s(t)) < 0\}$  the free boundary  $s$  is as smooth as the function  $f$ . This can be shown by a bootstrap argument, using standard regularity results for the heat equation on a domain with curved boundaries. We believe that the Theorem can be extended to a general monotone constitutive function  $\phi$  with  $\phi'$  discontinuous at 0 and with  $\phi'(\xi) \geq c > 0$ ,  $\xi \in \mathbf{R}_+$ ; the corresponding value of  $\kappa$  will depend on  $\phi'(0^+)$ .

The Theorem is proved in §3 by solving an integral equation for the function  $r$  derived in §2.

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**2. The integral equation for the free boundary.** Let

$$\Gamma(x, t) := \frac{1}{2\sqrt{\pi}} t^{-1/2} \exp\left(-\frac{x^2}{4t}\right)$$

denote the fundamental solution of the heat equation. Let  $v := u_x$  be the solution of the problem

$$(4a') \quad \begin{cases} v_t = v_{xx}, & (x, t) \in \Omega_T := \{(x, t): x > s(t), t \in (0, T)\}, \\ v(s(t), t) = 0, \\ v(\cdot, 0) = f' \end{cases}$$

and assume that the free boundary  $s$  satisfies  $s \in C[0, T] \cap C^1(0, T]$ . Integrating Green's identity

$$\frac{\partial}{\partial \xi} \left( \Gamma(x - \xi, t - \tau) v_\xi(\xi, \tau) - \frac{\partial}{\partial \xi} \Gamma(x - \xi, t - \tau) v(\xi, \tau) \right) - \frac{\partial}{\partial \tau} (\Gamma(x - \xi, t - \tau) v(\xi, \tau)) = 0$$

over the domain  $\Omega_t$  we obtain, for  $x > s(t)$ , the representations

$$(12) \quad v(x, t) = \int_0^\infty \Gamma(x - \xi, t) f'(\xi) d\xi - \int_0^t \Gamma(x - s(\tau), t - \tau) v_\xi(s(\tau), \tau) d\tau,$$

(13)

$$v_x(x, t) = \int_0^\infty \Gamma(x - \xi, t) f''(\xi) d\xi - \int_0^t \Gamma_x(x - s(\tau), t - \tau) v_\xi(s(\tau), \tau) d\tau.$$

Passing to the limit  $x \rightarrow s(t)^+$  in (13) yields

$$(14) \quad r(t) = 2 \int_0^\infty \Gamma(s(t) - \xi, t) f''(\xi) d\xi - 2 \int_0^t \Gamma_x(s(t) - s(\tau), t - \tau) r(\tau) d\tau,$$

where (see (6) and (7))  $r(t) = df(s(t))/dt = v_x(s(t), t)$ . The justification for this passage to the limit is contained in the following result.

LEMMA 1. *If  $s \in C([0, T]) \cap C^1((0, T])$  and  $r \in C([0, T])$ , we have for  $t < T$*

$$\lim_{x \searrow s(t)} \int_0^t [\Gamma_x(s(t) - s(\tau), t - \tau) - \Gamma_x(x - s(\tau), t - \tau)] r(\tau) d\tau = \frac{1}{2} r(t).$$

PROOF. We write

$$\begin{aligned} \int_0^t [\dots] r(\tau) d\tau &= \frac{1}{4\sqrt{\pi}} \int_0^t \frac{s(t) - s(\tau)}{(t - \tau)^{3/2}} \left[ \exp\left(-\frac{(x - s(\tau))^2}{4(t - \tau)}\right) \right. \\ &\quad \left. - \exp\left(-\frac{(s(t) - s(\tau))^2}{4(t - \tau)}\right) \right] r(\tau) d\tau \\ &+ \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x - s(t)}{(t - \tau)^{3/2}} \left[ \exp\left(-\frac{(x - s(\tau))^2}{4(t - \tau)}\right) - \exp\left(-\frac{(x - s(t))^2}{4(t - \tau)}\right) \right] r(\tau) d\tau \\ &+ \frac{1}{4\sqrt{\pi}} \int_0^t \frac{x - s(t)}{(t - \tau)^{3/2}} \exp\left(-\frac{(x - s(t))^2}{4(t - \tau)}\right) r(\tau) d\tau =: \sum_{\nu=1}^3 \int_0^t I_\nu. \end{aligned}$$

In view of the assumptions on  $s$  and  $r$  it is easy to see that, for  $\nu = 1, 2$ ,

$$\left| \int_0^t I_\nu \right| \leq \left| \int_{t-\delta}^t I_\nu \right| + \left| \int_0^{t-\delta} I_\nu \right| \leq O(\sqrt{\delta}) + c_\delta o(|x - s(t)|)$$

which implies that

$$\lim_{x \searrow s(t)} \int_0^t I_\nu = 0, \quad \nu = 1, 2.$$

Finally,

$$\frac{1}{4\sqrt{\pi}} \int_{-\infty}^t \frac{y}{(t - \tau)^{3/2}} \exp\left(-\frac{y^2}{4(t - \tau)}\right) d\tau = \frac{1}{2}$$

implies that

$$\lim_{x \searrow s(t)} \int_0^t I_3 = \frac{1}{2} r(t).$$

**3. Proof of the Theorem.** We write the integral equation (14) in the form

$$\begin{aligned}
 (15) \quad r(t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\frac{1}{4}\left(\frac{s(t)}{\sqrt{t}} - \xi\right)^2\right) f''(\xi\sqrt{t}) \, d\xi \\
 &\quad + \frac{1}{\sqrt{\pi}} \int_0^1 \frac{A(s, t, \tau)}{1 - \tau} \exp(-A(s, t, \tau)^2) r(t\tau) \, d\tau \\
 &=: (Fr)(t) + (Kr)(t),
 \end{aligned}$$

where

$$A(s, t, \tau) := \frac{s(t) - s(t\tau)}{2(t - t\tau)^{1/2}}.$$

It will be convenient to introduce the class of functions  $\mathfrak{C}^\alpha[0, T]$ ,  $0 < \alpha < 1$ , defined by

$$\mathfrak{C}^\alpha[0, T] = \left\{ \rho: [0, T] \rightarrow \mathbf{R} : |\rho|_\alpha := \sup_{0 < t \leq T} t^{1-\alpha} |\rho'(t)| < \infty \right\}.$$

The class  $\mathfrak{C}^\alpha$  is obviously contained in the Hölder-class with exponent  $\alpha$ .

The Theorem is a consequence of

**PROPOSITION.** *For any  $\alpha \in (0, 1/2)$ , the integral equation (15), with  $s$  related to  $r$  by (7), has a solution  $r \in \mathfrak{C}^\alpha[0, T]$  for some  $T > 0$ . The constant  $\kappa := \sqrt{r(0)} / \beta$  ( $\beta^2 = \frac{1}{2}f''(0)$ ) does not depend on  $f$  and is implicitly determined by the equation*

$$\begin{aligned}
 (16) \quad &\frac{4}{\sqrt{\pi}} \int_0^\infty \exp\left(-\left(\frac{\kappa}{2} + \xi\right)^2\right) \, d\xi \\
 &= \kappa^2 \left( 1 + \frac{1}{\sqrt{\pi}} \int_0^1 \frac{\kappa}{2} \frac{1}{\sqrt{1-\tau}(1+\sqrt{\tau})} \exp\left(-\frac{\kappa^2}{4} \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) \, d\tau \right);
 \end{aligned}$$

the numerical value of  $\kappa$  is 0.9034...

**REMARK.** The Proposition does not assert uniqueness of the function  $r$  (hence of the free boundary  $s$ ) which could be established by showing that the operator  $F + K$  in (15) is a strict contraction; this is technically even more complicated than our proof. However, the uniqueness of  $r$  is a consequence of the uniqueness of solutions of the original problem (1) discussed in the Introduction.

We prove the Proposition by iterating the integral equation (15) in the form

$$(17) \quad r_{n+1} = Fr_n + Kr_n, \quad n \in \mathbf{N},$$

with  $r(0) = \kappa^2\beta^2$ , where  $\kappa$  is the solution of (16) and  $\beta^2 = \frac{1}{2}f''(0)$ .

We shall show as a consequence of Lemmas 2 and 3 that, for  $r \in \mathfrak{C}^\alpha$  with  $r(0) = \kappa^2\beta^2$ ,

$$(18) \quad \lim_{t \rightarrow 0^+} (Fr)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\frac{1}{4}(\kappa + \xi)^2\right) 2\beta^2 \, d\xi,$$

$$(19) \quad \lim_{t \rightarrow 0^+} (Kr)(t) = -\frac{1}{\sqrt{\pi}} \int_0^1 \frac{\kappa}{2} \frac{1}{\sqrt{1-\tau}(1+\sqrt{\tau})} \exp\left(-\frac{\kappa^2}{4} \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) \kappa^2\beta^2 \, d\tau.$$

Since  $\kappa$  is the solution of (16), this implies that  $r_n(0) = \kappa^2\beta^2$  for  $n \in \mathbf{N}$ .

Moreover, we shall establish the a priori estimates: for  $r \in \mathcal{C}^\alpha[0, T], 0 < \alpha < 1/2,$

$$(20) \quad |Fr|_\alpha \leq c(T) + (c_1(\alpha) + c(T))|r|_\alpha,$$

where  $c_1(\alpha) = (\kappa\sqrt{\pi}(1 + \alpha))^{-1} \exp(-\frac{1}{4}\kappa^2),$  and

$$(21) \quad |Kr|_\alpha \leq (c_2(\alpha) + c(T))|r|_\alpha,$$

where  $c_2(\alpha) = c_{21}(\alpha) + c_{22}(\alpha)$  with

$$c_{21}(\alpha) = \frac{\kappa}{2\sqrt{\pi}} \int_0^1 \frac{\tau^\alpha}{\sqrt{1-\tau}(1+\sqrt{\tau})} \exp\left(-\frac{\kappa^2}{4} \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) d\tau,$$

$$c_{22}(\alpha) = \frac{\kappa(1+1/(2+2\alpha))}{\sqrt{\pi}(2+4\alpha)} \int_0^1 \frac{1-\tau^{1/2+\alpha}}{(1-\tau)^{3/2}} \left(1 - \frac{\kappa^2}{2} \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) \times \exp\left(-\frac{\kappa^2}{4} \frac{1-\sqrt{\tau}}{1+\sqrt{\tau}}\right) d\tau,$$

and where  $c(T)$  is a constant such that  $c(T) \rightarrow 0$  as  $T \rightarrow 0^+,$  uniformly for  $r \in \{\rho: |\rho(0)| + |\rho|_\alpha \leq \text{const}\}.$

We first use the estimates (20), (21) to complete the proof of the Proposition. Combining the estimates (20) and (21) one has

$$(22) \quad |r_{n+1}|_\alpha \leq c(T) + (c_1(\alpha) + c_2(\alpha) + c(T))|r_n|_\alpha.$$

Crucial for the following argument is the fact that

$$c_1\left(\frac{1}{2}\right) + c_2\left(\frac{1}{2}\right) = 0.339\dots + 0.453\dots =: \omega < 1.$$

Set  $\bar{\omega} := (1 + \omega)/2 < 1$  and choose  $\alpha \in (0, 1/2)$  close to  $1/2$  and  $T > 0$  such that for all  $r \in \mathcal{C}^\alpha$  with  $r(0) = \kappa^2\beta^2$  and  $|r|_\alpha \leq 1/(1 - \bar{\omega})$

$$c_1(\alpha) + c_2(\alpha) + c(T) \leq \bar{\omega}.$$

It should be observed that if one chooses  $\alpha \geq 1/2$  then one cannot prove the crucial estimate (20), cf. e.g. (24). By (22), we have

$$|r_n|_\alpha \leq 1/(1 - \bar{\omega}), \quad n \in \mathbf{N}.$$

Hence we can select a subsequence of  $r_n$  which converges in  $C[0, T]$  to a function  $r_\infty \in \mathcal{C}^\alpha[0, T]$  with  $r_\infty(0) = \kappa^2\beta^2.$  Set  $s_n := g(\sqrt{R_n}).$  To pass to the limit in (17) note that by Lemmas 2 and 3 the expressions  $\exp(-\frac{1}{4}(s_n(t)/\sqrt{t} - \xi)^2)$  and  $(A(s_n, t, \tau)/(1 - \tau))\exp(-A(s_n, t, \tau)^2)$  converge pointwise (for  $n \rightarrow \infty$ ) and are majorized by integrable functions, uniformly in  $n \in \mathbf{N}.$  This completes the proof of the Proposition and the Theorem.

It remains to establish the assertions (18)–(21). We require two auxiliary results. We denote by  $c$  a generic constant which may depend on  $\alpha, |r|_\alpha$  and  $T,$  and we assume throughout that  $T = T(|r|_\alpha, \alpha)$  is sufficiently small.

**LEMMA 2.** For  $r \in \mathcal{C}^\alpha, \alpha \in (0, 1/2),$  with  $r(0) = \kappa^2\beta^2$  we have

$$|s(t) + \kappa\sqrt{t}| \leq ct^{1/2+\alpha}.$$

PROOF. Note that  $|r(t) - r(0)| \leq ct^\alpha$  and therefore  $|R(t) - r(0)t| \leq ct^{1+\alpha}$ . Using (10), (11) and this inequality one has

$$|s(t) + \kappa\sqrt{t}| = |g(\sqrt{R(t)}) + \kappa\sqrt{t}| \leq |-\beta^{-1}\sqrt{R(t)} + \kappa\sqrt{t}| + ct \leq ct^{1/2+\alpha} + ct.$$

LEMMA 3. For  $r \in \mathcal{H}^\alpha$  with  $r(0) = \kappa^2\beta^2$  we have

$$|A(s, t, \tau)| \leq c \frac{1 - \sqrt{\tau}}{\sqrt{1 - \tau}} = c \frac{\sqrt{1 - \tau}}{1 + \sqrt{\tau}}.$$

PROOF. Using  $f'(s)s' = r$ , (9) and Lemma 2, we obtain

$$|s(t) - s(t\tau)| = \left| \int_{t\tau}^t \frac{r(\sigma)}{f'(s(\sigma))} d\sigma \right| \leq c \int_{t\tau}^t (2\beta^2\kappa\sqrt{\sigma} - c\sigma^{1/2+\alpha})^{-1} d\sigma \leq c(\sqrt{t} - \sqrt{t\tau});$$

this establishes the claim by the definition of  $A(s, t, \tau)$ .

Lemma 3 shows that the kernel corresponding to the operator  $K$  in (15) is integrable. Moreover, we see from Lemma 2 that

$$(23) \quad A_0(\tau) := \lim_{t \rightarrow 0^+} A(s, t, \tau) = -\frac{\kappa}{2} \frac{1 - \sqrt{\tau}}{\sqrt{1 - \tau}}.$$

Using this and Lemma 2, we can pass to the limit in (15), thus establishing (18) and (19).

PROOF OF (20). To estimate the norm of  $Fr$ , use the definition in (15) to form

$$\begin{aligned} \frac{d(Fr)(t)}{dt} &= \frac{1}{\sqrt{\pi}} \int_0^\infty \exp\left(-\frac{1}{4}\left(\frac{s(t)}{\sqrt{t}} - \xi\right)^2\right) \frac{1}{2} t^{-1/2} \xi f'''(\xi\sqrt{t}) d\xi \\ &\quad - \left[ \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{2} \left(\frac{s(t)}{\sqrt{t}} - \xi\right) \exp\left(-\frac{1}{4}\left(\frac{s(t)}{\sqrt{t}} - \xi\right)^2\right) f''(\xi\sqrt{t}) d\xi \right] \times \frac{d}{dt} \frac{s(t)}{\sqrt{t}}. \end{aligned}$$

As  $t \searrow 0$ , the term in square brackets tends (use (9)) to

$$\begin{aligned} \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{2} (-\kappa - \xi) \exp\left(-\frac{1}{4}(-\kappa - \xi)^2\right) 2\beta^2 d\xi \\ = -\frac{2}{\sqrt{\pi}} \beta^2 \exp\left(-\frac{1}{4}\kappa^2\right) = 2(1 + \alpha)\kappa\beta^2 c_1(\alpha). \end{aligned}$$

Therefore,

$$(24) \quad \left| \frac{d(Fr)(t)}{dt} \right| \leq ct^{-1/2} + (2(1 + \alpha)\kappa\beta^2 c_1(\alpha) + \bar{c}(t)) \left| \frac{d}{dt} \left( \frac{s(t)}{\sqrt{t}} \right) \right|.$$

It remains to estimate  $(d/dt)(s(t)/\sqrt{t})$ . Using (7), Lemma 2, (9) and (10) we have

$$\begin{aligned} \left| \frac{s'(t)}{\sqrt{t}} - \frac{1}{2} \frac{s(t)}{t^{3/2}} \right| &= t^{-3/2} \left| \frac{1}{f'(s(t))} \right| \left| tr(t) - \frac{1}{2}s(t)f'(s(t)) \right| \\ &\leq t^{-3/2} t^{-1/2} \left( \frac{1}{2}\beta^{-2}\kappa^{-1} + \bar{c}(t) \right) \left[ |tr(t) - \beta^2 s(t)^2| + \bar{c}t^{3/2} \right] \\ &\leq t^{-1} \left( \frac{1}{2}\beta^{-2}\kappa^{-1} + \bar{c}(t) \right) \left[ |tr(t) - R(t)| + \bar{c}t^{3/2} \right]. \end{aligned}$$



A simple calculation shows that

$$(25) \quad |tr(t) - R(t)| \leq \frac{1}{1 + \alpha} t^{1+\alpha} |r|_{\alpha},$$

and this yields

$$(26) \quad \left| \frac{d}{dt} \left( \frac{s(t)}{\sqrt{t}} \right) \right| \leq t^{\alpha-1} \left( \frac{1}{2} \beta^{-2\kappa-1} + \bar{c}(t) \right) \left( \frac{1}{1 + \alpha} + \bar{c}(t) \right) |r|_{\alpha}.$$

Combining (24) and (26) proves (20).

We next turn to the *proof of (21)*. We write (cf. (15))

$$\begin{aligned} \frac{d}{dt}(Kr)(t) &= \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{1 - \tau} A \exp(-A^2) \tau r'(t\tau) d\tau \\ &\quad + \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1}{1 - \tau} (1 - 2A^2) \exp(-A^2) \frac{dA}{dt} r(t\tau) d\tau \\ &=: (K_1r)(t) + (K_2r)(t) \end{aligned}$$

and estimate each term separately.

(i) Since  $|r'(t\tau)| \leq (t\tau)^{\alpha-1} |r|_{\alpha}$ , it follows from (23) that

$$(27) \quad |(K_1r)(t)| \leq (c_{21} + c(t)) t^{\alpha-1} |r|_{\alpha}.$$

(ii) To estimate  $K_2r$  we first consider the term  $dA(s, t, \tau)/dt$ . Using the definition of  $A$  and (7), we obtain

$$\begin{aligned} &2(t - t\tau)^{1/2} t \frac{d}{dt} \left( \frac{1}{2} \frac{s(t) - s(t\tau)}{(t - t\tau)^{1/2}} \right) \\ &= ts'(t) - (t\tau)s'(t\tau) - \frac{1}{2}s(t) + \frac{1}{2}s(t\tau) \\ &= \int_{t\tau}^t \frac{d}{d\sigma} \left( \sigma s'(\sigma) - \frac{1}{2}s(\sigma) \right) d\sigma \\ &= \int_{t\tau}^t \left( \frac{1}{2}s'(\sigma) + \sigma \frac{d}{d\sigma} \left( \frac{r(\sigma)}{f'(s(\sigma))} \right) \right) d\sigma, \end{aligned}$$

i.e.

$$(28) \quad \frac{dA}{dt} = \frac{1}{2} t^{-1} (t - t\tau)^{-1/2} \int_{t\tau}^t (Q_1(\sigma) + Q_2(\sigma)) d\sigma$$

with

$$\begin{aligned} Q_1(\sigma) &:= \sigma \frac{r'(\sigma)}{f'(s(\sigma))}, \\ Q_2(\sigma) &:= s'(\sigma) \left( \frac{1}{2} - \sigma \frac{r(\sigma)f''(s(\sigma))}{f'(s(\sigma))^2} \right). \end{aligned}$$

We estimate each term separately. By Lemma 2 and (9), we have

$$(29) \quad \int_{t\tau}^t |Q_1(\sigma)| d\sigma \leq \int_{t\tau}^t \sigma \frac{\sigma^{\alpha-1} |r|_\alpha}{2\beta^2 \kappa \sqrt{\sigma} - c\sigma^{1/2+\alpha}} d\sigma$$

$$\leq \left( \frac{1}{2} \frac{1}{1/2 + \alpha} \beta^{-2\kappa^{-1}} + c(t) \right) t^{1/2+\alpha} (1 - \tau^{1/2+\alpha}) |r|_\alpha.$$

We write  $Q_2$  in the form

$$Q_2(\sigma) = \frac{r(\sigma)}{f'(s(\sigma))^3} \left( \frac{1}{2} \left( f'(g(\sqrt{R(\sigma)})) \right)^2 - \sigma r(\sigma) f''(s(\sigma)) \right).$$

Since by (9), (10) and Lemma 2,

$$\left. \begin{aligned} & \left| \frac{1}{2} \left( f'(g(\sqrt{R(\sigma)})) \right)^2 - 2\beta^2 R(\sigma) \right| \\ & \left| \sigma r(\sigma) f''(s(\sigma)) - 2\beta^2 \sigma r(\sigma) \right| \end{aligned} \right\} \leq c\sigma^{3/2},$$

we obtain, also using (25),

$$(30) \quad \int_{t\tau}^t |Q_2(\sigma)| d\sigma \leq \int_{t\tau}^t \frac{r(0)}{(2\beta^2 \kappa \sqrt{\sigma})^3} (1 + c(\sigma)) \left( 2\beta^2 \frac{1}{1 + \alpha} \sigma^{1+\alpha} |r|_\alpha \right) d\sigma$$

$$\leq \left( \frac{1}{4} \frac{1}{(1 + \alpha)(1/2 + \alpha)} \beta^{-2\kappa^{-1}} + c(t) \right) t^{1/2+\alpha} (1 - \tau^{1/2+\alpha}) |r|_\alpha.$$

Combining (29) and (30) with (28), it follows that

$$(31) \quad |(K_2 r)(t)| \leq \frac{1}{4} \frac{1}{1/2 + \alpha} \beta^{-2\kappa^{-1}} \left( 1 + \frac{1}{2} \frac{1}{1 + \alpha} \right) (1 + c(t)) t^{\alpha-1} |r|_\alpha$$

$$\times \frac{1}{\sqrt{\pi}} \int_0^1 \frac{1 - \tau^{1/2+\alpha}}{(1 - \tau)^{3/2}} (1 - 2A_0(\tau)^2) \exp(-A_0(\tau)^2) (1 + c(t)) r(0) d\tau.$$

Adding the estimates (27) and (31) proves (21).

**REMARK.** We conjecture that, for smooth initial data  $f$ , the function  $r(t^2)$  is smooth, i.e.

$$(32) \quad r(t) = \kappa^2 \beta^2 + r_{1/2} \sqrt{t} + r_1 t + \dots$$

Assuming an expansion of the form (32), we can calculate the coefficients  $r_{1/2}, r_1, \dots$  from the integral equation (15). In particular,  $f'''(0) \neq 0$  implies that  $r_{1/2} \neq 0$ . This shows that (8) is, in general, not valid for  $\alpha > 1/2$ .

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MATHEMATICS RESEARCH CENTER, UNIVERSITY OF WISCONSIN, MADISON, WISCONSIN 53706