A new ground state energy model

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This note gives a summary of corresponding papers in

www.riemann-hypothesis.de
www.navier-stokes-equations.com

Abstract Let $H=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u\in H:=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu:=\frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

enable the definitions of the norms (e.g. [ILi] 11.1.5, [KBr0])

$$\|u\|_\beta^2:=\sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 .$$

We propose to replace current re-normalization techniques to overcome certain convergence issue concerning today's ground state energy model by a modified (less regular) Hilbert space framework than current $H:=L_2^*(\Gamma)$ Hilbert space.

We propose an alternative ground state energy model based on the Hilbert space H_{-1}. The orthogonal projection from $H_{-1}\to H_0$ ensures consistency with today's standard L_2-model. The mathematical framework and the notation are given in [KBr3]. It is built on the operators

$$(S_{-1}u)(x):=-\oint\log 2\sin\frac{x-y}{2}u(y)dy=:Au(x) \quad , \quad (S_0u)(x):=\oint\frac{1}{2}\cot\frac{x-y}{2}u(y)dy=:Hu(x)$$

$$(S_1u)(x):=\oint\frac{1}{4\sin^2\frac{x-y}{2}}u(y)dy$$

The Dirichlet integral $D(u,v):=(u',v')$ defines the inner product of the "standard" "energy space". The proposed potential model concept of J. Plemelj ([JPl] §8) in combination with the Hilbert transform operator S_0 in the form

$$(S_1u)(x)=\oint\frac{1}{2}\cot\frac{x-y}{2}d_yu$$

is applied to define an alternative inner product (and an alternative Hilbert space domain, which is less regular than H_1) by

$$(u,v)_E:=((du,dv)):=(S_1u,S_1v) \quad , \quad u,v\in D(S_1)=H_{-1} .$$

In general, in this new Hilbert space framework the reverse Legendre transformation

$$d(g)=yd\psi-\frac{\partial f}{\partial x}dx+(d\psi dy)$$

is no longer valid. Therefore, in general the Hamiltonian and the Lagrangian formalisms are no longer equivalent, i.e. while the concept of "energy" of a mass element dm ([JPl] p. 12) in the form $\|dm\|$ is a valid definition in the sense of above, the concept of "force" may no longer be defined in corresponding (quantum mechanics) models.

Notations

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

enable the definitions of the norms (see e.g. [ILi] Remark 11.1.5, [KBr0])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \; .$$

There is a natural representation of the Fourier decomposition

$$u(x) = \frac{a_0}{2} + \sum_1^\infty a_\nu \cos(\nu x) + \sum_1^\infty b_\nu \sin(\nu x) := \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \in L_2$$

as Laurent series description in terms of a complex variable, defined on a circle $z = e^{ix}$:

$$u(z) := \tilde{u}(z) := u(x) = \sum_{-\infty}^{\infty} u_\nu z^\nu \in H := L_2^*(\Gamma) \; .$$

with $$u_0 := \frac{a_0}{2} \;,\; u_\nu := \frac{1}{2}(a_\nu - ib_\nu) \;,\; c_{-\nu} := \frac{1}{2}(a_\nu + ib_\nu) \;,\; \nu > 0 \, .$$

Then H is the space of L_2-periodic function in R.

Remark: From [DGa] pp.63 and [SGr] 1.441, we recall

$$\frac{1}{2\pi}\oint_{0\to 2\pi}\begin{Bmatrix}\sin n\vartheta \\ \cos n\vartheta\end{Bmatrix}\cot\frac{\varphi-\vartheta}{2}d\vartheta = \begin{Bmatrix}-\cos(n\varphi) \\ \sin(n\varphi)\end{Bmatrix} \;,\; \frac{1}{2\pi}\oint_{0\to 2\pi}\cot\frac{\varphi-\vartheta}{2}d\vartheta = 0$$

resp.

$$\frac{1}{2\pi}\oint_{0\to 2\pi} e^{in\varphi}\cot\frac{\varphi-\vartheta}{2}d\vartheta = \begin{cases} -ie^{in\varphi} & n = 1,2,3,... \\ 0 & n = 0 \\ ie^{in\varphi} & n = -1,-2,.. \end{cases}$$

From [ILi] (1.2.34) we note the identity with a hyper singular integral equation of kernel of Hilbert type

$$-n(a_n \cos nx_0 + b_n \sin nx_0) = \frac{1}{4\pi}\int_0^{2\pi}\frac{a_n \cos nx + b_n \sin nx}{\sin^2\frac{x_0 - x}{2}}dx \; .$$

This identity is related to the following integral operators ([ILi] (1.2.31)-(1.2.33), [Ili1])

(A) $$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2}u(y)dy =: \oint k(x-y)u(y)dy \quad \text{and} \quad D(A) = H = L_2^*(\Gamma)$$

(H) $$(Hu)(x) := [u](x) := \frac{1}{2}\oint \cot\frac{x-y}{2}u(y)dy = -\lim_{\varepsilon\to 0}\frac{1}{2}\int_{\varepsilon}^{1}[u(x+y)-u(x-y)]\cot\frac{y}{2}dy \quad .$$

the following properties are valid:

Lemma

i) The operator H is skew symmetric in the space $L_2(0,2\pi)$ (e.g. [DGa], [BPe]) and maps the space $H := L_2(0,2\pi) - R$ isometric onto itself, and it holds

$$\|Hu\| = \|u\| \text{ and } H^2 = -I \quad , \; (Hu,v) = -(u,Hv) \quad , \quad [u'](x) = [u]'(x)$$

$$(Hu)_\nu = -isign(\nu)u_\nu \quad , \quad (Hu)(x) = i\sum_1^\infty [u_{-\nu}e^{-i\nu x} - u_\nu e^{i\nu x}] \in L_2 \quad \text{for } u \in L_2$$

iii) The operator A is symmetric in its domain $D(A)$ and the Fourier coefficients of the convolutions of both operators are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|}u_\nu \quad , D(A) \subseteq H_A = H_{-1/2}(\Gamma) \; .$$

J. Plemelj's suggestion ([JPl] XV, p. 12, p. 17), see also [JAh], [JNi]) is about a relationship between the differential form calculus and its application in physics (e.g. [HCa], [HFl]) and a modified representation of the potential in the form

$$(*) \quad v(s) = -\frac{1}{\pi}\oint \log|\zeta(s)-\zeta(t)|u(t)dt \qquad \text{by} \qquad (**) \quad v(s) = -\frac{1}{\pi}\oint \log|\zeta(s)-\zeta(t)|du(t)$$

Plemelj's quote: "*Bisher war es üblich, für das Potential die Form (*) zu nehmen. Eine solche Einschränkung erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (**) verwendbar.*"

The Dirichlet integral

$$D(u,v) := (u',v') \quad , \qquad u,v \in H_1$$

defines the inner product of the "standard" "energy space". We apply the concept of J. Plemelj to the Hilbert transform operator S_0 in the form

$$(S_1 u)(x) = \oint \frac{1}{2}\cot\frac{x-y}{2} d_y u$$

in order to define a Dirichlet integral like inner product (with alternative Hilbert space domain, which requires less regularity assumptions than $\in H_1$) by

$$(u,v)_E := ((du,dv)) := (S_1 u, S_1 v) \,, \qquad u,v \in D(S_1) = H_{-1} \quad .$$

The "extension" of the Hilbert transform for $n>1$ is given by the Riesz transforms ([BPe], [ESt]). Those transforms enable the corresponding definition of the Dirichlet integral "extension" for $n>1$ in the form

$$\left(\left(\begin{pmatrix} du_1 \\ du_2 \\ \dots \\ du_n \end{pmatrix}, \begin{pmatrix} dv_1 \\ dv_2 \\ \dots \\ dv_n \end{pmatrix}\right)\right) := \sum_{j=1}^{n} (R_j(du_j), R_j(dv_j))$$

For corresponding physical models (e.g. Navier-Stokes equations, Maxwell equations and Einstein's field equations) to apply variation theory in combination with this kind of inner product for differential forms (see e.g. [HFl]) to the physical science we refer to e.g. [ESc], [RSe]. For the relationship between differential forms and mathematical concepts like differential forms of geodesic curvature, total curvature and parallel transport, as well as the calculation of the total curvature of a surface by means of the first fundamental form we refer to e.g. [HCa]. For the method of Pfaffians in the theory of curves and surfaces in the context of conformal mapping and minimal surfaces we refer e.g. to [DSt].

For nonlinear functional analysis we refer to e.g. [KDe].

A differential form is a tensor field, Therefore its Lie derivative can be built enabled by a vector field. The Lie derivative of a differential form can be described by the exterior (Cartan) derivative. The Lie derivative of a differential form is again a differential form. The Lie derivative of a vector field Y with a vector field X is given by the Lie bracket of X and Y. Therefore, knowing the exterior derivative of a 1-form is the same as knowing the Lie bracket on a vector fields (see e.g. [SDo] 1.2).

As a kind of conceptual counterpart to the operators S_{-1}, S_0, S_1 ([KBr3]) we note Hodge's main result ([HFl] 8.4).

The Hodge theorem: Let ω be any $p-$form and η any $(p+1)-$form. Putting $\delta\omega := (-1)^{np+n+1} * d * \omega$ then $(d\omega,\eta) = (\omega,\delta\eta)$ and there is a $(p-1)-$form α, a $(p+1)-$form β and a harmonic $p-$form γ such that

$$\omega = d\alpha + \delta\beta + \gamma \ .$$

The forms $d\alpha, \delta\beta, \gamma$ are unique.

As an application of nonlinear variation of an "energy" functional we note the variation of total curvature of the Hilbert-Einstein functional:

Let (M,g) be a compact Riemann manifold with volume element $dV_g = \sqrt{\det g_{ij}}\, dx^1 \wedge dx^2 .... \wedge dx^n$ with varying metric g. The (volume and total curvature (Hilbert-Einstein)) functionals of g are given by

$$Vol(g) := \int_M\int dV_g \quad , \quad S(g) := \int_M\int S_g dV_g$$

Let f,h be symmetric $(0,2)-$tensor, $(e_1, e_2, ...e_n)$ an orthogonal basis with respect to g, $g_t = g + t \cdot h$ be a variation of the metric g and S_t the scalar curvature of g_t with $S = S_0$.

Putting

$$\langle f,h \rangle_g := \int_M \sum_{i,j} f(e_i,e_j)h(e_i,e_j)dV_g$$

it holds with the Ricci tensor Ric

$$\frac{d}{dt}\Big|_{t=0} S(g_t) = \frac{d}{dt}\Big|_{t=0} \int_M S_t dV_t = \left\langle \frac{S}{2} g - Ric, h \right\rangle_g \ .$$

In in contrast to the Ricci tensor the tensor $Sg/2 - Ric$ is divergence free, which is also the case for $\Lambda g + Sg/2 - Ric$ with Einstein's very small cosmological constant Λ, which he introduced to enable a static universe model. The proposed ground state energy for "objects" dm with its corresponding ("energy") inner product might provide an additional rational for such a constant Λ.

We note, that the Einstein field equations, which state that the matter, described by the energy-momentum tensor is generated by the curvature of the space-time, is an AXIOM.

For the equivalence of extremal problems for nonlinear problems, built on symmetric bilinear form and convex functionals and corresponding variational equations built on Gateaux differentials for nonlinear problems we refer to e.g. [WVe].

Proposition: The mathematical relationship between and the concept of “force” (modeled by the Lagrangian formalism) and the concept of “energy” (modeled by the Hamiltonian formalism) is given by the Legendre transformation, defined by

$$g := g(x,y) := \psi y - f = y\psi(x,y) - f(x,y)$$

and its reverse operation

$$d(g) = y d\psi - \frac{\partial f}{\partial x} dx + (d\psi dy) \; .$$

In case f' is well defined, the Hamiltonian and the Lagrangian formalism are equivalent. As a consequence, in the proposed new Hilbert space framework above, where on df is required to be an element of the domain of the Hilbert transform operator, the concept of force the above is no longer valid, resp. only then “existing”, when f'. According to J. PLemelj’s quote above this *“dispossess the potential of his biggest efficiency*”.

The Hilbert spaces $H_{-1/2}, H_{-1}$ are characterized by

$$H_{-1/2} = \left\{ \psi \middle| \|\psi\|_{-1/2}^2 = (A\psi,\psi)_0 < \infty \right\} , \; H_{-1} = \left\{ \psi \middle| \|\psi\|_{-1}^2 = (A\psi, A\psi)_0 < \infty \right\} .$$

The related representation of the Dirac function is given in the form

$$\delta(x) := \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} dk = \frac{1}{\pi} \int_0^{\pi} \cos(kx) dk \in H_{-n/2-\varepsilon} \subset H_{-1} \qquad \text{for } n = 1.$$

From [ESt1] IV 6.3, we note that a periodic function on R in the form

$$u(x) = \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \quad \text{with} \quad |u_\nu| \leq \frac{1}{\nu}$$

is an element of the function space of bounded mean oscillation, i.e. $u \in BMO(R)$.

Suppose

$$\sum_{-\infty}^{\infty} v_\nu e^{i\nu x} \in BMO(R) \; , \; v_\nu \geq 0 \text{ then } \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \in BMO(R) \quad \text{whenever } |u_\nu| \leq v_\nu \; .$$

The Hardy spaces H^p are equivalent to L_p for $p > 1$. For $p = 1$ we note $H^1 \cong BMO^*$, which can be seen as proper substitution of L_1 and L_∞ ([HAb] 4.7). Our concept above is about the alternative duality $H_{-1} \cong H_1^*$ and $H_{-1/2} \cong H_{1/2}^*$ of Hilbert spaces, embedded in a Hilbert scale framework with corresponding spectral theory instead of Banach spaces only.

Remark: There are several other relationships in the context of Fourier transforms and Euler's formula (see ([ETi] 2.1), [BPe]): Let $[x]$ denote the largest integer not exceeding the real number x and let $\rho(x):=\{x\}:=x-[x]$ be the fractional part of x.

i) $$\rho(x)=\{x\}=x-[x]=\frac{1}{2}-\sum_1^\infty \frac{\sin 2\pi\nu x}{2\pi|\nu|} \quad , \quad -i\pi sign(x)=-2i\int_0^\infty \frac{\sin(tx)dt}{t}=2\int_0^\infty \frac{\sinh(tx)dt}{t}=\left[P.v.(\frac{1}{x})\right]^\wedge \ .$$

ii) $$\sum_1^\infty \frac{\sin \nu x}{\nu}=\frac{\pi-x}{2} \quad , \quad \sum_1^\infty \frac{\cos \nu x}{\nu}=\frac{1}{2}\log\frac{1}{2(1-\cos x)} \ , \quad 0<x<2\pi \ .$$

Remark: In ([AZy], 5.28, 7.2, 13.11) the concept of "logarithmic", $\alpha-$capacity" of sets and convergence of Fourier series to functions with

$$\sum_1^\infty n\left[a_n^2+b_n^2\right]<\infty$$

is given. In ([AZy]) the following two examples to the above are provided (see also [HEd] 9.7):

i) $$\lambda(x)\approx\sum_1^\infty \frac{\cos 2\pi\nu x}{\nu}=-\log 2\sin(\pi x) \qquad \text{whereby} \qquad \left|\sum_1^N \frac{\cos \nu x}{\nu}\right|\leq\log(\frac{1}{x})+C \ ,$$

ii) $$\lambda(x)\approx\sum_1^\infty \frac{\cos \nu x}{\nu^{1-\alpha}}\cong c_\alpha|x|^{-\alpha} \ , \quad (x\to 0; 0<\alpha<1) \ .$$

In [CBe] 8, Entry17(iv) its relationship to Ramanujan's divergent series technique is mentioned: "*Ramanujan informs us to note that*

$$\sum_1^\infty \sin(2\pi\nu x)=\frac{1}{2}\cot(\pi x) \ ,$$

which also is devoid of meaning" .... "may be formally established by differentiating the well known equality"

$$\sum_1^\infty \frac{\cos 2\pi\nu x}{\nu}=-\log 2\sin(\pi x) \ .$$

We further note that in harmonic analysis the energy of the harmonic continuation $h=\mathrm{E}(\varphi)$ to the boundary is given by

$$[\varphi]^2:=\frac{\pi}{2}\sum_1^\infty \nu(a_\nu^2+b_\nu^2)=\frac{1}{2}\iint|dh(z)|^2 dxdy=\frac{1}{4\pi}\int_{\partial B}\int_{\partial B}\frac{|\varphi(w)-\varphi(\zeta)|^2}{|w-\zeta|^2}ds(w)d\zeta<\infty \ .$$

A relationship to the Gamma function and the Euler constant is given by ([CBe] 8, entry 17(iv), ([NNi] chapter II, §33)):

i) $$\log\sin\pi x=\log\frac{\pi}{\Gamma^2(x)}+\frac{2}{\pi}\sum_1^\infty(\gamma+\log(2\pi k))\frac{\sin(2\pi kx)}{k} \qquad \text{for } 0<x<1$$

ii) $$\gamma=\frac{1}{2}+\sum_{n=1}^\infty\int_n^\infty \cos(2\pi t)\frac{dt}{t} \ .$$

The Helmholtz Free Energy

In this chapter we recall the mathematical background of the Helmholtz free energy of a quantum harmonic oscillator. Our proposal is to move current quantum theory models from a L_2- based to a $H_{-1}-$ based Hilbert space environment, applying spectral theory to a corresponding self-adjoint and bounded (singular integral) operator.

The function

$$L(x) := -\log 2\sinh(x)$$

plays a key role in the context of free energy, vacuum energy of electromagnetic fields, the density matrix for a one-dimensional harmonic oscillator and the Planck black body radiation law (concerning the notations we refer to [RFe]):

the exact value of the free energy F of a linear system of harmonic oscillators is given by

$$\beta F := \sum_{k=1}^{\infty} L(\beta\lambda_k) \qquad \text{with} \quad \frac{1}{\beta} := k_B T \qquad \text{and} \quad \lambda_k := \frac{\hbar\omega_k}{2}$$

with the related probability values in the form $\quad a_k = e^{-\beta(\lambda_k - F)}$.

Due to convergence issues in order to calculate a normalization factor Z the ground state zero term $\beta\lambda_0$ is omitted and F is replaced by

$$\beta F^* = \sum_{k=1}^{\infty} \log(1 - e^{-2\beta\lambda_k}) = -\sum_{k=1}^{\infty} K(2\beta\lambda_k)$$

leading to $\quad a_k^* = e^{-\beta(\lambda_k - F^*)} = \frac{1}{Z^*} e^{-\beta\lambda_k}$ and $\quad \varphi^* := \sum_{k=1}^{\infty} a_k^* \varphi_k \in H_0$.

We propose the shift from the underlying Hilbert space H_0 into H_{-1} while keeping the information about the ground state term as part of the physical models, but applying the analysis of this paper to e.g.

$$a_k = e^{-\beta(\lambda_k - F)} = \frac{1}{Z} e^{-\beta\lambda_k} \quad \text{and} \quad \varphi := \sum_{k=1}^{\infty} a_k \varphi_k \in H_{-1} \quad \text{and} \quad Z := \|\varphi\| .$$

For

$$\varphi_\lambda(x) := -\frac{1}{2\pi}\log\left[2\sin\frac{x-\lambda}{2}\right] \quad , \qquad \lambda \in [0,x].$$

In combination with

$$\psi = \sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n + \int \varphi_\lambda(\varphi_\lambda,\psi)d\lambda$$

and the relations (see e.g. [JNe])

$$(\varphi_n,\varphi_m) = \delta_{n,m} \ , \ (\varphi_\lambda,\varphi_n) = A\varphi_n(\lambda)$$

it follows

$$\psi = \sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n + \int \varphi_\lambda[A\psi](\lambda)d\lambda = \sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n + A^2\psi \ .$$

The spectrum for a self-adjoint operator is real and closed. If the operator is additionally compact, then the spectrum is discrete. In case the operator is not compact, but bounded (continuous), there is a spectral representation built on Riemann-Stieltjes integral over projection operator valued step functions (see also [KBr2], Lommel polynomials). In case of unbounded operators the closed graph theorem can be applied to build bounded operators with respect to the graph norm. The below indicates to analyze the graph norm for the momentum operator for those physical states, represented by the elements out of

$$H_0^\perp := \left\{\psi \in H_{-1} := \overline{H_0}^{\|\ \|} \,\middle|\, ((\psi,\varphi)) = 0, \varphi, H\varphi \in H_0\right\}$$

whereby

$$\|\|\psi\|\|^2 := \sum_{n=1}^{\infty}|(\psi,\varphi_n)|^2 + \|A\psi\|^2 = \sum_{n=1}^{\infty}|(\psi,\varphi_n)|^2 + \|\psi\|_{-1}^2 \ .$$

Remark: The equivalent norm

$$\|\|\psi\|\|_*^2 = \sum_{n=1}^{\infty}|(\psi,\varphi_n)|^2 + |(\psi,\varphi_n)_{-12}|^2 + \|\psi\|_{-1}^2$$

is proposed to be used to model spin effects.

Remark: For $\psi \in H_0^\perp$ it holds

$$(([\frac{\partial}{\partial x}x - x\frac{\partial}{\partial x}]\psi,\varphi)) \cong (A\psi',\varphi) \cong -(H\psi,\varphi) = (\psi,H\varphi) = 0 \qquad \text{for all } \varphi \in H_0 \ .$$

Remark: We note that e.g. in case of the harmonic quantum oscillator it holds in the L_2 – framework

$$\overline{E}_0=\frac{1}{2}\sum\hbar\omega_n\approx c\sum\hbar n=\infty\quad ,$$

which leads to the concept/requirement of “re-normalization” to ensure the existence of *bounded* Hermitian operators $\overline{H}_{renorm}$, with

$$\overline{H}=\overline{H}_{renorm}+\overline{E}_0\quad .$$

This is the analogue a priori representation of a physical state of a particle in the form

$$\psi=\sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n+\int\varphi_\lambda[A\psi](\lambda)d\lambda=\sum_{n=1}^{\infty}(\psi,\varphi_n)\varphi_n+A^2\psi\quad .$$

The later one can be interpreted as “ideal number” or “non-standard number” as analogue to a real number r represented in the form $r+i$, whereby i denotes an infinitely small, finite non-real number, which is not equal zero, but smaller than any positive real number $\varepsilon\in R^+$ ([WLu]).

Remark: The relationship of Hermitean commutators properties with respect to the norm $\|\psi\|^2$ and the weaker $\|\psi\|_{-1}$ – norm is given by (appendix resp. [SGr] 4.384, 1.441):

i) the norms $\|HA\psi\|_0^2\cong\|A\psi\|_0^2\cong\|\psi\|_{-1}^2$ are equivalent

ii) the range of a “constant” operator is zero according to

$$\frac{1}{2\pi}\oint_{0\to 2\pi}\log 2\sin\frac{y}{2}dy=0\quad , \quad \frac{1}{2\pi}\oint_{0\to 2\pi}\cot\frac{\lambda-y}{2}dy=0\quad .$$

Remark: For the commutator $[P,Q]$ it holds

$$(([P,Q]\psi,\chi))=c((\psi,\chi))\qquad \text{for all}\quad \chi\in H_{-1}\ .$$

Therefore for the Ritz projection ([KBr3])

$$R_{-1.0}:H_{-1}\to H_0$$

$$[P,Q]\psi\to\psi_R:=[P,Q]_R\psi:=R_{-1.0}([P,Q]\psi)$$

it holds

$$(\psi_h,\chi)=0\ \text{ for all }\quad \chi\in H_0\subset H_{-1}\ .$$

Appendix

From lecture notes, internet and literature

The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|...\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on K are:

i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$

ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,

iii) K is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} \quad .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} \quad .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$of eigenelements and eigenvalues

$K\varphi_i = \lambda_i \varphi_i$ with the properties

i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$

ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$

iii) the generalized Fourier sums $S_n := \sum_{i=1}^{n} (x, \varphi_i)\varphi_i \to x$ with $n \to \infty$ for all $x \in H$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|...\|$ and A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigenelement of K is also an eigenelement of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i \varphi_i \quad , \quad (\varphi_i, \varphi_k) = \delta_{i,k} \text{ and } \lim_{i\to\infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty} (x, \varphi_i)\varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2 .$$

Lemma: Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i)\varphi_i \quad , \quad \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2 ,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i) .$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, ...)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i) .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x}, \hat{y})_\alpha = \sum_i^\infty \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x}, \hat{x})_\alpha$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite $\alpha-$ norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_α : they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x, y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha \cdot$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax, Ax)_0 \ , \ H_2 = D(A) \cdot$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $\lim \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, ...)$ with

$$\|\hat{x}\|_2^2 < \infty \ , \ \|\hat{x}\|_0^2 = \infty \ \cdot$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \quad \text{for} \quad x \in H_\gamma$$

with $\mu = \dfrac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \dfrac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \quad \text{and} \quad \|x\|_2 = \|Ax\|.$$

We give three examples of differential operator and singular integral operators, whereby the integral operators are related to each other by partial integration:

Example 1: Let $H = L^2(0,1)$ and

$$Au := -u''$$

with

$$D(A) = \dot{W}_2^2(0,1) := \overset{\circ}{W}{}_2^1(0,1) \cap W_2^2(0,1)\,.$$

Building on the orthogonal set of eigenpairs $\{\lambda_i, \varphi_i\}$ of A_i, i.e.

$$-\varphi_i'' = \lambda_i \varphi_i \qquad \varphi_i(0) = \varphi_i(1) = 0$$

it holds the inclusion

$$D(A) \subseteq H_A = H_1 = \overset{\circ}{W}{}_2^1(0,1) \subseteq L^2(0,1)\,.$$

Example 2: Let $H = L_{22}^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Then H is the space of integrable periodic function in R. Let

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2} u(y)dy =: \oint k(x-y)u(y)dy$$

and

$$D(A) = H = L_{22}^*(\Gamma)\ .$$

The Fourier coefficients of this convolution are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|} u_\nu$$

i.e. it holds $D(A) \subseteq H_A = H_{-1/2}(\Gamma)$.

A relation of this Fourier representation to the fractional function is given by

$$x - [x] - \frac{1}{2} = -\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$$

Remark: We give some further background and analysis of the even function

$$k(x) := -\ln\left|2\sin\frac{x}{2}\right| =: -\log\left|2\sin\frac{x}{2}\right| .$$

Consider the model problem

$$-\Delta U = 0 \qquad \text{in } \Omega$$

$$U = f \qquad \text{on } \Gamma := \partial\Omega \;,$$

whereby the area Ω is simply connected with sufficiently smooth boundary. Let $y = y(s) - s \in (0,1]$ be a parametrization of the boundary $\partial\Omega$. Then for fixed $\bar{z}$ the functions

$$U(\bar{x}) = -\log|\bar{x} - \bar{z}|$$

are solutions of the Lapace equation and for any $L_1(\partial\Omega)-$ integrable function $u = u(t)$ the function

$$(Au)(\bar{x}) := \oint_{\partial\Omega} \log|\bar{x} - u(t)|dt$$

is a solution of the model problem. In an appropriate Hilbert space H this defines an integral operator, which is coercive for certain areas Ω and which fulfills the Garding inequality for general areas Ω . We give the Fourier coefficient analysis in case of $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $x(s) := (\cos(s), \sin(s))$ be a parametrization of $\Gamma := S^1(R^2)$ then it holds

$$|x(s) - x(t)|^2 = \left|\begin{pmatrix}\cos(s) - \cos(t)\\ \sin(s) - \sin(t)\end{pmatrix}\right|^2 = 2 - 2\cos(s-t) = 2(1 - \cos(2\frac{s-t}{2})) = 2\left[2\sin^2\frac{s-t}{2}\right] = 4\sin^2\frac{s-t}{2}$$

and therefore

$$-\log|x(s) - x(t)| = -\log 2\left|\sin\frac{s-t}{2}\right| = k(s-t) \;.$$

The Fourier coefficients k_ν of the kernel $k(x)$ are calculated as follows

$$k_\nu := \frac{1}{2\pi}\oint k(x)e^{-i\nu x}dx = \frac{1}{2\pi}\int_0^{2\pi}\log\left|2\sin\frac{t}{2}\right|e^{-i\nu t}dt = \frac{2}{2\pi}\int_0^{\pi}\log\left|2\sin\frac{t}{2}\right|\cos(\nu t)dt = k_{-\nu}$$

As $\varepsilon\log 2\sin\frac{\varepsilon}{2} \underset{\varepsilon\to 0}{\to} 0$ partial integration leads to

$$k_\nu = \frac{1}{\nu\pi}\sin(\nu t)\Big|_0^{\pi} - \frac{1}{\nu\pi}\int_0^{\pi}\frac{2\sin(\nu t)\cos\frac{t}{2}}{2\sin\frac{t}{2}}dt = -\frac{1}{\nu\pi}\int_0^{\pi}\frac{\sin(\frac{2\nu+1}{2}t) - \sin(\frac{2\nu-1}{2}t)}{2\sin\frac{t}{2}}dt$$

$$k_\nu = -\frac{1}{\nu\pi}\int_0^{\pi}(\left[\frac{1}{2} + \cos(t)..+\cos(\nu t)\right] - \left[\frac{1}{2} + \cos(t)..+\cos((\nu-1)t)\right])dt = -\frac{1}{\nu} \;.$$

Extension and generalizations

For $t>0$ we introduce an additional inner product resp. norm by

$$(x,y)_{(t)}^2=\sum_{i=1} e^{-\sqrt{\lambda_i}t}(x,\varphi_i)(y,\varphi_i)$$

$$\|x\|_{(t)}^2=(x,x)_{(t)}^2 \ .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i}t}$ instead of a polynomial decay in case of λ_i^α.
Obviously we have

$$\|x\|_{(t)}\le c(\alpha,t)\|x\|_\alpha \text{ for } x\in H_\alpha$$

with $c(\alpha,t)$ depending only from α and $t>0$. Thus the $(t)-norm$ is weaker than any $\alpha-norm$. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha<0$, is bounded by the $0-norm$ and the newly introduced $(t)-norm$. It holds:

Lemma: Let $\alpha>0$ be fixed. The $\alpha-norm$ of any $x\in H_0$ is bounded by

$$\|x\|_{-\alpha}^2\le\delta^{2\alpha}\|x\|_0^2+e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta>0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha-norm$, which can be reformulated in the form ($\mu,\nu>0$, $\mu+\nu>1$)

$$\|x\|_\beta^2\le\nu\varepsilon\|x\|_\gamma^2+\mu e^{-\nu/\mu}\|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2\le(\|x\|_\alpha^2)^\mu(\|x\|_\gamma^2)^\nu \ .$$

The counterpart of lemma 4 above is

Lemma: Let $t,\delta>0$ be fixed. To any $x\in H_0$ there is a $y=y_t(x)$ according to

i) $\|x-y\|\le\|x\|$

ii) $\|y\|_1\le\delta^{-1}\|x\|$

iii) $\|x-y\|_{(t)}\le e^{-t/\delta}\|x\|$.

Eigenfunctions and Eigendifferentials

Let H *be* a (infinite dimensional) Hilbert space with inner product$(.,.)$, the norm$\|...\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu(\varphi_\mu,*)d\mu \quad , \quad \mu \in [\lambda_0,\infty) \; ,$$

i.e.

$$dE_\lambda = \varphi_\lambda(\varphi_\lambda,*)d\lambda \; .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ.
The set E_λ fulfills the following properties:

i) E_λ is a projection operator for all $\lambda \in R$

ii) for $\lambda \le \mu$ it follows $E_\lambda \le E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

iii) $\lim_{\lambda \to -\infty} E_\lambda = 0$ and $\lim_{\lambda \to \infty} E_\lambda = Id$

iv) $\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f:[a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda)d(E_\lambda x, x) \; .$$

Symbolically one writes

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda \; .$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax,y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \quad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2x,y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \; , \; \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \; .$$

The function $\sigma(\lambda):=\|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function.

It hold the following eigenpair relations

$$A\varphi_i = \lambda_i\varphi_i \qquad A\varphi_\lambda = \lambda\varphi_\lambda \qquad \|\varphi_\lambda\|^2 = \infty \ , \ (\varphi_\lambda,\varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu) \ .$$

The φ_λ are not elements of the Hilbert space. The so-called eigendifferentials, which play a key role in quantum mechanics, are built as superposition of such eigenfunctions.

Let I be the interval covering the continuous spectrum of A. We note the following representations:

$$x = \sum_1^\infty (x,\varphi_i)\varphi_i + \int_I \varphi_\mu(\varphi_\mu, x)d\mu \ , \quad Ax = \sum_1^\infty \lambda_i(x,\varphi_i)\varphi_i + \int_I \lambda\varphi_\mu(\varphi_\mu, x)d\mu$$

$$\|x\|^2 = \sum_1^\infty |(x,\varphi_i)|^2 + \int_I |(\varphi_\mu, x)|^2 d\mu \ ,$$

$$\|x\|_1^2 = \sum_1^\infty \lambda_i |(x,\varphi_i)|^2 + \int_I \lambda|(\varphi_\mu, x)|^2 d\mu$$

$$\|x\|_2^2 = \|Ax\|^2 = \sum^\infty \lambda_i^2|(x,\varphi_i)|^2 + \int_I \lambda^2|(\varphi_\mu, x)|^2 d\mu \ .$$

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda\varphi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\varphi_\lambda(x)$ it results into elements of H. The practical way to find Eigen-differentials is looking for solutions of a distribution equation.

Hermitian Operator and Physical Observabales

The spectrum of a hermitian, positive definite operator

$$A : D(A) \to H$$

with domain $D(A)$ in a complex-valued Hilbert space H is discrete. This property enables an axiomatic building of the quantum mechanics, whereby, roughly speaking, physical states are modeled by the elements of the Hilbert space, observables of states by the operator A and the mean value of the observable A at the state ψ with $\|\psi\|$ is given by

$$\langle A\psi, \psi\rangle \; .$$

In other words, the expectation value of an operator $\hat{A}$ is given by

$$\langle A\rangle = \int \psi^*(\vec{r})\hat{A}\psi(\vec{r})d\vec{r}$$

and all physical observables are represented by such expectation values. Obviously, the value of a physical observable such as energy or density must be real, so it's required $\langle A\rangle$ to be real. This means that it must be $\langle A\rangle = \langle A\rangle^*$, or

$$\langle A\rangle = \int \psi^*(\vec{r})\hat{A}\psi(\vec{r})d\vec{r} = \int \left[\hat{A}\psi(\vec{r})\right]^* \psi(\vec{r})d\vec{r} = \langle A\rangle^*$$

An operator $\hat{A}$, which satisfy this condition are called *Hermitian*. One can also show that for a Hermitian operator,

$$\int \psi_1^*(\vec{r})\hat{A}\psi_2(\vec{r})d\vec{r} = \int \left[\hat{A}\psi_1(\vec{r})\right]^* \psi_2(\vec{r})d\vec{r}$$

for any two states ψ_1 and ψ_2.

For the eigenvalue problem of a self-adjoint, positive operator A

$$A\varphi = \lambda\varphi$$

the eigenvalues $\{\lambda\}$ are the discrete spectrum λ_n with either finite or countable infinite set of values

$$A\varphi_n = \lambda\varphi_n \quad , \quad \|\varphi_n\|^2 = 1$$

In this case the mean value of A is given by

$$\overline{A} := \langle \psi, A\psi\rangle \; .$$

Let w_n the probability, that the eigenvalue occurs of a measurement of the observables A then the mean value of A is defined by

$$\overline{A} := \sum_n w_n \lambda_n = \sum_n w_n \langle \varphi_n, A\varphi_n \rangle \qquad \varphi = \sum_n \alpha_n \varphi_n$$

and it holds

$$\overline{A} := \langle \psi, A\psi \rangle = \left\langle \sum_n \alpha_n \varphi_n, A(\sum_n \alpha_n \varphi_n) \right\rangle = \sum_n \alpha_n{}^* \alpha_n \langle \varphi_n, A\varphi_n \rangle$$

$$\overline{A} := \langle \psi, A\psi \rangle = \sum_n \alpha_n{}^* \alpha_n \lambda_n \langle \varphi_n, \varphi_n \rangle = \sum_n |\alpha_n|^2 \lambda_n \quad ,$$

i.e. $$w_n = |\alpha_n|^2 = |\langle \varphi_n, \varphi \rangle|^2 \quad .$$

The general solution of the Schrödinger equation is given by

$$\varphi(\vec{x}, t) = \sum_n c_n e^{-i\lambda_n \hbar t} \varphi_n(\vec{x}) \quad .$$

In case the operator A is only hermitian (without being positive definite) Hilbert, von Neumann and Dirac developed a corresponding spectral theory. This leads to a continuous spectrum $\lambda(\nu)$, indexed by a continuous ν. In this case $\psi(x;\nu)$ denotes an eigen function to the eigen value $\lambda(\nu)$. The norm of this function is infinite, i.e. the function is not an element of the Hilbert space. An approximation to this function with finite norm is given (with sufficiently small $\Delta\nu$) by the eigen differential

$$\Phi_{\Delta\nu}(x;\nu) = \frac{1}{\sqrt{\Delta\nu}} \int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2} \phi(x;\nu') d\nu' \quad .$$

All for the Hilbert space related properties are valid for the eigen differentials, but not for the eigenfunction itself. The scalar product of the eigenfunction is normed to a Dirac δ-function:

$$\langle \phi(x;\nu'), \phi(x;\nu'') \rangle = \delta(\nu' - \nu'') \quad .$$

The norm of the eigen differentials is given by:

$$\langle \Phi_{\Delta\nu}(x;\nu), \Phi_{\Delta\nu}(x;\nu') \rangle = \frac{1}{\Delta\nu} \int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2} \int_{\nu'-\Delta\nu/2}^{\nu'+\Delta\nu/2} \langle \phi(x;\eta') d\nu' \phi(x;\eta'') \rangle d\eta' d\eta''$$

$$\langle \Phi_{\Delta\nu}(x;\nu), \Phi_{\Delta\nu}(x;\nu') \rangle = \frac{1}{\Delta\nu} \int_{\nu-\Delta\nu/2}^{\nu+\Delta\nu/2} \int_{\nu'-\Delta\nu/2}^{\nu'+\Delta\nu/2} \delta(\eta' - \eta'') d\eta' d\eta''$$

The integral is 1 for $\nu = \nu'$ (with appropriate norm factor) and 0 if $|\nu - \nu'| > \Delta\nu$.

In case if ν is a momentum the eigendifferential gives a wave package with finite distance $\Delta\nu$ in the momentum space and therefore with finite distance $\Delta x \approx \frac{1}{\Delta\nu}$ in the particle space. Such a package can normed to the value 1 (1 particle). Δx (and correspondingly $\Delta\nu$) has to be larger than all other typical distances of the problem. In this sense eigendifferentials correspond to the formalism of wave package modeling.

The eigenfunctions of the discrete and continuous spectrum build an extended Hilbert space to ensure that for every ψ it holds

$$\psi(x) = \sum_n c_n \phi_n(x) + \int c(\nu')\phi(x;\nu')d\nu'$$

with

$$c_n = \langle \phi_n(x), \psi(x) \rangle$$

and

$$c(\nu) = \langle \phi(x;\nu), \psi(x) \rangle$$

It holds the Parceval identity:

$$\langle \psi, \psi \rangle = \sum_n |c_n|^2 + \int |c(\nu')|^2 d\nu'$$

and the eigendifferential are orthogonal wave packages.

If for every function $\in L_2$ such a representation is possible, one calls the system. $\{\phi\}$ a complete orthogonal system. Such a complete orthogonal system is not uniquely defined. There is always the degree of freedom

- to choose arbitrarily the phase of each eigenfunction

- the set of the non-standard eigenvalues can be orthogonized on different ways

- to replace the index ν of the continuous spectrum by an index $\mu(\nu)$ with $\mu(\nu)$ – differentiable, monotone function of ν. Then

$$\phi(x;\mu) = \frac{\phi(x;\nu)}{\sqrt{d\mu/d\nu}}.$$

Not all existing hermitian operators are built on a complete orthogonal system of eigenfunctions. For all operators, which represent physical observables, there exists such a complete orthogonal system.

An alternative polynomial system to the Hermite polynomials

We propose to apply the Lommel polynomials $g_{n.\nu+1}(x)$ as corresponding polynomial orthogonal system framework to build a (negatively scaled) Hilbert space. D. Dickinson's proof ([DDi]) of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral, enabled by the density function

$$d\psi_\nu = \frac{J_{\nu+1}(2\sqrt{x})}{\sqrt{x}J_\nu(2\sqrt{x})}dx \quad \text{with} \quad \frac{J_{\nu+1}(2\sqrt{x})}{\sqrt{x}J_\nu(2\sqrt{x})} = \lim_{n\to\infty}\frac{g_{n.\nu+1}(x)}{g_{n+1.\nu}(x)} \ ,$$

which is analytic outside any circle that contains the finite zeros of $J_\nu(1/x)$. The prize to be paid to build the orthogonality relation is an only stepwise density (bounded variation) function $d\psi_\nu$.

The Lommel polynomials $g_n(x) := g_{n.0}(x)$, defined by ([GWa] 9-6)

$$g_n(x) = \sum_0^{\le[n/2]}(-1)^m \frac{(\mathrm{n-m})!}{\mathrm{m!(n-2m)!}}\frac{\Gamma(\mathrm{n+1-m})}{\mathrm{m!}}x^m \ ,$$

$$h_n(\frac{1}{2x}) := x^{-n}g_n(x^2) \quad h_n(\frac{1}{2\sqrt{x}}) := x^{-n/2}g_n(x)$$

fulfill the recurrence relations

$$g_{n+1.\nu}(x) = (\nu+n+1)g_{n.\nu}(x) - xg_{n-1.\nu+1}(x) \ , \ g_0(x) := g_1(x) := 1 \ .$$

$$h_{n+1.\nu}(x) = 2x(n+\nu)h_{n.\nu}(x) - h_{n-1.\nu}(x) \ , \ h_{-1.\nu}(x) = 0 \ \ h_{0.\nu}(x) := 1$$

A relation between the modified Lommel polynomials and the Bessel function is given by Hurwitz's asymptotic formula ([GWa] 9-65):

$$J_0(\frac{1}{x}) = \lim_{n\to\infty}\frac{(2x)^{-n}}{n!}h_n(x) \ , \ J_0(\frac{2}{\sqrt{x}}) = \lim_{n\to\infty}\frac{x^{-n/2}}{n!}h_n(\frac{\sqrt{x}}{2})$$

From the above and [GWa] 9-65, it follows:

$$J_0(2\sqrt{x}) = \lim_{n\to\infty}\frac{g_n(x)}{n!} = \lim_{n\to\infty}\frac{x^{n/2}}{n!}h_n(\frac{1}{2\sqrt{x}}) = \lim_{n\to\infty}\frac{x^{n/2}}{n!}\frac{1}{\sqrt{n+1}}L_n(x)$$

$$\frac{J_1(2\sqrt{x})}{\sqrt{x}} = \lim_{n\to\infty}\frac{g_{n.1}(x)}{(n+1)!} .$$

Favards's theorem (([TCh] 7, II, theor. 6.4) implies that the Lommel polynomials are orthogonal polynomials with respect to a positive weighted, bounded variation measure function. We recall from [DDi]

$$(*) \qquad \sum_{k=1}^{\infty}\frac{1}{j_k{}^2}h_m(\frac{\pm 1}{j_k})h_n(\frac{\pm 1}{j_k}) = \frac{\delta_{n,m}}{2(n+1)} \quad .$$

With the relations above it follows

Proposition: For the Lommel polynomials the following orthogonality relation holds true

$$(**)\quad \sum_{k=1}^{\infty}\frac{g_n(\alpha_k)}{2\alpha_k^{(n+1)/2}}\frac{g_m(\alpha_k)}{2\alpha_k^{(m+1)/2}}=\frac{\delta_{n,m}}{2(n+1)}\ .$$

Orthogonal polynomials have only real zeros and are eigenfunctions of corresponding self-adjoint differential operators. Following the arguments from §2, [DBu] and [GPo3] this property implies that the zeros of its Mellin transforms lie all on the critical line.

The proof of the orthogonality of the modified Lommel polynomials is built on a properly defined Riemann-Stieltjes integral [DDi], enabled by the term

$$\frac{d\rho}{dx}:=\left[J_1(\frac{1}{x})\right]/\left[J_0(\frac{1}{x})\right]\ ,$$

which is analytic outside any circle that contains the finite zeros of $J_0(1/x)$. Hence it possesses a Laurent expansion about the origin that converges uniformly on and in any annulus, whose inside boundary has the finite zeros of $J_0(1/x)$ in its interior: Let C be the contour that encircles the origin in a positive direction and that lies within the annulus.

Then it holds [DDi]

$$\frac{1}{2\pi i}\int_C x^k h_n(x)d\rho=\begin{cases}0 & k<n\\ \dfrac{1}{2^{n+1}(n+1)} & k=n\end{cases}$$

Let $\alpha(x)$ the non-decreasing step function having increase of

$$\frac{1}{j_k^2}=\frac{1}{4\alpha_k}\qquad \text{at the point}\qquad x=\frac{\pm 1}{j_k}=\frac{1}{2\sqrt{\alpha_k}}\qquad \text{for } k=1,2,3,...$$

then it holds [DDi]

$$\int h_n(x)h_m(x)d\tilde{\alpha}(x)=\frac{\delta_{n,m}}{2^{n+1}(n+1)}\ .$$

Black-body radiation

A famous usage of Dirichlet's series is in the context of Planck's black-body radiation function

$$\frac{dR(\lambda,T)}{d\lambda}=\frac{c_1}{\lambda^5}\frac{1}{e^{c_2/\lambda T}-1}=\frac{c_1}{\lambda^5}\sum_1^\infty e^{-nc_2/\lambda T}$$

with $c_1=2\pi hc^2$ and $c_2=hc/k$. The relation to the Zeta function

$$\zeta(s)\Gamma(s)=\int_0^{+\infty}\frac{x^s}{e^x-1}\frac{dx}{x}$$

is given by

$$\frac{\pi^4}{90}=\zeta(4)\Gamma(4)=\int_0^{+\infty}x^4(\sum_1^\infty e^{-nx})\frac{dx}{x}=\int_0^{+\infty}x^{-4}(\sum_1^\infty e^{-\frac{n}{x}})\frac{dx}{x} \ .$$

This describes the total radiation and its spectral density at the same time, i.e.

$$g(x)dx=\frac{x^{-4}}{e^{1/x}-1}\frac{dx}{x}=\frac{x^4}{e^x-1}\frac{dx}{x}=g(\frac{1}{x})dx \ .$$

The weak formulation (and the positive Berry conjecture answer) should enable an alternative model for the total radiation and its spectral density.

Further information from internet

A. Einstein developed his quantum/photon concept motivated by the question: „if one moves exactly in parallel to a light signal (a photon or a wave?), how the light signal looks like? In principle it should be that the signal of light is a sequence of stationary waves, which are fixed in the time, i.e. the light signal should look like without any movement. If one follows it, it looks like a non-moving, oscillating, electromagnetic field. But something like this seems to be not existed neither caused by observation, nor by the Maxwell-equations model. The later ones exclude the existence of stationary, inelastic waves. Based on the Maxwell equations the electrons would have to lose its energy within nearly no time.

In any relativistic theory the vacuum, the state of lowest energy, if it exists in „reality", has to have the energy zero.

In the same way for any free particle with momentum $\vec{p}$ and mass m the energy has to be

$$E = \sqrt{m^2c^4 + \vec{p}^2c^2} .$$

In the literature the ground state energy of the harmonic operator is mostly defined by $\frac{1}{2}\hbar\omega$.

Already M. Planck knew that this cannot be, when deriving his radiation formula: he assigned states with n photons the energy $n\hbar\omega$, but not the value

$$(n+\frac{1}{2})\hbar\omega \quad ,$$

which is not compatible with the relativistic co-variant description of photons.

The ground state energy is not measurable. Its chosen value is therefore arbitrarily, triggered only by the fact, to keep calculations as easily as possible, and, mainly, to ensure convergent integrals/series. Energies of freely composed systems should be additive. For photons in a box section (cavity) there are infinite numbers of frequencies ω_i . If one assigns any frequency a ground state energy value $\hbar\omega_i \;/\, 2$, then the ground state energy without photons has the infinite energy

$$\frac{1}{2}\sum_i \hbar\omega_i = \infty .$$

The **miss understanding**, that the **ground state energy is fixed** and uniquely defined, starts already in the classical physics: The definition of the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}\omega^2 x^2 =: T + V$$

defines the non-measurable ground state energy in that way, that the state of lowest energy, the point $(x=0, p=0)$ in the phase space, that the energy is zero:

the kinetic energy of strings with mass ρ are given by

$$T = \rho \int_0^l \frac{1}{2} u_x^2(x,t) dx \ .$$

The internal forces of strings (being looked at as mechanical systems) are built on strains, depending proportionally from its lengths:

$$L = \int_0^l \sqrt{1 + u_x^2(x,t)} dx \ .$$

For small displacements this is replaced by

$$L = l + \Delta l = \int_0^l \left[1 + \frac{1}{2} u_x^2(x,t) + \ldots \right] dx \quad \text{with} \quad \Delta l \approx \int_0^l \frac{1}{2} u_x^2(x,t) dx \ .$$

Correspondingly the potential energy $V(x)$ is approximately defined by

$$V(L) = V(l + \Delta l) \approx V(l) + \Delta l \frac{dV}{dL}\Big|_{L=l} \ .$$

Putting

$$\sigma_s := \frac{dV}{dL}\Big|_{L=l}$$

as "tension" or "strain constant", the choice

$$V(l) := 0$$

simplifies the algebraic term for the potential energy V in the form:

$$V \approx \sigma_s \int_0^l \frac{1}{2} u_x^2(x,t) dx \ .$$

For example for the "string velocity"

$$c_s := \sqrt{\frac{\sigma_s}{\rho}}$$

the wave equation of strings is given by

$$u_{tt} - c_s^2 u_{xx} = 0 \ .$$

Alternatively to $V(x)$ in case of the harmonic oscillator one could have chosen instead e.g.

$$V(x)=\frac{1}{2}\omega^2x^2-\hbar\omega/2$$

or (with reference to the theory of minimal surfaces, using $1+\sinh^2 x=\cosh^2 x$)

$$1+V(x)=\kappa\cosh x\,.$$

For a single particle in a potential energy $V(x,t)$ the Schrödinger equation is ([RFe] 4-1)

$$\psi_t(x,t)=-\frac{i}{\hbar}\overline{H}\psi(x,t)$$

with

$$\overline{H}\psi(x,t):=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x}+V(x,t)\ .$$

With respect to our proposal above we note

$$\overline{H}x-x\overline{H}=-\frac{\hbar^2}{m}\frac{\partial}{\partial x} \qquad \text{resp.} \qquad \left\|(\overline{H}x-x\overline{H})\psi\right\|_{-1}=c\left\|\psi\right\|_0 \quad .$$

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