

A NONLINEAR HEAT EQUATION WITH SINGULAR INITIAL DATA

By

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1. Introduction

We consider the question of local existence, for the model problem

$$(1) \quad \begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } (0, T) \times \Omega, \\ u = 0 & \text{on } (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain and $p > 1$. It is well-known that if $u_0 \in L^\infty(\Omega)$, there is a unique solution defined on a maximal interval $[0, T_{\max})$. This solution satisfies:

- (2)
 - u is a classical solution of (1) on $(0, T_{\max}) \times \bar{\Omega}$
(u is C^1 in $t \in (0, T_{\max})$ and C^2 in $x \in \bar{\Omega}$),
- (3)
 - $u \in L^\infty((0, T) \times \Omega)$ for all $T < T_{\max}$,
- (4)
 - $\|u(t) - T(t)u_0\|_{L^\infty} \rightarrow 0$ as $t \downarrow 0$,

where $(T(t))_{t \geq 0}$ is the linear heat flow (see e.g. [6]).

We are concerned with the question of what happens if $u_0 \notin L^\infty(\Omega)$. More precisely, we assume that $u_0 \in L^q(\Omega)$ for some $1 \leq q < \infty$. This type of problem was first considered by F. B. Weissler [17, 18] who has obtained interesting existence and uniqueness results. Further results have been obtained by Y. Giga [12] and by W.-M. Ni and P. Sacks [16]. Our purpose is to shed some new light on the problem; in particular, we prove uniqueness in a larger, more natural, class (see the precise statements in Theorems 1 and 4 below).

The value

$$q = \frac{N(p-1)}{2}$$

plays a critical role, and one has to distinguish two cases:

Case 1: $q \geq N(p-1)/2$.

Case 2: $q < N(p-1)/2$.

Roughly speaking, in case 1 we obtain the existence and uniqueness of a local solution for any $u_0 \in L^q(\Omega)$. In case 2, it seems that there exists no local solution in any reasonable sense for some initial conditions $u_0 \in L^q(\Omega)$ (see Weissler [18, 19] and Section 7.6 below).

Our main existence and uniqueness result is the following.

Theorem 1 *Assume $q > N(p-1)/2$ (resp. $q = N(p-1)/2$) and $q \geq 1$ (resp. $q > 1$), $N \geq 1$. Given any $u_0 \in L^q(\Omega)$, there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0, T], L^q(\Omega))$ with $u(0) = u_0$, which is a classical solution of (1) on $(0, T) \times \bar{\Omega}$ (in the sense of (2)).*

Moreover, we have:

(i) *Smoothing effect and continuous dependence, namely*

$$(5) \quad \|u(t) - v(t)\|_{L^q} + t^{N/2q} \|u(t) - v(t)\|_{L^\infty} \leq C \|u_0 - v_0\|_{L^q},$$

for all $t \in (0, T]$ where $T = \min\{T(u_0), T(v_0)\}$ and C can be estimated in terms of $\|u_0\|_{L^q}$ and $\|v_0\|_{L^q}$.

$$(ii) \quad \lim_{t \downarrow 0} t^{N/2q} \|u(t)\|_{L^\infty} = 0.$$

(iii) *If $u_0 \geq 0$, then $u(t) \geq 0$ for all $t \in [0, T(u_0)]$.*

Furthermore, for any bounded set (resp. compact set) \mathcal{K} in $L^q(\Omega)$, there is a (uniform) time $T = T(\mathcal{K})$ such that for any $u_0 \in \mathcal{K}$ the solution of (1) exists on $[0, T]$.

The uniqueness part in Theorem 1 seems totally new. Previously, many people have established uniqueness results for nonlinear evolution equations with singular initial conditions, in particular the Navier-Stokes and the Euler equations (see e.g. Kato and Fujita [15], Kato [14], Ben-Artzi [4], Weissler [18]). In all these works it is assumed that $u \in L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$ and **also** that $\lim_{t \downarrow 0} t^\alpha \|u(t)\|_{L^\infty} = 0$ for some appropriate $\alpha > 0$. Our main point is that such an assumption is redundant. A similar observation was first made in [5].

Remark 2 Since u is a classical solution on $(0, T) \times \bar{\Omega}$, the usual blow up alternative holds: either $T_{\max} = +\infty$ or else $T_{\max} < \infty$ and $\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^\infty} = +\infty$.

Remark 3 The “doubly critical” case, $q = N(p-1)/2$ and $q = 1$, in Theorem 1 is delicate and widely open (see Section 7.5). For example, when $N = 1$, the very simple equation

$$u_t - u_{xx} = u^3,$$

with an initial condition $u_0 \in L^1(\Omega)$, enters in this category.

It seems that for some $u_0 \in L^1(\Omega)$ there is not even a local solution. See Open Problem 2. We are, at least, able to find some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that equation (1) has no **nonnegative** solution $u \in C([0, T], L^1(\Omega)) \cap L^\infty_{loc}((0, T), L^\infty(\Omega))$. See Section 7.5 below.

When $q \geq p$, it makes sense to talk about weak solutions $u \in C([0, T], L^q(\Omega))$ in the integral sense, i.e.

$$(6) \quad u(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s) ds,$$

for all $t \in [0, T]$. Uniqueness holds in that class:

Theorem 4 *Assume $q > N(p-1)/2$ (resp. $q = N(p-1)/2$) and $q \geq p$ (resp. $q > p$), $N \geq 1$. Then uniqueness for (6) holds in the class $C([0, T], L^q(\Omega))$.*

The conclusion of Theorem 4 (at least when $q > N(p-1)/2$ and $q > p$) is essentially contained in a work of Weissler [17].

Remark 5 In the “doubly critical” case $q = N(p-1)/2$ and $q = p$, i.e. $q = p = N/(N-2)$ with $N \geq 3$, the conclusion of Theorem 4 fails, i.e. uniqueness fails in the class $C([0, T], L^q(\Omega))$. See Ni and Sacks [16] and Section 7.4 below.

Remark 6 The solution u of (1) given in Theorem 1 also satisfies (6); here, there is no restriction about q except for the assumptions of Theorem 1. This is not completely obvious since the integral on the right hand side of (6) need not be well-defined. To establish the convergence of this integral, we rely on the smoothing effect (5). Clearly we have

$$(7) \quad u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma) d\sigma,$$

for all $0 < s < t < T$. We let $s \downarrow 0$ in (7); to justify this passage to the limit it suffices to check that

$$\int_0^t \|T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\|_{L^q} d\sigma < \infty.$$

The only difficulty is when σ is near 0. But

$$\|T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\|_{L^q} \leq (t-\sigma)^{-\frac{N}{2}(\frac{q-1}{q})} \|u(\sigma)\|_{L^p}^p.$$

(See e.g. Lemma 7 below.) We may always assume that $p > q$ (the case $q \geq p$ has been handled above); and so,

$$\|u(\sigma)\|_{L^p}^p \leq \|u(\sigma)\|_{L^q}^q \|u(\sigma)\|_{L^\infty}^{p-q} \leq C\sigma^{-\frac{N(p-q)}{2q}}.$$

The result follows, since $\frac{N(p-q)}{2q} = 1 - \frac{1}{q} \left(q - \frac{N(p-1)}{2} \right) - \frac{N}{2q}(q-1) < 1$.

In several places, it is convenient to view the nonlinear equation (1) as a linear problem

$$u_t - \Delta u = au,$$

and we have collected in the Appendix some useful facts about this linear heat equation with a potential.

The plan of the paper is the following. In Section 2, we prove Theorem 4. In Sections 3 and 4, we prove the existence part of Theorem 1 in the super-critical and critical cases. In Section 5 we prove the uniqueness part of Theorem 1, and in Section 6 we prove the stability and continuous dependence (properties (i)–(iii)). Finally, Section 7 is devoted to additional results and open problems.

2. Proof of Theorem 4

We will frequently use the standard smoothing effect of $(T(t))_{t \geq 0}$:

Lemma 7 *Let $1 \leq \beta \leq \gamma \leq \infty$. Then*

$$(8) \quad \|T(t)\varphi\|_{L^\gamma(\Omega)} \leq \frac{1}{t^{\frac{N}{2}(\frac{1}{\beta} - \frac{1}{\gamma})}} \|\varphi\|_{L^\beta(\Omega)},$$

for all $t > 0$ and all $\varphi \in L^\beta(\Omega)$.

We consider separately two cases:

Case A: $q > N(p-1)/2$ and $q \geq p$,

Case B: $q = N(p-1)/2$ and $q > p$.

Case A Let u and v be two solutions, $u, v \in C([0, T], L^q(\Omega))$. We have

$$(9) \quad u(t) - v(t) = \int_0^t T(t-s) (|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)) ds.$$

Thus, by the smoothing effect of $T(t) : L^{q/p}(\Omega) \rightarrow L^q(\Omega)$,

$$\begin{aligned} \|u(t) - v(t)\|_{L^q} &\leq C \int_0^t (t-s)^{-\alpha} \|(|u|^{p-1} + |v|^{p-1})|u - v|\|_{L^{q/p}} ds \\ &\leq C \int_0^t (t-s)^{-\alpha} (\|u\|_{L^q}^{p-1} + \|v\|_{L^q}^{p-1}) \|u - v\|_{L^q} ds, \end{aligned}$$

where $\alpha = N(p-1)/2q < 1$ since we are in case A. Let

$$M = \sup_{0 \leq t \leq T} \|u(t)\|_{L^q} + \|v(t)\|_{L^q} \quad \text{and} \quad \psi(t) = \sup_{0 \leq s \leq t} \|u(s) - v(s)\|_{L^q},$$

for $t \in [0, T]$. We deduce that

$$\psi(t) \leq CM^{p-1} \frac{T^{1-\alpha}}{1-\alpha} \psi(t).$$

Hence $\psi(t) = 0$ for t sufficiently small. Repeating the same argument, we see that $\psi(t) = 0$ for $t \in [0, T]$.

Case B Note that $q = N(p - 1)/2 > p$, thus $N \geq 3$. Let u, v be two solutions and let $w = u - v$. We set

$$(10) \quad a(t, x) = \begin{cases} \frac{|u|^{p-1}u - |v|^{p-1}v}{u - v} & \text{if } u \neq v, \\ p|u|^{p-1} & \text{if } u = v, \end{cases}$$

so that

$$w(t) = \int_0^t T(t - s)a(s)w(s) ds.$$

We claim that

$$(11) \quad a \in C([0, T], L^{N/2}(\Omega)).$$

We may then apply Theorem A2 (in the Appendix at the end of this paper) to conclude that $w \equiv 0$. Note that (since we are in case B) $q > N/(N - 2)$. \square

Proof of (11) We have $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0, T), L^{N/2}(\Omega))$. We now establish (11) by contradiction. Otherwise, there exist $\varepsilon > 0$, $t \in [0, T]$ and a sequence $(t_n)_{n \geq 0} \in [0, T]$ such that $t_n \rightarrow t$ and $\|a(t_n, \cdot) - a(t, \cdot)\|_{L^{N/2}} \geq \varepsilon$. On the other hand, by possibly extracting a subsequence, we may assume that $u(t_n) \rightarrow u(t)$ and $v(t_n) \rightarrow v(t)$ in $L^q(\Omega)$ and almost everywhere, and that there exists $\varphi \in L^q(\Omega)$ such that $|u(t_n)| + |v(t_n)| \leq \varphi$ almost everywhere. It follows easily that $a(t_n) \rightarrow a(t)$ almost everywhere and that $|a(t_n)| \leq C|\varphi|^{p-1} \in L^{N/2}(\Omega)$. By dominated convergence, we deduce $a(t_n) \rightarrow a(t)$ in $L^{N/2}(\Omega)$, which is absurd. \square

3. Proof of the existence part in Theorem 1 when $q > N(p - 1)/2$ and $q \geq 1$

We first establish the existence of a solution

$$u \in L^\infty((0, T), L^q(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^{pq}(\Omega)).$$

We use the contraction mapping principle in a somewhat unusual space (this idea is due to F. B. Weissler [17]). Fix $M \geq \|u_0\|_{L^q}$ and let

$$E = L^\infty((0, T), L^q(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^{pq}(\Omega)),$$

and

$$K = K(T) = \{u \in E; \|u(t)\|_{L^q} \leq M + 1 \text{ and } t^\alpha \|u(t)\|_{L^{pq}} \leq M + 1 \text{ for } t \in (0, T)\},$$

with $\alpha = N(p-1)/2pq < 1/p < 1$. We equip K with the distance

$$d(u, v) = \sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^{pq}},$$

so that (K, d) is a nonempty complete metric space. Given $u \in K$, we set

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s) ds.$$

For $u \in K$, we have

$$\begin{aligned} \|\Phi(u)(t)\|_{L^q} &\leq \|u_0\|_{L^q} + \int_0^t \|u(s)\|_{L^{pq}}^p ds \\ &\leq \|u_0\|_{L^q} + \left(\sup_{0 < t < T} t^\alpha \|u(t)\|_{L^{pq}}\right)^p \int_0^t s^{-p\alpha} ds \\ &\leq \|u_0\|_{L^q} + \frac{T^{1-p\alpha}}{1-p\alpha} (M+1)^p. \end{aligned}$$

Next,

$$\begin{aligned} t^\alpha \|\Phi(u)(t)\|_{L^{pq}} &\leq \|u_0\|_{L^q} + t^\alpha \int_0^t (t-s)^{-\alpha} \|u(s)\|_{L^{pq}}^p ds \\ &\leq \|u_0\|_{L^q} + t^\alpha (M+1)^p \int_0^t (t-s)^{-\alpha} s^{-p\alpha} ds \\ &\leq \|u_0\|_{L^q} + T^{1-p\alpha} (M+1)^p \int_0^1 (1-\sigma)^{-\alpha} \sigma^{-p\alpha} d\sigma. \end{aligned}$$

Similarly, one shows that for $u, v \in K$,

$$t^\alpha \|\Phi(u)(t) - \Phi(v)(t)\|_{L^{pq}} \leq CT^{1-p\alpha} (M+1)^{p-1} d(u, v).$$

It follows from the above estimates that if T is small enough (depending on M), then $\Phi : K \rightarrow K$ is a strict contraction. Thus Φ has a unique fixed point in K .

To complete the argument, it suffices to show that $u \in C([0, T], L^q(\Omega)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$ (once $u \in L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$, it must be a classical solution on $(0, T) \times \bar{\Omega}$). Since $u \in K$ and $p\alpha < 1$, we have $|u|^{p-1}u \in L^1((0, T), L^q(\Omega))$. This implies that $u \in C([0, T], L^q(\Omega))$. (Recall that, in a general setting, if $f \in L^1((0, T), X)$ and $u(t) = \int_0^t T(t-s)f(s) ds$, then $u \in C([0, T], X)$.)

Next, we prove that $u \in L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$. Indeed, we have $u \in L^\infty_{\text{loc}}((0, T), L^{pq}(\Omega))$. Therefore, we may apply Theorem A1 in the Appendix (with $r = pq$ and $\sigma = pq/(p - 1)$) on every interval $(\varepsilon, T - \varepsilon)$, with $a = |u|^{p-1}$.

Note that the choice of T depends only on M . This establishes the last assertion in Theorem 1. □

4. Proof of the existence part in Theorem 1 when $q = N(p - 1)/2$ and $q > 1$

We will use the following lemma.

Lemma 8 *Given a compact set $\mathcal{K} \subset L^q(\Omega)$ and $q < r \leq \infty$, there exists a function $\gamma : (0, 1] \rightarrow (0, \infty)$ with*

$$\lim_{t \downarrow 0} \gamma(t) = 0,$$

such that

$$t^\alpha \|T(t)u_0\|_{L^r} \leq \gamma(t),$$

for all $t \in (0, 1)$ and all $u_0 \in \mathcal{K}$, where $\alpha = \frac{N}{2} \left(\frac{1}{q} - \frac{1}{r} \right)$.

Proof If \mathcal{K} is reduced to a single point u_0 , the result is clear. Indeed, for any $v_0 \in L^\infty(\Omega)$

$$\begin{aligned} t^\alpha \|T(t)u_0\|_{L^r} &\leq t^\alpha \|T(t)(u_0 - v_0)\|_{L^r} + t^\alpha \|T(t)v_0\|_{L^r} \\ &\leq \|u_0 - v_0\|_{L^q} + Ct^\alpha \|v_0\|_{L^\infty}; \end{aligned}$$

and so

$$\limsup_{t \downarrow 0} t^\alpha \|T(t)u_0\|_{L^r} \leq \|u_0 - v_0\|_{L^q}.$$

The assertion follows since v_0 is arbitrary.

In the general case, given any $\rho > 0$, there is a finite covering of \mathcal{K} by balls $B(u_i, \rho)$ in $L^q(\Omega)$. Any $u_0 \in \mathcal{K}$ belongs to some $B(u_i, \rho)$, and we then write

$$\begin{aligned} t^\alpha \|T(t)u_0\|_{L^r} &\leq t^\alpha \|T(t)(u_0 - u_i)\|_{L^r} + t^\alpha \|T(t)u_i\|_{L^r} \\ &\leq \|u_0 - u_i\|_{L^q} + t^\alpha \|T(t)u_i\|_{L^r} \\ &\leq \rho + t^\alpha \|T(t)u_i\|_{L^r}. \end{aligned}$$

The conclusion of the lemma then follows from the first assertion. □

We now return to the proof of the theorem. The strategy is the same as in Section 3 with some minor technical differences. Fix any $r \in (q, pq)$, $r \geq p$, and set

$$\tilde{E} = L^\infty((0, T), L^q(\Omega)) \cap \{u \in L^\infty_{\text{loc}}((0, T), L^r(\Omega)); t^\alpha u \in L^\infty((0, T), L^r(\Omega))\},$$

and

$$E = L^\infty((0, T), L^q(\Omega)) \cap \{u \in L^\infty_{\text{loc}}((0, T), L^r(\Omega)); t^\alpha u \in C_0([0, T], L^r(\Omega))\},$$

with $\alpha = \frac{N}{2} \left(\frac{1}{q} - \frac{1}{r} \right) < \frac{1}{p} < 1$ (since $r < pq$). Here C_0 means that we consider functions which vanish at $t = 0$. Fix $M \geq \|u_0\|_{L^q}$. Given $\delta > 0$ to be chosen later, let

$$\tilde{K} = \tilde{K}(T) = \{u \in \tilde{E}; \|u(t)\|_{L^q} \leq M + 1 \text{ and } t^\alpha \|u(t)\|_{L^r} \leq \delta \text{ for } t \in (0, T)\},$$

and

$$K = K(T) = \tilde{K} \cap E.$$

We equip \tilde{K} with the distance

$$d(u, v) = \sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^r},$$

so that (\tilde{K}, d) and (K, d) are nonempty complete metric spaces. Consider the same mapping Φ as in Section 3. Let $a = \frac{N}{2} \left(\frac{p}{r} - \frac{1}{q} \right)$. For $u \in \tilde{K}$, we have by using the smoothing effect $L^{r/p} \rightarrow L^q$ (note that $r < pq$, so that $r/p < q$)

$$\begin{aligned} \|\Phi(u)(t)\|_{L^q} &\leq \|u_0\|_{L^q} + \int_0^t (t-s)^{-a} \|u(s)\|_{L^r}^p ds \\ (12) \quad &\leq \|u_0\|_{L^q} + \left(\sup_{0 < t < T} t^\alpha \|u(t)\|_{L^r} \right)^p \int_0^t (t-s)^{-a} s^{-p\alpha} ds \\ &\leq \|u_0\|_{L^q} + C_1 \delta^p, \end{aligned}$$

since $a + p\alpha = 1$. Here the constant C_1 (and the constants C_2, C_3 below) depends only on p, q, r, N . Therefore,

$$\|\Phi(u)(t)\|_{L^q} \leq M + 1,$$

provided

$$(13) \quad C_1 \delta^p \leq 1.$$

Next, using the smoothing effect $L^{r/p} \rightarrow L^r$, we have

$$\begin{aligned}
 t^\alpha \|\Phi(u)(t)\|_{L^r} &\leq \sup_{0 < t < T} t^\alpha \|T(t)u_0\|_{L^r} + t^\alpha \int_0^t (t-s)^{-N(p-1)/2r} \|u(s)\|_{L^r}^p ds \\
 &\leq \sup_{0 < t < T} t^\alpha \|T(t)u_0\|_{L^r} \\
 &\quad + \left(\sup_{0 < t < T} t^\alpha \|u(t)\|_{L^r} \right)^p t^\alpha \int_0^t (t-s)^{-N(p-1)/2r} s^{-p\alpha} ds \\
 &\leq \sup_{0 < t < T} t^\alpha \|T(t)u_0\|_{L^r} + C_2 \delta^p,
 \end{aligned}
 \tag{14}$$

since $p\alpha + N(p-1)/2r = \alpha + 1$. Therefore,

$$\sup_{0 < t < T} t^\alpha \|\Phi(u)(t)\|_{L^r} \leq \sup_{0 < t < T} t^\alpha \|T(t)u_0\|_{L^r} + \delta/2,
 \tag{15}$$

provided

$$C_2 \delta^{p-1} \leq \frac{1}{2}.
 \tag{16}$$

Similarly, one shows that for $u, v \in \tilde{K}$,

$$\sup_{0 < t < T} t^\alpha \|\Phi(u)(t) - \Phi(v)(t)\|_{L^r} \leq C_3 \delta^{p-1} d(u, v) \leq \frac{1}{2} d(u, v),
 \tag{17}$$

provided

$$C_3 \delta^{p-1} \leq \frac{1}{2},
 \tag{18}$$

for some constant C_3 . It follows from the above estimates that $\Phi : \tilde{K} \rightarrow \tilde{E}$.

We fix any $\delta > 0$ small enough so that (13), (16) and (18) are satisfied. The choice of δ depends only on N, p, q, r .

Next, we fix $T > 0$ such that

$$\sup_{0 < t < T} t^\alpha \|T(t)u_0\|_{L^r} \leq \delta/2.
 \tag{19}$$

In view of Lemma 8, the choice of T depends only on the compact set $\mathcal{K} \subset L^q(\Omega)$. This establishes the last assertion in Theorem 1.

By (17), (15) and (19), $\Phi : \tilde{K} \rightarrow \tilde{K}$ is a strict contraction, and thus has a unique fixed point in \tilde{K} .

Next, we claim that this fixed point belongs to K . For this purpose, it suffices to verify that $\Phi : K \rightarrow K$. We have to check that $\Phi(u) \in C((0, T], L^r(\Omega))$ and that $\lim_{t \downarrow 0} t^\alpha \Phi(u)(t) = 0$ in $L^r(\Omega)$. Since by Lemma 8, $T(t)u_0$ satisfies the above

requirements, we may always assume that $u_0 = 0$. It is clear that $\Phi(u) \in K$ when $u \in C([0, T], L^\infty(\Omega))$. Since $K \cap C([0, T], L^\infty(\Omega))$ is dense in K equipped with the metric d , the result follows from (17).

We now show that $u \in L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$. Indeed, we have $u \in L^\infty_{\text{loc}}((0, T), L^r(\Omega))$. Therefore, we can apply Theorem A1 (with $\sigma = r/(p - 1)$) on every interval $(\varepsilon, T - \varepsilon)$, with $a = |u|^{p-1}$. Indeed, $r \geq p > p - 1$, so that $\sigma > 1$; $r > q = N(p - 1)/2$, so that $\sigma > N/2$; and $r \geq p$, so that $r \geq \sigma'$.

Finally, we show that $u \in C([0, T], L^q(\Omega))$. Indeed, we have $u \in K$, so that in particular $u \in C((0, T], L^r(\Omega)) \subset C((0, T], L^q(\Omega))$. Therefore, it remains to show that $u(t) - T(t)u_0 \xrightarrow[t \downarrow 0]{} 0$ in $L^q(\Omega)$. As in (12) we have

$$\|u(t) - T(t)u_0\|_{L^q} \leq C_1 \sup_{0 < s < t} (s^\alpha \|u(s)\|_{L^r})^p \xrightarrow[t \downarrow 0]{} 0,$$

since $u \in E$. □

5. Proof of the uniqueness in Theorem 1

For every $u_0 \in L^q(\Omega)$, we denote by $U(t)u_0$ the solution constructed via the above contraction argument (in Section 3 or 4) on some interval $[0, T(u_0)]$. We shall need the following lemma.

Lemma 9 *Let $u_0 \in L^\infty(\Omega)$ and consider the classical solution \tilde{u} of (1) defined on the maximal interval $[0, T_{\max}(u_0))$ and given by (2)–(4). Then $T(u_0) < T_{\max}(u_0)$ and $\tilde{u}(t) = U(t)u_0$ for all $t \in [0, T(u_0)]$.*

Proof It is clear that $\tilde{u} \in K(\tau)$ for some $0 < \tau \leq T(u_0)$ sufficiently small. By uniqueness in $K(\tau)$ we have

$$\tilde{u}(t) = U(t)u_0, \quad \text{for } 0 \leq t \leq \tau.$$

After time τ , both $\tilde{u}(t)$ and $U(t)u_0$ are classical solutions. Hence the result. □

We now return to the proof of uniqueness. Here we use the same idea as in [5]. We give the proof only in the critical case $q = N(p - 1)/2$ and $q > 1$; the other case is simpler. Let $v \in C([0, T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$ be a solution of (1) with $v(0) = u_0$. Recall that v is a classical solution of (1) on $(0, T) \times \bar{\Omega}$. We are going to prove that $v(t) = U(t)u_0$ on some interval $[0, T')$. Then, $v(t) = U(t)u_0$ as long as both exist, by standard uniqueness in $L^\infty(\Omega)$.

Set

$$\mathcal{K} = v([0, T]),$$

and

$$M = \sup_{0 \leq t \leq T} \|v(t)\|_{L^q}.$$

Since \mathcal{K} is a compact set in $L^q(\Omega)$, there is a uniform $T_1 > 0$ such that $U(t)v_0$ is well defined for all $v_0 \in \mathcal{K}$ and all $t \in [0, T_1]$. Moreover, since $U(t)v(s) \in K(T_1)$ (considered as a function of t), we have

$$(20) \quad \begin{aligned} \|U(t)v(s)\|_{L^q} &\leq M + 1, \\ t^\alpha \|U(t)v(s)\|_{L^r} &\leq \delta, \end{aligned}$$

for all $s \in (0, T)$ and all $t \in (0, T_1)$.

Fix any $0 < s < T$. It follows from Lemma 9 that

$$(21) \quad v(t + s) = U(t)v(s) \quad \text{for } 0 \leq t \leq \min\{T - s, T_1\}.$$

Combining (20) and (21) we obtain

$$\begin{aligned} \|v(t + s)\|_{L^q} &\leq M + 1, \\ t^\alpha \|v(t + s)\|_{L^r} &\leq \delta, \end{aligned}$$

for $t + s < T$ and $t < T_1$. Passing to the limit as $s \downarrow 0$, we deduce that

$$\begin{aligned} \|v(t)\|_{L^q} &\leq M + 1, \\ t^\alpha \|v(t)\|_{L^r} &\leq \delta, \end{aligned}$$

for $0 < t < \min\{T, T_1\}$. Therefore, $v(t) \in \tilde{K}(T')$ where $T' = \min\{T, T_1\}$. We may now argue as in Remark 6 to assert that

$$(22) \quad v(t) = T(t)u_0 + \int_0^t T(t - s)|v(s)|^{p-1}v(s) ds,$$

i.e. $v = \Phi(v)$. By (17) we deduce $v(t) = U(t)u_0$ on $[0, T']$. □

6. Smoothing effect and stability: proof of (i), (ii) and (iii) in Theorem 1

To prove (5) we consider three cases. The methods are essentially the same in all three cases with some minor technical changes.

Case a: $q > N(p - 1)/2$, $q \geq p - 1$ and $q \geq 1$;

Case b: $q > N(p - 1)/2$ and $1 \leq q < p - 1$;

Case c: $q = N(p - 1)/2$ and $q > 1$;

Case a: $q > N(p - 1)/2$ and $q \geq p - 1$. We apply Theorem A1 with a given by (10). We have $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0, T), L^\sigma(\Omega))$ with $\sigma = q/(p - 1) > N/2, \sigma \geq 1$. By (A3) we have

$$\|u(t) - v(t)\|_{L^\infty} \leq C e^{Ct \|a\|_{L^\infty((0,T),L^\sigma)}^\sigma} (t^{-N/2q} + 1) \|u_0 - v_0\|_{L^q},$$

with $\alpha = N(p - 1)/2pq$. By construction, $u, v \in K$ where M is chosen such that $M \geq \|u_0\|_{L^q}$ and $M \geq \|v_0\|_{L^q}$; and thus, the L^∞ estimate of (5) follows.

On the other hand, we have

$$\|u(t) - v(t)\|_{L^q} \leq \|u_0 - v_0\|_{L^q} + C \int_0^t (\|u\|_{L^{pq}}^{p-1} + \|v\|_{L^{pq}}^{p-1}) \|u - v\|_{L^{pq}}.$$

Since $u, v \in K$, we have

$$\|u(s)\|_{L^{pq}}^{p-1} + \|v(s)\|_{L^{pq}}^{p-1} \leq \frac{C}{s^{\alpha(p-1)}} (M + 1)^{p-1}.$$

Therefore,

$$(23) \quad \sup_{0 < t < T} \|u(t) - v(t)\|_{L^q} \leq \|u_0 - v_0\|_{L^q} + C(M + 1)^{p-1} \sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^{pq}}.$$

Here, we use the fact that $\alpha p < 1$. Furthermore, by using the $L^q(\Omega) \rightarrow L^{pq}(\Omega)$ smoothing effect, we have

$$\|u(t) - v(t)\|_{L^{pq}} \leq t^{-\alpha} \|u_0 - v_0\|_{L^q} + CA \int_0^t (t - s)^{-\alpha} s^{-\alpha(p-1)} \|u - v\|_{L^{pq}},$$

where $A = \sup_{0 < s < T} s^{\alpha(p-1)} (\|u(s)\|_{L^{pq}}^{p-1} + \|v(s)\|_{L^{pq}}^{p-1})$. From the singular Gronwall lemma below, we deduce

$$(24) \quad \sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^{pq}} \leq C \|u_0 - v_0\|_{L^q}.$$

Combining (23) and (24) we obtain the L^q estimate of (5).

A singular Gronwall lemma *Let $T > 0, A \geq 0, 0 \leq \alpha, \beta \leq 1$ and let f be a nonnegative function with $f \in L^p(0, T)$ for some $p > 1$ such that $p' \max\{\alpha, \beta\} < 1$. Consider a nonnegative function $\varphi \in L^\infty(0, T)$ such that*

$$\varphi(t) \leq A t^{-\alpha} + \int_0^t (t - s)^{-\beta} f(s) \varphi(s) ds, \quad \text{for almost all } t \in [0, T].$$

Then there exists C , depending only on T, α, β, p and $\|f\|_{L^p}$, such that

$$\varphi(t) \leq A C t^{-\alpha},$$

for almost all $t \in [0, T]$.

For the proof, see e.g. [6].

Case b. Here, we cannot apply Theorem A1 since $\sigma = q/(p - 1) < 1$. However, the second part of the proof is unchanged and we deduce as above that

$$\|u(t) - v(t)\|_{L^q} + t^\alpha \|u(t) - v(t)\|_{L^{pq}} \leq C \|u_0 - v_0\|_{L^q},$$

for all $t \in [0, T]$. This establishes the L^q estimate of (5).

We now turn to the proof of the L^∞ estimate of (5). First note that

$$\|u(t/2) - v(t/2)\|_{L^{pq}} \leq C t^{-\alpha} \|u_0 - v_0\|_{L^q}.$$

We now apply Theorem A1 on the interval $(t/2, t)$ with $r = pq$ and $\sigma = pq/(p - 1) > 1$ and with a given by (10). Since $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, we have that $a \in L^\infty((t/2, t), L^\sigma(\Omega))$; and $\|a\|_{L^\infty((t/2, t), L^\sigma)} \leq C(M + 1)t^{-\alpha(p-1)}$. It follows that

$$\begin{aligned} \|u(t) - v(t)\|_{L^\infty} &\leq C \exp\left(Ct(M + 1)^{\frac{2\sigma(p-1)}{2\sigma-N}} (t^{-\alpha(p-1)})^{\frac{2\sigma}{2\sigma-N}}\right) \\ &\quad \times (1 + t^{-\frac{N}{2pq}}) \|u(t/2) - v(t/2)\|_{L^{pq}}; \end{aligned}$$

and so

$$\|u(t) - v(t)\|_{L^\infty} \leq C \exp\left(C(M + 1)^{\frac{2\sigma(p-1)}{2\sigma-N}} t^\gamma\right) (1 + t^{-\frac{N}{2pq}}) \|u(t/2) - v(t/2)\|_{L^{pq}},$$

with $\gamma = 1 - \frac{2\sigma\alpha(p-1)}{2\sigma-N}$. Since $\gamma > 0$, we obtain

$$\|u(t) - v(t)\|_{L^\infty} \leq C t^{-\frac{N}{2pq}} t^{-\alpha} \|u(t/2) - v(t/2)\|_{L^{pq}}.$$

Therefore,

$$\|u(t) - v(t)\|_{L^\infty} \leq C t^{-\frac{N}{2pq}} t^{-\alpha} \|u_0 - v_0\|_{L^q} = C t^{-\frac{N}{2q}} \|u_0 - v_0\|_{L^q},$$

which is the L^∞ estimate of (5).

Case c: $q = N(p - 1)/2$ and $q > 1$. Since $u - v = T(t)(u_0 - v_0) + \Phi(u) - \Phi(v)$, it follows from (17) that

$$\sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^r} \leq \sup_{0 < t < T} t^\alpha \|T(t)(u_0 - v_0)\|_{L^r} + \frac{1}{2} \sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^r},$$

with $\alpha = N(r - q)/2qr$; and so

$$(25) \quad \sup_{0 < t < T} t^\alpha \|u(t) - v(t)\|_{L^r} \leq 2 \sup_{0 < t < T} t^\alpha \|T(t)(u_0 - v_0)\|_{L^r} \leq 2 \|u_0 - v_0\|_{L^q}.$$

Furthermore (as in (12)) we have

$$\begin{aligned} \|u(t) - v(t)\|_{L^q} &\leq \|u_0 - v_0\|_{L^q} \\ &\quad + C \int_0^t (t - s)^{-a} (\|u(s)\|_{L^r}^{p-1} + \|v(s)\|_{L^r}^{p-1}) \|u(s) - v(s)\|_{L^r} ds \\ &\leq \|u_0 - v_0\|_{L^q} + C \delta^{p-1} \sup_{0 < s < t} t^\alpha \|u(s) - v(s)\|_{L^r} \int_0^t (t - s)^{-a} s^{-\alpha p} ds \\ &\leq C \|u_0 - v_0\|_{L^q}, \end{aligned}$$

by (25). This establishes the L^q estimate of (5).

To prove the L^∞ estimate, we apply Theorem A1 on the interval $(t/2, t)$ with $\sigma = r/(p - 1) > N/2$, $\sigma > 1$, and with a given by (10). We have $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((t/2, t), L^\sigma(\Omega))$; and since $u, v \in K$, $\|a\|_{L^\infty((t/2, t), L^\sigma)} \leq Ct^{-\alpha(p-1)}$. It follows that

$$\|u(t) - v(t)\|_{L^\infty} \leq C \exp\left(Ct(t^{-\alpha(p-1)})^{\frac{2\sigma}{2\sigma-N}}\right) (1 + t^{-\frac{N}{2r}}) \|u(t/2) - v(t/2)\|_{L^r}.$$

But $1 - \frac{2\sigma\alpha(p-1)}{2\sigma-N} = 0$; and so

$$(26) \quad \|u(t) - v(t)\|_{L^\infty} \leq C(1 + t^{-\frac{N}{2r}}) \|u(t/2) - v(t/2)\|_{L^r}.$$

Combining (25) and (26), we obtain

$$\|u(t) - v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2r}} t^{-\alpha} \|u_0 - v_0\|_{L^q} = Ct^{-\frac{N}{2q}} \|u_0 - v_0\|_{L^q},$$

which is the desired estimate. In fact, in this case the constant C in (5) is independent of $\|u_0\|_{L^q}$ and $\|v_0\|_{L^q}$.

Finally, (ii) and (iii) are clearly true when $u_0 \in L^\infty(\Omega)$, and the general case follows by continuous dependence (5).

7. Additional results and open problems

7.1. The above results hold for more general nonlinearities with similar proofs. More precisely, one can replace $|u|^{p-1}u$ by $g(u)$ where $g : \mathbf{R} \rightarrow \mathbf{R}$ verifies $|g(x) - g(y)| \leq C(1 + |x|^{p-1} + |y|^{p-1})|x - y|$.

7.2. Even when $q < p$, uniqueness holds in a class larger than $C([0, T], L^q(\Omega)) \cap L^\infty_{loc}((0, T), L^\infty(\Omega))$. More precisely, uniqueness holds in $C([0, T], L^q(\Omega)) \cap L^\infty_{loc}((0, T), L^p(\Omega))$. Indeed, note first that the equation makes sense in that class (since $|u|^{p-1}u \in L^\infty_{loc}((0, T), L^1(\Omega))$). Furthermore, if $u \in L^\infty_{loc}((0, T), L^p(\Omega))$, then $|u|^{p-1} \in L^\infty_{loc}((0, T), L^{p/(p-1)}(\Omega))$. We have $p > q = N(p - 1)/2$, so that $p/(p - 1) > N/2$. Therefore, it follows from Theorem A1 that $u \in L^\infty_{loc}((0, T), L^\infty(\Omega))$.

7.3. Open problem 1 Recall that in the critical case $q = N(p - 1)/2$ and $q > 1$, the time of existence $T(u_0)$ depends (in our proof of Theorem 1) on u_0 and not only on $\|u_0\|_{L^q}$. It would be interesting to clarify this point. In particular, is it possible to construct a sequence of initial conditions $(u_0^n)_{n \geq 0}$ which is bounded in $L^q(\Omega)$ and such that $T_{\max}(u_0^n) \xrightarrow{n \rightarrow \infty} 0$?

7.4. The “doubly critical” case in Theorem 4 As we have already mentioned in Remark 5, if $q = N(p - 1)/2$ and $q = p$, i.e. $q = p = N/(N - 2)$ ($N \geq 3$), then the conclusion of Theorem 4 fails, i.e. uniqueness fails in the class $C([0, T], L^q(\Omega))$. This is a result of Ni and Sacks [16], and we sketch their argument. First, a simple lemma.

Lemma 10 Let $\varphi, f \in L^1(\Omega)$ satisfy the equation

$$\begin{cases} -\Delta\varphi = f & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$(27) \quad - \int_{\Omega} \varphi \Delta\zeta = \int_{\Omega} f\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Then

$$(28) \quad \varphi = T(t)\varphi + \int_0^t T(s)f \, ds,$$

for all $t \geq 0$.

Proof The conclusion is trivial if φ is smooth. In the general case, let $(f_n)_{n \geq 0} \subset \mathcal{D}(\Omega)$ be such that $f_n \xrightarrow{n \rightarrow \infty} f$ in $L^1(\Omega)$ and let φ_n be the corresponding solution of (27).

Then $\varphi_n \rightarrow \varphi$ in $L^1(\Omega)$ (see e.g. Lemma 1 in [7]) and one passes to the limit in (28). □

In the case $\Omega =$ the unit ball of \mathbf{R}^N , Ni and Sacks [16] have constructed a radial function $\psi \in C^2(\overline{\Omega} \setminus \{0\})$, $\psi > 0$ in Ω , $\psi = 0$ on $\partial\overline{\Omega}$, $\psi \in L^p(\Omega)$, $\lim_{x \rightarrow 0} \psi(x) = +\infty$, satisfying the equation

$$\begin{cases} -\Delta\psi = \psi^p & \text{in } \Omega \quad \text{with } p = N/(N - 2), \\ \psi = 0 & \text{in } \partial\Omega, \end{cases}$$

in the sense that

$$-\int_{\Omega} \psi \Delta\zeta = \int_{\Omega} \psi^p \zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. In view of Lemma 10, $v(t) \equiv \psi$ is a solution of (22) in $C([0, \infty), L^p(\Omega))$. On the other hand, the solution u of (1) (with initial condition ψ) given by Theorem 1 has a smoothing effect. Hence, the two solutions are distinct.

7.5. The “doubly critical” case in Theorem 1 If $q = N(p - 1)/2$ and $q = 1$, i.e. $p = (N + 2)/N$ ($N \geq 1$) Theorem 1 does not apply and we suspect that the conclusions fail. (This concerns, for example, the simple case $N = 1$, $p = 3$, $q = 1$.) More precisely:

Open problem 2 Is there some $u_0 \in L^1(\Omega)$ for which there is no (local) solution of (1) ? This means that for every $T > 0$ there is no function $u \in C([0, T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$ satisfying (1) in the sense of Theorem 1.

Open problem 3 Is there some $u_0 \in L^1(\Omega)$ for which uniqueness fails in the class $C([0, T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$ for some $T > 0$?

Open problem 4 Could there be failure of the maximum principle? More precisely, is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ and a solution $u \in C([0, T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$ for some $T > 0$ which does not preserve the positivity ?

Open problem 5 Is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that problem (1) with the “truncated” initial condition

$$u_0^n = \min\{u_0, n\}$$

has a (classical) solution u^n on some maximal interval $[0, T_{\max}(u_0^n))$ satisfying $T_{\max}(u_0^n) \xrightarrow{n \rightarrow \infty} 0$?

Alternatively, consider the “truncated” problem

$$\begin{cases} u_t^n - \Delta u^n = g^n(u^n) & \text{in } (0, \infty) \times \Omega, \\ u^n = 0 & \text{on } (0, \infty) \times \partial\Omega, \\ u^n(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $g^n(t) = \min\{|t|^p, n\} \text{ sign } t$. Is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that $u^n(t, x) \xrightarrow[n \rightarrow \infty]{} +\infty$ for all $x \in \Omega$ and all $t > 0$?

Here is some evidence suggesting that the answers to the above questions might be positive.

Theorem 11 *Assume again $p = (N + 2)/N$, $q = 1$. There is some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that for every $T > 0$ problem (1) has no nonnegative solution $u \in C([0, T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0, T), L^\infty(\Omega))$.*

Here $N \geq 1$ and Ω can be arbitrary.

Proof Fix any open ball $\omega \subset \Omega$ with $\bar{\omega} \subset \Omega$.

Claim There is some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that $v(t) = T(t)u_0$ satisfies

$$(29) \quad \int_0^1 \int_\omega v^p(t, x) \, dx \, dt = +\infty.$$

(Note that $v \geq 0$ by the maximum principle.)

Proof of claim We argue by contradiction. Suppose that for every $u_0 \in L^1(\Omega)$, $u_0 \geq 0$,

$$\int_0^1 \int_\omega v^p(t, x) \, dx \, dt < \infty.$$

It follows that for every $u_0 \in L^1(\Omega)$,

$$\int_0^1 \int_\omega |v|^p(t, x) \, dx \, dt < \infty.$$

Applying the closed graph theorem to the operator $u_0 \mapsto v|_{(0,1) \times \omega}$, we see that there is a constant C such that

$$(30) \quad \int_0^1 \int_\omega |v|^p(t, x) \, dx \, dt \leq C \|u_0\|_{L^1}^p,$$

for every $u_0 \in L^1(\Omega)$. Without loss of generality, we may assume that $0 \in \omega$ and we consider a sequence $(u_0^n)_{n \geq 0} \in \mathcal{D}(\omega)$ such that $\|u_0^n\|_{L^1} \leq 1$ and $u_0^n \xrightarrow[n \rightarrow \infty]{} \delta$ (= the Dirac mass at 0) in the weak* topology of measures. The corresponding solutions

$v^n(t, x)$ converge to $v = T(t)\delta$. From (30) applied to u_0^2 , we deduce (using Fatou's lemma) that

$$\int_0^1 \int_\omega |T(t)\delta|^p(x) \, dx \, dt < \infty.$$

On the other hand, $|T(t)\delta - E(t, \cdot)| \leq C$ on $(0, 1) \times \omega$, where E is the fundamental solution of the heat equation in \mathbf{R}^N . In particular,

$$(31) \quad \int_0^1 \int_\omega |E(t, x)|^p(t, x) \, dx \, dt < \infty.$$

Since $E(t, x) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$, a direct computation shows that (31) does not hold. □

We now return to the proof of Theorem 11 and we choose u_0 as in the claim. Assume by contradiction that for some $T > 0$ there is a nonnegative solution $u \in C([0, T], L^1(\Omega)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$ of (1). We have

$$u(t + s) \geq T(t)u(s),$$

for all $t \geq 0, s > 0, t + s < T$. As $s \downarrow 0$ we find

$$(32) \quad u(t) \geq T(t)u_0 = v(t),$$

for all $t \in [0, T]$. Since u is a classical solution of (1) for $t \in (0, T)$, we may multiply (1) by $\zeta \in \mathcal{D}(\Omega), \zeta \geq 0$ on $\Omega, \zeta \geq 1$ on ω and we obtain

$$\frac{d}{dt} \int_\Omega u\zeta + \int_\Omega u(-\Delta\zeta) = \int_\Omega u^p\zeta \geq \int_\omega u^p.$$

Integrating on (ε, T) and letting $\varepsilon \downarrow 0$ (since $u \in C([0, T], L^1(\Omega))$), we deduce that

$$\int_0^T \int_\omega u^p < \infty,$$

which contradicts (29) and (32). □

7.6. The “subcritical” case $1 \leq q < N(p - 1)/2$

Open problem 6 Is there some $u_0 \in L^q(\Omega)$ for which there is no (local) solution of (1)? This means that given any $T > 0$ (as small as we please) there is no function $u \in C([0, T], L^q(\Omega)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$ satisfying (1).

Here is a suggestion how to construct such a u_0 . Let Ω be the unit ball in \mathbf{R}^N , and let $\varphi = \varphi(r)$ with $r = |x|, \varphi \in C^1(\bar{\Omega}), \varphi > 0$ in $\Omega, \varphi = 0$ on $\partial\Omega, \varphi'(r) < 0$ for

$r \in (0, 1)$, $\varphi''(0) < 0$ and $\Delta\varphi + \varphi^p \geq 0$ in Ω be such that the solution v of (1) with the initial condition $v(0) = \varphi$ blows up in finite time T_{\max} . (It is well known that such a φ exists.) By Theorem 2.4 of Friedman and McLeod [11],

$$\sup_{0 \leq t < T_{\max}} \|v(t)\|_{L^q} < \infty \quad \text{for all } 1 \leq q < \frac{N(p-1)}{2}.$$

Set

$$u_0 = \lim_{t \uparrow T_{\max}} v(t).$$

This u_0 belongs to $L^q(\Omega)$ for all $1 \leq q < N(p-1)/2$. We suspect that for such an initial condition u_0 , there exists no local solution of (1) in any reasonable sense. That there is no nonnegative solution follows from Baras and Cohen [2]. Indeed, suppose there is a nonnegative solution u of (1) with the initial condition $u(0) = u_0$ on $[0, T]$ for some $T > 0$. Set

$$w(t) = \begin{cases} v(t) & \text{for } 0 < t \leq T_{\max}, \\ u(t - T_{\max}) & \text{for } T_{\max} \leq t \leq T_{\max} + T. \end{cases}$$

This is an integral solution of (1) in the sense of Baras–Cohen [2] and Baras–Pierre [3] which blows up at $t = T_{\max}$. From [2], one knows that the only way to continue a solution beyond blow up time is by $+\infty$ everywhere.

Remark 12 Baras [1] has given examples showing that uniqueness for problem (1) fails in the class $C([0, T], L^q(\Omega)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$ for $1 \leq q < N(p-1)/2$. Here, the initial condition can be any smooth function u_0 , for example $u_0 = 0$. Such a phenomenon had been observed earlier by Haraux and Weissler [13] when $\Omega = \mathbf{R}^N$.

7.7. Blow up and L^q norms Let $u_0 \in L^\infty(\Omega)$ and let u be the corresponding solution of (1). Assume that $T_{\max} < \infty$ (it is well known that $T_{\max} < \infty$ if u_0 is “large” enough). By the blow up alternative, we know that u blows up in $L^\infty(\Omega)$, i.e.

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^\infty} = +\infty.$$

It is natural to ask whether $\|u(t)\|_{L^q}$ also blows up as $t \uparrow T_{\max}$ for some $q < \infty$. It turns out that

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^q} = +\infty,$$

for $q \geq 1$, $q > N(p-1)/2$ (see Weissler [18, 19]). We can also derive this result as a simple consequence of Theorem 1 (see below). In fact, F. Weissler [20] (see also [6]) has a rate of blow up:

$$\liminf_{t \uparrow T_m} (T_m - t)^\delta \|u(t)\|_{L^q} > 0,$$

$$\text{with } \delta = \frac{1}{p-1} - \frac{N}{2q}.$$

Corollary 13 *Let $u_0 \in L^\infty(\Omega)$, let u be the corresponding solution of (1) and assume that $T_{\max} < \infty$. Then*

$$(33) \quad \lim_{t \uparrow T_{\max}} \|u(t)\|_{L^q} = +\infty,$$

for any $q \geq 1$, $q > N(p-1)/2$.

Suppose in addition that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and that

$$(34) \quad \Delta u_0 + |u_0|^{p-1} u_0 \geq 0,$$

a.e. in Ω . If

$$(35) \quad \frac{N(p-1)}{2} > 1,$$

then

$$(36) \quad \lim_{t \uparrow T_{\max}} \|u(t)\|_{L^{N(p-1)/2}} = +\infty.$$

Proof of (33) Suppose by contradiction that $\liminf_{t \uparrow T_{\max}} \|u(t)\|_{L^q} < \infty$ and let $(t_n)_{n \geq 0}$ be a sequence such that $t_n \uparrow T_{\max}$ as $n \rightarrow \infty$ and $\sup_{n \geq 0} \|u(t_n)\|_{L^q} < \infty$. Applying Theorem 1 with $u(t_n)$ as initial condition, we obtain a uniform $T > 0$. Thus $T_{\max} \geq t_n + T$. This is impossible as $n \rightarrow \infty$. \square

Proof of (36) We argue as above, except that since we are in a critical case, we must know that $u(t_n)$ is contained in a compact set of $L^{N(p-1)/2}(\Omega)$. This is indeed the case, since, by the maximum principle, $u_t(t, x) \geq 0$ on $(0, T_{\max}) \times \Omega$. And thus $(u(t_n))_{n \geq 0}$ is a nonincreasing sequence and has a limit in $L^{N(p-1)/2}(\Omega)$. \square

Open problem 7 We do not know if the assumptions (34) and (35) in Corollary 13 are essential. In other words, if $N(p-1)/2 \geq 1$, does $\|u(t)\|_{L^{N(p-1)/2}}$ blow up as $t \uparrow T_{\max}$ for any initial condition $u_0 \in L^\infty(\Omega)$?

There is much evidence in favor of a positive answer:

- Assuming $N \geq 3$, $N(p-1)/2 = 1$ and (34), then (36) holds (see [6]).
- Assuming $N(p-1)/2 = 2$, i.e. $p = 1 + 4/N$, then (36) holds without having to assume (34) (see [6]).

Remark 14 The second assertion in Corollary 13 was already known (see Weissler [19]) under further restrictions on p : $p > 1 + 4/N$ and $(N - 2)p < N + 2$.

Since we are discussing the blow up of L^q norms as $t \uparrow T_{\max}$, we want to call attention to an interesting open problem connected with the result of Friedman and McLeod [11] mentioned above.

Open problem 8 Does $\|u(t)\|_{L^q}$ remain bounded as $t \uparrow T_{\max}$, for any $u_0 \in L^\infty(\Omega)$ with $T_{\max} < \infty$ and any q , $1 \leq q < N(p - 1)/2$?

Appendix: The linear heat equation with a potential

Here we consider the equation

$$(A1) \quad \begin{cases} u_t - \Delta u - a(t, x)u = 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases}$$

under various assumptions on the potential a . We write the equation (A1) in the form

$$(A2) \quad u(t) = T(t)u_0 + \int_0^t T(t-s)a(s)u(s) ds.$$

Theorem A1 Let $0 < T < \infty$, let $\sigma > N/2$, $\sigma \geq 1$, and let $a \in L^\infty((0, T), L^\sigma(\Omega))$. Given $u_0 \in L^r(\Omega)$, $1 \leq r < \infty$, there exists a unique solution $u \in C([0, T], L^r(\Omega)) \cap L^\infty_{loc}((0, T), L^\infty(\Omega))$ of equation (A1). Moreover, there is a constant C depending only on $N, \sigma, r, |\Omega|$ such that u satisfies

$$(A3) \quad \|u(t)\|_{L^\infty} \leq C e^{Ct \|a\|_{L^\infty((0,t), L^\sigma)}^\beta} t^{-N/2r} \|u_0\|_{L^r},$$

for all $t \in (0, T]$, with $\beta = 2\sigma/(2\sigma - N)$.

Uniqueness also holds in the class $L^\infty((0, T), L^r(\Omega))$ provided $r \geq \sigma'$ (without having to assume $u \in L^\infty_{loc}((0, T), L^\infty(\Omega))$).

Proof By a solution $u \in C([0, T], L^r(\Omega)) \cap L^\infty_{loc}((0, T), L^\infty(\Omega))$ of equation (A1), we mean that

$$(A4) \quad \begin{cases} u(t) = T(t-\varepsilon)u(\varepsilon) + \int_\varepsilon^t T(t-s)(a(s)u(s) + f(s)) ds & \text{for } 0 < \varepsilon \leq t \leq T, \\ u(t) \xrightarrow{t \downarrow 0} u_0 & \text{in } L^r(\Omega). \end{cases}$$

If $r \geq \sigma'$ and $u \in L^\infty((0, T), L^r(\Omega))$, then $au \in L^\infty((0, T), L^1(\Omega))$, so that the equation (A2) makes sense in $L^1(\Omega)$ and is equivalent to (A4). We now proceed in seven steps.

Step 1 Given $u_0 \in L^\infty(\Omega)$, there exists a unique solution $u \in L^\infty((0, T), L^\infty(\Omega))$ of (A2) on $(0, T)$, and it satisfies

$$(A5) \quad \|u(t)\|_{L^\infty} \leq 2e^{Ct\|a\|_{L^\infty((0,t),L^\sigma)}^\beta} \|u_0\|_{L^\infty},$$

for all $t \in (0, T)$.

We apply the contraction mapping principle to the map $\Psi : L^\infty((0, T), L^\infty(\Omega)) \rightarrow L^\infty((0, T), L^\infty(\Omega))$ defined by

$$\Psi(u)(t) = T(t)u_0 + \int_0^t T(t-s)a(s)u(s) ds,$$

for $t \in (0, T)$. Note that

$$\begin{aligned} \|\Psi(u)(t) - \Psi(v)(t)\|_{L^\infty} &\leq \int_0^t (t-s)^{-\frac{\beta}{2\sigma}} \|a(s)\|_{L^\sigma} \|u(s) - v(s)\|_{L^\infty} ds \\ &\leq CT^{1-\frac{\beta}{2\sigma}} \|a\|_{L^\infty((0,T),L^\sigma)} \|u - v\|_{L^\infty((0,T),L^\infty)}. \end{aligned}$$

Hence, Ψ is a strict contraction for example if

$$C\|a\|_{L^\infty((0,T),L^\sigma)} T^{1-\frac{\beta}{2\sigma}} \leq \frac{1}{2}.$$

Therefore, Ψ has a fixed point, which is a solution of (A2). In this case, the conclusion of the lemma follows with $\|u(t)\|_{L^\infty} \leq 2\|u_0\|_{L^\infty}$. The general case follows by a standard iteration argument.

Step 2 There exists C such that for every $r \in [1, \infty]$ and every $u_0 \in L^\infty(\Omega)$, the corresponding solution u of (A2) given by Step 1 verifies

$$(A6) \quad \|u(t)\|_{L^r} \leq 2e^{Ct\|a\|_{L^\infty((0,t),L^\sigma)}^\beta} \|u_0\|_{L^r},$$

for all $t \in (0, T)$.

By duality, we deduce from (A5) that

$$(A7) \quad \|u(t)\|_{L^1} \leq 2e^{Ct\|a\|_{L^\infty((0,t),L^\sigma)}^\beta} \|u_0\|_{L^1}.$$

The general case $1 < r < \infty$ now follows from (A5), (A7) and Riesz–Thorin’s interpolation theorem.

Step 3 There exists C such that for every $u_0 \in L^\infty(\Omega)$, the corresponding solution u of (A2) given by Step 1 verifies

$$(A8) \quad \|u(t)\|_{L^2} \leq C e^{Ct \|a\|_{L^\infty((0,t),L^\sigma)}^\beta} t^{-N/4} \|u_0\|_{L^1},$$

for all $t \in (0, T)$.

We assume $u_0 \neq 0$. Multiplying the equation (A1) by u , we obtain

$$(A9) \quad \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 = \int_{\Omega} a u^2.$$

By Hölder’s and Gagliardo–Nirenberg’s inequalities, we have

$$\int_{\Omega} a u^2 \leq \|a\|_{L^\sigma} \|u\|_{L^{2\sigma'}}^2 \leq C \|a\|_{L^\sigma} \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{N}{2\sigma}} \left(\int_{\Omega} u^2 \right)^{\frac{2\sigma-N}{2\sigma}}.$$

By Young’s inequality,

$$\begin{aligned} C \|a\|_{L^\sigma} \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{N}{2\sigma}} \left(\int_{\Omega} u^2 \right)^{\frac{2\sigma-N}{2\sigma}} &= C \left(\int_{\Omega} |\nabla u|^2 \right)^{\frac{N}{2\sigma}} \left(\|a\|_{L^\sigma}^\beta \int_{\Omega} u^2 \right)^{\frac{2\sigma-N}{2\sigma}} \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{C}{2} \|a\|_{L^\sigma}^\beta \int_{\Omega} u^2. \end{aligned}$$

Therefore, we deduce from (A9) that

$$(A10) \quad \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} |\nabla u|^2 \leq C \|a\|_{L^\sigma}^\beta \int_{\Omega} u^2.$$

On the other hand, by Gagliardo–Nirenberg’s inequality and (A7) we have

$$(A11) \quad \left(\int_{\Omega} u^2 \right)^{\frac{N+2}{N}} \leq C \int_{\Omega} |\nabla u|^2 \left(\int_{\Omega} |u| \right)^{\frac{4}{N}} \leq C \int_{\Omega} |\nabla u|^2 \left(C e^{Ct \|a\|_{L^\infty((0,t),L^\sigma)}^\beta} \|u_0\|_{L^1} \right)^{\frac{4}{N}}.$$

It now follows from (A10) and (A11) that the function

$$f(t) = \int_{\Omega} u^2(t, x) dx$$

satisfies the differential inequality

$$f'(t) + A f(t)^{1+\frac{1}{N}} \leq B f(t),$$

with

$$A = \left(C e^{Ct \|a\|_{L^\infty((0,t),L^\sigma)}^\beta} \|u_0\|_{L^1} \right)^{-\frac{4}{N}},$$

and

$$B = C \|a\|_{L^\infty((0,t),L^\sigma)}^\beta.$$

This yields

$$f(t) \leq \left(\frac{2}{NA t} \right)^{\frac{N}{2}} e^{Bt},$$

which is the desired estimate.

Step 4 If $u_0 \in L^\infty(\Omega)$, then for every $1 \leq r \leq \infty$ the estimate (A3) holds.

By duality, we deduce from (A8) that

$$(A12) \quad \|u(t)\|_{L^\infty} \leq C e^{Ct \|a\|_{L^\infty((0,t),L^\sigma)}^\beta} t^{-N/4} \|u_0\|_{L^2}.$$

Combining (A8) and (A12), we deduce that

$$(A13) \quad \|u(t)\|_{L^\infty} \leq C e^{Ct \|a\|_{L^\infty((0,t),L^\sigma)}^\beta} t^{-N/2} \|u_0\|_{L^1}.$$

(A3) now follows from (A5), (A13) and Riesz–Thorin’s interpolation theorem. (The arguments used in Steps 3 and 4 use an idea of Fabes and Stroock [10].)

Step 5 Existence in the class $C([0, T], L^r(\Omega)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$. Let $u_0 \in L^r(\Omega)$, and let $(u_0^n)_{n \geq 0} \subset L^\infty(\Omega)$ be such that $u_0^n \xrightarrow{n \rightarrow \infty} u_0$ in $L^r(\Omega)$. Let u^n be the corresponding solutions of (A2). It follows from (A6) and (A3) that u^n converges to a limit u in $C([0, T], L^r(\Omega))$ and in $C([\varepsilon, T], L^\infty(\Omega))$ for every $0 < \varepsilon < T$. Therefore, u solves the equation (A4) and satisfies the estimate (A3).

Step 6 Uniqueness in the class $C([0, T], L^r(\Omega)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\Omega))$. Suppose $u_0 = 0$. It follows from Step 4 that

$$\|u(t + \varepsilon)\|_{L^\infty} \leq C e^{Ct \|a\|_{L^\infty((0,t),L^\sigma)}^\beta} t^{-N/2r} \|u(\varepsilon)\|_{L^r},$$

for all $t \in (0, T - \varepsilon)$. Letting $\varepsilon \downarrow 0$, we obtain $u(t) = 0$ for all $t \in (0, T)$.

Step 7 Uniqueness in the class $L^\infty((0, T), L^{\sigma'}(\Omega))$. If u and v are two solutions, we have

$$\begin{aligned} \|u(t) - v(t)\|_{L^{\sigma'}} &\leq \int_0^t (t-s)^{-\frac{N}{2}(1-\frac{1}{\sigma'})} \|a(u-v)\|_{L^1} ds \\ &\leq \int_0^t (t-s)^{-\frac{N}{2\sigma}} \|a\|_{L^\sigma} \|u-v\|_{L^{\sigma'}} ds \\ &\leq \|a\|_{L^\infty((0,T),L^\sigma)} \int_0^t (t-s)^{-\frac{N}{2\sigma}} \|u-v\|_{L^{\sigma'}} ds, \end{aligned}$$

and it follows from the singular Gronwall lemma that $u = v$. This completes the proof. \square

Finally, we present a uniqueness result when $a \in C([0, T], L^{N/2}(\Omega))$.

Theorem A2 *Assume $N \geq 3$. Let $T > 0$ and $a \in C([0, T], L^{N/2}(\Omega))$. If $u \in L^\infty((0, T), L^q(\Omega))$ with $q > N/(N - 2)$ satisfies*

$$(A14) \quad u(t) = \int_0^t T(t-s)a(s)u(s) ds,$$

for all $t \in [0, T]$, then $u(t) \equiv 0$.

Proof We have $au \in L^\infty((0, T), L^{r_0}(\Omega))$, with $1/r_0 = 1/q + 2/N$. In particular, $1 < r_0 < \infty$, so that by maximal regularity (see [9]) we have $u \in L^p((0, T), W^{2,r_0}(\Omega) \cap W_0^{1,r_0}(\Omega))$ for every $p < \infty$, and u satisfies

$$(A15) \quad u_t - \Delta u = au,$$

in $L^{r_0}\Omega$ for almost all $t \in (0, T)$.

We now use a duality argument. Fix $t_0 \in (0, T)$, and $\psi \in \mathcal{D}(\Omega)$. Let $a_n = \min\{n, \max\{a, -n\}\}$. We have $(a_n)_{n \geq 0} \subset C([0, T], L^{N/2}(\Omega)) \cap L^\infty((0, T) \times \Omega)$. Moreover (as in the proof of (11)) $a_n \rightarrow a$ in $C([0, T], L^{N/2}(\Omega))$ as $n \rightarrow \infty$.

Let v_n be the solution of

$$\begin{cases} -(v_n)_t - \Delta v_n = a_n v_n, & \text{in } (0, t_0) \times \Omega, \\ v_n = 0 & \text{on } (0, t_0) \times \partial\Omega, \\ v_n(t_0) = \psi & \text{in } \Omega. \end{cases}$$

We now multiply the equation (A15) by v_n and integrate on $(0, t_0) \times \Omega$. We obtain

$$\begin{aligned} \left[\int_\Omega u v_n \right]_0^{t_0} &= \int_0^{t_0} \int_\Omega (u(v_n)_t + u_t v_n) = \int_0^{t_0} \int_\Omega (u(-\Delta v_n - a_n v_n) + v_n(\Delta u + au)) \\ &= \int_0^{t_0} \int_\Omega (a - a_n) u v_n. \end{aligned}$$

Therefore,

$$\int_\Omega u(t_0)\psi = \int_0^{t_0} \int_\Omega (a - a_n) u v_n.$$

Hence

$$(A16) \quad \left| \int_\Omega u(t_0)\psi \right| \leq t_0 \|a - a_n\|_{C([0, t_0], L^{N/2})} \|u\|_{L^\infty(0, t_0), L^q} \|v_n\|_{L^\infty((0, t_0), L^q)},$$

with $1/\theta = 1 - 1/q - 2/N > 0$. In particular, we have $\theta < \infty$. We claim that for every $2 \leq r < \infty$ there exists a constant C (C depends on r) such that

$$(A17) \quad \sup_{n \geq 0} \|v_n\|_{L^\infty((0,t_0),L^r)} \leq C\|\psi\|_{L^r}.$$

Assuming the claim, we let $n \rightarrow \infty$ in (A16) and we obtain

$$\int_{\Omega} u(t_0)\psi = 0.$$

Since $t_0 \in (0, T)$ and $\psi \in \mathcal{D}(\Omega)$ are arbitrary, we deduce that $u \equiv 0$. □

Proof of Claim (A17) We use the same method as in Brezis and Kato [8]. It is convenient to introduce $w_n(t) = v_n(t_0 - t)$ so that w_n satisfies

$$(A18) \quad \begin{cases} (w_n)_t - \Delta w_n = b_n w_n & \text{in } (0, t_0) \times \Omega, \\ w_n = 0 & \text{in } (0, t_0) \times \partial\Omega, \\ w_n(0) = \psi & \text{in } \Omega, \end{cases}$$

with $b_n(s) = a_n(t_0 - s)$. We multiply the equation (A18) by $|w_n|^{r-2}w_n$ to obtain

$$(A19) \quad \frac{1}{r} \frac{d}{dt} \int_{\Omega} |w_n(t)|^r + \frac{4(r-1)}{r^2} \int_{\Omega} |\nabla |w_n|^{r/2}|^2 \leq \int_{\Omega} |b_n| |w_n|^r \leq \int_{\Omega} |b| |w_n|^r,$$

where $b(t) = a(t_0 - t)$ for $0 \leq t \leq t_0$. Given $j \geq 0$ to be chosen large enough, we write $b = b - b_j + b_j$, and we estimate

$$(A20) \quad \begin{aligned} \int_{\Omega} |b| |w_n|^r &\leq \int_{\Omega} |b - b_j| |w_n|^r + \int_{\Omega} |b_j| |w_n|^r \\ &\leq \|b - b_j\|_{L^{N/2}} \|w_n\|_{L^{Nr/(N-2)}}^r + j \|w_n\|_{L^r}^r \\ &\leq C \|b - b_j\|_{L^{N/2}} \int_{\Omega} |\nabla |w_n|^{r/2}|^2 + j \|w_n\|_{L^r}^r, \end{aligned}$$

where the last inequality follows from Sobolev's inequality. We now choose j large enough (independent of n) so that

$$C \|b - b_j\|_{L^{N/2}} \leq \frac{4(r-1)}{r^2}.$$

(Recall that $b_j \xrightarrow{j \rightarrow \infty} b$ in $C([0, t_0], L^{N/2}(\Omega))$. It is here that we use the assumption $a \in C([0, T], L^{N/2}(\Omega))$; $a \in L^\infty((0, T), L^{N/2}(\Omega))$ would not be sufficient.) It now follows from (A19) and (A20) that

$$\frac{1}{r} \frac{d}{dt} \int_{\Omega} |w_n(t)|^r \leq j \|w_n\|_{L^r}^r,$$

from which we deduce $\|w_n(t)\|_{L^r}^r \leq \|\psi\|_{L^r}^r e^{jrt}$. \square

Remark A3 The conclusion of Theorem A2 fails if $q = N/(N-2)$. To construct such an example, we use the technique of Ni and Sacks [16]. Let ψ be as in Section 7.4 and let v be the solution of (1) with the initial condition $v(0) = \psi$. Set $u = v - \psi$ and

$$a = \begin{cases} \frac{v^p - \psi^p}{v - \psi} & \text{if } v \neq \psi, \\ v^{p-1} & \text{if } v = \psi. \end{cases}$$

u satisfies (A14) but $u \neq 0$.

Open problem 9 Can one replace the assumption $a \in C([0, T], L^{N/2}(\Omega))$ by $a \in L^\infty((0, T), L^{N/2}(\Omega))$ in Theorem A2 ? The problem is open even under the additional assumption $u \in C_c^\infty((0, T) \times \Omega)$.

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