Parabolic Equations with Continuous Initial Data

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Declaration

The work in this thesis is my own except where otherwise stated.

Julie Clutterbuck

Thanks

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Abstract

The aim of this thesis is to derive new gradient estimates for parabolic equations. The gradient estimates found are independent of the regularity of the initial data. This allows us to prove the existence of solutions to problems that have non-smooth, continuous initial data. We include existence proofs for problems with both Neumann and Dirichlet boundary data.

The class of equations studied is modelled on mean curvature flow for graphs. It includes anisotropic mean curvature flow, and other operators that have no uniform non-degeneracy bound.

We arrive at similar estimates by three different paths: a 'double coordinate' approach, an approach examining the intersections of a solution and a given barrier, and a classical geometrical approach.

Contents

Chapter 1

Preface

Mean curvature flow

Let $\{M_t\}$ be a family of hypersurfaces, each smoothly embedded in $\mathbb{R}^{n+1}$ and indexed by $t \in [0, T]$. We say M_t is moving by mean curvature flow when

$$\frac{\partial x}{\partial t} = \mathbf{H},$$

where $\mathbf{H}$ is the mean curvature vector at $x \in M_t$.

In the last twenty-five years, this flow has been the subject of concerted study, as have other geometric flows such as the Ricci flow and Gauss curvature flow. Notable results have included Grayson's proof that the curve shortening flow (which is mean curvature flow, reduced to one space dimension) shrinks embedded curves to a spherical point [17] and Huisken's proof that convex surfaces become spherical under mean curvature flow [18].

If we observe that $\mathbf{H} = \Delta_{M_t} x$, where Δ_{M_t} is the Laplace-Beltrami operator on the manifold M_t, then it seems natural to consider mean curvature flow as the heat flow for manifolds.

Unlike classical heat flow, this is a nonlinear operator, but it still has some of the same attributes; in particular, this flow exhibits a *smoothing property*. In [14], Ecker and Huisken showed that if the initial surface is given by a locally Lipschitz graph, then there exists a smooth solution for positive times. In this thesis, this result is extended to non-Lipschitz initial conditions.

Mean curvature flow and parabolic differential equations

Chapter 2 is a short introduction to parabolic differential operators and some key results in mean curvature flow.

If M_t is locally represented as $\operatorname{graph} u$ for some $u : \Omega \times [0, T] \to \mathbb{R}$, then u will satisfy the parabolic equation

$$\frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2}\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \left(\delta^{ij} - \frac{D_i u D_j u}{1 + |Du|^2}\right) D_{ij} u.$$

This places mean curvature flow in the setting of classical parabolic partial differential equations, the framework for most of this thesis.

Motivated by this setting, we examine other parabolic operators that have similar diffusion properties to mean curvature flow. This includes anisotropic mean curvature flow, a generalization of mean curvature flow arising in many physical applications.

Gradient estimates using a ‘double coordinates’ approach

In this thesis three distinct methods are used for deriving gradient estimates. The first of these is introduced in Chapter 3. It originated with Kružkov in [22]. Given a parabolic equation in one space dimension

$$u_t = F\left(u_{xx}, u_x, u, x, t\right)$$

one can form an evolution equation for the difference $w(x,y,t) = u(x,t) - u(y,t)$,

$$w_t = F\left(u_{xx}, u_x, u(x), x, t\right) - F\left(u_{yy}, u_y, u(y), y, t\right).$$

Under favourable conditions on F, one can use the maximum principle and an appropriate barrier to find estimates for w that depend on $|x-y|$. Letting $|x-y| \to 0$ will then give a gradient estimate for u.

In this thesis, Kružkov's method is extended by making use of the full Hessian

$$[D^2 w] = \begin{bmatrix} w_{xx} & w_{xy} \\ w_{yx} & w_{yy} \end{bmatrix}$$

rather that just the diagonal elements w_{xx} and w_{yy}. In the one-dimensional case this is of little importance, but in the higher-dimensional case it gives us greater scope to choose barriers.

In Chapter 3, we also describe a barrier which begins with an unbounded gradient, but instantly becomes smooth.

Such barriers allow estimates that are independent of initial gradient bounds, and that therefore may be used to prove the existence of solutions to parabolic equations with continuous initial data. This is one of the main features of the gradient estimates in this thesis.

The ‘double coordinate’ method is extended to higher dimensions in Chapter 5 for a class of operators that have similar diffusion to the mean curvature flow.

As this class includes anisotropic curvature flow , this is a significant improvement to the existing regularity theory.

Gradient estimates are found for both entire periodic solutions and boundary value problems.

Existence results

The gradient estimates of Chapters 3 and 5 may be used to extend standard existence results.

We do this for a class of parabolic equations in the one dimensional case in Chapter 4; for mean curvature flow with Neumann boundary conditions in Chapter 6; and for mean curvature flow with Dirichlet boundary conditions in Chapter 7.

Gradient estimates by counting intersections

The second technique for finding gradient estimates is found in Chapter 8, and it involves examination of the intersections between a given solution u and a barrier φ.

In [5], Angenent proved that the number of points in the *zero set* — the set where $u(x,t) = 0$ — of a solution to a parabolic equation in one space dimension is non-increasing.

This is applied to the difference $u - \varphi$. The intersections of u and φ are the zeroes of $u - \varphi$, and so Angenent's results allow us to show that u and φ do not develop new intersections as they evolve.

When two functions intersect only once, the gradient of one of them dominates the gradient of the other at that point. Tailoring the barriers gives us estimates for $u_x(x,t)$ in terms of the height of $u(x,t)$, the time t, and (in the case of bounded domains) the distance of x from the boundary. Again, there is no dependence on an initial gradient estimate.

As Angenent's results are limited to equations in one space dimension — it is difficult to imagine what a generalization of these results to higher dimensions would look like — this technique applies only to parabolic operators in one space dimension.

These methods will apply to a wide range of parabolic operators, provided that suitable barriers exist. In particular, we can apply this method to the class of operators studied in Chapter 4.

Gradient estimates using a geometric approach

The third method for finding gradient estimates, in Chapter 9, is a rather geometrical approach found in the classic Ecker–Huisken curvature flow papers [13], [14], [18], [19] and [20].

A maximum principle is applied to the difference $Z = v - \varphi$, where v is a "gradient function" (for example, $v = \sqrt{1 + |Du|^2}$), and φ is a barrier. In contrast to the earlier two approaches, this creates a direct estimate for the gradient Du itself, rather than for the difference $u(x) - u(y)$.

We apply this to the mean curvature flow (re-creating some of the results obtained in earlier chapters) and also to the anisotropic mean curvature flow, under some restrictions on the degree of anisotropy allowed. Results for entire periodic solutions and strictly interior results are found in both cases. The estimates found are again independent of initial gradient bounds, but dependent on the height.

Appendices

An inventory of standard results for parabolic partial differential equations, relevant to the existence results in Chapters 4, 6 and 7, is included for the convenience of the reader. There is also a nomenclature listing.

Chapter 2

Mean curvature flow and parabolic equations

2.1 Parabolic equations

An operator $\boldsymbol{P} : \mathbb{R} \times \mathbb{S}^n \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R} \times [0, T]$ is considered parabolic on a domain when

$$\boldsymbol{P}(z + \sigma, r + \eta, p, q, x, t) > \boldsymbol{P}(z, r, p, q, x, t)$$

for any positive definite $\eta \in \mathbb{S}^n$, positive number σ, and any (z, r, p, q, x, t) in the domain. Here, $\mathbb{S}^n$ is the set of $n \times n$ symmetric matrices.

In this thesis we look only at operators of the form

$$\boldsymbol{P}u = \boldsymbol{P}(-u_t, D^2u, Du, u, x, t) = -u_t + F(D^2u, Du, u, x, t).$$

If F is differentiable with respect to the first variable, then $\boldsymbol{P}$ will be parabolic if the matrix of derivatives $F_r = \left[\dfrac{\partial F}{\partial r_{ij}}\right]$ is positive definite.

If we can write $F(r, p, q, x, t) = a^{ij}(p, q, x, t)r_{ij} + b(p, q, x, t)$ for some symmetric $a : \mathbb{R}^n \times \mathbb{R} \times \Omega \times [0, T] \to \mathbb{S}^n$, then we call the operator *quasilinear*. It is to quasilinear operators that we will pay most attention in the following pages.

In this case, $F_r = [a^{ij}]$ and so the operator is parabolic on $\mathcal{S}$ if $[a^{ij}(p, q, x, t)]$ is positive definite for all $(p, q, x, t) \in \mathcal{S}$, where $\mathcal{S}$ is a subset of $\mathbb{R}^n \times \mathbb{R} \times \Omega \times [0, T]$.

We denote the maximum and minimum eigenvalues of $[a^{ij}(p, q, x, t)]$ (or F_r) by $\lambda(p, q, x, t)$ and $\Lambda(p, q, x, t)$.

If the ratio Λ/λ is bounded on $\mathcal{S}$, then $\boldsymbol{P}$ is called *uniformly parabolic* on $\mathcal{S}$.

An operator is parabolic with respect to a function u when $\boldsymbol{P}(D^2u, Du, u, x, t)$ is parabolic.

When $\lambda \not> 0$ — for example, when $a^{ij}(Du, u, x, t) = |Du|^{p-1}\delta^{ij}$, as in the parabolic p-Laplacian equation — such an operator is called *degenerate*.

The key to much of the theory used here is the *Comparison Principle*. As presented in [25]:

Theorem 2.1 (Quasilinear comparison principle 1). *Suppose that $\boldsymbol{P}$ is a quasilinear parabolic operator*

$$\boldsymbol{P}u = -u_t + a^{ij}(Du, u, x, t)D_{ij}u + b(Du, u, x, t).$$

Let u and v be in $C^{2,1}(\overline{\Omega} \setminus \mathcal{P}\Omega) \cap C(\overline{\Omega})$ and let $\boldsymbol{P}$ be parabolic with respect to either u or v. Then if $\boldsymbol{P}u > \boldsymbol{P}v$ in $\overline{\Omega} \setminus \mathcal{P}\Omega$ and if $u < v$ on $\mathcal{P}$, then $u < v$ in $\overline{\Omega}$.

Here $\mathcal{P}$ denotes the *parabolic boundary*

$$\mathcal{P}\left(\Omega \times [0, T]\right) := \Omega \times \{0\} \cup \partial\Omega \times [0, T].$$

The proof of this theorem is simple, and is an excellent illustration of later arguments.

Proof: Suppose that there is an interior point x_0 at time $t_0 > 0$ where $u = v$ for the first time. Since this is an internal maximum of $w = u - v$, $Dw = Du(x_0, t_0) - Dv(x_0, t_0) = 0$ and $D_{ij}w = D_{ij}u - D_{ij}v$ must be negative semi-definite. Now,

$$\begin{aligned} w_t &= u_t - v_t \\ &= -\boldsymbol{P}u + a^{ij}(Du, u, x, t)D_{ij}u + b(Du, u, x, t) \\ &\qquad\qquad + \boldsymbol{P}v - a^{ij}(Dv, v, x, t)D_{ij}v - b(Dv, v, x, t) \\ &< a^{ij}(Du, u, x, t)D_{ij}w. \end{aligned}$$

However, as this is the first such maximum we must have $w_t \geq 0$, which give us a contradiction. It follows that $u < v$. ☐

We also use the following form, again as in [25] :

Theorem 2.2 (Quasilinear comparison principle 2). *Suppose that $\boldsymbol{P}$ is a quasilinear parabolic operator*

$$\boldsymbol{P}u = -u_t + a^{ij}(Du, x, t)D_{ij}u + b(Du, u, x, t),$$

and that there is an increasing function $k(M)$ such that $b(p, q, x, t) + k(M)q$ is a decreasing function of q on $\Omega \times [-M, M] \times \mathbb{R}^n$ for any $M > 0$. If u and v are functions in $C^{2,1}(\Omega) \cap C(\overline{\Omega})$ with $\boldsymbol{P}u \geq \boldsymbol{P}v$ in $\overline{\Omega} \setminus \mathcal{P}\Omega$ and $u \leq v$ on $\mathcal{P}\Omega$, and if $\boldsymbol{P}$ is parabolic with respect to either u or v, then $u \leq v$ in $\overline{\Omega}$.

2.2 Mean curvature flow

If our family of hypersurfaces M_t is also a family of embeddings $\mathbf{F}_t : M^n \to \mathbb{R}^{n+1}$, then we can write mean curvature flow as

$$\frac{\partial}{\partial t}\mathbf{F}_t = \mathbf{H}, \tag{2.1}$$

where $\mathbf{H}$ is the mean curvature vector at $\mathbf{F}_t(x) \in M_t$. In the case that M_t can be written locally as a graph over a set $\Omega \in \mathbb{R}^n$, we write $\mathbf{F}_t(x) = (x, u(x, t))$, and can calculate geometric quantities such as the upwards unit normal

$$\nu = \frac{(-Du, 1)}{\sqrt{1+|Du|^2}},$$

the metric on the surface

$$g_{ij} = \delta_{ij} + D_i u D_j u,$$

the second fundamental form

$$h_{ij} = -\frac{D_{ij}u}{\sqrt{1+|Du|^2}},$$

and the mean curvature

$$H = g^{ij}h_{ij} = -\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

The mean curvature vector is $\mathbf{H} = H\nu$, and so (if we remove movement tangential to the surface) mean curvature flow for graphs is given by

$$\frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) D_{ij}u.$$

In the case when $n = 1$, this reduces to curve-shortening flow

$$\frac{\partial u}{\partial t} = \frac{u_{xx}}{1+u_x^2}.$$

With reference to the previous section, note that the largest and smallest eigenvalues for mean curvature flow for graphs are $\Lambda = 1$ and $\lambda = (1+|p|^2)^{-1}$, so it will be uniformly parabolic only when the gradient is bounded.

Whether studied in a geometric setting as (2.1), or as a special case of a quasilinear parabolic differential equation as (2.2), the comparison principle has been crucial. Applied to mean curvature flow, it gives:

Theorem 2.3. *Let M_t and M_t' be two smooth compact surfaces moving under mean curvature flow. If they are disjoint at the initial time, they are disjoint at later times.*

We can also make similar comparisons between surfaces with boundaries, and between other quantities (such as the *gradient function* $v = \sqrt{1+|Du|^2}$). The following theorem from [14] is one such result.

Theorem 2.4 (Interior gradient estimate). *Suppose that u satisfies the mean curvature flow equation* (2.2) *on a cylinder $B_R(y_0) \times [0,T]$. Then we have an estimate for the gradient at the center of the ball at later times:*

$$\sqrt{1+|Du(y_0,t)|^2} \le C_1 \sup_{y\in B_R(y_0)} \sqrt{1+|Du(y,0)|^2} \exp\left[\frac{C_2}{R^2}\left(\sup_{B_R(y_0)\times[0,T]} u(y,t) - u(y_0,t)\right)^2\right],$$

where C_1 *and* C_2 *depend only on* n*.*

We will also use the following *a priori* estimates for higher derivatives, from the same paper:

Theorem 2.5 (C^2 interior estimate for mean curvature flow). *Suppose that* u *satisfies* (2.2) *on* $B_R(y_0) \times [0,T]$*. Then for arbitrary* $0 \leq \theta < 1$ *the estimate*

$$\sup_{B_{\theta R}(y_0)} |A|^2(t) \leq c(n)(1-\theta^2)^{-2}\left(\frac{1}{R^2}+\frac{1}{t}\right) \sup_{B_R(y_0)\times[0,t]} (1+|Du|^2)^2$$

holds for all $0 \leq t \leq T$*. Here* $|A|^2 = h_{ij}h_{kl}g^{ik}g^{kl}$ *.*

Theorem 2.6 (C^k interior estimate for mean curvature flow). *Suppose that* u *satisfies* (2.2) *on* $B_R(y_0) \times [0,T]$*. Then for* $m \geq 0$ *and arbitrary* $0 \leq \theta < 1$ *we have the estimate*

$$\sup_{B_{\theta R}(y_0)} |\nabla^m A|^2(t) \leq c_m(1-\theta^2)^{-2}\left(\frac{1}{R^2}+\frac{1}{t}\right)^{m+1},$$

where c_m *is a constant depending on* n, m *and* $\sup_{B_R(y_0)\times[0,t]}(1+|Du|^2)^{1/2}$*.*

A bound on $|A|$ gives a bound on $|u|_{C^2}$, since (using coordinates in which h is diagonal)

$$\begin{aligned}
|A|^2 &= h_{ij}h_{kl}g^{ik}g^{kl} \\
&= h_{ii}g^{ik}g^{kl}h_{ll} \\
&\geq \frac{1}{(1+|Du|^2)^2}\sum(h_{ii})^2 \\
&\geq \frac{1}{(1+|Du|^2)^3}\sup_{ij}|D_{ij}u|^2,
\end{aligned}$$

where we have used that the smallest eigenvalue of g^{ij} is $(1+|Du|^2)^{-1}$. So,

$$|D_{ij}u| \leq (1+|Du|^2)^{3/2}|A|,$$

and in a similar manner, bounds on derivatives of $|A|$ give bounds on higher derivatives of u.

These estimates may be used to show long-time existence results, such as the following from [13]:

Theorem 2.7. *If* u_0 *is a locally Lipschitz, entire graph over* $\mathbb{R}^n$*, then there is smooth solution to* (2.2) *for all* $t > 0$*.*

In the following pages, we will derive new existence results of this sort.

Chapter 3

Gradient estimates for parabolic equations of curve shortening flow type in one space dimension

In this chapter we outline gradient estimates for a class of parabolic equations in one space dimension.

This chapter takes inspiration from the work of Huisken in [21], where he investigated embedded plane curves evolving by curve shortening flow by looking at the evolution equation for the quotient of $d(p, q)$, the distance between two points p and q in the metric of the plane, and $l(p, q)$, the length of curve between p and q. This introduced a double set of space coordinates (those around the point p and those around the point q). At a maximum point, the first and second derivative conditions give strong conditions at both p and q, allowing close examination of all possible situations. An application of the maximum principle resulted in a new proof of Grayson's theorem regarding the evolution of embedded curves.

In this chapter, we follow the approach of Kružkov in [22] (and well described in Lieberman's book [25], chapter XI, section 6).

If $u(x, t)$ solves a parabolic partial differential equation in one space variable, then $v(x, y, t) = u(x, t) - u(y, t)$ solves a parabolic equation in two space variables, for which we can seek a barrier.

In the paper cited, Kružkov was interested in fully nonlinear equations

$$u_t = F(u_{xx}, u_x, u, x, t)$$

with uniform parabolicity condition

$$\frac{\partial}{\partial r} F(r, p, q, x, t) \geq A > 0.$$

In this section, we do not require uniform parabolicity, but in order to show existence of the barriers, we will require a scaling similar to that of the curve shortening flow equation

$$u_t = \frac{u_{xx}}{1 + {u_x}^2},$$

in that $\dfrac{\partial F}{\partial r} \sim |p|^{-2}$ for large $|p|$.

We begin with a description of the ideas motivating the method. The notation $'$ will indicate derivatives with respect to the space variable, which I hope I will use only where this is unambiguous.

3.1 Outline of the 'double coordinate' method

Consider a smooth $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfying

$$\begin{aligned} u_t &= a(u_x, u, x, t)u_{xx} + b(u_x) \\ u(x,0) &= u_0. \end{aligned}$$

Let $Z : \mathbb{R} \times \mathbb{R} \times (0,T) \to \mathbb{R}$ be given by

$$Z(x,y,t) = u(y,t) - u(x,t) - \phi(|y-x|,t),$$

where ϕ is some smooth function.

Suppose now that Z attains a maximum at some point (x,y,t), with $y > x$. At the maximum point, the first derivatives are zero, and so

$$\begin{aligned} 0 = Z_x &= -u'(x,t) + \phi'(y-x,t) \\ 0 = Z_y &= u'(y,t) - \phi'(y-x,t). \end{aligned} \tag{3.1}$$

Similarly, the matrix of second order partial derivatives is non-positive, by which we mean that for all $v \in \mathbb{R}^2$, $v^T[D^2Z]v \le 0$, where $[D^2Z]$ is the Hessian matrix

$$\begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} = \begin{bmatrix} -u''(x,t) - \phi''(y-x,t) & \phi''(y-x,t) \\ \phi''(y-x,t) & u''(y,t) - \phi''(y-x,t) \end{bmatrix}. \tag{3.2}$$

If we now consider the evolution equation satisfied by Z,

$$\begin{aligned} \frac{\partial Z}{\partial t} &= u_t(y,t) - u_t(x,t) - \phi_t(|y-x|,t) \\ &= a\left(u_y, u(y,t), y, t\right) u_{yy}(y,t) - a\left(u_x, u(x,t), x, t\right) u_{xx}(x,t) \\ &\qquad + b(u_y) - b(u_x) - \phi_t(|y-x|,t); \end{aligned}$$

and if we take this at the local maximum we have

$$\begin{aligned} \frac{\partial Z}{\partial t} &= a\left(\phi', u(y,t), y, t\right)\left(Z_{yy} + \phi''\right) - a\left(\phi', u(x,t), x, t\right)\left(-Z_{xx} - \phi''\right) \\ &\quad + b(\phi') - b(\phi') - \phi_t(|y-x|,t) \\ &= \operatorname{trace}\left(\begin{bmatrix} a\left(\phi', u(x,t), x, t\right) & c_1 \\ c_2 & a\left(\phi', u(y,t), y, t\right) \end{bmatrix}\begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}\right) \\ &\quad + a\left(\phi', u(y,t), y, t\right)\phi'' + a\left(\phi', u(x,t), x, t\right)\phi'' - (c_1 + c_2)\phi'' - \phi_t, \end{aligned}$$

for some c_1 and c_2. If the first matrix above is positive semi-definite, then as $[D^2Z]$ is negative semi-definite, the trace above is non-positive and

$$\frac{\partial Z}{\partial t} \leq \phi'' \left[a\left(\phi', u(x,t), x, t\right) + a\left(\phi', u(y,t), y, t\right) - c_1 - c_2 \right] - \phi_t.$$

A useful choice for c_1 and c_2 that makes the first matrix positive semi-definite is $c_1 = -a\left(\phi', u(x,t), x, t\right)$, $c_2 = -a\left(\phi', u(y,t), y, t\right)$; then

$$\frac{\partial Z}{\partial t} \leq 2\big(a\left(\phi', u(x,t), x, t\right) + a\left(\phi', u(y,t), y, t\right)\big)\phi'' - \phi_t. \tag{3.3}$$

The idea now is to choose ϕ in a way so that at the local maximum, $Z_t \leq 0$. We begin by observing that for simple equations, a solution to a simplified version of the equation itself is acceptable for use as the barrier ϕ.

Remark: We could simplify the method by choosing $c_1 = c_2 = 0$, in which case the factor of 2 is absent from (3.3). The use of cross-derivatives will be important when we extend this method to higher dimensions.

3.2 An estimate for periodic solutions

Theorem 3.1. *Suppose $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ is a H_2, periodic and bounded solution of*

$$u_t = a(u_x)u_{xx} + b(u_x) \tag{3.4}$$

with initial condition

$$u(\cdot, 0) = u_0,$$

with $u(x,t) = u(x+L,t)$ and $\operatorname{osc} u(\cdot,t) \leq M$.

If $\varphi : \mathbb{R} \times (0,T) \to \mathbb{R}$ is a solution of

$$\varphi_t \geq a(\varphi_x)\varphi_{xx},$$

with the initial and boundary conditions

$$\begin{aligned} &\varphi(x,t) \to 1 \text{ as } t \to 0 \text{ for } x > 0, \\ &\varphi(0,t) = 0 \text{ for } t > 0, \\ &\varphi(x,t) \to 1 \text{ as } x \to \infty \text{ for } t > 0, \end{aligned}$$

then

$$|u(x,t) - u(y,t)| \leq M\varphi\left(\frac{|y-x|}{M}, \frac{4t}{M^2}\right).$$

Proof: Following on from the previous remarks, we set

$$Z(x,y,t) := u(y,t) - u(x,t) - \phi(|y-x|, t)$$

and choose $\phi(z,t) = M\varphi(z/M, 4t/M^2)$.

As $t \to 0$, $\phi \to M$ for all $z \neq 0$, and so $Z(x,y,0) \leq 0$.

As u is periodic, $Z(x,y,t) = Z(x+L, y+L, t)$ and so Z is periodic over strips

$$\{ (x,y) : 2nL \le y + x \le 2(n+1)L \} .$$

Within each strip, $Z(y,y,t) = 0$ and

$$Z(x,y,t) = u(y,t) - u(x,t) - \phi \le M - \phi \to 0$$

as $|y-x| \to \infty$, so Z attains its spatial maximum in each strip, and hence in the entire plane, for each $t > 0$. We can calculate

$$\begin{aligned} \phi_t &= \frac{4}{M}\varphi_t\left(\frac{z}{M}, \frac{4t}{M^2}\right) \\ &\ge \frac{4}{M}a(\varphi')\varphi'' \\ &= \frac{4}{M}a(\phi')M\phi'' \\ &= 4a(\phi')\phi'', \end{aligned}$$

and in particular at a maximum point (x,y,t) with $x \ne y$ and Z non-negative, equation (3.3) becomes

$$\frac{\partial Z}{\partial t} \le 4a(\phi')\phi'' - \phi_t \le 0.$$

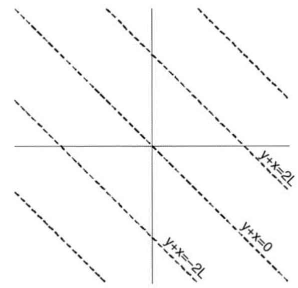


Figure 3.1: Z is periodic over strips

Therefore, at such a maximum point Z is non-increasing in time and so $Z \le 0$.

The reason for the restriction $x \ne y$ is that ϕ is not differentiable here. When the maximum *is* attained at such a point, then $Z(x,x,t) = 0$, so in either case $Z \le 0$ for all t. The result follows. □

We can find explicit estimates for more general equations by choosing an explicit barrier.

3.3 Description of a barrier

This barrier, ψ, will be used often.

Let Φ be the fundamental solution to the heat equation,

$$\Phi(y,t) = \frac{1}{\sqrt{t}} \exp\left(\frac{-cy^2}{t}\right),$$

so that $\Phi_{yy} = 4c\Phi_t$, where c is a positive constant and $t > 0$. Implicitly define $\psi(z,t)$ by

$$z = \Phi(\psi - 1, t) - \Phi(\psi + 1, t).$$

This function has the property that as $t \to 0$,

$$\psi(z,t) \to \begin{cases} 1 & z > 0, \\ -1 & z < 0. \end{cases}$$

and that $\psi(0,t) = 0$ for all $t > 0$.

We can calculate

$$\psi' = \frac{1}{\Phi_y(\psi - 1, t) - \Phi_y(\psi + 1, t)},$$
$$\psi'' = -\left(\psi'\right)^3 \left[\Phi_{yy}(\psi - 1, t) - \Phi_{yy}(\psi + 1, t)\right],$$

and (third derivatives are included for completeness but not used until a later chapter)

$$\psi''' = 3\frac{(\psi'')^2}{\psi'} - (\psi')^4 \left[\Phi_{yyy}(\psi - 1, t) - \Phi_{yyy}(\psi + 1, t)\right],$$

while

$$\frac{\partial \psi}{\partial t} = -\frac{\Phi_t(\psi - 1, t) - \Phi_t(\psi + 1, t)}{\Phi_y(\psi - 1, t) - \Phi_y(\psi + 1, t)}.$$

Routine calculations yield

$$\begin{aligned} \Phi_y &= -\frac{2cy}{t^{3/2}} \exp\left(-\frac{cy^2}{t}\right), \\ \Phi_{yy} &= \frac{2c}{t^{3/2}} \left[\frac{2cy^2}{t} - 1\right] \exp\left(-\frac{cy^2}{t}\right), \\ \Phi_{yyy} &= \frac{4c^2 y}{t^{5/2}} \left[3 - \frac{2cy^2}{t}\right] \exp\left(-\frac{cy^2}{t}\right), \\ \Phi_t &= \frac{1}{2t^{3/2}} \left[-1 + \frac{2cy^2}{t}\right] \exp\left(-\frac{cy^2}{t}\right). \end{aligned} \tag{3.5}$$

The partial differential equation satisfied by ψ is

$$\frac{\partial \psi}{\partial t} = \frac{1}{4c} \frac{\psi''}{\psi'^2}.$$

3.4 An explicit estimate for periodic equations

Theorem 3.2. *Let $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ be a H_2 solution of*

$$u_t = a(u_x, u, x, t) u_{xx} + b(u_x)$$
$$u(\cdot, 0) = u_0$$

where u_0 is continuous; both u_0 and a are periodic, $u_0(x+L) = u_0(x)$, $a(p,q,x,t) = a(p,q,x+L,t)$ (and therefore u is also periodic); $\operatorname{osc} u(\cdot,t) \le M$; and where we can find positive constants A and P such that

$$a(p,q,x,t)p^2 \ge A \text{ for all } |p| \ge P. \tag{3.6}$$

Then there is a $T' > 0$ such that for $t \in (0, T']$,

$$|u_x| \le C_1\sqrt{t}\,(1+t)\exp(C_2/t),$$

where T', C_1 and C_2 are dependent on M, A and P.

Proof: Let Z be as before, with $\phi(z,t) := 2M\psi(z/2M, t/4M^2)$ for the ψ defined in Section 3.3, with the constant c given by $c = \max\left(\frac{1}{16A}, CP^2\right)$, where C will be chosen later.

Consider the region

$$G := \left\{ (x,y,t) : 0 \le y-x \le z_M(t), 0 \le t \le 2cM^2/3 \right\},$$

where $z_M(t)$ satisfies $\phi(z_M(t),t) = M$. Explicitly,

$$z_M(t) = \frac{4M^2}{\sqrt{t}}\left[\exp(-cM^2/t) - \exp(-9cM^2/t)\right]. \tag{3.7}$$

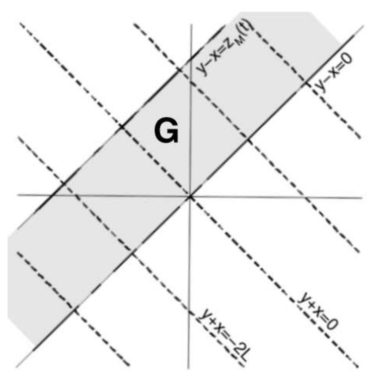


Figure 3.2: The (periodic) region G at some time t

As before, Z is periodic over strips parallel to $y+x=0$ and so it attains its maximum on G. We first show that $Z \le 0$ on the boundary of G.

For $y-x \ne 0$, as $t \to 0$, $\phi \to 2M$ and so $Z < 0$.

At $y-x=0$, $\phi(0,t) = 0$ for all $t > 0$, and so $Z(x,x,t) = 0$. At $y-x = z_M(t)$, $\phi(z_M,t) = M$ and so $Z \le 0$.

Now suppose that Z attains a maximum on the interior of G.

It follows from (3.3) that at the maximum,

$$\begin{aligned}
\frac{\partial Z}{\partial t} &\leq 2\left[a\left(\phi', u(x,t), x, t\right) + a\left(\phi', u(y,t), y, t\right)\right]\phi'' - \phi_t \\
&= 2\left[a\left(\psi', u(x,t), x, t\right) + a\left(\psi', u(y,t), y, t\right)\right]\frac{\psi''}{2M} - \frac{\psi_t}{2M} \\
&= \frac{\psi''}{2M}\left[2a\left(\psi', u(x,t), x, t\right) + 2a\left(\psi', u(y,t), y, t\right) - \frac{1}{4c\psi'^2}\right].
\end{aligned}$$

The second derivative

$$\psi'' = -\left(\psi'\right)^3\left[\phi_{yy}(\psi-1, t/4M^2) - \phi_{yy}(\psi+1, t/4M^2)\right]$$

is negative, as ψ' is positive, and the part inside the square brackets $[\cdot]$ is positive in G if $t \leq 2cM^2/3 = T'$.

We can estimate

$$\begin{aligned}
\psi'^2 &\geq \inf_G \psi'^2 \\
&\geq \left(\sup_{\substack{0\leq\psi\leq 1/2\\ 0\leq t\leq T'}} \Phi_y(\psi-1, t/4M^2) - \Phi_y(\psi+1, t/4M^2)\right)^{-2} \\
&\geq \left(\sup_{0\leq\psi\leq 1/2}\ \sup_{0\leq t\leq T'} \left|\Phi_y(\psi-1, t/4M^2)\right| + \left|\Phi_y(\psi+1, t/4M^2)\right|\right)^{-2} \\
&\geq \left(\sup_{0\leq\psi\leq 1/2}\left|\Phi_y\left(\psi-1, \frac{2}{3}c(\psi-1)^2\right)\right| + \left|\Phi_y\left(\psi+1, \frac{2}{3}c(\psi+1)^2\right)\right|\right)^{-2} \\
&= \left(\sup_{0\leq\psi\leq 1/2} \frac{2e^{-3/2}}{(2/3)^{3/2}\sqrt{c}|\psi-1|^2} + \frac{2e^{-3/2}}{(2/3)^{3/2}\sqrt{c}|\psi+1|^2}\right)^{-2} \\
&\geq c\frac{2e^3}{3^3 5^2} \\
&= P^2,
\end{aligned}$$

where the last line follows by choosing $C = 3^3 5^2/(2e^3)$ and recalling that $c \geq CP^2$.

Now we can use the condition (3.6), controlling the degeneracy of a, to estimate

$$\begin{aligned}
\frac{\partial Z}{\partial t} &\leq \frac{\psi''}{2M}\left[2a\left(\psi', u(x,t), x, t\right) + 2a\left(\psi', u(y,t), y, t\right) - \frac{1}{4c\psi'^2}\right] \\
&\leq \frac{\psi''}{2M}\left[4\frac{A}{\psi'^2} - \frac{1}{4c\psi'^2}\right] \\
&\leq 0
\end{aligned}$$

as $c \geq (16A)^{-1}$. So, $Z_t \leq 0$ at an internal maximum, $Z \leq 0$ on the boundary, and so $Z \leq 0$ on G.

Explicitly, for $(x, y, t) \in G$, $Z \leq 0$ means

$$\begin{aligned} u(x,t) - u(y,t) &\leq 2M\psi\left(\frac{|y-x|}{2M}, \frac{t}{4M^2}\right) \\ &\leq |y-x| \sup_z \psi'\left(\frac{z}{2M}, \frac{t}{4M^2}\right) \\ &= |y-x|\psi'\left(0, \frac{t}{4M^2}\right) \\ &= |y-x|\frac{t^{3/2}}{4c(2M)^3}\exp\left(\frac{4cM^2}{t}\right), \end{aligned} \tag{3.8}$$

which is an estimate for the difference quotient $|u(x,t) - u(y,t)|/|x-y|$.

We can obtain an estimate for $(x, y, t) \notin G$, $y > x$ by observing that for $z > z_M(t)$,

$$u(y,t) - u(x,t) - \frac{Mz}{z_M(t)} \;\leq\; u(y,t) - u(x,t) - M \;\leq\; 0$$

so that as $|y - x| > z_M(t)$,

$$u(y,t) - u(x,t) \;\leq\; \frac{M|y-x|}{z_M(t)} \;\leq\; |y-x|\frac{\sqrt{t}}{2M}\exp\left(\frac{cM^2}{t}\right). \tag{3.9}$$

So far, we have estimates for when $y \geq x$. We can find identical estimates for the region where $y < x$ by reflecting in the line $x - y = 0$.

Letting $y \to x$ in the estimates for the difference quotients (3.8) and (3.9) gives the result. □

Comparing φ with the special barrier ψ gives us a gradient estimate for solutions of quasilinear equations with this scaling. We will use this estimate in Chapter 5.

Corollary 3.3 (Gradient estimates for the barrier φ). *Let φ be a smooth solution of*

$$\varphi_t \leq a(\varphi', \varphi, z, t)\varphi'',$$

on $(0, \infty) \times (0, T)$, with the initial and boundary conditions

$$\begin{aligned} &\varphi(z, 0) = 1 \text{ for } t \geq 0 \text{ and } z \neq 0, \\ &\varphi(0, t) = 0 \text{ for } t > 0, \\ &\varphi(x, t) \to 1 \text{ as } x \to \infty \text{ for } t > 0. \end{aligned}$$

If

$$a(p, r, x, t)p^2 \geq A > 0 \text{ for all } |p| \geq P > 0,$$

then there is a $T' > 0$ such that for $t \in (0, T']$

$$\varphi'(0, t) \leq C_1\sqrt{t}(1 + t)\exp\left(C_2/t\right)$$

where T', C_1 and C_2 depend on A and P.

Proof: We apply the comparison principle to φ and $\psi^\epsilon(x, t) := \epsilon(1+t) + 2\psi(x/2, t/4)$, where ψ is as in Section 3.3 with the constant $c = (4A)^{-1}$.

The barrier ψ^ϵ dominates φ on $(0,\infty)\times\{0\}$ and $\{0\}\times[0,T]$.

Choose $T'>0$ and z_1 so that on $(0,z_1)\times(0,T')$, $\psi^{\epsilon\prime\prime}\le 0$, $\psi^{\epsilon\prime}\ge P$ and at z_1, $\psi^\epsilon(z_1,t)>1\ge\varphi(z_1,t)$.

Then $\boldsymbol{P}\varphi=-\varphi_t+a(\varphi',\varphi,z,t)\varphi''\ge 0$ and

$$\begin{aligned}\boldsymbol{P}\psi^\epsilon &= -\epsilon-\psi_t+a(\psi',\psi^\epsilon,x,t)\psi'' \\ &= -\epsilon-\frac{\psi''}{(\psi')^2}\left[\frac{1}{4c}-a(\psi',\psi^\epsilon,x,t)\right] \\ &< 0,\end{aligned}$$

so Theorem 2.1 implies that $\psi^\epsilon>\varphi$ on $(0,z_1)\times(0,T')$ for all $\epsilon>0$, and as $\epsilon\to 0$, $\psi^0\ge\varphi$. Since $\psi^0(0,t)=\varphi(0,t)$ this gives us the boundary gradient estimate

$$\varphi'(0,t)\le\psi'(0,t/4)\le C_1\sqrt{t}(1+t)\exp\left(C_2/t\right).$$

□

3.5 Interior estimates for non-periodic equations with $b=0$

Theorem 3.4. *Let $u:\Omega\times[0,T]\to\mathbb{R}$ be a H_2 solution to*

$$u_t=a(u_x,u,x,t)u_{xx},$$

where $\Omega\subset\mathbb{R}$ is an open interval and where there are positive constants A and P so that

$$a(p,q,x,t)p^2\ge A \quad \text{for all } |p|\ge P. \tag{3.10}$$

If $\operatorname{osc}_{\Omega\times[0,T]}u\le M$, then we can find an estimate for $0<t<T'$

$$|u'(x,t)|\le C_1\sqrt{t}(1+t)\exp\left(\frac{C_2}{t}\right),$$

where T', C_1 and C_2 are dependent on A, P, M and $\operatorname{dist}(x,\partial\Omega)$.

We modify the previous proof, introducing a new boundary in the x,y coordinates, since we no longer have compactness of the domain through periodicity. We will seek to avoid a maximum of Z occurring on the new boundary.

Proof: Firstly, suppose that $\Omega=(-1,1)$.

We consider the sub-region

$$G:=\{\,(x,y,t)\in(-1,1)^2\times[0,T']:0\le t\le T'\le T,|y+x|<1,0\le y-x<z_M(t)\,\},$$

where $z_M(t)$ is as before in (3.7) and T' is chosen so that $z_M(t)<1$ for $t\le T'$.

Define Z on G by

$$Z(x,y,t):=u(y,t)-u(x,t)-\phi(x,y,t).$$

In order to avoid a positive maximum of Z on the boundary, we will ensure that ϕ satisfies

- $\phi(x, y, 0) \geq M$ for $y \neq x$
- $\phi(x, y, t) \geq M$ for $|y - x| = z_M(t)$
- $\phi(x, x, t) \geq 0$ for $t > 0$
- $\phi(x, y, t) \geq M$ for $|y + x| = 1$.

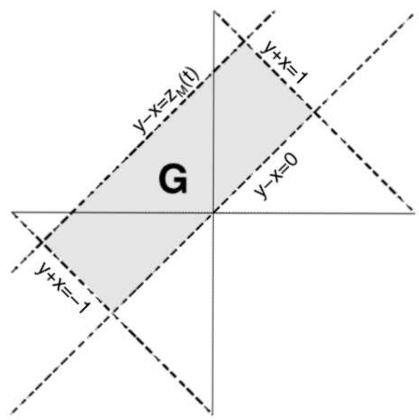


Figure 3.3: The region G at some time t

We choose

$$\phi = 2M\psi\left(\frac{y-x}{2M}, \frac{t}{4M^2}\right) + \gamma(x+y-2\beta)^2,$$

where ψ is the explicit barrier defined in Section 3.3, for positive constants c and γ. We will also choose β with $|\beta| < 1/2$ later.

This ϕ satisfies the first three conditions. In order to fulfill the final condition, choose $\gamma = M/(1-2|\beta|)^2$. Then at $|x+y| = 1$,

$$\gamma(x+y-2\beta)^2 \geq (\gamma\,(|x+y| - 2|\beta|)^2 = \gamma\,(1-2|\beta|)^2 = M.$$

Now suppose that Z first reaches a positive maximum at an internal point $(x, y, t) \in G$. As usual, we calculate that at this point first derivatives are zero

$$0 = Z_x = -u'(x,t) + \psi' - 2\gamma(x+y-2\beta),$$
$$0 = Z_y = u'(y,t) - \psi' - 2\gamma(x+y-2\beta),$$

and the matrix of second derivatives is negative semi-definite

$$0 \geq \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix} = \begin{bmatrix} -u''(x,t) - \frac{1}{2M}\psi'' - 2\gamma & \frac{1}{2M}\psi'' - 2\gamma \\ \frac{1}{2M}\psi'' - 2\gamma & u''(y,t) - \frac{1}{2M}\psi'' - 2\gamma \end{bmatrix}.$$

We use these in the evolution equation for Z

$$\frac{\partial Z}{\partial t} = a(u_y, u(y), y, t)u_{yy} - a(u_x, u(x), x, t)u_{xx} - \phi_t$$

$$
\begin{aligned}
&= a(u_y, u(y), y, t)\left[Z_{yy} + \frac{\psi''}{2M} + 2\gamma\right] - a\left(u_x, u(x), x, t\right)\left[-Z_{xx} - \frac{\psi''}{2M} - 2\gamma\right] \\
&\qquad\qquad - \frac{\psi_t}{2M} - \left(a(u_y, u(y), y, t) + a(u_x, u(x), x, t)\right) Z_{xy} \\
&\qquad\qquad + \left(a(u_y, u(y), y, t) + a(u_x, u(x), x, t)\right)\left(\frac{\psi''}{2M} - 2\gamma\right) \\
&= \operatorname{trace}\left(\begin{bmatrix} a(u_x, u(x), x, t) & -a(u_x, u(x), x, t) \\ -a(u_y, u(y), y, t) & a(u_y, u(y), y, t)\end{bmatrix}\begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy}\end{bmatrix}\right) \\
&\qquad\qquad + 2a(u_y, u(y), y, t)\frac{\psi''}{2M} + 2a(u_x, u(x), x, t)\frac{\psi''}{2M} - \frac{\psi_t}{2M} \\
&\le \left[2a(u_y, u(y), y, t) + 2a(u_x, u(x), x, t) - \frac{1}{4c\psi'^2}\right]\frac{\psi''}{2M},
\end{aligned}
$$

the last line applying at the internal maximum . As before, $\psi'' \le 0$ for $t \le T' \le 2cM^2/3$. The first derivative condition for Z implies that $u_x = \psi' - 2\gamma(x + y - 2\beta)$ and $u_y = \psi' + 2\gamma(x + y - 2\beta)$ and so

$$
\max(|u_x|, |u_y|) \ge |\psi'| \ge P, \tag{3.11}
$$

where the final inequality comes from choosing $c \ge CP^2$ as in the previous section.

We can then exploit (3.10), the condition on a;

$$
\begin{aligned}
2a\left(u_y, u(y), y, t\right) + 2a\left(u_x, u(x), x, t\right) &\ge \frac{2A}{\max\left(|u_x|, |u_y|\right)^2} \\
&\ge \frac{2A}{(|\psi'| + 2\gamma|x + y - 2\beta|)^2}.
\end{aligned}
$$

This is greater than $1/4c\psi'^2$ whenever $(\sqrt{8Ac} - 1)|\psi'| \ge 4\gamma$, so we use (3.11), the lower bound on $|\psi'|$, and choose

$$
c \ge \frac{(4\gamma + P)^2}{8AP^2} = \left(\frac{4M}{(1 - 2|\beta|)^2} + P\right)^2 \frac{1}{8AP^2}.
$$

Now $Z_t \le 0$ at interior maxima, and so the parabolic maximum principle ensures that $Z \le 0$ on G.

For points outside G, but in the rectangle $|y + x| < 1$, $z_M(t) \le y - x < 1$,

$$
\begin{aligned}
u(y, t) - u(x, t) \le M &\le \frac{M|y - x|}{z_M(t)} \\
&\le \frac{\sqrt{t}}{2M}\exp\left(\frac{cM^2}{t}\right)|y - x|.
\end{aligned}
$$

We can repeat both these estimates for a reflected region where $x > y$; putting them all together gives

$$
|u(x, t) - u(y, t)| \le 2M\psi\left(\frac{|y - x|}{2M}, \frac{t}{4M^2}\right) + \frac{M(x - y)^2}{(1 - 2|\beta|)^2} + \frac{\sqrt{t}}{2M}\exp\left(\frac{cM^2}{t}\right)|y - x|.
$$

Now, for $|y| < 1/2$, set $\beta = y$ and let $x \to y$ to give a gradient estimate at y

$$\begin{aligned} |u'(y,t)| &\leq \psi'\left(0, \frac{t}{4M^2}\right) + \frac{\sqrt{t}}{2M} \exp\left(\frac{cM^2}{t}\right) \\ &\leq \frac{t^{3/2}}{24cM^3} \exp\left(\frac{4cM^2}{t}\right) + \frac{\sqrt{t}}{2M} \exp\left(\frac{cM^2}{t}\right). \end{aligned}$$

For the case of a general interval $\Omega = [x_1, x_2]$, we can rescale around a point $y \in \Omega$ by using scaled coordinates $\tilde{x} = 2(x-y)/\mathrm{dist}(y, \partial\Omega)$. We obtain the estimate

$$|u'(y,t)| \leq \frac{2}{\mathrm{dist}(y,\partial\Omega)} \left[\frac{t^{3/2}}{24cM^3} \exp\left(\frac{4cM^2}{t}\right) + M\sqrt{t} \exp\left(\frac{cM^2}{t}\right) \right],$$

where c depends on A, P, M, and $\mathrm{dist}(y, \partial\Omega)$. □

3.6 A generalisation to fully nonlinear equations

In this section we consider equations of the form

$$u_t = F(u_{xx}, u_x, u, x, t), \tag{3.12}$$

where $F : \mathbb{R}^3 \times \Omega \times [0,T] \to \mathbb{R}$ is C^1.

Let u be a smooth solution to (3.12). As before, define

$$Z(x,y,t) := u(y,t) - u(x,t) - \phi(|y-x|, t),$$

and suppose it first becomes non-negative at some point (x, y, t), with $y > x$. First and second derivatives of Z will satisfy (3.1) and (3.2), but the evolution equation for Z will be given by

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= u_t(y,t) - u_t(x,t) - \phi_t(|y-x|,t) \\
&= F(u_{yy}, u_y, u(y), y, t) - F(u_{xx}, u_x, u(x), x, t) - \phi_t(|y-x|,t) \\
&= \int_0^1 \frac{\partial F}{\partial s}(su_{yy} + (1-s)u_{xx}, su_y + (1-s)u_x, su(y) + (1-s)u(x), sy + (1-s)x, t)ds \\
&\qquad\qquad - \phi_t(|y-x|,t) \\
&= [u_{yy} - u_{xx}] \int_0^1 \frac{\partial F}{\partial r}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds \\
&\qquad\qquad + [u_y - u_x] \int_0^1 \frac{\partial F}{\partial p}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds \\
&\qquad\qquad + [u(y) - u(x)] \int_0^1 \frac{\partial F}{\partial q}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds \\
&\qquad\qquad + [y-x] \int_0^1 \frac{\partial F}{\partial x}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds - \phi_t(|y-x|,t) \\
&= \operatorname{trace}\left(\begin{bmatrix} a & c_1 \\ c_2 & a \end{bmatrix} \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}\right) + 2\phi'' \int_0^1 \frac{\partial F}{\partial r}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds \\
&\qquad\qquad - (c_1 + c_2)\phi'' - \phi_t + \phi \int_0^1 \frac{\partial F}{\partial q}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds \\
&\qquad\qquad + [y-x] \int_0^1 \frac{\partial F}{\partial x}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds,
\end{aligned}$$

where we have used that $u_y = u_x$ at a spatial maximum, and have abbreviated

$$\int_0^1 \frac{\partial F}{\partial r}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds = a$$

and where we have added and subtracted $(c_1 + c_2)Z_{xy}$ for some c_1, c_2. If we choose

$$c_1 = c_2 = -a$$

the first matrix above is positive semi-definite. Since the matrix of second derivatives is negative semi-definite, we have

$$\begin{aligned}
\frac{\partial Z}{\partial t} \leq 4\phi'' \int_0^1 \frac{\partial F}{\partial r}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds - \phi_t \\
+ \phi \int_0^1 \frac{\partial F}{\partial q}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds \\
+ [y-x] \int_0^1 \frac{\partial F}{\partial x}\left(su_{yy} + (1-s)u_{xx}, \dots\right) ds.
\end{aligned}$$

If we make quite harsh restrictions on F, then we can use our explicit barrier to find an analogue of Theorem 3.2 for periodic nonlinear equations.

Theorem 3.5 (Nonlinear version of Theorem 3.2). *Let* $u : \mathbb{R} \times (0, T) \to \mathbb{R}$ *be a* C^2

solution of

$$u_t = F(u_{xx}, u_x, u, t)$$
$$u(\cdot, 0) = u_0$$

where u_0 *is continuous and periodic,* $u_0(x + L) = u_0(x)$*, (and therefore* u *is also periodic);* $\operatorname{osc} u(\cdot, t) \le M$*; where we can find positive constants* A *and* P *such that*

$$\frac{\partial F}{\partial r}(r, p, q, t)p^2 \ge A \text{ for all } |p| \ge P;$$

and where $\dfrac{\partial F}{\partial q} \le 0$ *.*

Then there is a $T' > 0$ *such that for* $t \in (0, T']$*,*

$$|u_x| \le C_1 \sqrt{t}\,(1 + t) \exp(C_2/t),$$

where T'*,* C_1 *and* C_2 *are dependent on* M*,* A *and* P*.*

Proof: The proof is the same as that of Theorem 3.2, including the choice $\phi(z, t) = 2M\psi(z/2M, t/4M^2)$, but instead of using inequality (3.3) for interior maximum points, we have

$$\begin{aligned}\frac{\partial Z}{\partial t} &\le 4\phi'' \int_0^1 \frac{\partial F}{\partial r}(\dots)ds + \phi \int_0^1 \frac{\partial F}{\partial q}(\dots)ds - \phi_t \\ &\le 4\phi'' \frac{A}{\phi'^2} - \frac{\phi''}{4c\phi'^2} \\ &\le 0,\end{aligned}$$

where the omitted argument of the derivatives of F, denoted by $(\dots)$, is $(su_{yy} + (1 - s)u_{xx}, su_y + (1 - s)u_x, su(y) + (1 - s)u(x), sy + (1 - s)x, t)$. □

Chapter 4

An existence result for a parabolic equation in one space dimension

Although this is a standard result (see Theorem 12.25 of [25]), for completeness we sketch a short time existence result in the one-dimensional case, where the spatial domain is $\Omega = (x_0, x_1) \subset \mathbb{R}$, and the initial and boundary data is continuous.

The parabolic equation is

$$u_t = a(u_x, u, x, t)u_{xx}, \tag{4.1}$$

with initial and boundary data prescribed by

$$u(x,t) = u_0(x,t) \text{ for } (x,t) \in \mathcal{P}(\Omega \times [0,T]). \tag{4.2}$$

We require that $a > 0$ is in $H_\alpha(\mathcal{K})$ for all bounded $\mathcal{K} \subseteq \mathbb{R} \times \mathbb{R} \times \Omega \times [0,T]$ and some $\alpha \in (0,1)$.

This implies that for every such $\mathcal{K}$ we can find positive $\lambda_\mathcal{K}$ and $\Lambda_\mathcal{K}$ such that

$$\lambda_\mathcal{K} \le a(p,q,x,t) \le \Lambda_\mathcal{K}, \text{ when } (p,q,x,t) \in \mathcal{K}. \tag{4.3}$$

When we can find bounds of this type that depend only on the gradient, we will write

$$\lambda(K) \le a(p,q,x,t) \le \Lambda(K), \text{ for } |p| \le K. \tag{4.4}$$

Suppose also that there are positive constants A and P such that

$$a(p,q,x,t)p^2 \ge A > 0, \text{ for } |p| \ge P. \tag{4.5}$$

The first part of this chapter is a survey of the main steps needed to find the existence result for the Cauchy-Dirichlet problem with $H_{1+\beta}$ initial and boundary data. We follow the treatment in Lieberman [25].

These results mean that when we approximate continuous initial data by smooth initial data, a solution will exist for the approximate initial data. In the later parts of the chapter, we use the gradient estimate established in Chapter 3 to find uniform gradient estimates for $t > 0$. This will gives us a solution for $t > 0$; in order to show that this approaches the initial data as $t \to 0$, we will need some displacement estimates which limit the distance a function can travel in a given time.

4.1 Existence of solutions with $H_{1+\beta}$ initial and boundary data

Theorem 4.1. *Consider the Cauchy-Dirichlet problem given by* (4.1) *and* (4.2)*, where* $0 < \beta < 1$*.*

Suppose that u_0 *is defined on the parabolic boundary* $\mathcal{P}(\Omega \times [0,T])$ *and* $u_0 \in H_{1+\beta}(\mathcal{P})$*. Also, suppose that either* u_0 *is time-independent, or else there are constants* A *and* P *such that* (4.5) *is satisfied.*

Then there is a smooth solution $u \in C^{2+1}(\Omega \times (0,T)) \cap C\left(\overline{\Omega} \times [0,T]\right)$*.*

This solution has a gradient bound $|u|_{1+\delta,\delta/2} \le C$ *where* C *depends on* $|u_0|_{1+\beta,\beta/2}$*,* β*,* $\lambda_{\mathcal{K}}$*,* $\Lambda_{\mathcal{K}}$ *and* $\operatorname{diam}\Omega$*.*

The proof of this result follows a standard pattern for showing existence — a bound on $\sup|u|$; a bound on $\sup|Du|$; a Hölder gradient bound $|Du|_\alpha$; and then the application of a fixed point theorem. These steps are sketched by the following results.

We begin by using the comparison principle to bound $|u|$.

Lemma 4.2 (A bound on $\sup|u|$). *If* u *is a smooth solution of* (4.1)*,* (4.2) *in* $\Omega \times [0,T]$*, then*

$$\sup_{\Omega\times[0,T]} |u(x,t)| \le \sup|u_0|.$$

Proof idea: Set $k = \sup {u_0}^+$ and apply the comparison principle (Theorem 2.1) to u and k on $E := \{\, (x,t) \in \Omega \times [0,T] : u(x,t) > 0 \,\}$. Since $k \ge u$ on ∂E, it follows that $k \ge u$ on all of E and so on all of $\Omega \times [0,T]$.

Similar steps can be followed to find that $\inf_{\Omega\times[0,T]} u \ge \inf {u_0}^-$, completing the result. ☐

We begin our gradient estimates with a boundary gradient estimate.

Lemma 4.3 (Boundary gradient estimate). *Let* $0 < \beta \le 1$*. If* u *is a smooth solution of* (4.1)*,* (4.2) *in* $\Omega \times [0,T]$*, and either* u_0 *is time-independent or else* a *satisfies the condition* (4.5)*, then*

$$\sup_{\substack{(x,t)\in\partial\Omega\times[0,T] \\ (y,s)\in\Omega\times[0,t]}} \frac{|u(x,t)-u(y,s)|}{|(x,t)-(y,s)|} \le L,$$

where L *depends only on* $\operatorname{osc} u_0$ *,* $|u_0|_{1+\beta,\beta/2}$ *and* β*.*

In fact, we can relax the regularity requirements on the initial and boundary data and still find a continuity estimate on the boundary.

A *modulus of continuity* is a concave, continuous function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, with $\omega(0) = 0$. This ω is a modulus of continuity for a function g at y if

$$|g(x) - g(y)| \le \omega(|x-y|)$$

for all x in the domain of g. It is a modulus of continuity for g if the above relationship also holds for all y in the domain of g.

A modulus of continuity can be defined for every continuous function on a closed bounded set.

Lemma 4.4 (Boundary continuity estimate). *Let u satisfy* (4.1), (4.2)*, where u_0 has modulus of continuity ω, and suppose there are positive constants μ and P so that*

$$|p|\Lambda(p,q,x,t) + 1 \leq \mu a(p,q,x,t)p^2 \tag{4.6}$$

whenever $|p| \geq P$.

Then u has a modulus of continuity on the boundary

$$|u(x,t) - u(y,t)| \leq \omega^*(|x-y|)$$

for $x \in \Omega$ and $y \in \partial\Omega$, where ω^* can be determined by ω, $\sup_{\mathcal{P}} |u_0|$, Ω, and a.

Equipped with the boundary gradient estimate, we can now find a global gradient estimate. In this one-dimensional case, the global gradient estimate is the result of Kružkov mentioned in Chapter 3.

Lemma 4.5 (Global gradient estimate). *If u is a smooth solution of* (4.1), (4.2) *in $\Omega \times [0,T]$ with an oscillation bound $\operatorname{osc} u = M$, and a Lipschitz estimate on the parabolic boundary $|u(x,t) - u(y,t)| \leq L|x-y|$ for all $(x,t) \in \mathcal{P}(\Omega \times [0,T])$ and $y \in \Omega$, then*

$$\sup_{\Omega\times[0,T]} |u_x| \leq 2L.$$

Lemma 4.6 (Global Hölder gradient estimate). *Suppose that u satisfies* (4.1), (4.2) *in $\Omega \times [0,T]$, where there are positive constants $\lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ such that whenever (p,q,x,t) in the set $\mathcal{K} := \{\, (p,q,x,t) : |p| \leq K, |q| \leq M, x \in \Omega, t \in [0,T] \,\}$,*

$$\lambda_{\mathcal{K}} \leq a(p,q,x,t) \leq \Lambda_{\mathcal{K}}.$$

If $u \in C^{2+1}(\Omega \times [0,T]) \cap C(\overline{\Omega} \times [0,T])$, set $M = \sup|u|$ and $K = \sup|Du|$. Then there are positive constants α and C determined by β, $\lambda_{\mathcal{K}}$, $\Lambda_{\mathcal{K}}$ and $\operatorname{diam}\Omega$ such that

$$[Du]_\alpha \leq C\left(\sup|u| + \sup|Du| + |u_0|_{1+\beta,\beta/2}\right).$$

Now that we have bounds for $|u|_{1+\alpha,\alpha/2}$, we can apply the following existence theorem, which is derived from a fixed point theorem.

Lemma 4.7 (Existence theorem). *Let u_0 be in $H_{1+\delta}$ for some $\delta \in (0,1)$*

If there is a constant M_δ independent of ϵ such that any solution of (4.1), (4.2) *on $\Omega \times [0,\epsilon)$ satisfies*

$$|u|_{1+\delta,\delta/2} \leq M_\delta,$$

then there is a solution of the Cauchy-Dirichlet problem (4.1), (4.2) *in $\Omega \times [0,T]$.*

4.2 Displacement estimates

The following estimates for the displacement suffered in a given interval of time by a function moving under a parabolic flow apply to any strictly parabolic operator satisfying bounds of the form (4.3) or (4.4).

Lemma 4.8 (Displacement estimate for Lipschitz initial data). *Let* $u : \Omega\times[0,T] \to \mathbb{R}$ *satisfy* (4.1)*, where* $\Omega \subseteq \mathbb{R}$*, and* a *has bounds of the form* (4.4)*.*

Suppose that u *has initial data whose graph lies below a cone centred at some point* h

$$u(x,0) \le L|x-h|$$

and, in the case that $\Omega \ne \mathbb{R}$*, whose boundary data lies below the same cone*

$$u(x,t) \le L|x-h|, \quad x \in \partial\Omega.$$

Then, at later times,

$$u(x,t) \le L(x-h)\mathcal{E}\mathrm{rf}\left(\frac{x-h}{2\sqrt{\Lambda t}}\right) + 2L\sqrt{\frac{\Lambda t}{\pi}}\exp\left(-\frac{(x-h)^2}{4\Lambda t}\right), \tag{4.7}$$

where $\Lambda = \Lambda(L)$ *is given by* (4.4)*.*

Proof: For some small $\epsilon > 0$, set

$$v(x,t) := L(x-h)\mathcal{E}\mathrm{rf}\left(\sqrt{\frac{c}{t+\epsilon}}(x-h)\right) + L\sqrt{\frac{t+\epsilon}{c\pi}}\exp\left(-\frac{c(x-h)^2}{t+\epsilon}\right),$$

which satisfies the heat equation

$$v_t = \frac{v_{xx}}{4c}$$

and approaches the cone of gradient L centred at h as $t+\epsilon \to 0$.

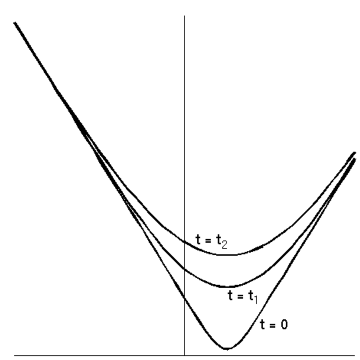


Figure 4.1: The barrier $v(\cdot,t)$

Note that $v(x,0) > L|x-h|$, that $|v_x(x,t)| = L\left|\mathcal{E}\mathrm{rf}\left(x\sqrt{c/(t+\epsilon)}\right)\right| < L$ and that $v_{xx} > 0$, so

$$\begin{aligned} v_t - a(v_x, v, x, t)v_{xx} &\ge v_t - \sup_{|p|\le L} a(p,v,x,t)v_{xx} \\ &\ge v_t - \Lambda(L)v_{xx} \\ &= 0 \end{aligned}$$

where we choose $c^{-1} = 4\Lambda(L)$.

The estimate follows by applying the comparison principle (Theorem 2.2) to show that $u(x,t) \leq v(x,t)$, and then letting $\epsilon \to 0$.

□

Now, we apply this to three different cases, firstly when u initially satisfies a Hölder condition and when we have polynomial growth in Λ, secondly when $u(\cdot, 0)$ has a modulus of continuity, and thirdly when u is initially bounded by a step function.

Corollary 4.9 (Displacement estimate for Hölder initial data). *Let $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ satisfy* (4.1)*. Suppose that a not only satisfies* (4.4)*, but more specifically has at most polynomial growth, so that*

$$a(p,q,x,t) \leq \bar{\Lambda}(1 + K^m) \text{ for } |p| \leq K \tag{4.8}$$

where $\bar{\Lambda}$ and m are positive constants. Also, suppose that $u(\cdot, 0)$ satisfies a Hölder condition around some point h

$$|u(h,0) - u(x,0)| \leq L|x-h|^{\alpha}, \quad 0 < \alpha < 1.$$

Then, at later times,

$$|u(h,0) - u(h,t)| \leq c(\alpha, m, L, \bar{\Lambda}) t^{\frac{\alpha}{2+m(1-\alpha)}}.$$

Proof: For simplicity, assume $h = 0$ and $u(h,0) = 0$. The initial data is bounded

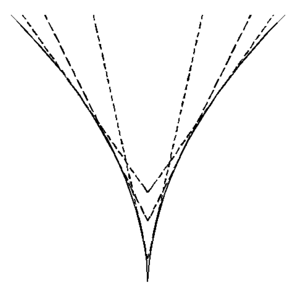


Figure 4.2: The bounding cusp is itself bounded above by cones

above by cones centred at h and indexed by k, the (positive) x-coordinate of the point of contact with the bounding cusp $L|x|^{\alpha}$, so

$$u(x,0) \leq L|x|^{\alpha} \leq \alpha L k^{\alpha-1}|x| + L(1-\alpha)k^{\alpha}.$$

The estimate (4.7) taken at $x = 0$ is then

$$u(0,t) \leq 2L\alpha k^{\alpha-1}\left(1 + \left(Lk^{\alpha-1}\right)^m\right)^{1/2}\sqrt{\frac{\bar{\Lambda}t}{\pi}} + L(1-\alpha)k^{\alpha}$$

and optimizing over k gives

$$u(0,t) \leq c(\alpha, m, L, \bar{\Lambda}) t^{\frac{\alpha}{2+m(1-\alpha)}}.$$

□

Corollary 4.10 (Displacement estimate for continuous initial data). *Suppose that $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ satisfies* (4.1) *and* (4.4)*, where u has initial data with a modulus of continuity ω at a point h*

$$|u(h,0) - u(x,0)| \leq \omega\left(|x-h|\right).$$

Then

$$|u(h,0) - u(h,t)| \leq c(t)$$

where c is dependent on ω and Λ, and where $c(t) \to 0$ as $t \to 0$.

Proof: For simplicity, assume $h = 0$ and $u(h,0) = 0$. Consider the cones

$$C_k(x) = c_k\left(|x| - k\right) + \omega(k)$$

indexed by $k > 0$, the (positive) x-coordinate of a point of contact with ω. As ω is concave it has both left and right derivatives, and we can choose the slope of the cone $c_k = \omega'_-(k)$. Then

$$u(x,0) = u(x,0) - u(0,0) \leq \omega(|x|) \leq \omega'_-(k)\left(|x| - k\right) + \omega(k) = C_k(x).$$

Now we have a cone as an upper boundary, we can use estimate (4.7) at $x = 0$

$$u(0,t) \leq 2\omega'_-(k)\sqrt{\frac{\Lambda t}{\pi}} - \omega'_-(k)k + \omega(k),$$

where $\Lambda = \Lambda(\omega'_-(k))$ is given by (4.4). Minimize this over k to get the displacement bound

$$c(t) := \inf_{k>0}\left(2\omega'_-(k)\sqrt{\frac{\Lambda t}{\pi}} - \omega'_-(k)k + \omega(k)\right).$$

In order to show that $c(t) \to 0$ as $t \to 0$, let $\delta > 0$. As ω is concave and positive, it has positive left derivative and for $k > 0$ we have

$$0 \leq \omega'_-(k)k \leq \omega(k).$$

And as ω is continuous,

$$0 \leq \lim_{k\to 0} \omega'_-(k)k \leq \lim_{k \to 0} \omega(k) = 0.$$

Choose $k = k_\delta$ so that $\omega(k_\delta) - \omega'_-(k_\delta)k_\delta < \delta$. Choose τ so that

$$\sqrt{\tau} \leq \frac{\delta\sqrt{\Lambda\left(\omega'_-(k_\delta)\right)}}{2\omega'_-(k_\delta)\sqrt{\pi}},$$

then for all $t \le \tau$,

$$c(t) \le 2\omega'_-(k_\delta)\sqrt{\frac{\Lambda t}{\pi}} - \omega'_-(k_\delta)k_\delta + \omega(k_\delta) \le 2\delta,$$

and so $c(t) \to 0$. □

Set σ to be the maximal monotone graph

$$\sigma(x) = \begin{cases} +1, & x > 0 \\ [-1, 1], & x = 0 \\ -1, & x < 0 \end{cases} \tag{4.9}$$

which we will refer to as the *step "function"*.

Corollary 4.11 (Displacement estimate for step functions). *If u satisfies* (4.1) *and* (4.4)*, and is initially bounded by a step function*

$$u(x, 0) \le c\sigma(x),$$

then for $x < 0$

$$u(x, t) \le \min\left\{\frac{4c}{|x|}\sqrt{\frac{\Lambda t}{\pi}} - c, c\right\},$$

where $\Lambda = \Lambda(2c/|x|)$ as in (4.4)*.*

Proof: Near some point $h < 0$, $u(\cdot, 0)$ satisfies a Lipschitz condition

$$u(x, 0) \le L^h|x - h| - c$$

where $L^h = 2c/|h|$.

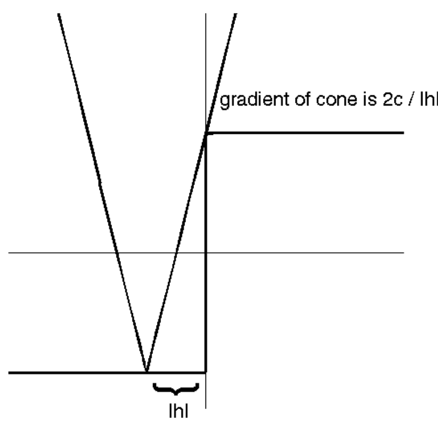


Figure 4.3: Cone bounding the step function

Lemma 4.8 then gives that

$$u(x,t) \leq \inf_{h<0} \frac{2c}{|h|}\mathcal{E}\mathrm{rf}\left(\frac{x-h}{2\sqrt{\Lambda t}}\right) + \frac{4c}{|h|}\sqrt{\frac{\Lambda t}{\pi}}\exp\left(-\frac{(x-h)^2}{4\Lambda t}\right) - c$$

and if we let $h = x$ then we find that for $x < 0$,

$$u(x,t) \leq \frac{4c}{|x|}\sqrt{\frac{\Lambda t}{\pi}} - c.$$

The final result is found by comparison to the constant function c. □

Remark: If a satisfies the condition (4.8), then

$$u(x,t) \leq C(c,\bar{\Lambda})c^{1+m/2}\sqrt{t}|x|^{-1-m/2} - c.$$

□

4.3 Existence of solutions with continuous initial data

Theorem 4.12. *Consider the Cauchy-Dirichlet problem given by* (4.1) *and* (4.2)*. If* $u_0 \in C\left(\mathcal{P}(\Omega \times [0,T])\right)$ *and if there are constants* A *and* P *such that* (4.5) *holds, then* (4.1), (4.2) *has a solution* $u \in C^{2+1}\left(\Omega \times (0,T)\right) \cap C\left(\overline{\Omega} \times [0,T]\right)$.

The first step in the proof of the above is to approximate u_0 by u_0^ϵ in C^∞, so that $\sup_{x\in\Omega}|u_0^\epsilon - u_0| < \epsilon$.

Lemma 4.13 (Existence of solutions with approximate boundary data). *For all* $\epsilon > 0$*, there exist solutions* $u^\epsilon : \Omega \times [0,T] \to \mathbb{R}$ *to* (4.1) *with boundary data* u_0^ϵ *These solutions are in* $C^{2+1}\left(\Omega \times [0,T]\right) \cap C\left(\overline{\Omega} \times [0,T]\right)$.

Proof: As u_0^ϵ is in the Hölder space $H_{1+\beta}$, this is a consequence of Theorem 4.1. □

Lemma 4.14 (Existence of uniform oscillation bound). *For all* $\epsilon > 0$*,*

$$\operatorname{osc} u^\epsilon \leq 4\left(\sup|u_0|\right).$$

Proof: For any fixed ϵ, set $k = \sup {u_0^\epsilon}^+$ and apply the comparison principle (Theorem 2.1) to k and u^ϵ on $E = \{\,(x,t) \in \Omega \times [0,T] : u^\epsilon(x,t) > 0\,\}$. Since $k \geq u^\epsilon$ on ∂E, it follows that $k \geq u^\epsilon$ on all of E and hence on all of $\Omega \times [0,T]$, and so

$$\sup_{\Omega\times[0,T]} |u^\epsilon(x,t)| \leq \sup|u_0^\epsilon| \leq 2\left(\sup|u_0|\right),$$

where the last inequality will hold for small enough ϵ.

This leads to a uniform oscillation bound for u^ϵ, which we denote by M —

$$\operatorname{osc} u^\epsilon \leq 4\sup|u_0| =: M.$$

□

Theorem 3.4 gives a uniform gradient bound on interior sets, up to some time $T' > 0$. For $t_0 \in (0, T'/2)$,

$$|u^\epsilon{}_x|_{\Omega' \times (t_0, T')} \le C_1 \sqrt{t_0}(1 + t_0) \exp\left(\frac{C_2}{t_0}\right) =: L(t_0),$$

where T', C_1 and C_2 are dependent on A, P, M and $\operatorname{dist}(\Omega', \partial\Omega)$.

Lemma 4.15 (Higher regularity on interior sets). *On interior sets $\Omega' \times (2t_0, T')$ we can estimate higher derivatives*

$$|u^\epsilon|_{2+k+\alpha} \le C$$

where C depends on $\operatorname{dist}(\Omega', \partial\Omega)$, $\operatorname{diam}(\Omega)$, t_0, A, P, $|a|_\alpha$ and M.

Proof: Once we have an oscillation bound M and a gradient bound $L(t_0)$, (4.3) implies uniform parabolicity. A uniform Hölder gradient bound on interior sets results from Theorem 12.2 of [25]. In particular, on interior sets and when $T'/2 > t_0 > 0$,

$$[u^\epsilon{}_x]_{\alpha;\Omega' \times (2t_0, T')} \le C \min\left\{\operatorname{dist}(\Omega', \partial\Omega), \sqrt{t_0}\right\}^{-\alpha},$$

where both α and C depend on $\lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$, given by (4.3), with

$$\mathcal{K} = \{\, (p, q, x, t) : |p| \le L(t_0), |q| \le M, x \in \Omega, \text{ and } t \in [0, T] \,\}$$

and C also depends on $L(t_0) + M$, and $\operatorname{diam}\Omega$.

Equipped with a Hölder gradient bound, we can treat the equation as a uniformly parabolic equation with Hölder continuous coefficients, and use standard results, such as Theorem A.4, to find that u^ϵ is uniformly bounded in $H_{2+\alpha}\left(\Omega' \times (2t_0, T')\right)$.

From here, it is possible to use the bootstrapping method to obtain interior estimates for all higher derivatives.

□

Corollary 4.16. *On any interior set $\Omega' \times (t_0, T')$, there exists a subsequence converging to some u that also solves the partial differential equation* (4.1).

In order for this u to be a solution of the Cauchy-Dirichlet problem, we need to show that u attains the initial and boundary data.

Lemma 4.17 (Convergence to initial data). *On any spatially interior set Ω',*

$$\sup_{x \in \Omega'} |u(x, t) - u_0(x)| \to 0 \text{ as } t \to 0.$$

Proof: Let x be any point in Ω'. Let ω be a modulus of continuity for u_0.

We can off-set u^ϵ by defining

$$w^\epsilon(y, t) := u^\epsilon(y, t) - u^\epsilon(x, 0) + u_0(x),$$

so that $w^\epsilon(x,0) = u_0(x)$. Let u be the limit of a subsequence u^ϵ, as in Corollary 4.16.

$$\begin{aligned}
|u(x,t) - u_0(x)| &= \lim_{\epsilon\to 0} |u^\epsilon(x,t) - u_0(x)| \\
&= \lim_{\epsilon\to 0} |w^\epsilon(x,t) + u^\epsilon(x,0) - 2u_0(x)| \\
&\le \lim_{\epsilon\to 0} \left(|w^\epsilon(x,t) - u_0(x)| + |u^\epsilon(x,0) - u_0(x)|\right) \\
&= \lim_{\epsilon\to 0} |w^\epsilon(x,t) - w^\epsilon(x,0)| + \lim_{\epsilon\to 0} |u^\epsilon(x,0) - u_0(x)|.
\end{aligned}$$

The second of these terms is zero. To estimate the first term, note that the approximations $u^\epsilon(\cdot,0)$ satisfy the same the same continuity condition as u_0, and therefore so does $w^\epsilon(\cdot,0)$, with $|w^\epsilon(0,x) - w^\epsilon(0,y)| \le \omega(|x-y|)$, for all $x, y \in \Omega$. Corollary 4.10 then gives the estimate

$$|w^\epsilon(x,t) - w^\epsilon(x,0)| \le c(t),$$

where c depends only on the exact forms of ω and $\Lambda_{\mathcal{K}}$ (given in (4.3)). In particular, c is independent of x and ϵ, and as $c(t) \to 0$ as $t \to 0$, the result follows. □

More specific continuity-in-time estimates are given by the continuity of the initial data and the upper growth bound of a. If, for example, the initial data is Hölder continuous

$$|u_0(x) - u_0(y)| \le L|x-y|^\alpha$$

and a has polynomial growth in the gradient term, satisfying (4.8) for constants $\bar\Lambda$ and m, then Corollary 4.9 indicates that $c(t) = C(\alpha, m, L, \bar\Lambda)|t|^{\frac{\alpha}{2+m(1-\alpha)}}$.

Lemma 4.18 (Convergence to boundary data). *We can continuously extend $u(\cdot,t)$, defined on the interior of Ω at time t, to $\overline{\Omega}$. Moreover, $u = u_0$ on the boundary.*

Proof: We need to show that for $y \in \partial\Omega$, $\lim_{x\to y} u(x,t) = u_0(y,t)$.

We note that our parabolic equation satisfies condition (4.6), since

$$|p|\Lambda(p,q,x) + 1 = |p||a(p,q,x)| + 1 \le \frac{2}{A} a|p|^2$$

for $|p| \ge P$, using (4.5).

Let ω be a modulus of continuity for u_0. As each u_0^ϵ has at least the same modulus of continuity as u_0, Lemma 4.4 gives us an estimate uniform in t and ϵ,

$$|u^\epsilon(x,t) - u_0^\epsilon(y,t)| \le \omega^*(|x-y|).$$

Then for a point $y \in \partial\Omega$ and fixed t,

$$\begin{aligned}
\sup_{B_r(y)\cap\Omega} |u(x,t) - u_0(y,t)| &= \sup_{B_r(y)\cap\Omega} |\lim_{\epsilon\to 0} u^\epsilon(x,t) - u_0^\epsilon(y,t) + u_0^\epsilon(y,t) - u_0(y,t)| \\
&\le \sup_{B_r(y)\cap\Omega} \omega^*\left(|x-y|\right) \\
&= \omega^*(r)
\end{aligned}$$

so as $|x-y| \to 0$, $\omega^*(|x-y|) \to 0$ and $u(x,t) \to u_0(y,t)$ — that is, we can continuously extend u to u_0 on $\partial\Omega$ for $t > 0$. □

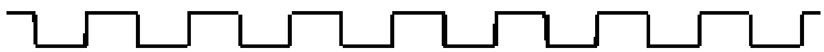
4.4 Existence of entire solutions with stepped initial conditions

Consider equation (4.1), under the conditions on a given by (4.3) and (4.5).

Lemma 4.19. *There exist entire solutions to this equation with the periodic, crenellated initial data*

$$g_R(x) = M\sigma\left(\sin(\pi x/R)\right),$$

where σ is given by (4.9)*.*



Proof: If we let g^ϵ be the smooth mollification of g_R, then for $|x| < R$,

$$M\sigma(x-\epsilon) \leq g^\epsilon(x) \leq M\sigma(x+\epsilon).$$

Theorem 4.1 ensures that there is a smooth solution u^ϵ to 4.1 with initial condition g^ϵ, with a Hölder gradient bound dependent on ϵ. The gradient bound in Theorem 3.2 is independent of ϵ; for $t \in (0, T')$,

$$|u^\epsilon_x| \leq C_1\sqrt{t}\,(1+t)\exp(C_2/t)$$

where T', C_1 and C_2 are dependent on M, A and P, but not R. Higher gradient bounds for $t > 0$ follow from the interior estimate (4.15) and we can find a subsequence converging to u_R which also solves the equation on $[t, T']$.

To show convergence of u_R to the initial data, suppose that $-R/2 < x < 0$. As in Section 4.2, we can bound the initial data g^ϵ by cones centred at $h \in (-R/2, -\epsilon)$ —

$$g^\epsilon(x) \leq \frac{M}{|h+\epsilon|}|x-h| - M.$$

Applying Lemma 4.8 to this, and setting $h = x$, we find that for $-R/2 < x < -\epsilon$,

$$u^\epsilon(x,t) \leq \frac{2M}{|x+\epsilon|}\sqrt{\frac{\Lambda t}{\pi}} - M,$$

where $\Lambda = \Lambda\left(M/|x+\epsilon|\right)$ as in (4.3) and so we have the estimate

$$\begin{aligned}|u_R(x,t) - g_R(x)| &= |\lim_{\epsilon\longrightarrow 0} u^\epsilon(x,t) + M| \\ &= \frac{2M}{|x+\epsilon|}\sqrt{\frac{\Lambda t}{\pi}}.\end{aligned}$$

A similar estimate holds for all $x \neq nR$, and so for all $\mu > 0$, we can find t (dependent on x) such that $|u_R(x,t) - g_R(x)| \leq \mu$. □

Corollary 4.20. *There exists an entire solution to this problem with the initial data*

$$u_0(x) = M\sigma(x).$$

This solution has a gradient estimate for $t < T'$*:*

$$|u_x| \leq C_1\sqrt{t}\,(1+t)\exp(C_2/t),$$

where T'*,* C_1 *and* C_2 *are dependent on* M*,* A *and* P*.*

Proof: Take the limit of the solutions u_R given by the previous lemma as $R \to \infty$. □

Chapter 5

Gradient estimates for parabolic equations in higher dimensions

In this chapter we extend the methods of Chapter 3 to higher dimensions.

Consider a smooth solution $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ to

$$u_t = a^{ij}(Du,t)D_{ij}u + b(Du,t), \tag{5.1}$$

where $A(p,t) = [a^{ij}(p,t)]$ is a symmetric, positive semi-definite $n \times n$ matrix that is smoothly dependent on $(p,t) \in \mathbb{R}^n \times [0,T]$.

Define

$$\alpha(p,t) := |p|^2 \inf_{v \in S^n, v\cdot p \neq 0} \frac{v^T A(p,t) v}{(v \cdot p)^2}. \tag{5.2}$$

Compare this definition to that of the *Bernstein $\mathcal{E}$ function*, (see Chapter 10 of [16])

$$\mathcal{E}(p,q,x,t) = a^{ij}(p,q,x,t)p_i p_j.$$

Clearly, $\alpha(p)|p|^2 \leq \mathcal{E}$ and if λ, Λ are the smallest and largest eigenvalues of A, then for $p \neq 0$,

$$\lambda \leq \alpha(p) \leq \frac{\mathcal{E}}{|p|^2} \leq \Lambda.$$

The middle inequality here becomes an equality when p is an eigenvector of A.

Our aim is to reduce the n-dimensional problem to a parabolic equation in one space dimension; we can do this if $\alpha(p)$ is bounded below by a positive function of $|p|$. We will call this

$$\tilde{\alpha}(s) := \inf_{p \in \mathbb{R}^n : |p| = s} \alpha(p). \tag{5.3}$$

For the existence of specific barriers we will require a control on the degeneracy of A — the existence of positive constants A_0 and P such that

$$\tilde{\alpha}(s)s^2 \geq A_0 \text{ for } s \geq P. \tag{5.4}$$

Example 1: If p is an eigenvector of $A(p)$, then $\alpha(p)$ is the associated eigenvalue.

Example 2: As a specific example of the above, if a^{ij} is of the form

$$a^{ij}(p) = a_\infty(p)\left(\delta^{ij} - \frac{p_i p_j}{|p|^2}\right) + a_0(p)\frac{p_i p_j}{|p|^2} \tag{5.5}$$

for functions $a_\infty, a_0 : \mathbb{R}^n \to \mathbb{R}$, with $a_0 > 0$ and $a_\infty \geq 0$, then $\alpha(p) = a_0(p)$.

In the mean curvature flow case, $a_\infty = 1$ and

$$a_0(p) = \alpha(p) = \frac{1}{1+|p|^2}.$$

Example 3: In the most general situation, if $v_k(p)$ are the non-null eigenvectors of $A(p)$ with eigenvalues $\lambda_k(p) > 0$, then

$$\alpha(p) = \begin{cases} |p|^2\left(\sum_k \frac{(v_k\cdot p)^2}{\lambda_k}\right)^{-1} & \text{if } p \in (\mathrm{Null}A(p))^\perp \\ 0 & \text{otherwise.}\end{cases}$$

Example 4: In the case that A is positive definite, all eigenvalues are positive and

$$\alpha(p) = \frac{|p|^2}{pA^{-1}p}.$$

As Example 2 shows, A need not be positive definite.

Example 5: An elliptic operator $[a^{ij}]$ is called of *mean curvature type* if there are positive constants λ, Λ so that

$$\lambda m^{ij}(p)\xi_i\xi_j \leq a^{ij}(p,q,x)\xi_i\xi_j \leq \Lambda m^{ij}(p)\xi_i\xi_j,$$

where m^{ij} are the coefficients of mean curvature flow [15, 27, 16]. For such equations, one can (under some conditions, particularly on the shape of the boundary) find *apriori* estimates on $|Du|$ in terms of $|u|$. It is therefore interesting to note that if $[a^{ij}]$ satisfies only the lower inequality above, then it also satisfies (5.4).

Example 6: If one may be forgiven for referring to a future section, note that if the flow is the *anisotropic mean curvature flow* (9.5) of Section 9.1, then Lemma 9.5 implies that $\tilde{\alpha}(|p|) \geq A_0|p|^2$ when $\bar{F}(p) \geq P$. The function $\bar{F}$ will be positive and homogeneous of order one, so that $c_1|p| \leq \bar{F}(p) \leq c_2|p|$.

Thus, anisotropic mean curvature flow satisfies conditions (5.3) and (5.4).

5.1 Reduction to a one-dimensional problem

Let $u : \mathbb{R}^n \times (0,T] \to \mathbb{R}$ satisfy (5.1), where $A = [a^{ij}]$ is a symmetric, positive semi-definite matrix with $\alpha > 0$.

In the following, we generalise the calculations of Section 3.1 to higher dimensions. As in the one-dimensional case, we begin our discussion by defining

$$Z(x,y,t) := u(y,t) - u(x,t) - \phi(|y-x|,t),$$

where $\phi : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a C^2 function that will be chosen later.

At an internal maximum point of Z, the first derivative conditions are

$$\begin{aligned} D_{x^i}Z &= -D_iu(x,t) - D_{x^i}\phi(|y-x|),t) = -D_iu(x,t) + \phi'\frac{y^i-x^i}{|y-x|} = 0 \\ D_{y^i}Z &= \quad D_iu(y,t) - D_{y^i}\phi(|y-x|,t) = \quad D_iu(y,t) - \phi'\frac{y^i-x^i}{|y-x|} = 0. \end{aligned} \tag{5.6}$$

The second derivatives of Z are

$$\begin{aligned} D_{x^ix^j}Z &= -u_{ij}(x,t) - D_{x^ix^j}\phi(|y-x|,t) \\ &= -u_{ij}(x,t) - \phi''\frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2} - \frac{\phi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right) \\ D_{y^iy^j}Z &= u_{ij}(y,t) - D_{y^iy^j}\phi(|y-x|,t) \\ &= u_{ij}(y,t) - \phi''\frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2} - \frac{\phi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right) \\ D_{x^iy^j}Z &= -D_{x^iy^j}\phi(|y-x|,t) \\ &= \phi''\frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2} + \frac{\phi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right) \end{aligned}$$

and at a maximum point, the matrix $[D^2Z]$ must be negative semi-definite.

The evolution equation for Z is

$$\begin{aligned} \frac{\partial Z}{\partial t} &= u_t(y,t) - u_t(x,t) - \phi_t \\ &= a^{ij}(D_yu,t)u_{ij}(y,t) + b(D_yu,t) - a^{ij}(D_xu,t)u_{ij}(x,t) - b(D_xu,t) - \phi_t \\ &= a^{ij}(D_yu,t)\left[D_{y^iy^j}Z + D_{y^iy^j}\phi(|y-x|,t)\right] + b(D_yu,t) \\ &\qquad - a^{ij}(D_xu,t)\left[-D_{x^ix^j}Z - D_{x^ix^j}\phi(|y-x|,t)\right] - b(D_xu,t) \\ &\qquad - \phi_t + 2c^{ij}D_{x^iy^j}Z + 2c^{ij}D_{x^iy^j}\phi(|y-x|,t) \\ &= a^{ij}(D_yu,t)\left[D_{y^iy^j}Z + \phi''\frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2} + \frac{\phi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right)\right] \\ &\qquad - a^{ij}(D_xu,t)\left[-D_{x^ix^j}Z - \phi''\frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right. \\ &\qquad\qquad \left. - \frac{\phi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right)\right] \\ &\qquad + b(D_yu,t) - b(D_xu,t) - \phi_t + 2c^{ij}D_{x^iy^j}Z \\ &\qquad + 2c^{ij}\left[-\phi''\frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2} - \frac{\phi'}{|y-x|}\left(\delta_{ij} - \frac{(y^i-x^i)(y^j-x^j)}{|y-x|^2}\right)\right], \end{aligned}$$

where we add and subtract cross derivative terms with yet-to-be-chosen coefficients c^{ij}. If we write $\xi = (y-x)/|y-x|$, and assume that we are at an internal maximum, then

(5.6) implies that $D_i u(x,t) = \phi' \xi_i = D_i u(y,t)$, and we can continue the calculation:

$$\begin{aligned}\frac{\partial Z}{\partial t} &= a^{ij}(\phi'\xi,t) D_{y^i y^j} Z + a^{ij}(\phi'\xi,t) D_{x^i x^j} Z + 2c^{ij} D_{x^i y^j} Z \\ &\qquad + \phi'' \left(2a^{ij}(\phi'\xi)\xi_i\xi_j - 2c^{ij}\xi_i\xi_j\right) + 2\frac{\phi'}{|y-x|}\left(a^{ij}(\phi'\xi) - c^{ij}\right)(\delta_{ij} - \xi_i\xi_j) \\ &\qquad\qquad + b(\phi'\xi,t) - b(\phi'\xi,t) - \phi_t \\ &= \text{trace}\left(\begin{bmatrix} A(\phi'\xi) & C \\ C^T & A(\phi'\xi)\end{bmatrix} D^2 Z\right) + 2\phi''\left(\xi^T A\xi - \xi^T C\xi\right) \\ &\qquad\qquad + 2\frac{\phi'}{|y-x|}\left(\text{trace} A - \xi^T A\xi - \text{trace} C + \xi^T C\xi\right) - \phi_t\end{aligned}$$

The idea now is to choose the off-diagonal block $C = [c^{ij}]$ in such a way that the $2n \times 2n$ matrix

$$A' = \begin{bmatrix} A & C \\ C^T & A\end{bmatrix}$$

is positive semi-definite, leaves the coefficient of ϕ'' positive and sets the coefficient of $\phi'/|y-x|$ to zero.

The first and third of these requirements imply that C is given by $c^{ij} = a^{ij}(\phi'\xi,t) - c\,\xi_i\xi_j$ for some $c > 0$. We can check that

$$\xi^T A\xi - \xi^T C\xi = c > 0$$

and that

$$\begin{aligned}\text{trace}\, C - \xi^T C\xi &= \text{trace}[a^{ij} - c\,\xi_i\xi_j] - \xi_j(a^{ij} - c\,\xi_i\xi_j)\xi_i \\ &= \text{trace}\, A - \xi^T A\xi.\end{aligned}$$

If we set $c = 2\alpha(\phi'\xi,t)$, with α defined by (5.2), this maximizes the coefficient of ϕ'', while keeping A' positive semi-definite. For any $v, w \in \mathbb{R}^n$,

$$\begin{aligned}(v^T, w^T)A'\begin{pmatrix} v \\ w\end{pmatrix} &= v^T Av + w^T Aw + 2v^T Cw \\ &= v^T Av + w^T Aw + 2v^T Aw - 4\alpha(\xi\cdot v)(\xi\cdot w) \\ &= (v+w)^T A(v+w) - \alpha\left[(\xi\cdot v+w)^2 - (\xi\cdot v-w)^2\right] \\ &\geq (v+w)^T A(v+w) - \alpha(\xi\cdot v+w)^2 \\ &\geq 0.\end{aligned}$$

At the maximum point of Z, we find that

$$\frac{\partial Z}{\partial t} \leq 4\alpha(\phi'\xi)\phi'' - \phi_t.$$

In this way, we have reduced our problem to finding ϕ that satisfies the above equation, or, if we can find a lower bound on α dependent only on $|p|$, as in (5.3), then

$$4\tilde{\alpha}(\phi')\phi'' - \phi_t \leq 0,$$

We will use the results of Chapter 3 to do this.

5.2 Estimates for periodic solutions

The following theorem is an analogue of Theorem 3.2 for higher dimensions. In the special case of mean curvature flow, this is joint work with Ben Andrews.

Let α be defined by (5.2).

Theorem 5.1. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a smooth solution to*

$$\frac{\partial u}{\partial t} = a^{ij}(Du,t)u_{ij} + b(Du,t)$$
$$u(x,0) = u_0(x)$$

where u_0 is smooth with oscillation bound $\operatorname{osc} u_0 \leq M$, and is also spatially periodic, $u_0(x) = u_0(x+\Gamma)$, for some lattice Γ.

Suppose that $\alpha(p) = \tilde{\alpha}(|p|) > 0$ for all p.

If $\varphi : \mathbb{R}^+ \times [0,T]$ is a smooth solution to the auxilliary one-dimensional equation

$$\varphi_t = 4\tilde{\alpha}(|\varphi'|,t)\varphi'', \tag{5.7}$$

and satisfies the boundary conditions

$$\begin{aligned} &\lim_{t\to 0} \varphi(z,t) = 1 \text{ for } z \neq 0 \\ &\varphi(0,t) = 0 \text{ for all } t > 0 \\ &\lim_{z\to\infty} \varphi(z,t) \to 1 \text{ for all } t > 0 \end{aligned} \tag{5.8}$$

then

$$|u(y,t) - u(x,t)| \leq M\varphi\left(\frac{|y-x|}{M}, \frac{t}{M^2}\right).$$

Corollary 5.2. *If there are positive constants A_0 and P so that*

$$\tilde{\alpha}(|p|)|p|^2 \geq A_0 \text{ for } |p| \geq P, \tag{5.9}$$

then there is a $T' > 0$ such that for $t \in (0, T']$,

$$|Du| \leq C_1\sqrt{t}(1+t)\exp(C_2/t),$$

where T', C_1 and C_2 depend on n, M, A_0, and P.

Proof of Theorem 5.1. The proof of this is substantially the same as the proof of Theorem 3.2, the gradient estimate for periodic, one-dimensional equations.

As in the previous pages, let

$$Z(x,y,t) := u(y,t) - u(x,t) - \phi(|y-x|,t),$$

and choose $\phi(z,t) = M\varphi\left(z/M, t/M^2\right)$, so that $Z(x,y,0) \leq 0$.

As u is periodic over the lattice $\Gamma = (L_1, \ldots, L_n)$, Z is periodic over regions

$$\{ (x, y, t) \in \mathbb{R}^{2n} \times [0, T] : 2nL_i - x_i \leq y_i \leq 2(n+1)L_i - x_i \}.$$

On any one of these regions, note that $Z(y, y, t) = 0$ and that $Z(x, y, t) \leq M - \phi \to 0$ as $|y_i - x_i| \to \infty$, so Z attains a spatial maximum on the region (and hence on the entire domain $\mathbb{R}^{2n}$).

If there is a maximum point (x, y) at some $t_0 \in (0, T')$ with $x \neq y$, then at this point Z is smooth and

$$\frac{\partial Z}{\partial t} \leq 4\tilde{\alpha}(\phi'\xi)\phi'' - \phi_t \leq 4\tilde{\alpha}(|\varphi'|)\frac{\varphi''}{M} - \frac{\varphi_t}{M} = 0.$$

If $x = y$ at the maximum point, then here $Z(x, x, t) = 0$ and in either case, $Z \leq 0$. The estimate for $|u(y, t) - u(x, t)|$ follows. □

Proof of Corollary 5.2: If $\tilde{\alpha}$ satisfies the degeneracy condition (5.9), then for small times the gradient of φ may be estimated by Corollary 3.3. Letting $x \to y$ gives that

$$\begin{aligned} |Du(y, t)| &\leq n\varphi'(0, t) \\ &\leq C_1\sqrt{t}(1 + t)\exp(C_2/t). \end{aligned}$$

□

5.3 Estimates for boundary value problems

In the special case of mean curvature flow the following theorem is joint work with Ben Andrews.

Theorem 5.3 (Neumann problem). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain with C^2 boundary, and let u be a smooth solution of*

$$\begin{aligned} \frac{\partial u}{\partial t} &= a^{ij}(Du)u_{ij} + b(Du, t) \\ D_\nu u(x, t) &= 0, \text{ for } x \in \partial\Omega, t > 0. \end{aligned}$$

If a^{ij} and φ satisfy the same conditions as in Theorem 5.1, then for any x and y in $\overline{\Omega}$,

$$|u(y, t) - u(x, t)| \leq \varphi(|y - x|, t),$$

where $\operatorname{osc} u_0 = M$.

Furthermore if a^{ij} satisfies the degeneracy condition (5.4)*, then for $t \in (0, T')$ a short-time gradient bound holds:*

$$|Du(x, t)| \leq C_1\sqrt{t}(1 + t)\exp(C_2/t) \text{ for } (x, t) \in \Omega \times (0, T'] \qquad (5.10)$$

where T', C_1 and C_2 depend on n, M, A_0, and P.

Proof: As in the previous proof, set

$$Z := u(y, t) - u(x, t) - M\varphi\left(|y - x|/M, t/M^2\right).$$

Note that $Z \le 0$ when $t = 0$.

For any $t > 0$, suppose that (x, y) is a spatial maximum of Z. We will consider the possibility that x and y are both interior points, that y is a boundary point and so is x, or that y is a boundary point while x is not (the converse follows without loss of generality).

If both x and y are interior points, then the arguments of Theorem 5.1 apply and $Z \le 0$ at this point.

Consider the case that y is on the boundary $\partial\Omega$. If we take derivatives at y that are in directions μ_y that have no component normal to the boundary, then as before $D_{\mu_y} Z(x, y, t) = 0$. On the other hand, let ν_y be the outward unit normal at y. The outwards-pointing derivative of Z here is

$$\begin{aligned}\frac{d}{ds} Z(x, y + s\nu_y, t)\bigg|_{s=0} &= \nu_y \cdot Du(y, t) - \varphi' \frac{y - x}{|y - x|} \cdot \nu_y \\ &= 0 - \varphi' \frac{y - x}{|y - x|} \cdot \nu_y \\ &\le 0,\end{aligned}$$

where we have used the boundary condition $D_{\nu_y} u(y, t) = 0$ and that as Ω is convex, $(y - x) \cdot \nu \ge 0$.

This inequality cannot be strict, for if it is, then there is a small $s > 0$ such that

$$Z(x, y - s\nu_y, t) > Z(x, y, t)$$

which would contradict that (x, y) is a maximum of Z. Therefore

$$\frac{d}{ds} Z(x, y + s\nu_y, t)\bigg|_{s=0} = 0,$$

and indeed $D_y Z(x, y, t) = 0$.

Now consider the position of x. If it is on the boundary, let ν_x be the outward unit normal at x, and so

$$\frac{d}{ds} Z(x + s\nu_x, y, t)\bigg|_{s=0} \;=\; -\nu_x \cdot Du(x, t) + \varphi' \frac{y - x}{|y - x|} \cdot \nu_x \;\le\; 0,$$

Again, this inequality cannot be strict if (x, y) is to be a maximum of Z, so the outward derivative $\frac{d}{ds} Z(x + s\nu_x, y, t)\Big|_{s=0} = 0$. As before, other non-normal derivatives are also zero, so $D_x Z(x, y, t) = 0$.

So, when both x and y are boundary points, $DZ = 0$ and $[D^2 Z]$ is negative semi-definite. We can argue as before that $Z_t \le 0$.

In the case that x is an interior point, $D_x Z = 0$ and so $DZ = 0$, $[D^2 Z]$ is negative semi-definite here, and $Z_t \le 0$.

It follows that $Z \le 0$ for all $t > 0$. □

The highly geometric nature of mean curvature flow allows us to relax the conditions on the convexity of the boundary. In the following theorem we consider domains that are merely mean-convex. This means that at every point on the boundary, the sum of the principal curvatures of $\partial\Omega$ is positive:

$$\sum_{i=1}^{n-1} \kappa_i \geq 0.$$

Under the assumption of convexity (rather than mean-convexity), the following theorem is joint work with Ben Andrews.

Theorem 5.4 (Dirichlet problem for mean curvature flow). *Let $\Omega \subset \mathbb{R}^n$ be a mean-convex domain with a C^2 boundary, and let u be a smooth solution of the mean curvature flow for graphs*

$$\frac{\partial u}{\partial t} = \left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) D_{ij} u,$$

with prescribed boundary values

$$u(x,t) = 0 \quad \text{for } x \in \partial\Omega,\ t > 0,$$
$$u(\cdot, 0) = u_0.$$

If φ is a smooth solution to curve shortening flow

$$\varphi_t = \frac{\varphi''}{1+(\varphi')^2},$$

with boundary conditions given by (5.8)*, then there is an estimate*

$$|u(y,t) - u(x,t)| \leq 2M\varphi\left(\frac{|y-x|}{2M}, \frac{t}{4M^2}\right),$$

where $M = \sup|u_0|$.

Proof: We find a boundary gradient estimate by defining a new Z_B on $\Omega \times (0,T)$ which incorporates the distance to the boundary

$$Z_B(y,t) := u(y,t) - 2M\varphi\left(\frac{d}{2M}, \frac{t}{4M^2}\right),$$

where $d(y) = \operatorname{dist}(y, \partial\Omega)$ is a C^2 function in the neighbourhood of the boundary

$$\Omega \setminus \Omega^R := \{\, y \in \Omega : \operatorname{dist}(y, \partial\Omega) \leq R \,\}.$$

Here, $R = (\sup_{\partial\Omega} \kappa_i)^{-1}$ and κ_i are the principal curvatures of $\partial\Omega$. This close to the boundary, each point y has a unique closest point $x \in \partial\Omega$.

Choose $T' > 0$ so that for $0 < t < T'$, if $0 < d < R$ then $\varphi'\left(d/(2M), t/(4M^2)\right) \geq 0$, and if $d \geq R$ then $\varphi\left(d/(2M), t/(4M^2)\right) \geq 1$.

At $t = 0$, $Z_B \leq 0$ for all $y \in \overline{\Omega}$. For y on the boundary, $Z_B(y,t) = 0 - 2M\varphi \leq 0$. For $t < T'$ and points at least distance R from the boundary, $y \in \Omega^R$, $Z_B(y,t) \leq u(y,t) - M \leq 0$.

We can find spatial derivatives for Z_B:

$$D_{y^i} Z_B = D_{y^i} u - \varphi' D_i d$$
$$D_{y^i y^j} Z_B = D_{y^i y^j} u - \frac{\varphi''}{2M} D_i d D_j d - \varphi' D_{ij} d.$$

Now suppose that y is an interior maximum of Z_B at some time $t < T'$. At this point, $DZ_B = 0$ and $[D^2 Z_B]$ is negative semi-definite, so

$$\begin{aligned}\frac{\partial Z_B}{\partial t} &= u_t - \frac{\varphi_t}{2M} \\ &= m^{ij}(\varphi' Dd)\left[D_{y^i y^j} Z_B + \frac{\varphi''}{2M} D_i d D_j d + \varphi' D_{ij} d\right] - \frac{\varphi_t}{2M} \\ &= m^{ij} D_{y^i y^j} Z_B + \frac{1}{2M}\frac{\varphi''}{1+\varphi'^2} + \varphi' D_{ii} d - \frac{\varphi_t}{2M},\end{aligned}$$

where we have used that $|Dd| = 1$ and $D_i d D_{ij} d = 0$.

As in Lemma 14.17 of [16],

$$\mathrm{trace}[D^2 d(y)] = \sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d},$$

where κ_i are the principal curvatures of $\partial\Omega$ at x, the closest point on $\partial\Omega$ to y. If $d < R$, then $\kappa_i d < 1$ and

$$\sum_{i=1}^{n-1} \frac{-\kappa_i}{1 - \kappa_i d} \;\le\; -\sum_{i=1}^{n-1} \kappa_i \;\le\; 0,$$

the last inequality resulting from the mean-convexity of $\partial\Omega$. Then

$$\frac{\partial Z_B}{\partial t} = m^{ij} D_{y^i y^j} Z_B + \frac{1}{2M}\frac{\varphi''}{1+\varphi'^2} + \varphi' D_{ii} d - \frac{\varphi_t}{2M} \le 0.$$

It follows that $Z_B \le 0$ for $t < T'$, and so for all $x \in \partial\Omega$ and $y \in B_R(x)$,

$$u(y,t) - u(x,t) \le 2M\varphi\left(\frac{\mathrm{dist}(y,\partial\Omega)}{2M}, \frac{t}{4M^2}\right) \le 2M\varphi\left(\frac{|x-y|}{2M}, \frac{t}{4M^2}\right). \qquad (5.11)$$

This gives us an estimate on the boundary. We complete our proof by using the same Z as before:

$$Z(x,y,t) = u(y,t) - u(x,t) - 2M\varphi\left(\frac{|x-y|}{2M}, \frac{t}{4M^2}\right).$$

Once again, $Z(x,y,0) \le 0$, and when both x and y are on the boundary, $Z(x,y,t) = -2M\varphi \le 0$. If x is a boundary point and y is an interior point (or vice-versa), then

$$Z(x,y,t) = u(y,t) - u(x,t) - 2M\varphi\left(\frac{|x-y|}{2M}, \frac{t}{4M^2}\right) \le 0$$

by (5.11).

Finally, if (x, y) is a maximum of Z at some time $t < T'$, where both x and y are interior, then as in Section 5.2 $\dfrac{\partial Z}{\partial t} \leq 0$ at this point, and so $Z \leq 0$ for all $t < T'$. The estimate follows. □

Remark: We can use these methods to find gradient estimates for equations of more general form.

For the Dirichlet problem with conditions on a^{ij} given in Theorem 5.1, and $u = 0$ on $\partial\Omega$, we can find estimates of the type in Theorem 5.4 for convex Ω.

If a^{ij} has the form (5.5), then we can find estimates of this type on domains that are merely mean-convex.

Chapter 6

Application of gradient estimates to the Neumann problem

In this chapter we use the gradient estimate derived previously to establish the existence of solutions to the mean curvature flow equation with Neumann boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) D_{ij} u \text{ in } \Omega \times (0,T], \\ D_\nu u(x,t) &= 0 \text{ for } x \in \partial\Omega \text{ and } t \in (0,T] \end{aligned} \tag{6.1}$$

$$u(\cdot, 0) = u_0, \tag{6.2}$$

where $\Omega \subset \mathbb{R}^n$ is a compact, open convex domain with $C^{2+\alpha}$ boundary $\partial\Omega$, and $u_0 \in C(\overline{\Omega})$. The outward unit normal on the boundary is ν.

This extends Huisken's result in [19] showing the existence of smooth solutions to (6.1) for initial data with greater regularity.

Theorem 6.1 (Huisken). *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2+\alpha}$. If $u_0 \in C^{2+\alpha}(\overline{\Omega})$ satisfies $D_\nu u_0 = 0$ on $\partial\Omega$, then* (6.1), (6.2) *has a smooth solution on $\Omega \times (0,T)$.*

Note that while this theorem makes no restriction on the convexity of Ω, the main result of this chapter does.

Theorem 6.2. *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded, open, convex domain, and let $u_0 \in C(\overline{\Omega})$. Then the Neumann problem* (6.1) *has a smooth solution for $t > 0$, which converges uniformly to u_0 as $t \to 0$, at a rate dependent on the modulus of continuity of u_0.*

6.1 Some remarks about changes of coordinates that straighten boundaries

A similar discussion of boundary curvatures and the distance function may be found in Appendix 14.6 of [16]. Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$. The boundary is said to be $C^{k+\alpha}$ if for each boundary point x_0 we can find a

$C^{k+\alpha}$ mapping $f : \mathbb{R}^{n-1} \to \mathbb{R}$ which has the boundary in a neighbourhood of x_0 as its graph.

Set

$$R := \frac{1}{2} \sup \{ r : \text{If } \operatorname{dist}(x, \partial\Omega) < r \text{ then } x \text{ has a unique closest point } x_0 \in \partial\Omega \} . \quad (6.3)$$

If Ω is convex, then we can take

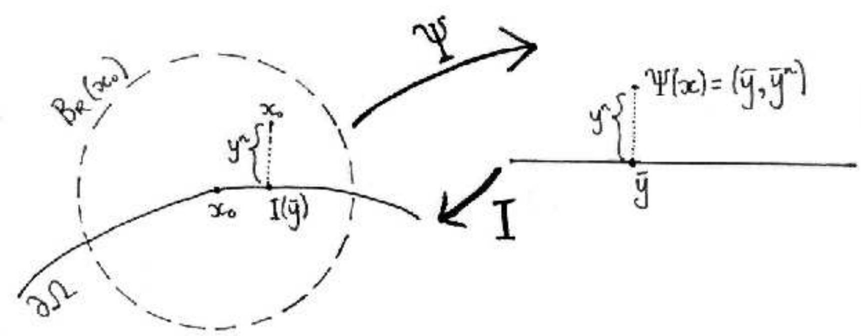
$$R := \frac{1}{2} \inf_{\substack{x \in \partial\Omega \\ 1 \le i \le n-1}} \frac{1}{\kappa_i(x)},$$

where κ_i are the principal curvatures of $\partial\Omega$.

On balls $B_R(x_0)$ centred on the boundary, we introduce a change of coordinates $\Psi : B_R \to \partial\Omega \times [-R, R]$ such that if the new coordinates are denoted $y = (\overline{y}, y^n) = (y^1, \dots, y^{n-1}, y^n)$, $\Psi(x) = (\overline{y}, y^n)$, and $I : \partial\Omega \to \mathbb{R}^n$ is the immersion of the boundary into $\mathbb{R}^n$, then $d(x, \partial\Omega) = d(x, I(\overline{y})) = y^n$. In other words, $I(\overline{y})$ is the closest point to x on $\partial\Omega$ and y^n is the *signed distance* between x and $\overline{y}$, being positive if $x \notin \Omega$, zero if $x \in \partial\Omega$, and negative otherwise.

The inverse transformation is easier to work with, being given by $\Psi^{-1}(\overline{y}, y^n) = I(\overline{y}) + \nu(\overline{y}) y^n$, where ν is the outward-pointing unit normal to $\partial\Omega$.

As the boundary in the new coordinates is simply $y^n = 0$, this is referred to as a boundary-straightening transformation.



If the graph $I(\overline{y}) = (y^1, \dots, y^{n-1}, f(\overline{y}))$ is a local immersion of the boundary, then the outward unit normal is given by

$$\nu(\overline{y}) = \frac{1}{\sqrt{1 + |Df|^2}} \left(-\frac{\partial f}{\partial y^1}, \dots, -\frac{\partial f}{\partial y^{n-1}}, 1 \right),$$

and on $\Psi(B_R(x_0))$ we have

$$[D\Psi^{-1}]_i^j = \frac{\partial \Psi^{-1j}}{\partial y^i} = \begin{cases} \delta_i^j + y^n \frac{\partial \nu^j}{\partial y^i} & \text{for } i, j = 1, \dots, n-1 \\ \frac{\partial f}{\partial y^i} + y^n \frac{\partial \nu^n}{\partial y^i} & \text{for } j = n \text{ and } i = 1, \dots, n-1; \\ \nu^j & \text{for } i = n, \end{cases} \quad (6.4)$$

so eigenvalues for $[D\Psi^{-1}]$ are $1-y^n\kappa_1(1+|Df|^2)^{-1/2},\ldots,1-y^n\kappa_{n-1}(1+|Df|^2)^{-1/2}$, and lie between $\frac{1}{2}$ and $\frac{3}{2}$ on B_R. The curvatures of the boundary, κ_i, are given by the eigenvalues of $[D^2 f]$.

Also, second derivatives are

$$[D^2\Psi^{-1}]^j_{ik} = \begin{cases} y^n \frac{\partial^2 \nu^j}{\partial y^i \partial y^k} & \text{for } i,j,k \neq n, \\ \frac{\partial^2 f}{\partial y^i \partial y^k} + y^n \frac{\partial^2 \nu^n}{\partial y^i \partial y^k} & \text{for } j = n \text{ and } i,k \neq n, \\ \frac{\partial \nu^j}{\partial y^i} & \text{for } i \neq n \text{ and } k = n, \\ \frac{\partial \nu^j}{\partial y^k} & \text{for } i = n \text{ and } k \neq n, \\ 0 & \text{for } i = n, k = n. \end{cases}$$

The smoothness of this change of coordinates is dependent on the smoothness of the boundary: if $\partial\Omega$ is $C^{k+\alpha+2}$ then $D\Psi^{-1}$ is $C^{k+\alpha}$.

When u is defined in the old coordinates on $\mathbb{R}^n$, in the new coordinates we can define a new function

$$v(y,t) := u(\Psi^{-1}(y),t).$$

First derivatives are related by $D_i v(y,t) = [D\Psi^{-1}]^k_i D_k u(\Psi^{-1}(y),t)$, and second derivatives by $[D^2 v]_{ij} = [D\Psi^{-1}]^m_j [D\Psi^{-1}]^l_i [D^2 u]_{ml} + (D_m u)[D^2\Psi^{-1}]^m_{ij}$.

Putting this all together, we notice that if u satisfies (6.1) then v satisfies

$$\begin{aligned} v_t(y,t) &= m^{ij}([D\Psi]Dv)\left([D\Psi]^l_j [D\Psi]^k_i [D^2 v]_{lk} + (D_s v)[D^2\Psi]^s_{ij}\right) \\ &= a^{kl}(Dv,y)[D^2 v]_{lk} + b(Dv,y) \qquad \text{for } y \in \Psi(Q_R)\,, \end{aligned} \tag{6.5}$$

$$D_n v(y,t) = 0 \qquad \text{at } y^n = 0. \tag{6.6}$$

where the mean curvature operator is abbreviated as $m^{ij}(p) = \delta^{ij} - \frac{\delta^{ik}\delta^{jl}p_k p_l}{1+|p|^2}$, and we write

$$\begin{aligned} a^{lk}(p,y) &:= m^{ij}([D\Psi]p)[D\Psi]^l_j [D\Psi]^k_i, \\ b(p,y) &:= m^{ij}([D\Psi]p)(p_s)[D^2\Psi]^s_{ij}. \end{aligned} \tag{6.7}$$

Once we have straightened out the boundary, we will find it useful later on to define a reflection ρ in the boundary that extends v outside Ω:

$$\tilde{v}(y,t) := v(\rho(y),t) \quad \text{for } y \in B_R$$

where $\rho(y) = (y^1,\ldots,y^{n-1},-|y^n|)$.

Let Q_R be the intersection of a parabolic cylinder with the domain of interest:

$$Q_R(x_0,t_0) = \{\,(x,t) \in \Omega \times [0,T] : x \in B_R(x_0), t \in (t_0 - R^2, t_0)\,\}.)$$

When v satisfies (6.5) on Q_R, $\tilde{v}$ will satisfy

$$\begin{aligned}\tilde{v}_t(y,t) &= v_t(\rho(y),t)\\ &= a^{kl}(Dv(\rho(y),t),\rho(y))[D^2v(\rho(y),t)]_{kl} + b(Dv(\rho(y),t),\rho(y))\\ &= a^{kl}([D\rho]D\tilde{v}(y,t),\rho(y))[D\rho]^i_l[D\rho]^j_k[D^2\tilde{v}(y,t)]_{ij} + b([D\rho]D\tilde{v}(y,t),\rho(y))\\ &= \tilde{a}^{kl}(D\tilde{v},y)[D^2\tilde{v}]_{kl} + \tilde{b}(D\tilde{v},y) \end{aligned} \tag{6.8}$$

on $B_R \times (t_1 - R^2, t_1)$ where $[D\rho] = \mathrm{diag}(1,\ldots,1,-y^n/|y^n|)$.

The regularity of the coefficients of the reflected equation is estimated:

Lemma 6.3. *If v is a $C^{1+\alpha}$ function on Q_R, with $D_n v(\overline{y},0) = 0$, then the coefficients for the reflected equation*

$$\begin{aligned}\tilde{a}^{ij}(D\tilde{v},y) &= a^{kl}([D\rho]D\tilde{v},\rho(y))[D\rho]^i_l[D\rho]^j_k,\\ \tilde{b}(D\tilde{v},y) &= b([D\rho]D\tilde{v},\rho(y))\end{aligned}$$

satisfy Hölder estimates

$$\begin{aligned}|\tilde{a}^{ij}|_{\alpha;B_R\times(t_1-R^2)} &\le 2|a^{ij}|_{C^1;Q_R}\left(1+|Dv|_{\alpha;Q_R}\right),\\ |\tilde{b}|_{\alpha;B_R\times(t_1-R^2)} &\le 2|b|_{C^1;Q_R}\left(1+|Dv|_{\alpha;Q_R}\right)\end{aligned}$$

for some $0<\alpha<1$.

Proof: In general, if a function is defined piecewise on a convex domain U divided into U_1 and $U_2 = U\backslash U_1$,

$$h(x) = \begin{cases} h_1(x) & x\in U_1\\ h_2(x) & x \in U_2\end{cases}$$

and is continuous across any shared boundary $\overline{U}_1\cap\overline{U}_2$ then if h_1 is C^α on U_1 and h_2 is C^α on U_2, it follows that h is C^α on $U = U_1\cup U_2$.

We can see this by letting x_1 and x_2 be in U_1 and U_2 respectively. We can find a point z in the shared boundary $\overline{U}_1\cap\overline{U}_2$ directly between the two, with $|y-x| = |y-z|+|z-x|$.

Then

$$\begin{aligned}|h(x)-h(y)| &\le |h(x)-h(z)|+|h(z)-h(y)|\\ &= |h_1(x)-h_1(z)|+|h_2(z)-h_2(y)|\\ &\le C|x-z|^\alpha + C|z-y|^\alpha\\ &= C\left(s^\alpha|x-y|^\alpha + (1-s)^\alpha|y-x|^\alpha\right)\\ &= 2C|x-y|^\alpha\end{aligned}$$

for $s = |x-z|/|x-y| < 1$.

This observation applies to both $\tilde{v}$ and $D\tilde{v}$ — as $D_n v = 0$ on the boundary, $D\tilde{v} = [D\rho][Dv]$ is continuous across the boundary, even though $[D\rho]$ itself is not — and so $D\tilde{v}$ is C^α.

It is clear that $\tilde{b}(D\tilde{v},y) = b\left([D\rho]D\tilde{v},\rho(y)\right) = b\left(Dv,\rho(y)\right)$ is continuous, and $\tilde{b}$ shares the same regularity as b.

As

$$\tilde{a}^{ij}(D\tilde{v},y)=\begin{cases}a^{ij}\left([D\rho]D\tilde{v}(y,t),\rho(y)\right) & \text{for } 1\le i,j\le n-1 \text{ or } i=j=n,\\ -\frac{y^n}{|y^n|}a^{ij}\left([D\rho]D\tilde{v}(y,t),\rho(y)\right), & \text{for } i\neq n \text{ and } j=n \text{ or vice-versa,}\end{cases}$$

we only need to check whether the terms in the off-diagonal block $\tilde{a}^{in}$ are continuous. These are given by

$$\begin{aligned}
&\tilde{a}^{in}(D\tilde{v},y)\\
&\quad=\frac{-y^n}{|y^n|}a^{in}\left([D\rho]D\tilde{v}(y,t),\rho(y)\right)\\
&\quad=\frac{-y^n}{|y^n|}m^{kl}\left([D\Psi][D\rho]D\tilde{v}\right)[D\Psi]^i_k[D\Psi]^n_l\\
&\quad=\frac{-y^n}{|y^n|}\left([D\Psi]^i_k[D\Psi]^n_k-\frac{\delta^{k\alpha}\delta^{l\beta}\left([D\Psi][D\rho]D\tilde{v}\right)_\alpha\left([D\Psi][D\rho]D\tilde{v}\right)_\beta[D\Psi]^i_k[D\Psi]^n_l}{1+|[D\Psi][D\rho]D\tilde{v}|^2}\right)\\
&\quad=\frac{y^n}{|y^n|}\left(\frac{\delta^{k\alpha}\delta^{l\beta}[D\Psi]^\gamma_\alpha[D\rho]^j_\gamma D_j\tilde{v}[D\Psi]^\mu_\beta[D\rho]^s_\mu D_s\tilde{v}[D\Psi]^i_k[D\Psi]^n_l}{1+|[D\Psi][D\rho]D\tilde{v}|^2}\right)\\
&\quad=\frac{y^n}{|y^n|}\frac{1}{1+|[D\Psi][D\rho]D\tilde{v}|^2}\sum_{k,l,\alpha,\beta}\delta^{k\alpha}\delta^{l\beta}\left(\sum_{\gamma=1}^{n-1}\sum_{\mu=1}^{n-1}[D\Psi]^\gamma_\alpha D_\gamma\tilde{v}[D\Psi]^\mu_\beta D_\mu\tilde{v}\right.\\
&\qquad\quad-\frac{y^n}{|y^n|}\sum_{\gamma=1}^{n-1}[D\Psi]^\gamma_\alpha D_\gamma\tilde{v}[D\Psi]^n_\beta D_n\tilde{v}-\frac{y^n}{|y^n|}\sum_{\mu=1}^{n-1}[D\Psi]^n_\alpha D_n\tilde{v}[D\Psi]^\mu_\beta D_\mu\tilde{v}\\
&\qquad\qquad\left.+[D\Psi]^n_\alpha D_n\tilde{v}[D\Psi]^n_\beta D_n\tilde{v}\right)[D\Psi]^i_k[D\Psi]^n_l,
\end{aligned}$$

(here there is no summation over n).

Between the third and the fourth line, we have used that $[D\Psi^{-1}]^\cdot_i$ is tangent to the boundary while $[D\Psi^{-1}]^\cdot_n$ is normal to the boundary (see equation (6.4)), so for $i\neq n$, we have $\sum_k[D\Psi^{-1}]^k_i[D\Psi^{-1}]^k_n=0$. It follows that $[D\Psi]^i_k[D\Psi]^n_k=0$.

In the last step, the second, third and fourth terms are zero (and so continuous) on the boundary, as $D_n\tilde{v}(\overline{y},0)=0$. The first term is zero due to the presence of

$$\sum_l\left([D\Psi]^\mu_l D_\mu v\right)[D\Psi]^n_l=0,$$

since $\mu\neq n$.

So, both $\tilde{a}^{ij}$ and $\tilde{b}$ are continuous across the boundary.

It follows from the first observation that the Hölder constant of $\tilde{a}^{ij}$ on $B_R(y_1)\times(t_1-R^2,t_1)$ is the same as that of a^{ij} on $Q_R(y_1,t_1)$; and if we consider $a^{ij}(Dv(y),y)$ as a function of y, then we find that

$$\begin{aligned}
|\tilde{a}^{ij}(D\tilde{v}(\cdot),\cdot)|_\alpha &\le 2|a^{ij}(Dv(\cdot),\cdot)|_\alpha \\
&= 2\sup_{z_1,z_2}\frac{|a^{ij}(Dv(z_1),z_1)-a^{ij}(Dv(z_2),z_2)|}{|z_1-z_2|^\alpha} \\
&\le 2\sup_{z_1,z_2}\frac{1}{|z_1-z_2|^\alpha}\Big[|a^{ij}(Dv(z_1),z_1)-a^{ij}(Dv(z_1),z_2)| \\
&\qquad\qquad + |a^{ij}(Dv(z_1),z_2)-a^{ij}(Dv(z_2),z_2)|\Big] \\
&\le 2\sup_{z_1}|a^{ij}(Dv(z_1),\cdot)|_\alpha \\
&\qquad + \sup_{z_2}|a^{ij}(\cdot,z_2)|_{C^1}\sup_{z_1,z_2}\frac{1}{|z_1-z_2|^\alpha}|Dv(z_1)-Dv(z_2)| \\
&\le 2|a^{ij}|_{C^1}\left(1+|Dv|_\alpha\right).
\end{aligned}$$

Similarly, if we consider $\tilde{b}(D\tilde{v},y)$ as a function of y, we find that

$$\begin{aligned}
|\tilde{b}(D\tilde{v}(\cdot),\cdot)|_\alpha &\le 2|b(Dv(\cdot),\cdot)|_\alpha \\
&\le 2|b|_{C^1}\left(1+|Dv|_\alpha\right).
\end{aligned}$$

□

6.2 Existence of solutions with continuous initial data and Neumann boundary conditions

We begin our proof of Theorem 6.2 by approximating the continuous initial data by mollified functions that will satisfy the requirements of Theorem 6.1, being smooth and satisfying the Neumann boundary condition.

Lemma 6.4. *There exists an approximating sequence* $u_0^\epsilon \in C^\infty(\overline{\Omega})$ *with* $u_0^\epsilon \to u_0$ *in* $C(\Omega)$*,* $D_\nu u_0^\epsilon = 0$ *on* $\partial\Omega$*, and* $u^{\epsilon_1} > u^{\epsilon_2} > u_0$ *whenever* $\epsilon_1 > \epsilon_2$*.*

Proof: Let B_R be a ball centred on the boundary. We work in the new coordinates on $\Psi(B_R)$, and write $v_0(y) = u_0(\Psi^{-1}(y))$.

Remembering that $\tilde{v}_0$ denotes the extension by reflection of v_0, define the mollified function

$$v_0^\epsilon(y) := \eta_\epsilon * \tilde{v}_0\ (y) = \int_{z\in\Psi(B_R)}\eta_\epsilon(y-z)\tilde{v}_0(z)dz,$$

where we use the usual mollifier

$$\eta(z) = \begin{cases} c\exp\left(\frac{1}{|z|^2-1}\right), & |z|\le 1 \\ 0 & \text{otherwise,}\end{cases}$$

and $\eta_\epsilon(z) = \frac{1}{\epsilon^n}\eta(\frac{z}{\epsilon})$.

This approximation has all the usual qualities of mollifications: $v_0^\epsilon \in C^\infty(\Psi(B_R)^\epsilon)$, where $\Psi(B_R)^\epsilon = \{\, y \in \Psi(B_R) : \mathrm{dist}(y,\partial\Psi(B_R)) > \epsilon \,\}$; and since $\tilde{v}_0 \in C(\Psi(B_R))$, $v_0^\epsilon \to \tilde{v}_0$ uniformly on compact subsets of $\Psi(B_R)$.

In addition, each v_0^ϵ satisfies the Neumann condition $D_n v_0^\epsilon(y) = 0$ for $y \in \partial\Omega \cap \Psi(B_R)^\epsilon$, since

$$D_n v_0^\epsilon(y) = D_n \int_{z\in B_\epsilon(0)} \eta_\epsilon(z)\tilde{v}_0(y-z)dz$$
$$= \int_{\substack{z\in B_\epsilon(0)\\ z^n\geq 0}} \eta_\epsilon(z) D_n\tilde{v}_0(y-z)dz + \int_{\substack{z\in B_\epsilon(0)\\ z^n<0}} \eta_\epsilon(z) D_n\tilde{v}_0(y-z)dz$$

Recalling the relationship between the reflected and original functions, $\tilde{v}_0(y-z) = v_0(y^1 - z^1, \dots, -|y^n - z^n|)$, we observe that when $y \in \partial\Omega$, $y^n = 0$ and $D_n\tilde{v}_0(y-z) = \frac{z^n}{|z^n|} D_n v_0(y^1 - z^1, \dots, -|z^n|)$. Consequently,

$$D_n v_0^\epsilon(y) = \int_{\substack{z\in B_\epsilon(0)\\ z^n>0}} \eta_\epsilon(z) D_n v_0(y^1 - z^1, \dots, -|z^n|)dz$$
$$- \int_{\substack{z\in B_\epsilon(0)\\ z^n<0}} \eta_\epsilon(z) D_n v_0(y^1 - z^1, \dots, -|z^n|)dz = 0$$

as the mollifier has the symmetry $\eta_\epsilon(z^1, \dots, z^n) = \eta_\epsilon(z^1, \dots, -z^n)$.

This is only a *local* approximation, but in the next step we extend it to the entire domain, taking care to preserve the Neumann boundary condition.

Let the set of boundary-centred balls $\{ B_R(x_i) \}_{i=1,N}$ be a finite cover of the boundary $\partial\Omega$ with the property that the set of balls of half the radius $\{ B_{R/2}(x_i) \}_{i=1,N}$ is also a cover. On each ball $B_R(x_i)$ we can define the approximation $v_{0,i}^\epsilon := v_0^\epsilon$ as described above.

Now, define a new cover of Ω by the sets

$$W^i := \{ x \in B_R(x_i) : \text{ if } \Psi(x) = (\overline{y}, y^n), \text{ then } \overline{y} \in \Psi(B_{R/2}(x_i)) \text{ and } |y^n| \leq \frac{R}{2} \}.$$

The cover is completed by $W^0 := \{ x \in \Omega : \text{dist}(x, \partial\Omega) > R/4 \}$.

Note that $B_{R/2}(x_i) \subseteq W^i \subseteq B_R(x_i)$ and so this is indeed a cover; also, $v_{0,i}^\epsilon$ is defined on W^i. On W^0 we define the usual mollification with no reflection, which we call $u_{0,0}^\epsilon$.

Let $\overline{\xi}_i$ be a partition of unity with respect to the sets $\{B_{R/2}(x_i) \cap \partial\Omega\}$ which cover the boundary; that is, $0 \leq \overline{\xi}_i(x) \leq 1$ for $x \in B_{R/2}(x_i) \cap \partial\Omega$, $\overline{\xi}_i \in C_0^\infty(B_{R/2} \cap \partial\Omega)$ (that is, compactly supported with respect to $\partial\Omega$), and $\sum \overline{\xi}_i(x) = 1$ for all $x \in \partial\Omega$.

In the new coordinates on $B_R(x_i)$, we could write $\overline{\xi}_i = \overline{\xi}_i(y^1, \dots, y^{n-1})$, since $\overline{\xi}_i$ is defined only on the boundary. We can extend $\overline{\xi}_i$ to *all* of $B_{R/2}(x_i)$ by setting $\xi_i(x) := \overline{\xi}_i(\Psi(x)^1, \dots, \Psi(x)^{n-1})$.

Let $\tilde{\zeta} : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function satisfying

$$\tilde{\zeta}(d) = \begin{cases} 1 & |d| < \frac{R}{4} \\ 0 & |d| \geq \frac{R}{2}. \end{cases}$$

We will set $\zeta(x) := \tilde{\zeta}(d(x, \partial\Omega))$ where $d(\cdot, \partial\Omega)$ is the signed distance function.

Now, we claim that the functions $\xi_i(x)\zeta(x)$ and $1 - \zeta(x)$ are a partition of unity with respect to the sets $\{W^i \cap \Omega\}$ and W^0. Firstly, all functions are smooth and compactly supported on their respective domains (but they are not zero on the external boundary $\partial\Omega$). Secondly, if $x \in \Omega$, then

$$\sum_i \xi_i(x)\zeta(x) + (1 - \zeta(x)) = 1.$$

This is because if $\mathrm{dist}(x, \partial\Omega) \geq R/2$ then $\zeta(x) = 0$, while if $\mathrm{dist}(x, \partial\Omega) < R/2$ then x has a unique closest boundary point x_0. In the latter case $\xi_i(x) = \overline{\xi}_i(x_0)$ and $\sum_i \overline{\xi}_i(x)\zeta(x) + (1 - \zeta(x)) = \zeta(x)\sum_i \overline{\xi}_i(x_0) + (1 - \zeta(x)) = 1$ as $\overline{\xi}_i$ is a partition of unity on the boundary.

This construction ensures that $D_\nu (\zeta(x)\xi_i(x)) = 0$ when $x \in \partial\Omega$ and so if we define our global approximation as

$$u_0^\epsilon(x) := \sum_{i=1}^{N} \xi_i(x)\zeta(x)v_{0,i}^\epsilon(\Psi(x)) + (1 - \zeta(x))u_{0,0}^\epsilon(x),$$

we find that $u_0^\epsilon \to u_0$ uniformly in $C(\Omega)$, and each $u_0^\epsilon \in C^\infty(\overline{\Omega})$ satisfies $D_\nu u_0^\epsilon = 0$ on $\partial\Omega$.

We can ensure that this sequence is monotone in ϵ, in the sense that $u_0^{\epsilon_1}(x) < u_0^{\epsilon_2}(x)$ whenever $\epsilon_1 < \epsilon_2$ by restricting to a subsequence and off-setting if necessary. □

The result of Huisken mentioned at the start of this chapter now implies that there is a smooth solution u^ϵ to (6.1) with $u^\epsilon(\cdot, 0) = u_0^\epsilon$.

Lemma 6.5. *The approximate solutions u^ϵ have a uniform height bound*

$$\sup_{\Omega\times[0,T]} |u^\epsilon| \leq \sup_{\Omega} |u_0|.$$

Proof: As the mollification u_0^ϵ is created by a local averaging of u_0,

$$\sup_{\Omega} |u^\epsilon(\cdot, 0)| \leq \sup_{\Omega} |u_0|.$$

Suppose at some time $t > 0$ and point x_1, u^ϵ equals $|u_0|$. From the Comparison Principle (Theorem 2.2), x_1 can be assumed to be a boundary point. The Neumann condition $D_\nu u^\epsilon = 0$ implies that $Du^\epsilon = 0$ and so $[D^2 u^\epsilon]$ is negative semi-definite at this point; it follows that $\dfrac{\partial u^\epsilon}{\partial t} \leq 0$ and so u^ϵ is not increasing at this point. □

This height estimate is of course also an oscillation bound

$$|u^\epsilon(x, t) - u^\epsilon(z, t)| \leq 2|u_0|.$$

We are now in a position to use the gradient estimate of Theorem 5.3. For some $T > 0$, there is a gradient bound

$$|Du^\epsilon(x, t)| \leq L(t) \text{ for } t \in (0, T) \text{ and } x \in \overline{\Omega}, \tag{6.9}$$

where $L(t)$ and T are dependent on n and $\sup_\Omega |u_0|$.

Lemma 6.6. *Higher derivatives of u^ϵ are uniformly bounded on the interior, with*

$$|D^k u^\epsilon|_{\Omega^r\times\{t_1\}} \leq c\,(n,k,L(t_0))\left(\frac{1}{r^2}+\frac{1}{t_1-t_0}\right)^{\frac{k-1}{2}} \tag{6.10}$$

for $t_1 > t_0 > 0$ and all $k = 1,2,\ldots$, where Ω^r is the interior set $\{\, x\in\Omega : \mathrm{dist}(x,\partial\Omega) > r\,\}$.

Proof: This is an application of the Ecker-Huisken interior curvature estimate described in Theorem 2.6, originally in the paper [14].

We apply it to the interior of Ω (with $\theta = 0$ and $k = m+2$) to find bounds on all higher derivatives.□

This estimate provides no information as we approach the boundary. However, our uniform gradient bound $L(t_0)$ ensures that the evolution equation is uniformly parabolic, since for $t > t_0$,

$$m^{ij}(Du^\epsilon)\xi_i\xi_j \;\geq\; \frac{1}{1+|Du^\epsilon|^2}|\xi|^2 \;\geq\; \frac{1}{1+L(t_0)^2}|\xi|^2.$$

As we have uniform parabolicity for strictly positive times, extending regularity up to the boundary is a routine application of known results. This is the subject of Lemma 6.7 – Lemma 6.9.

We begin by showing that a function with a Hölder estimate on the boundary of a region, and a strictly interior gradient estimate, has a global Hölder estimate. We plan to apply this to finding a Hölder estimate for the gradient Du.

Lemma 6.7. *Let Ω be a convex domain. If $f : \Omega\times[0,T]\to\mathbb{R}$ has a Hölder oscillation bound on the boundary*

$$\operatorname*{osc}_{Q_r(x_0,t_0)} f \leq C(t_0)r^\alpha \text{ for all } x_0\in\partial\Omega \text{ and } t_0 > r^2,$$

where $C(t)$ is non-increasing in t; and gradient bounds on the interior

$$|Df(x,t)| \leq c\left(\frac{1}{\mathrm{dist}(x,\partial\Omega)^2}+\frac{1}{t}\right)^{1/2};$$

and

$$|f_t(x,t)| \leq c\left(\frac{1}{\mathrm{dist}(x,\partial\Omega)^2}+\frac{1}{t}\right);$$

then we can find an $\alpha' > 0$ such that for all $x,y\in\Omega$ and $s,t\in(0,T]$,

$$|f(x,t)-f(y,s)| \leq C(|x-y|+|t-s|^{1/2})^{\alpha'}$$

where C depends on $\min\{t,s\}$, $\mathrm{diam}\,\Omega$, c and α, and α' depends on α.

Proof: We split the difference in the obvious way

$$|f(x,t)-f(y,s)| \leq |f(x,t)-f(y,t)|+|f(y,t)-f(y,s)|, \tag{6.11}$$

and look at the first term.

Without loss of generality, set $d = \operatorname{dist}(y, \partial\Omega) \leq \operatorname{dist}(x, \partial\Omega)$, and y_0 to be the closest point to y in $\partial\Omega$, so that $|y - y_0| = d$.

If we are close to the boundary, so that $d \leq |x - y|$, then

$$\begin{aligned}
|f(x,t) - f(y,t)| &\leq |f(x,t) - f(y_0,t)| + |f(y_0,t) - f(y,t)| \\
&\leq \operatorname*{osc}_{Q_{|x-y_0|}(y_0, t+|x-y_0|^2)} f + \operatorname*{osc}_{Q_{|y-y_0|}(y_0, t+|y-y_0|^2)} f \\
&\leq C(t + |x-y_0|^2)|x-y_0|^\alpha + C(t + |y-y_0|^2)|y-y_0|^\alpha \\
&\leq C(t)\left[|x-y|^\alpha + |y-y_0|^\alpha\right] + C(t)|y-y_0|^\alpha \\
&\leq 2C(t)|x-y|^\alpha + C(t)|y-x|^\alpha \\
&\leq c_1|x-y|^\alpha
\end{aligned}$$

where we have used that $|y - y_0| = \operatorname{dist}(y, \partial\Omega) \leq |x - y|$, and have set $c_1 = 3C(t)$.

If we are further from the boundary, so that $d > |x - y|$, then for some $\epsilon \in (0, 1)$,

$$\begin{aligned}
|f(x,t) - f(y,t)| &= |f(x,t) - f(y,t)|^\epsilon \, |f(x,t) - f(y,t)|^{1-\epsilon} \\
&\leq \left[\operatorname*{osc}_{Q_{|x-y_0|}(y_0, t+|x-y_0|^2)} f\right]^\epsilon \left[c\left(\frac{1}{\operatorname{dist}(y,\partial\Omega)^2} + \frac{1}{t}\right)^2 |x-y|\right]^{1-\epsilon} \\
&\leq \left[C(t + |x-y_0|^2)|x-y_0|^\alpha\right]^\epsilon c^{1-\epsilon}\left(\frac{1}{d^2} + \frac{1}{t}\right)^{\frac{1-\epsilon}{2}} |x-y|^{1-\epsilon} \\
&\leq C(t)^\epsilon\left[|x-y|^\alpha + |y-y_0|^\alpha\right]^\epsilon c^{1-\epsilon}\left(d^{\epsilon-1} + t^{(\epsilon-1)/2}\right)|x-y|^{1-\epsilon} \\
&\leq 2^\epsilon C(t)^\epsilon\left(d^{\alpha\epsilon+\epsilon-1} + d^{\alpha\epsilon}t^{(\epsilon-1)/2}\right)|x-y|^{1-\epsilon} \\
&\leq c_2|x-y|^{1-\epsilon},
\end{aligned}$$

setting $\epsilon = 1/(\alpha + 1)$ so that $\alpha\epsilon + \epsilon - 1 = 0$, and $c_2 = 2^\epsilon C(t)^\epsilon \left(1 + (\operatorname{diam}\Omega)^{\alpha\epsilon} t^{(\epsilon-1)/2}\right)$.

Now consider the second term of (6.11), and suppose without loss of generality that $s < t$. As before, set $d = \operatorname{dist}(y, \partial\Omega)$ and $y_0 \in \partial\Omega$.

If we are close to the boundary, so that $d \leq \sqrt{t - s}$, then

$$\begin{aligned}
|f(y,t) - f(y,s)| &\leq \operatorname*{osc}_{Q_{\sqrt{t-s}}(y_0,t)} f \\
&\leq C(t)|t-s|^{\alpha/2} \\
&= c_3|t-s|^{\alpha/2}
\end{aligned}$$

Otherwise, if $d > \sqrt{t-s}$, then

$$\begin{aligned} |f(y,t)-f(y,s)| &= |f(y,t)-f(y,s)|^{\mu}\,|f(y,t)-f(y,s)|^{1-\mu} \\ &\le \left[\operatorname*{osc}_{Q_d(y_0,t)} f\right]^{\mu} \left[c\left(\frac{1}{d^2}+\frac{1}{s}\right)|t-s|\right]^{1-\mu} \\ &\le [C(t)d^{\alpha}]^{\mu}\, c^{1-\mu}\left(\frac{1}{d^2}+\frac{1}{s}\right)^{1-\mu} |t-s|^{1-\mu} \\ &\le C(t)^{\mu}c^{1-\mu}\left(d^{\alpha\mu-2(1-\mu)}+\frac{d^{\alpha\mu}}{s^{1-\mu}}\right)|t-s|^{1-\mu} \\ &\le c_4|t-s|^{(2-2\mu)/2} \end{aligned}$$

where $\mu = 2/(\alpha+2)$ so that $\alpha\mu - 2(1-\mu) = 0$, and $c_4 = C(s)^{\mu}c^{1-\mu}\left(1+(\operatorname{diam}\Omega)^{\alpha\mu}s^{\mu-1}\right)$.

We find the final estimate by choosing $C = \sup\{c_1, c_2, c_3, c_4\}$ and $\alpha' = \min\{1-\epsilon, 2-2\mu\} = \min\{\alpha/(\alpha+1), 2\alpha/(\alpha+2)\} = \alpha/(\alpha+1)$. □

Lemma 6.8. *The gradient Du^{ϵ} is Hölder continuous, with bound*

$$|Du^{\epsilon}|_{\alpha,\alpha/2} \le C$$

on $\Omega\times[t,T]$, for $t > t_0 > 0$ and some $\alpha > 0$, where $C = C(n, L(t_0), |t-t_0|, \Omega, |\partial\Omega|_{C^2})$.

Proof: We can use the Hölder gradient estimate near a flat boundary from Theorem A.1, but we will need to work locally with $v^{\epsilon}(y,t) = u^{\epsilon}(\Psi^{-1}(y),t)$ in the flat-boundary coordinates.

The gradient bound (6.9) for u^{ϵ} implies that

$$|Dv^{\epsilon}|_{Q_r} \le L(t_0)|D\Psi^{-1}|_{B_r} \le L(t_0)|\partial\Omega|_{C^2}$$

on the cylinder $Q_r(y_1,t_1)$ for some $r < R$ to be chosen later, and where $y_1 \in \partial\Omega$ and $t_1 > t_0 + r^2$.

On Q_r, v^{ϵ} satisfies the evolution equation (6.5). To check that the coefficients a^{lk} and b (given by (6.7)) satisfy the conditions of Theorem A.1, we note that: a^{lk} is uniformly parabolic, since for $\xi\in\mathbb{R}^n$,

$$\begin{aligned} a^{lk}\xi_l\xi_k &= m^{ij}([D\Psi]Dv^{\epsilon})[D\Psi]^l_j[D\Psi]^k_i\xi_l\xi_k \\ &\ge \frac{1}{1+|[D\Psi]Dv^{\epsilon}|^2}|[D\Psi]\xi|^2 \\ &\ge \frac{1}{1+\Lambda^2_{[D\Psi]}}\lambda^2_{[D\Psi]}|\xi|^2 \\ &\ge \frac{1}{1+4L(t_0)^2}\left(\frac{2}{3}\right)^2|\xi|^2 \\ &\ge c(L(t_0))|\xi|^2 \end{aligned}$$

where $\lambda_{[D\Psi]}$ and $\Lambda_{[D\Psi]}$ are the smallest and largest eigenvalues of $[D\Psi]$, which are bounded between $2/3$ and 2 on B_R; $|a^{lk}|$ is bounded above, as

$$|m^{ij}([D\Psi]Dv^{\epsilon})[D\Psi]^l_j[D\Psi]^k_i| \le |\Lambda_{[D\Psi]}|^2;$$

and a^{lk} has bounded derivative with respect to the gradient, for if we write $q = [D\Psi]p$, then $a^{lk}(p,y) = m^{ij}(q)[D\Psi]^l_j[D\Psi]^k_i$ and

$$\begin{aligned}\frac{\partial a^{lk}(p,y)}{\partial p^\alpha} &= [D\Psi]^l_j[D\Psi]^k_i \frac{\partial m^{ij}(q)}{\partial q^\beta}\frac{\partial q^\beta}{\partial p^\alpha} \\ &= [D\Psi]^l_j[D\Psi]^k_i \left[-\frac{(\delta^\beta_i q_j + \delta^\beta_j q_i)}{1+|q|^2} + \frac{2q_\beta q_i q_j}{(1+|q|^2)^2}\right][D\Psi]^\beta_\alpha \\ &\leq 4|D\Psi|^3 \\ &\leq 4|\partial\Omega|^3_{C^2}.\end{aligned}$$

It is straightforward to bound the lower-order term in the equation —

$$|m^{ij}([D\Psi]Dv^\epsilon)[D^2\Psi]^s_{ij}D_s v^\epsilon| \;\leq\; L(t_0)|D^2\Psi| \;\leq\; L(t_0)|\partial\Omega|_{C^3}.$$

Finally, we need an oscillation bound smaller than σ for $a^{lk}(p,\cdot)$ on Q_r, but since

$$\begin{aligned}\frac{\partial a^{lk}(p,y)}{\partial y} &= \frac{\partial m^{ij}([D\Psi]p)}{\partial q^l}\frac{\partial([D\Psi]p)^l}{\partial y}[D\Psi]^l_j[D\Psi]^k_i + m^{ij}\frac{\partial}{\partial y}\left([D\Psi]^l_j[D\Psi]^k_i\right) \\ &\leq C|D^2\Psi||D\Psi| \\ &\leq C|\partial\Omega|^2_{C^3},\end{aligned}$$

we find that $\operatorname{osc}_{Q_r(y_1,t_1)} a^{ij}(p,\cdot) \leq rC|D^2\Psi||D\Psi|$. By choosing r small enough, we can ensure that this is less than σ.

Now Theorem A.1 implies that for all $s < r$ there is some $\alpha' > 0$ so that

$$\begin{aligned}\operatorname*{osc}_{Q_s} Dv^\epsilon &\leq c\left(\frac{s}{r}\right)^{\alpha'}\left(\operatorname*{osc}_{Q_r} Dv^\epsilon + L(t_0)|\partial\Omega|_{C^2}r\right) \\ &\leq c\left(\frac{s}{r}\right)^{\alpha'}\left(L(t_0) + L(t_0)|D^2\Psi|r\right) \\ &\leq cs^{\alpha'}r^{-\alpha'}(1+r).\end{aligned}$$

where $c(L(t_0), n, |D\Psi|, |D^2\Psi|)$.

This boundary oscillation estimate for Dv^ϵ on $Q_s(y_1,t_1)$ for all $s < r < R$ and $t_1 > t_0 + r^2$, together with the interior gradient bounds given by Lemma 6.6, means we can use Lemma 6.7 to give a global Hölder bound for Dv^ϵ and hence for Du^ϵ. □

Lemma 6.9. *We can find bounds for u^ϵ in $H_{2+\alpha}(\Omega \times (t,T])$*

$$|u^\epsilon|_{2+\alpha,1+\frac{\alpha}{2}} \leq C,$$

for $t > t_0 > 0$, where $C = C(n, L(t_0), |t-t_0|, |\partial\Omega|_{C^2}, \alpha)$.

Proof: To establish this, it is possible to use boundary estimates for the Neumann problem, but instead our approach is to use the reflection $\tilde{v}^\epsilon$ on $B_R(y_1) \times [t_0, T]$ — a domain that extends beyond Ω — which satisfies the reflected evolution equation (6.8), and apply the interior estimate from Theorem A.4.

We need to check that equation (6.8) satisfies the conditions of Theorem A.4.

In Lemma 6.3, we showed that the coefficients in equation (6.8) have regularity estimates

$$|\tilde{a}^{ij}|_{\alpha;B_r\times(t_1-r^2)} \leq 2|a^{ij}|_{C^1;Q_r}\,(1+|Dv^\epsilon|_{\alpha;Q_r})$$
$$|\tilde{b}|_{\alpha;B_r\times(t_1-r^2)} \leq 2|b|_{C^1;Q_r}\,(1+|Dv^\epsilon|_{\alpha;Q_r})\,;$$

and in Lemma 6.8 we found a uniform global bound for $|Dv^\epsilon|_{\alpha,\alpha/2;\Omega\times(t,T)}$ for $t > t_0$.

Our gradient estimate ensures that $\tilde{a}$ is uniformly parabolic for $t > t_0$.

Applying Theorem A.4 results in the bound

$$|\tilde{v}^\epsilon|_{2+\alpha,1+\alpha/2;Q_{R/3}} \leq C|\tilde{v}^\epsilon|_{0;Q_{2R/3}},$$

where C is dependent on the dimension n, R (which is determined by $|\partial\Omega|_{C^2}$), the ellipticity constant of $\tilde{a}^{ij}$ (dependent on $L(t_0)$), the Hölder exponent α, and the bound on the α norm of the coefficients (which is bounded by $|Dv^\epsilon|_{\alpha,\alpha/2}$). That is, $C = C(n, L(t_0), |\partial\Omega|_{C^2}, \alpha)$.

We can repeat this over the entire boundary, and together with the interior estimate (6.10), this gives us the claim. □

Lemma 6.10. *The sequence of approximate solutions u^ϵ converges*

$$\lim_{\epsilon\to 0} |u^\epsilon - u|_{2+\alpha',1+\alpha'/2} \to 0$$

to some $u \in H_{2+\alpha'}$ on $\Omega \times (t_0, T)$, for all $t_0 > 0$.

Proof: The uniform $H_{2+\alpha}(\overline{\Omega}\times(t_0,T))$ bounds on the u^ϵ ensure that there is a convergent subsequence (for a slightly smaller $\alpha' < \alpha$); the disjointness of initial data u_0^ϵ (and hence the disjointness of $u^\epsilon(\cdot,t)$) implies that the entire sequence must converge, and so this limit is unique. The limit u is in $H_{2+\alpha'}(\Omega\times(t_0,T))$ (and is C^∞ on the interior, by virtue of the interior estimate in Lemma 6.6).

It also satisfies $D_\nu u = 0$ on the boundary, and so is a solution to the Neumann problem given by (6.1).□

Lemma 6.11. *This solution u converges to u_0 in $C(\Omega)$ as $t \to 0$. The convergence is uniform in time, and if $u_0 \in C^\alpha(\Omega)$ then $u \in H_\alpha(\Omega \times [0,T])$.*

Proof: Let $\delta > 0$ be fixed. Our aim is to show that we can find t_δ so that for all $t \in (0, t_\delta)$,

$$\sup_{x\in\Omega} |u(x,t) - u_0(x)| \leq \delta.$$

The modulus of continuity for u_0 is

$$\omega(r) := \sup_{|x-y|=r} |u_0(x) - u_0(y)|.$$

Recall the approximate solutions u^ϵ, converging uniformly

$$|u^\epsilon(\cdot,t) - u(\cdot,t)| \to 0 \text{ as } \epsilon \to 0$$

for all $t > 0$. The approximations at the initial time have (at least) the same modulus of continuity as u_0:

$$|u^\epsilon(x,0) - u^\epsilon(y,0)| \le \omega(|x-y|).$$

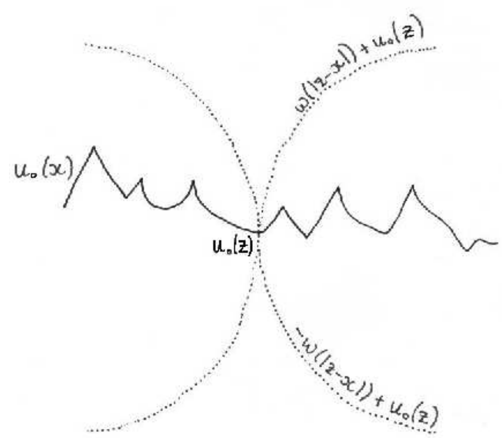


Figure 6.1: The modulus of continuity bounds u_0

Fix z to be any point in the interior of Ω. We can define a new solution to (6.1) by off-setting u^ϵ around $u_0(z)$:

$$w^\epsilon(x,t) := u^\epsilon(x,t) - u^\epsilon(z,0) + u_0(z),$$

so that $w^\epsilon(z,0) = u_0(z)$.

Then, for $t > 0$, we have

$$\begin{aligned} |u(z,t) - u_0(z)| &= \lim_{\epsilon \to 0} |u^\epsilon(z,t) - u_0(z)| \\ &= \lim_{\epsilon \to 0} |w^\epsilon(z,t) + u^\epsilon(z,0) - 2u_0(z)| \\ &\le \lim_{\epsilon \to 0} \left(|w^\epsilon(z,t) - u_0(z)| + |u^\epsilon(z,0) - u_0(z)|\right). \end{aligned} \tag{6.12}$$

To estimate the first term of this, we observe that every $w^\epsilon(\cdot,0)$ is inside the 'envelope' given by the continuity condition,

$$-\omega(|z-x|) + u_0(z) \le w^\epsilon(x,0) \le \omega(|z-x|) + u_0(z).$$

Above and below this envelope, we can place two spheres of radius r centred at $(z, u_0(z) \pm [r + \omega(r)])$. At $t = 0$, the spheres and the graph of $w^\epsilon(\cdot,0)$ are completely disjoint. The spheres are also completely disjoint from the graph of u_0.

The evolution of spheres under mean curvature flow is well-known — the centre remains fixed and the radius shrinks from the initial radius $r(0) = r_0$, with

$$r(t) = \sqrt{r_0^2 - 2nt}$$

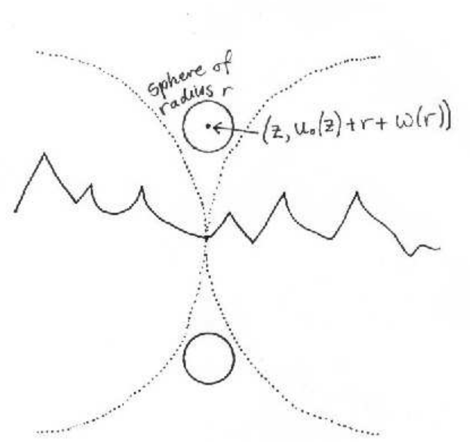


Figure 6.2: Spheres above and below w^ϵ or u_0

until the sphere disappears at time $t = r_0^2/2n$.

The parts of these spheres closest to the graph of w^ϵ — the lower part of the upper sphere and the upper part of the lower sphere — are

$$S^+(x,t) := u_0(z) + r_0 + \omega(r_0) - \sqrt{r(t)^2 - |x-z|^2},$$
$$S^-(x,t) := u_0(z) - r_0 - \omega(r_0) + \sqrt{r(t)^2 - |x-z|^2}.$$

Suppose that one of these spheres and w^ϵ first touch at some time $t > 0$. From the Comparison Theorem 2.2, we know that at this time there must be an intersection occurring on the boundary of Ω, say at $x_1 \in \partial\Omega$ (this doesn't rule out other intersections occurring simultaneously on the interior).

This intersection on the boundary is an extreme point of $S^{+/-}(\cdot,t) - w^\epsilon(\cdot,t)$ (either a minimum of $S^+ - w^\epsilon$ or a maximum of $S^- - w^\epsilon$).

Therefore the sign on the outward derivative of the intersecting sphere at this point is known — either

$$D_\nu \left(S^+(x_1,t) - w^\epsilon(x_1,t)\right) \leq 0 \text{ and so } D_\nu S^+(x_1,t) \leq 0,$$

or else

$$D_\nu \left(S^-(x_1,t) - w^\epsilon(x_1,t)\right) \geq 0 \text{ and so } D_\nu S^-(x_1,t) \geq 0,$$

where we have used that w^ϵ satisfies the Neumann condition $D_\nu w^\epsilon = 0$ on the boundary.

On the other hand, we can explicitly calculate the gradients of the spheres —

$$D_i S^+(x,t) = \frac{2(x^i - z^i)}{\sqrt{r^2 - |x-z|^2}}, \qquad D_i S^-(x,t) = \frac{-2(x^i - z^i)}{\sqrt{r^2 - |x-z|^2}}.$$

The convexity of Ω means that for any $z \in \overline{\Omega}$ and x_1 on the boundary, $\nu\cdot(x_1 - z) \geq 0$, and the inequality is strict if z is in the interior of Ω. Hence the sign of the normal derivative is known —

$$D_\nu S^+(x_1,t) \geq 0, \qquad D_\nu S^-(x_1,t) \leq 0.$$

Between these two observations, it must be the case that the intersecting sphere has a flat normal gradient —

$$D_\nu S^+(x_1,t) = 0 \quad \text{or} \quad D_\nu S^-(x_1,t) = 0,$$

and so $\nu\cdot(x_1 - z) = 0$, which in turn implies that z is on the boundary of Ω, contradicting our original assumption that z was an interior point. It follows that such spheres, centred on interior points, never touch the graph of w^ϵ for the duration of the spheres' existence, until $t = r_0^2/2n$.

In particular, above the point z the surfaces move by no more than $r + \omega(r)$ in the time $t \in (0, r^2/2n)$. We can choose $r > 0$ so that $\delta = r + \omega(r)$, and a corresponding $t_\delta = r^2/2n$, ensuring that

$$|w^\epsilon(z,t) - u_0(z)| \leq r + f(r) = \delta \text{ for } 0 \leq t \leq t_\delta,$$

where t is dependent on δ and ω alone.

This estimate is independent of z and ϵ, so

$$\lim_{\epsilon\to 0} |w^\epsilon(z,t) - u_0(z)| \leq \delta.$$

We still need to estimate the second part of equation (6.12), $|u^\epsilon(z,0) - u_0(z)|$. However the convergence here is uniform (for $z \in \Omega$), so that

$$\sup_{z\in\Omega} |u(z,t) - u_0(z)| \leq \delta + \sup_{z\in\Omega} \lim_{\epsilon\to 0} |u^\epsilon(z,0) - u_0(z)| = \delta$$

for $0 < t \leq t_\delta$, and so u is in $C([0,T); C(\Omega))$, with $u(\cdot,0) = u_0$.

This is also an estimate for the smoothness in time of the convergence; we can consider $\delta = \delta(t)$, with

$$\sup_{z\in\Omega} |u(z,t) - u_0(z)| \leq \delta(t)$$

by setting $t = r^2/2n$, so that $r = \sqrt{2nt}$ and thus $\delta(t) = \sqrt{2nt} + \omega(\sqrt{2nt})$. The convergence in time that this gives is *at best* like $t^{1/2}$ — which is in concordance with the result for initial data with a Lipschitz bound. In that case, $\omega(r) = |r|$ and the convergence is $C^{0+1/2}$ in time (Theorem 3.5 of [20]).

In the case that the initial data has a Hölder gradient bound, $\omega(r) = |r|^\alpha$, then the convergence to the initial data is as $t^{1/2} + t^{\alpha/2} \sim t^{\alpha/2}$. □

Remark: While we have mined the rich theory arising from mean curvature flow to find this result, there are similar results for other equations of the type studied in Chapter 5, and we expect to be able to find similar existence results.

In particular, one can find short-time existence results for anisotropic mean curvature flow with a zero Neumann boundary condition and continuous initial data on $\Omega \times [0,T]$, for convex Ω.

Chapter 7

Existence of solutions to the Dirichlet problem for mean curvature flow

In this chapter we use the gradient estimate to establish existence of solutions to the Cauchy-Dirichlet problem with zero boundary data.

Theorem 7.1 (Existence of solutions to the Dirichlet problem). *Let Ω be a domain in $\mathbb{R}^n$, with C^2 boundary $\partial\Omega$ that has non-negative mean curvature. If $u_0 \in C^0(\overline{\Omega})$ and $u_0 = 0$ on $\partial\Omega$, then the problem*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\delta_{ij} - \frac{D_i u D_j u}{1+|Du|^2}\right) D_{ij} u, \qquad (7.1) \\ u(x,t) &= 0 \quad \text{for } x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) &= u_0, \end{aligned}$$

has a smooth solution for $t > 0$ which converges uniformly to u_0 as $t \to 0$.

The existence of solutions to the mean curvature flow problem with prescribed boundary values was considered by Lieberman in [24] (and by Huisken in [19], where the long-time behaviour of solutions was also studied).

Lieberman considered time-dependent boundary data $u_0 \in H_{1+\alpha}(\mathcal{P}(\Omega \times [0,T]))$, with a Lipschitz bound (in time) on $\partial\Omega \times [0,T]$. The following theorem may be found as Theorems 12.10 and 12.18 of [25].

Theorem 7.2 (Lieberman). *Let $\Omega \subset \mathbb{R}^n$ be a domain with C^2 boundary, and let u_0 be a function defined on the boundary, with $u_0 \in H_{1+\alpha}(\mathcal{P}(\Omega \times [0,T]))$ for some $0 < \alpha < 1$. Then if the mean curvature H of $\partial\Omega$ is non-negative, there exists a solution to* (7.1) *with initial and boundary data*

$$u(x,t) = u_0(x,t) \text{ for } (x,t) \in \mathcal{P}(\Omega \times [0,T]).$$

Moreover, such a solution satisfies

$$[Du]_\beta \leq c,$$

where c and β depend on $|u_0|_{1+\alpha,\alpha/2}$.

The proof of Theorem 7.1 is very similar to that for the Neumann problem, Theorem 6.2. We will use the boundary-straightening change of coordinates described in Section 6.1, and the corresponding $v_0(y,t) := u_0(\Psi^{-1}(y),t)$.

Lemma 7.3. *There exists an approximating sequence $u_0^\epsilon \in C^\infty(\overline{\Omega})$ with $u_0^\epsilon = 0$ on $\partial\Omega$ and $u_0^\epsilon \to u_0$ in $C(\Omega)$.*

Proof: Let B_R be a boundary centred ball, where R is given by (6.3). We will define a local approximation on B_R and then put similar local approximations together to give a global one.

In Section 6.1 we defined $\tilde{v}_0$ to be a reflection across the boundary; this time, we let $\tilde{v}_0$ be the *odd* reflection over the boundary

$$\tilde{v}_0(y) := -\frac{y^n}{|y^n|} v_0\left(\rho(y)\right) = \begin{cases} u_0\left(\Psi^{-1}(\overline{y}, y^n)\right) & \text{if } y^n < 0, \\ 0 & \text{if } y^n = 0, \\ -u_0\left(\Psi^{-1}(\overline{y}, -y^n)\right) & \text{if } y^n > 0\,. \end{cases} \tag{7.2}$$

Mollifying this in the standard way

$$v_0^\epsilon(y) := \eta_\epsilon * \tilde{v}_0 = \int_{z\in\Psi(B_R)} \eta_\epsilon(y-z)\tilde{v}_0(z)dz, \tag{7.3}$$

we note that $v_0^\epsilon \to v_0$ uniformly on subsets of $\Psi(B_R)$, and we can check that if $y^n = 0$, then $v^\epsilon(y) = 0$. Returning to the original coordinates, set $u_1^\epsilon(x) := v_0^\epsilon(\Psi(x))$ on $\overline{\Omega} \cap B_R(x_1)$.

Now, cover $\partial\Omega$ by N such boundary-centred balls $B_R(x_1), \ldots, B_R(x_N)$ on which are defined approximations $u_1^\epsilon, \ldots, u_N^\epsilon$. Complete the cover of Ω by the set $\Omega^{R/2} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > R/2\}$. On this interior set let u_{N+1}^ϵ be the usual mollification of u_0.

If $\{\xi_i\}_{i=1,N+1}$ is a partition of unity with respect to these sets, then the sum $u_0^\epsilon = \sum_{i=1}^{N+1} \xi_i u_i^\epsilon$ converges uniformly to u_0 on $\overline{\Omega}$ and also has $u_0^\epsilon = 0$ on $\partial\Omega$.

We can restrict this to a subsequence $u_0^{\epsilon_i}$, where $|u_0 - u_0^{\epsilon_i}| < 2^{-i}$ (but retain the notation u_0^ϵ for the subsequence). If we off-set each member of the subsequence, by replacing u^{ϵ_i} by $u^{\epsilon_i} + 3(2^{-i})$, then this is a completely disjoint subsequence that still converges to u_0 as $i \to \infty$. □

As the boundary values

$$u^\epsilon = u_0^\epsilon \text{ on } \Omega \times \{0\},$$
$$u^\epsilon = 0 \text{ on } \partial\Omega \times [0,T],$$

are in $H_{1+\alpha}$ on $\mathcal{P}(\Omega \times [0,T])$, Theorem 7.2 ensures that a solution with these boundary values for (7.1) exists. Denote these approximate solutions by u^ϵ.

The gradient estimate derived for the Dirichlet problem in Theorem 5.4 implies that

$$|Du^\epsilon(\cdot,t)| \le C_1\sqrt{t}\,(1+t)\exp(C_2/t) := L(t),$$

for constants C_1 and C_2 dependent only on $\operatorname{osc} u_0^\epsilon \le \operatorname{osc} u_0$.

Lemma 7.4. *The approximate solutions u^ϵ have a Hölder gradient bound*

$$|u^\epsilon(\cdot,t)|_{C^\alpha} \leq C,$$

for $t > t_0 > 0$, where C is dependent on $L(t_0)$,$|t_1 - t_0|$, n, $|\partial\Omega|_{C^2}$, and $\operatorname{diam}\Omega$.

Proof: If we revert to v^ϵ in the straightened-boundary coordinates satisfying (6.5) on Q_R, Theorem A.2 gives an oscillation bound for the gradient on the boundary-centred cylinder $Q_r(x_1,t_1)$ for $t_1 > t_0 + r^2$ and $r < R$ —

$$\operatorname*{osc}_{Q_r} Dv^\epsilon \leq cr^\alpha,$$

where c and α depend on n, $|a^{ij}|_{C^1}$, $|b|$, $\lambda_{a^{ij}}$ and $\Lambda_{a^{ij}}$, where a^{ij} and b are the coefficients of (6.5), as in (6.7). These last four are in turn dependent on $L(t_0)$ and $|\partial\Omega|_{C^2}$.

On the interior we have the bounds on higher derivatives given by Lemma 6.6

$$|D^{m+2}u^\epsilon(x,t)| \leq c\,(n,m,L(t_0))\left(\frac{1}{\operatorname{dist}(x,\partial\Omega)^2} + \frac{1}{t-t_0}\right)^{\frac{m+1}{2}}. \tag{7.4}$$

With $m = 0$, this is a gradient bound for Du.

These can be linked together using Lemma 6.7 to find a global Hölder bound

$$|Du|_{\alpha';Q_r(x_1,t_1)} \leq C,$$

where $\alpha' = \alpha/(\alpha+1)$ and C additionally depends on $|t_0 - t_1|$ and $\operatorname{diam}\Omega$. □

Now, if we consider the approximate solutions to begin at some time $t_1 > t_0$ with the initial data $u^\epsilon(\cdot,t_1)$, the uniform Hölder gradient bound on $u^\epsilon(\cdot,t_1)$ means that Theorem 7.2 gives a Hölder gradient bound

$$|u^\epsilon|_{1+\beta;\Omega\times[t_1,T]} \leq C,$$

for all $t_1 > t_0 > 0$, where C and β depend on $L(t_0)$,$|t_1 - t_0|$, n, $|\partial\Omega|_{C^2}$, and $\operatorname{diam}\Omega$.

With these uniform estimates for positive times, there must be a convergence subsequence in $H_{1+\beta'}(\Omega\times[t_1,T])$ for some $\beta' < \beta$. If we off-set the initial data as mentioned in Lemma 7.3, the convergence of the subsequence implies the convergence of the entire sequence to a limit u, defined on all Ω for $t > 0$. Finally, the interior bounds (7.4) mean that on the interior, u is smooth.

We now need to show that $u(\cdot,t) \to u_0$ as $t \to 0$.

Lemma 7.5. *As $t \to 0$,*

$$\sup_{x\in\Omega} |u(x,t) - u_0(x)| \to 0.$$

Furthermore, u has a modulus of continuity in time dependent only on that of u_0.

Proof: As in the proof of Lemma 6.11, fix z to be any point in the interior of Ω and set

$$w^\epsilon(x,t) := u^\epsilon(x,t) - u^\epsilon(z,0) + u_0(z),$$

so that w^ϵ is a solution to (7.1) with $w^\epsilon(z,0) = u_0(z)$ and $w^\epsilon(x,t) = -u^\epsilon(z,0) + u_0(z)$ on the boundary.

Let ω be a modulus of continuity for u_0 and hence for $w^\epsilon(\cdot, 0)$, so that

$$-\omega(|z-x|) + u_0(z) \le w^\epsilon(x,0) \le \omega(|z-x|) + u_0(z).$$

Above and below these two bounds, we can place two spheres of radius r centred at $(z, u_0(z) \pm [r + \omega(r)])$. At $t = 0$, the spheres and the graph of $w^\epsilon(\cdot, 0)$ are completely disjoint. The spheres are also completely disjoint from the graph of $u_0(\cdot)$.

As they evolve under mean curvature, the parts of these spheres closest to the graph of w^ϵ — the lower part of the upper sphere and the upper part of the lower sphere — are

$$S^+(x,t) := u_0(z) + r + \omega(r) - \sqrt{r^2 - 2nt - |x-z|^2},$$
$$S^-(x,t) := u_0(z) - r - \omega(r) + \sqrt{r^2 - 2nt - |x-z|^2}.$$

Initially $S^+ - w^\epsilon$ is positive: suppose that $t \in (0, r^2/2n)$ is the first time that $S^+ - w^\epsilon$ decreases to zero. The comparison principle (Theorem 2.2) means that this occurs at some point x_1 on the boundary. On the boundary,

$$\frac{d}{dt}\left(S^+ - w^\epsilon\right) \;=\; \frac{n}{\sqrt{r^2 - 2nt - |x_1 - z|^2}} - 0 \;>\; 0$$

(since w^ϵ is constant on the boundary) and so this cannot be the first zero point; it follows that $S^+ - w^\epsilon > 0$ for the duration of the sphere's existence.

The same argument shows that $S^- - w^\epsilon < 0$.

We can conclude that the w^ϵ move at most by $r + \omega(r)$ in the time $t \in (0, r^2/2n)$.

This estimate is independent of z and ϵ, so this implies that $u^\epsilon(\cdot, t) \to u_0$ and that $u \in C\left(\Omega \times [0,T]\right)$. Furthermore, if $u_0 \in C^\alpha(\Omega)$, then $u \in H_\alpha\left(\Omega \times [0,T]\right) \cup C^\infty\left(\Omega \times (0,T]\right)$. $\square$

Chapter 8

Gradient estimates found by counting intersections

In the paper [5], Angenent proved a series of results regarding the finiteness and non-proliferation of the zeroes of a parabolic equation in one space dimension.

A *zero* of $v(\cdot, t)$ is simply a point x where $v(x,t) = 0$. A *multiple zero* is a point where both v and v_x vanish. In contrast to earlier results, Angenent did not exclude multiple zeroes from the zero set, defining the zero set as

$$z(t) = \{\, x \in \mathbb{R} : v(x,t) = 0 \,\}.$$

In the following, $z(t)$ is often used as shorthand for the counting measure $\mathcal{H}^0\left(z(t)\right)$.

These zero-counting results have been influential in many different areas, and have been used for geometric flows by Angenent himself, in [6], [8], [7], and [2], the last with Altshuler and Giga. Many others working in the area have also used these results.

Unlike approaches that depend more explicitly on the maximum principle, this technique seems limited to equations in one dimension. The gradient estimates found do not depend on the initial gradient, but do depend explicitly on the height: the smallest gradient estimates are found for when the height is largest.

This work originates in an idea of Ben Andrews; also, this approach to finding gradient estimates has been independently used by Nagase and Tonegawa in the forthcoming paper [26].

8.1 Counting zeroes

The estimates in this chapter rely on Theorem D of Angenent's paper:

Theorem 8.1 (Angenent). *Let* $v : [x_0, x_1] \times [0, T] \to \mathbb{R}$ *be a solution of*

$$v_t = a(x,t)v_{xx} + b(x,t)v_x + c(x,t)v.$$

such that there are no zeroes on the boundaries

$$v(x_i, t) \neq 0, \qquad i = 0, 1.$$

Let a, b, c satisfy

a *positive;*
$a,\ a^{-1},\ a_t,\ a_x,$ *and* a_{xx} *bounded;*
$b,\ b_t$ *and* b_x *bounded;*
c *bounded.*

Then if v_t, v_x and v_{xx} are continuous on $(x_0, x_1) \times [0, T]$,

- *for $t > 0$, $z(t)$ is finite*

- *if $\tilde{x}$ is a multiple zero of v at $\tilde{t}$ then for all $t_1 < \tilde{t} < t_2$ we have $z(t_1) > z(t_2)$.*

Consider a fully nonlinear equation on a domain $\Omega \times [0, T]$, where Ω is a connected subset of $\mathbb{R}$,

$$u_t = F(u_{xx}, u_x, u, x, t), \tag{8.1}$$

where F is parabolic, by which we mean that

$$\frac{\partial}{\partial r} F(r, p, q, x, t) > 0$$

for all $(r, p, q, x, t) \in \mathbb{R}^3 \times \overline{\Omega} \times [0, T]$.

Suppose that u and φ are smooth solutions of (8.1), with

$$|u|, |u_x|, |u_{xx}|, |u_t| \le C_1$$

and

$$|\varphi|, |\varphi_x|, |\varphi_{xx}|, |\varphi_t| \le C_1.$$

Then we can form the difference $w := u - \varphi$ satisfying the evolution equation

$$\begin{aligned} w_t &= u_t - \varphi_t \\ &= F(u_{xx}, u_x, u, x, t) - F(\varphi_{xx}, \varphi_x, \varphi, x, t) \\ &= \int_0^1 \frac{d}{ds} F(su_{xx} + (1-s)\varphi_{xx}, su_x + (1-s)\varphi_x, su + (1-s)\varphi, x, t) ds \\ &= \int_0^1 \frac{\partial}{\partial r} F(\dots) ds \ (u_{xx} - \varphi_{xx}) + \int_0^1 \frac{\partial}{\partial p} F(\dots) ds \ (u_x - \varphi_x) \\ &\qquad\qquad + \int_0^1 \frac{\partial}{\partial q} F(\dots) ds \ (u - \varphi) \\ &= A(x,t) w_{xx} + B(x,t) w_x + C(x,t) w, \end{aligned} \tag{8.2}$$

where the omitted argument of the derivatives of F, denoted by $(\dots)$, is always

$(su_{xx} + (1-s)\varphi_{xx}, su_x + (1-s)\varphi_x, su + (1-s)\varphi, x, t)$. In the last line,

$$A(x,t) := \int_0^1 \frac{\partial}{\partial r} F(su_{xx} + (1-s)\varphi_{xx}, su_x + (1-s)\varphi_x, su + (1-s)\varphi, x, t)ds \tag{8.3}$$

$$B(x,t) := \int_0^1 \frac{\partial}{\partial p} F(su_{xx} + (1-s)\varphi_{xx}, su_x + (1-s)\varphi_x, su + (1-s)\varphi, x, t)ds \tag{8.4}$$

and

$$C(x,t) := \int_0^1 \frac{\partial}{\partial q} F(su_{xx} + (1-s)\varphi_{xx}, su_x + (1-s)\varphi_x, su + (1-s)\varphi, x, t)ds. \tag{8.5}$$

In order to use Angenent's theorem, we need to establish that:

- A, A^{-1}, A_t, A_x, and A_{xx} are bounded,
- B, B_t and B_x are bounded
- and C is bounded on $\Omega \times [0,T]$.

Let $\mathcal{K} = \left\{ (r,p,q,x,t) \in \mathbb{R}^3 \times \Omega \times [0,T] : |r|, |p|, |q| \le C_1 \right\}$.
If $\dfrac{\partial F}{\partial r}$ is continuous, then there are positive constants $\lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ for which

$$0 < \lambda_{\mathcal{K}} \le \frac{\partial F}{\partial r} \le \Lambda_{\mathcal{K}} \text{ for all } (r,p,q,x,t) \in \mathcal{K}. \tag{8.6}$$

Bounds on A and A^{-1} follow from this:

$$A \le \Lambda_{\mathcal{K}}, \qquad A^{-1} \le \lambda_{\mathcal{K}}^{-1}. \tag{8.7}$$

A_t is given by

$$A_t = \int_0^1 \frac{\partial^2 F}{\partial r^2}(\dots)\left[su_{xxt} + (1-s)\varphi_{xxt}\right] + \frac{\partial^2 F}{\partial r \partial p}(\dots)\left[su_{xt} + (1-s)\varphi_{xt}\right] + \frac{\partial^2 F}{\partial r \partial q}(\dots)\left[su_t + (1-s)\varphi_t\right] + \frac{\partial^2 F}{\partial r \partial t}(\dots)ds,$$

and so

$$|A_t| \le \left|\frac{\partial F}{\partial r}\right|_{C^1(\mathcal{K})} \left(C_1 + |u_{xxt}, u_{xt}, \varphi_{xxt}, \varphi_{xt}|_{\Omega \times [0,T]}\right). \tag{8.8}$$

Similarly,

$$A_x = \int_0^1 \frac{\partial^2 F}{\partial r^2}(\dots)\left[su_{xxx} + (1-s)\varphi_{xxx}\right] + \frac{\partial^2 F}{\partial r \partial p}(\dots)\left[su_{xx} + (1-s)\varphi_{xx}\right] + \frac{\partial^2 F}{\partial r \partial q}(\dots)\left[su_x + (1-s)\varphi_x\right] + \frac{\partial^2 F}{\partial r \partial x}(\dots)ds,$$

so

$$|A_x| \le \left|\frac{\partial F}{\partial r}\right|_{C^1(\mathcal{K})} (C_1 + |u_{xxx}, \varphi_{xxx}|_{\Omega \times [0,T]}). \tag{8.9}$$

A_{xx} is given by

$$\begin{aligned}A_{xx} = \int_0^1 \frac{\partial^3 F}{\partial r^3}(\dots)\,[su_{xxx} + (1-s)\varphi_{xxx}]^2 \\ + 2\frac{\partial^3 F}{\partial^2 r \partial p}(\dots)\,[su_{xxx} + (1-s)\varphi_{xxx}]\,[su_{xx} + (1-s)\varphi_{xx}] \\ + 2\frac{\partial^3 F}{\partial^2 r \partial q}(\dots)\,[su_{xxx} + (1-s)\varphi_{xxx}]\,[su_{x} + (1-s)\varphi_{x}] \\ + 2\frac{\partial^3 F}{\partial^2 r \partial x}(\dots)\,[su_{xxx} + (1-s)\varphi_{xxx}] + R_1 \\ + \frac{\partial^2 F}{\partial r^2}(\dots)\,[su_{xxxx} + (1-s)\varphi_{xxxx}] + R_2\,ds,\end{aligned}$$

where R_1 and R_2 are combinations of terms involving second and first derivatives of $\dfrac{\partial F}{\partial r}$ respectively. Consequently,

$$\begin{aligned}|A_{xx}| \le \left|\frac{\partial F}{\partial r}\right|_{C^2(\mathcal{K})} \left(|u_{xxx}, \varphi_{xxx}|_{\Omega\times[0,T]} + C_1\right)^2 \\ + \left|\frac{\partial F}{\partial r}\right|_{C^1(\mathcal{K})} \left(|u_{xxxx}, \varphi_{xxxx}, u_{xxx}, \varphi_{xxx}|_{\Omega\times[0,T]} + C_1\right). \quad (8.10)\end{aligned}$$

The bounds for B, its derivatives, and C follow in a similar manner:

$$\begin{aligned}B_t = \int_0^1 \frac{\partial^2 F}{\partial p \partial r}(\dots)\,[su_{xxt} + (1-s)\varphi_{xxt}] + \frac{\partial^2 F}{\partial p^2}(\dots)\,[su_{xt} + (1-s)\varphi_{xt}] \\ + \frac{\partial^2 F}{\partial p \partial q}(\dots)\,[su_{t} + (1-s)\varphi_{t}] + \frac{\partial^2 F}{\partial p \partial t}(\dots)ds,\end{aligned}$$

so that

$$|B_t| \le \left|\frac{\partial F}{\partial p}\right|_{C^1(\mathcal{K})} \left(C_1 + |u_{xxt}, \varphi_{xxt}|_{\Omega\times[0,T]}\right); \quad (8.11)$$

while

$$\begin{aligned}B_x = \int_0^1 \frac{\partial^2 F}{\partial p \partial r}(\dots)\,[su_{xxx} + (1-s)\varphi_{xxx}] + \frac{\partial^2 F}{\partial p^2}(\dots)\,[su_{xx} + (1-s)\varphi_{xx}] \\ + \frac{\partial^2 F}{\partial p \partial q}(\dots)\,[su_{x} + (1-s)\varphi_{x}] + \frac{\partial^2 F}{\partial p \partial x}(\dots)ds,\end{aligned}$$

so that

$$|B_x| \le \left|\frac{\partial F}{\partial p}\right|_{C^1(\mathcal{K})} \left(C_1 + |u_{xxx}|_{\Omega\times[0,T]} + |\varphi_{xxx}|_{\Omega\times[0,T]}\right). \quad (8.12)$$

and finally

$$
\begin{aligned}
|C(x,t)| &= \left| \int_0^1 \frac{\partial}{\partial q} F(su_{xx} + (1-s)\varphi_{xx}, su_x + (1-s)\varphi_x, su + (1-s)\varphi, x, t) ds \right| \\
&\le \left| \frac{\partial F}{\partial q} \right|_{0;\mathcal{K}}. \qquad (8.13)
\end{aligned}
$$

It is clear that we will be able to apply Angenent's result to a smooth solution of a nonlinear parabolic equation $u_t = F$, when F satisfies the parabolicity condition (8.6) and both $\dfrac{\partial F}{\partial r}$ and F are C^2 on the bounded domain $\mathcal{K}$.

These conditions are not optimal — for example, in estimate (8.13) above, it is sufficient if $\dfrac{\partial F}{\partial q}$ is L^1 along line segments in $\mathcal{K}$ — however, they are enough to allow a theorem for *intersections of two solutions* rather than *zeroes* of one solution.

Theorem 8.2 (Intersection-counting theorem). *Let u and $\varphi : [x_0, x_1] \times [0, T] \to \mathbb{R}$ be solutions of*

$$u_t = F(u_{xx}, u_x, u, x, t),$$

which do not intersect on the boundaries

$$u(x_i, t) \ne \varphi(x_i, t), \quad i = 0, 1, \quad t \in [0, T].$$

If u and φ are C^2 on $(x_0, x_1) \times [0, T]$

$$|u|_{C^2}, |\varphi|_{C^2} \le c_1,$$

and if F is parabolic

$$\frac{\partial}{\partial r} F(r, p, q, x, t) > 0$$

and if both F and $\dfrac{\partial F}{\partial r}$ are C^2 on

$$\mathcal{K} = \left\{ (r, p, q, x, t) \in \mathbb{R}^3 \times \Omega \times [0, T] : |r|, |p|, |q| \le c_1 \right\},$$

then for $t > 0$ the number of intersections of u and φ are finite; and if $\tilde{x}$ is an intersection of u and φ at $\tilde{t}$ then for all $t_1 < \tilde{t} < t_2$, the number of intersections at t_1 is strictly less than the number of intersections at t_2.

Proof: We apply Theorem (8.1) to the difference $w = u - \varphi$ which satisfies equation (8.2). □

8.2 Gradient estimates for equations in one space dimension

In this chapter we seek interior estimates for bounded solutions of parabolic equations on connected domains $\Omega \times [0, T]$. When two functions intersect at a single point, then the gradient of the one that is smaller on the left of the intersection will dominate the gradient of the one that is smaller on the right.

The main idea is that optimal regularity of $u(x,t)$ is found by comparison to the solution of the same parabolic equation with initial data $C\sigma$, where σ is the maximal monotone graph

$$\sigma(x) = \begin{cases} +1, & x > 0 \\ [-1,1], & x = 0 \\ -1, & x < 0 \end{cases} \tag{8.14}$$

which we will refer to as the *step "function"*.

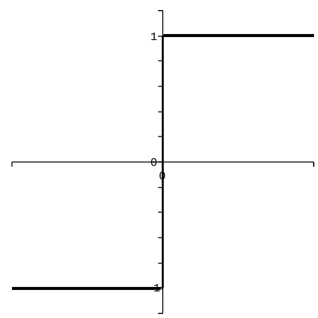


Figure 8.1: The step function σ

The method can be broken into the following steps:

- Creation of family of barriers $\{\varphi^{\epsilon,s}\}$ with $\varphi^{\epsilon,s}(x,t)$ approaching $C\sigma(x-s)$ as $t,\epsilon \to 0$
- Show that for all (x,t) in a subdomain of $\Omega \times [0,T]$, and for all $k \in [-M,M]$, we can find an s such that $\varphi^{\epsilon,s}(x,t) = k$
- Show that $|\varphi^{\epsilon,s}| > M$ at the boundaries of Ω
- Then use the Angenent result to count the intersections of u and $\varphi^{\epsilon,s}$. For small enough ϵ, there will be only one intersection, and so a gradient bound will follow.

We begin by looking at a simple estimate for entire solutions on $\mathbb{R}$, then find more specific estimates, firstly for the heat equation

$$u_t = \frac{1}{4c} u_{xx}, \tag{8.15}$$

and then for a nonlinear problem.

The following theorem says that if a solution with the step function as initial condition exists, then it will serve as a barrier for other solutions.

Theorem 8.3. *Consider the parabolic equation*

$$u_t = F(u_{xx}, u_x, u), \tag{8.16}$$

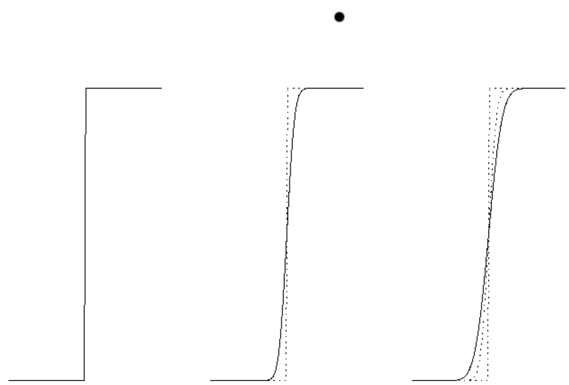


Figure 8.2: Evolution of a step function under curve shortening flow

where F satisfies (8.6)*. Let u be a solution of* (8.16) *on $\mathbb{R} \times (0, T]$ that has a bound*

$$|u| \leq M,$$

and has a uniform gradient bound at $t = 0$.

Suppose there exists a solution to (8.16) *on $\mathbb{R} \times (0, T]$ which is smooth for $t > 0$, and has initial condition*

$$\varphi(x, 0) = (M + 1)\sigma(x)$$

and boundary condition

$$\lim_{|x| \to \infty} \varphi(x, t) = (M + 1)\sigma(x).$$

Then there is a gradient estimate

$$u_x(x, t) \leq \varphi_x(z, t)$$

where z is chosen so that $\varphi(z, t) = u(x, t)$.

Proof: Let a family of barriers indexed by (z, τ) be given by $\varphi^{z,\tau}(x, t) := \varphi(x - z, t + \tau)$ for all $z \in \mathbb{R}$ and $\tau > 0$. Each of these satisfies (8.16), and is smooth on $\mathbb{R} \times [0, T]$.

As $u(\cdot, 0)$ has a uniform gradient bound, there exists a $\tau' > 0$ such that not only do $u(\cdot, 0)$ and $\varphi^{z,\tau'}(\cdot, 0)$ intersect only once, but also, $u(\cdot, 0)$ and $\varphi^{z,\tau}(\cdot, 0)$ intersect only once for all $\tau \in (0, \tau']$.

Let (x_1, t_1) be fixed. For each $\tau \leq \tau'$, there exists z such that $\varphi^{z,\tau}(x_1, t_1) = u(x_1, t_1)$.

Now, apply Angenent's theorem to $w = u - \varphi^{z,\tau}$ on some region $[-R, R] \times [0, t_1]$ containing x_1 and which is sufficiently large enough that for all $t \in [0, t_1]$, $\varphi^{z,\tau}(R, t) \geq M$ and $\varphi^{z,\tau}(-R, t) \leq -M$. The last conditions ensure that w has no zeroes on the boundary.

As w has only one zero at $t = 0$, it has no more than one zero for all t; as w is positive at $x = -R$ and negative at $x = -R$, it has exactly one zero for all t. In particular the zero at (x_1, t_1) is the only zero, and

$$\text{for } x > x_1, \quad \varphi^{z,\tau}(x, t_1) > u(x, t_1),$$
$$\text{for } x < x_1, \quad \varphi^{z,\tau}(x, t_1) < u(x, t_1),$$

from which we find a gradient estimate:

$$u_x(x_1, t_1) \le {\varphi^{z,\tau}}_x(x_1, t_1).$$

This holds for all $\tau \in (0, \tau_1)$ and so letting $\tau \to 0$ gives the result. □

The following theorem describes an explicit barrier in the case of the heat equation.

Theorem 8.4 (Gradient estimate for the heat equation). *Let* $\Omega = [x_0, x_1]$ *and* $u : \Omega \times [0, T] \to \mathbb{R}$ *be a smooth solution to the heat equation* (8.15) *with a height bound* $|u| < M$ *and Lipschitz bound* $\operatorname{Lip} u(\cdot, 0) < \infty$.

Then for $t > 0$,

$$u_x(x, t) \le 2N\sqrt{\frac{c}{\pi t}} \exp\left(-\mathcal{E}\mathrm{rf}^{-1}\left(\frac{u}{N}\right)^2\right),$$

where

$$N = M\left[\mathcal{E}\mathrm{rf}\left(\frac{\sqrt{c}\ \operatorname{dist}(x, \partial\Omega)}{2\sqrt{t}}\right)\right]^{-1}.$$

This leads to an estimate for an entire solution.

Corollary 8.5. *Let* $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ *be a smooth solution of the heat equation* (8.15) *with* $|u| < M$.

Then for $t > 0$,

$$u_x(x, t) \le 2M\sqrt{\frac{c}{\pi t}} \exp\left(-\mathcal{E}\mathrm{rf}^{-1}\left(\frac{u}{M}\right)^2\right).$$

Proof: Theorem 8.4 applies on any interval $[-R, R]$; let $R \to \infty$ to find the result. □

Proof of Theorem 8.4: Without loss of generality, let $\Omega = [-d, d]$.

For some $\epsilon \in (0, \epsilon_0)$, $|s| < d$ and $N \ge M$ to be chosen later, we can define the barrier

$$\varphi^{\epsilon,s}(x, t) := N\mathcal{E}\mathrm{rf}\left((x - s)\sqrt{\frac{c}{t + \epsilon}}\right) = \frac{2N}{\sqrt{\pi}} \int_0^{(x-s)\sqrt{\frac{c}{t+\epsilon}}} e^{-y^2} dy,$$

which satisfies the heat equation (8.15) on $\Omega \times [0, T]$. As $t + \epsilon \to 0$, $\varphi^{\epsilon,s}(x, t) \to N\sigma(x - s)$, where σ is the step function (8.14).

Choose ϵ_0 small enough, such that for any $|s| < d$ and $\epsilon < \epsilon_0$ there is only one intersection of $u(\cdot, 0)$ and $\varphi^{\epsilon,s}(\cdot, 0)$. This is possible as $u(\cdot, 0)$ has a uniform gradient bound.

Now, let $(\tilde{x}, \tilde{t}) \in \Omega \times (0, T]$ be fixed, and consider $\tilde{u} := u(\tilde{x}, \tilde{t})$. The bound on u implies that $\tilde{u} \in (-M, M)$.

If we choose $s = \tilde{x} - \sqrt{\frac{\tilde{t}+\epsilon}{c}}\mathcal{E}\mathrm{rf}^{-1}\left(\tilde{u}/N\right)$, then $\varphi^{\epsilon,s}(\tilde{x},\tilde{t}) = \tilde{u}$. Also choose

$$N = M\left[\mathcal{E}\mathrm{rf}\left(\frac{\sqrt{c}(d-|\tilde{x}|)}{2\sqrt{\tilde{t}+\epsilon}}\right)\right]^{-1}. \tag{8.17}$$

With these choices, we can check that

$$\begin{aligned}
|s| &\leq |\tilde{x}| + \sqrt{\frac{\tilde{t}+\epsilon}{c}}\left|\mathcal{E}\mathrm{rf}^{-1}\left(\tilde{u}/N\right)\right| \\
&= |\tilde{x}| + \sqrt{\frac{\tilde{t}+\epsilon}{c}}\left|\mathcal{E}\mathrm{rf}^{-1}\left(\frac{\tilde{u}}{M}\mathcal{E}\mathrm{rf}\left(\frac{\sqrt{c}(d-|\tilde{x}|)}{2\sqrt{\tilde{t}+\epsilon}}\right)\right)\right| \\
&< |\tilde{x}| + \sqrt{\frac{\tilde{t}+\epsilon}{c}}\left|\frac{\sqrt{c}(d-|\tilde{x}|)}{2\sqrt{\tilde{t}+\epsilon}}\right| \\
&= |\tilde{x}| + \frac{d-|\tilde{x}|}{2} \\
&< d;
\end{aligned}$$

and that on the boundaries $|\varphi^{\epsilon,s}| \geq M$ whenever $t < \tilde{t}$, since

$$\begin{aligned}
|\varphi^{\epsilon,s}(\pm d,t)| &= N\mathcal{E}\mathrm{rf}\left(|\pm d - s|\sqrt{\frac{c}{t+\epsilon}}\right) \\
&= N\mathcal{E}\mathrm{rf}\left(\left|\pm d - \tilde{x} + \sqrt{\frac{\tilde{t}+\epsilon}{c}}\mathcal{E}\mathrm{rf}^{-1}\left(\tilde{u}/N\right)\right|\sqrt{\frac{c}{t+\epsilon}}\right) \\
&\geq N\mathcal{E}\mathrm{rf}\left(|\pm d - \tilde{x}|\sqrt{\frac{c}{t+\epsilon}} - \sqrt{\frac{\tilde{t}+\epsilon}{t+\epsilon}}\left|\mathcal{E}\mathrm{rf}^{-1}\left(\tilde{u}/N\right)\right|\right) \\
&\geq N\mathcal{E}\mathrm{rf}\left((d-|\tilde{x}|)\sqrt{\frac{c}{t+\epsilon}} - \sqrt{\frac{\tilde{t}+\epsilon}{t+\epsilon}}\mathcal{E}\mathrm{rf}^{-1}\left(M/N\right)\right)
\end{aligned}$$

and if we use (8.17) for $(d-|\tilde{x}|)$, then this is

$$\begin{aligned}
&= N\mathcal{E}\mathrm{rf}\left(2\sqrt{\frac{\tilde{t}+\epsilon}{t+\epsilon}}\mathcal{E}\mathrm{rf}^{-1}\left(M/N\right) - \sqrt{\frac{\tilde{t}+\epsilon}{t+\epsilon}}\mathcal{E}\mathrm{rf}^{-1}\left(M/N\right)\right) \\
&\geq M.
\end{aligned}$$

We can now apply the intersection counting Theorem 8.2 with $w = u - \varphi^{\epsilon,s}$, with the previous calculation ensuring that there are no intersections on the boundary, and with the coefficients in equation (8.2) given by $A = 1$, $B = 0$ and $C = 0$. Since there is only one intersection at the initial time, there is never more than one intersection at later times (in particular, for our given $(\tilde{x}, \tilde{t})$, there is no other intersection at time $\tilde{t}$ than the one at $\tilde{x}$).

It follows that

$$\begin{aligned} &\text{for } y > \tilde{x}, \quad \varphi^{\epsilon,s}(y, \tilde{t}) > u(y, \tilde{t}), \\ &\text{for } y < \tilde{x}, \quad \varphi^{\epsilon,s}(y, \tilde{t}) < u(y, \tilde{t}), \end{aligned}$$

from which we find a gradient estimate:

$$u_x(\tilde{x}, \tilde{t}) \leq {\varphi^{\epsilon,s}}_x(\tilde{x}, \tilde{t}).$$

This holds for any smaller $\epsilon > 0$, so letting $\epsilon \to 0$ gives the final result. ☐

This method applies to all parabolic operators for which we can find solutions that have the step function as the initial condition.

When a quasilinear parabolic equation satisfies the conditions of Section 4.4, we can use the solutions with stepped initial data, whose existence was shown in that chapter.

Let $a > 0$ be in $H_\alpha(\mathcal{K})$ for all bounded $\mathcal{K} \subseteq \mathbb{R} \times \mathbb{R} \times \Omega \times [0, T]$, and some $\alpha \in (0, 1)$. This implies that for every such $\mathcal{K}$ we can find positive $\lambda_{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ such that $\lambda_{\mathcal{K}} \leq a(p, q, x, t) \leq \Lambda_{\mathcal{K}},$ when $(p, q, x, t) \in \mathcal{K}$.

Since we will be looking at bounded solutions in Ω, the bound on the gradient is the pertinent bound on $\mathcal{K}$; we highlight this by writing:

$$\lambda(K) \leq a(p, q, x, t) \leq \Lambda(K), \text{ when } |p| \leq K, \tag{8.18}$$

where we assume that $|q| \leq M$, $x \in \Omega$ and $t \in [0, T]$.

Also, suppose that there are positive constants A and P such that

$$a(p, q, x, t)p^2 \geq A > 0, \text{ for } |p| \geq P. \tag{8.19}$$

Theorem 8.6. *Let $u : \Omega \times [0, T] \to \mathbb{R}$ be a smooth solution to*

$$u_t = a(u_x, u, x, t)u_{xx}, \tag{8.20}$$

where a satisfies (8.18) *and* (8.19).

Let u be bounded, $|u(x, t)| < M$.

Let φ^s solve (8.20) *on $\mathbb{R} \times (0, T]$, with $\varphi^s(\cdot, t) \to 2M\sigma(x - s)$ as $t \to 0$, where σ is the step function* (8.14)*, and where s is chosen so that $u(x, t) = \varphi^s(x, t)$.*

If $t \leq c\ \mathrm{dist}(x, \partial\Omega)^2 / \Lambda\left(cM/\mathrm{dist}(x, \partial\Omega)\right)$, where $\Lambda(\cdot)$ is given by (8.18)*, then*

$$u_x(x, t) \leq \varphi^s_x(x, t).$$

That is, the gradient of u is bounded by the gradient of the barriers, at the same height.

Remark 1: We can replace $\sigma(x - s)$ by $\sigma(s - x)$ here, in which case we find that

$$u_x(x, t) \geq \varphi^s_x(x, t).$$

Remark 2: If a has polynomial growth, so that $\Lambda(K) \leq c(1+K^q)$ for some $q \geq 0$, then the interior region on which we can find bounds of this form is given by $t \leq c\ \operatorname{dist}(x, \partial\Omega)^{2+q} M^{-q/2}$, for some constant c.

Remark 3: If $\Omega = \mathbb{R}$ in Theorem 8.6, then the gradient bound applies for all $t \in (0, T]$.

Proof of Theorem 8.6:

We initially assume that $\Omega = [-d, d]$. We will derive a gradient bound at a single point $(0, t)$, and then generalize it to interior points on a general domain.

As u is smooth there are bounds on the first derivative and on higher derivatives

$$|u_x| \leq c_1, \qquad |u_{xx}, u_t, u_{xt}, u_{xxx}, u_{xxxx}, u_{xxt}| \leq c_2.$$

For $0 < \epsilon \leq \epsilon_0 << d/4$, let φ^ϵ be the standard mollification of φ

$$\varphi^\epsilon := 2M\eta_\epsilon * \sigma,$$

where σ is the step function (8.14).

Choose ϵ_0 small enough so that for all $\epsilon \leq \epsilon_0$, φ^ϵ satisfies a gradient estimate from below:

$$|{\varphi^\epsilon}_x(x)| \geq c_1 \text{ whenever } |\varphi^\epsilon(x)| \leq M. \tag{8.21}$$

Now, for fixed ϵ, define a family $\{\varphi^{\epsilon,s}\}_{|s| \leq d/2}$ of barriers, each of which solves (8.20) on $\mathbb{R} \times [0, \tau]$ (for some τ to be decided later) with initial condition

$$\varphi^{\epsilon,s}(x, 0) = \varphi^\epsilon(x - s).$$

The existence of such solutions follows from Corollary 4.20, which applies as a satisfies (8.19).

Standard results give

$$|\varphi_x^{\epsilon,s}| \leq c_3(\epsilon)$$
$$|\varphi_{xx}^{\epsilon,s}, \varphi_{xxx}^{\epsilon,s}, \varphi_{xxxx}^{\epsilon,s}, \varphi_t^{\epsilon,s}, \varphi_{tx}^{\epsilon,s}, \varphi_{xxt}^{\epsilon,s}| \leq c_4(\epsilon).$$

To avoid intersections of u and the barriers occurring on the boundary, we need to show that $|\varphi^{\epsilon,s}| \geq M$ when $x \in \{-d, d\}$.

Each barrier in the family is initially bounded above by a step function

$$\varphi^{\epsilon,s}(x, 0) \leq 2M\sigma(x - s + \epsilon)$$

and so Corollary 4.11 provides an estimate for $x < s - \epsilon$

$$\varphi^{\epsilon,s}(x, t) \leq \frac{8M}{|x - s + \epsilon|}\sqrt{\frac{\Lambda t}{\pi}} - 2M,$$

where $\Lambda = \Lambda\left(4M/|x - s + \epsilon|\right)$.

In particular, at $x = -d$,

$$\varphi^{\epsilon,s}(-d, t) \leq \frac{8M}{|-d - s + \epsilon|}\sqrt{\frac{\Lambda t}{\pi}} - 2M.$$

As $|s| < d/2$ and $\epsilon << d/4$,

$$\frac{1}{|-d-s+\epsilon|} < \frac{4}{d} \quad \text{and} \quad \Lambda\left(\frac{4M}{|-d-s+\epsilon|}\right) \leq \Lambda\left(\frac{16M}{d}\right).$$

If we choose $\tau = \dfrac{d^2\pi}{32^2\Lambda}$ where $\Lambda = \Lambda(16M/d)$, then

$$\begin{aligned}\varphi^{\epsilon,s}(-d,t) &\leq \frac{32M}{d}\sqrt{\frac{\Lambda t}{\pi}} - 2M \\ &\leq -M,\end{aligned}$$

whenever $t \leq \tau$.

A similar calculation for the other boundary point $x = d$ gives that

$$\varphi^{\epsilon,s}(d,t) \geq M$$

when $t \leq \tau$.

Let ϵ be fixed.

For each $s \in [-d/2, d/2]$, we can define $w := u - \varphi^{\epsilon,s}$ satisfying

$$w_t = Aw_{xx} + Bw_x + Cw$$

on $[-d,d] \times [0,\tau]$. Here, A is given by setting $\dfrac{\partial F}{\partial r}(r,p,q,x,t) = a(p,q,x,t)$ in (8.3), B by setting $\dfrac{\partial F}{\partial p} = r\dfrac{\partial}{\partial p}a(p,q,x,t)$ in (8.4), and C by setting $\dfrac{\partial F}{\partial q} = r\dfrac{\partial}{\partial q}a(p,q,x,t)$ in (8.5).

Let $\mathcal{K} = \{\,(p,q,x,t) : |p| \leq c_1 + c_3(\epsilon), |q| \leq M, x \in \Omega, t \in [0,\tau]\,\}$.

$A, A^{-1}, A_t, A_x, A_{xx}, B, B_t, B_x$, and C are bounded by constants dependent on c_1, c_2, $c_3(\epsilon)$, $c_4(\epsilon)$, $\lambda\left(c_1 + c_3(\epsilon)\right)$, $\Lambda\left(c_1 + c_3(\epsilon)\right)$ and $|a|_{C^2(\mathcal{K})}$, as in (8.7)–(8.13).

Since $|u| < M$, and at the boundary $|\varphi^{\epsilon,s}| \geq M$, w is *never* zero on the boundary.

In particular, $w(-d,t) = u(-d,t) - \varphi^{\epsilon,s}(-d,t) > 0$ and $w(d,t) = u(d,t) - \varphi^{\epsilon,s}(d,t) < 0$, so there is always *at least* one zero of w. The lower gradient bound (8.21) implies that there is *at most* one zero of w at the initial time.

Then the intersection counting theorem (Theorem 8.2) implies there is *exactly* one zero of w for *all* $t \leq \tau$.

In the following lemma we show that given (x,t) — or more specifically, $(0,t)$ — we can find s such that $(0,t)$ is a zero of $w = u - \varphi^{\epsilon,s}$. We will then return to the proof of Theorem 8.6.

Lemma 8.7. *Let $\epsilon \leq \epsilon_0$ and $t \leq \tau$ be fixed. For each $k \in [-M,M]$, there exists an $s \in [-d/2, d/2]$ such that*

$$\varphi^{\epsilon,s}(0,t) = k.$$

Proof: Firstly, we check that $\varphi^{\epsilon,d/2}(0,t) \leq -M$ and $\varphi^{\epsilon,-d/2}(0,t) \geq M$. Using Corollary 4.11,

$$\varphi^{\epsilon,d/2}(x,t) \leq \frac{8M}{|x - d/2 + \epsilon|}\sqrt{\frac{\Lambda t}{\pi}} - 2M$$

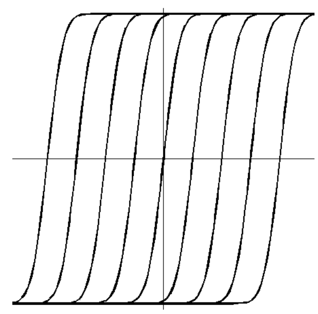


Figure 8.3: A family of barriers

for $x < d/2 - \epsilon$, where $\Lambda = \Lambda(4M/|x - d/2 - \epsilon|)$. Since $|d/2 - \epsilon|^{-1} \leq 4/d$, at $x = 0$ we have

$$\begin{aligned} \varphi^{\epsilon,d/2}(0,t) &\leq \frac{32M}{d}\sqrt{\frac{\Lambda t}{\pi}} - 2M \\ &\leq -M. \end{aligned}$$

It can similarly be shown that $\varphi^{\epsilon,-d/2}(0,t) \geq M$.

As $\varphi^{\epsilon,s}(\cdot,0)$ is continuous in s, and a is a continuous operator, $\varphi^{\epsilon,s}(0,t)$ is also continuous in s. In particular, $\{\varphi^{\epsilon,s}(0,t)\}_{|s|\leq d/2}$ is onto $[-M, M]$. □

Continuation of the proof of Theorem 8.6: Let $t \leq \tau$ be given. From the previous lemma, there exists s such that $\varphi^{\epsilon,s}(0,t) = u(0,t)$. This is the only intersection point of u and $\varphi^{\epsilon,s}$, and so

$$\begin{aligned} &\text{for } y > 0, \quad \varphi^{\epsilon,s}(y,t) > u(y,t), \\ &\text{for } y < 0, \quad \varphi^{\epsilon,s}(y,t) < u(y,t), \end{aligned}$$

from which we find the gradient estimate:

$$\begin{aligned} u_x(0,t) &= \lim_{y\to 0} \frac{u(y,t) - u(0,t)}{y} \\ &\leq \lim_{y\to 0} \frac{\varphi^{\epsilon,s}(y,t) - \varphi^{\epsilon,s}(0,t)}{y} \\ &= {\varphi^{\epsilon,s}}_x(0,t). \end{aligned}$$

This estimate holds for all $\epsilon \in (0, \epsilon_0]$.

If we let $\epsilon \to 0$, we firstly have that

$$\varphi^{\epsilon,s} \to \varphi^s,$$

where φ^s is the solution to (8.20) with discontinuous initial data $2M\sigma(x - s)$; and secondly that for all $t \leq \tau$,

$$u_x(0,t) \leq \varphi^s_x(0,t),$$

where s is chosen so that $u(0,t) = \varphi^s(0,t)$.

Now we turn to the general domain with $\Omega = [x_0, x_1]$. Given $x \in \Omega$, set $d := \mathrm{dist}(x, \partial\Omega)$. If $t \leq \dfrac{d^2\pi}{32^2\Lambda(24M/d)}$ then we can repeat the same calculation on the small domain $[x-d, x+d]$ to find the given result. □

Chapter 9

Estimates for isotropic and anisotropic mean curvature flow

9.1 A gradient estimate for mean curvature flow

This section follows the style established in papers such as [13] and [14], in particular the local gradient estimates of Section 2 of the latter paper. Ecker and Huisken consider the evolution of a hypersurface by mean curvature

$$\frac{d}{dt}\mathbf{F}(p,t) = \mathbf{H}(p,t), \qquad p \in M, \tag{9.1}$$

where $\mathbf{F} : M^n \times [0,T] \to \mathbb{R}^{n+1}$ is the immersion of the manifold M^n at each time t and $\mathbf{H}$ is the mean curvature vector.

M can be written as a graph when a fixed vector $\omega \in \mathbb{R}^{n+1}$ can be found so that for a choice of unit normal ν,

$$\langle \nu, \omega \rangle > 0$$

everywhere. Equivalently, $\langle \nu, \omega \rangle^{-1}$ is bounded above. The existence of an upper bound for this quantity and its analogue for anisotropic mean curvature flow is the subject of this chapter.

Given the image $\mathbf{F}(p,t)$ of a point $p \in M$, its coordinate vector is $\mathbf{x}(p,t)$. The *height* of M above the hyperplane defined by ω is denoted by

$$u = \langle \mathbf{x}, \omega \rangle.$$

The *gradient function* is given by

$$v = \langle \nu, \omega \rangle^{-1} = \sqrt{1 + |Du|^2}$$

We recall the evolution equations from [13]:

Lemma 9.1. *If M_t satisfies* (9.1) *then*

$$\left(\frac{d}{dt} - \Delta\right)|\mathbf{x}|^2 = -2n,$$
$$\left(\frac{d}{dt} - \Delta\right)u = 0,$$
$$\left(\frac{d}{dt} - \Delta\right)v = -|A|^2 v - 2v^{-1}|\nabla v|^2,$$

where Δ is the Laplace-Beltrami operator on M_t and $A = \{h_{ij}\}$ is the second fundamental form.

We derive a gradient estimate for periodic entire graphs, followed by an interior estimate. Estimates of this type have also been recently found by Colding and Minicozzi [12] in the isotropic case using similar techniques, although without the explicit dependence on the height of the graph that the following estimates display.

Theorem 9.2 (Estimate for periodic mean curvature flow). *Let $\mathbf{F}$ be a smooth, entire solution to mean curvature flow* (9.1) *which is a periodic graph over a hyperplane, in that $u(y,t) = u(y+L,t)$ for a fixed point L in the hyperplane, and has a height bound $|u| < M$. Then*

$$v \le t^{1/2} \exp\left(\frac{c(|u| - 2M)^2}{4t}\right)$$

for $0 < t \le T'$, where c and T' depend on M.

Proof: Define a new quantity

$$Z := v - \varphi(u,t)$$

where φ is a smooth positive function on $[-M,M] \times (0,T')$ chosen so that $\varphi \to \infty$ as $t \to 0$. This means that Z is strictly negative initially, regardless of the initial gradient.

The evolution equation for φ is given by

$$\left(\frac{d}{dt} - \Delta\right)\varphi = \varphi_t - \varphi_{uu}\left|\nabla u\right|^2, \tag{9.2}$$

and we can use this with the identities from Lemma 9.1 to find that

$$\left(\frac{d}{dt} - \Delta\right)Z = -|A|^2 v - 2v^{-1}|\nabla v|^2 - \varphi_t + \varphi_{uu}|\nabla u|^2.$$

Now, suppose (x,t) is the first point at which Z becomes non-negative. Since Z is periodic, this is an internal spatial maximum, and (spatial) first deriatives are zero:

$$0 = \nabla Z = \nabla v - \nabla\varphi,$$

so

$$v^{-1}|\nabla v|^2 = \frac{\left|\nabla\varphi\right|^2}{\varphi} = \frac{\varphi_u^2\left|\nabla u\right|^2}{\varphi},$$

and the evolution equation at this point is

$$\left(\frac{d}{dt} - \Delta\right) Z = -|A|^2 v - 2\frac{\varphi_u^2}{\varphi} |\nabla u|^2 - \varphi_t + \varphi_{uu} |\nabla u|^2.$$

A good choice for φ that will allow us to make the final terms negative is $\varphi(u,t) = 1/\Phi(u,t)$, where Φ solves the heat equation $\Phi_t = c\Phi''$ for some $c < 1$. In this case

$$\begin{aligned} \varphi' &= -\Phi^{-2}\Phi', \\ \varphi'' &= 2\Phi^{-3}(\Phi')^2 - \Phi^{-2}\Phi'', \\ \varphi_t &= -\Phi^{-2}\Phi_t \end{aligned}$$

and the equation satisfied by φ is

$$\varphi_t = c\varphi'' - 2c\frac{\varphi'^2}{\varphi}.$$

The final three terms of the evolution equation have become

$$\begin{aligned} -2\frac{\varphi_u^2}{\varphi} |\nabla u|^2 - \varphi_t + \varphi_{uu} |\nabla u|^2 &= -2\frac{\Phi'^2}{\Phi^3} |\nabla u|^2 + c\frac{\Phi''}{\Phi^2} - \frac{\Phi''}{\Phi^2} |\nabla u|^2 + 2\frac{\Phi'^2}{\Phi^3} |\nabla u|^2 \\ &= \frac{\Phi''}{\Phi^2}\left(c - |\nabla u|^2\right) \\ &= \frac{\Phi''}{\Phi^2}\left(c - 1 + \Phi^2\right). \end{aligned}$$

In the last line we have used that, with respect to a local orthonormal frame on M_t, $\nabla u = \langle e_i, \omega\rangle e_i$, while ω has unit length with $1 = |\omega|^2 = \sum_{i=1}^n |\langle e_i, \omega\rangle e_i|^2 + |\langle \nu, \omega\rangle \nu|^2$: it follows that

$$|\nabla u|^2 = \sum_{i=1}^n \langle e_i, \omega\rangle^2 = 1 - \langle \nu, \omega\rangle^2 = \left(1 - \frac{1}{v^2}\right) = \left(1 - \Phi^2\right),$$

the last equality holding only at a maximum point.

If we let Φ be a fundamental solution of the heat equation

$$\Phi(u,t) = \frac{1}{\sqrt{t}} \exp\left(-c\frac{(u \pm 2M)^2}{4t}\right), \tag{9.3}$$

we can choose T' and c depending only on M so that $\Phi'' \geq 0$ and $c - 1 + \Phi^2 \leq 0$ for $t < T'$.

So, at the first interior point where $Z = 0$, $Z_t \leq 0$ and so $Z \leq 0$ for $t < T'$. □

Theorem 9.3 (Interior estimate for mean curvature flow). *Let* $\mathbf{F}$ *be a smooth solution to mean curvature flow* (9.1) *which is a graph over a ball in the hyperplane* $B_R(0)$. *Then we have the interior estimate*

$$v \leq t^{q/2} \exp\left(\frac{cq(u + 2M)^2}{4t}\right)\left(R^2 - 2nt - |\mathbf{x}|^2 + u^2\right)^{-1}$$

for $0 \le t \le T'$, where $q > 1$, c, and T' depend on M and R.

Proof: We replace φ in our previous definition of Z by φ/η:

$$Z := v - \frac{\varphi(u,t)}{\eta},$$

where a smooth positive function η is chosen so that $Z < 0$ on the boundary of a ball of shrinking radius $B_{\sqrt{R^2-2nt}}$, and, as before, $\varphi \ge 0$ is chosen so that $Z < 0$ at the initial time.

In particular, choose $\eta = R^2 - 2nt - |\mathbf{x}|^2 + u^2$. The evolution equation for η is given by

$$\left(\frac{d}{dt} - \Delta\right)\eta = -2\,|\nabla u|^2\,,$$

and we can use this, the identities from Lemma 9.1, and the evolution equation for φ (9.2) to find that

$$\left(\frac{d}{dt} - \Delta\right) Z = -|A|^2 v - 2v^{-1}|\nabla v|^2 - \frac{1}{\eta}\left(\varphi_t - \varphi_{uu}|\nabla u|^2\right) \\ - 2\frac{\varphi}{\eta^2}\,|\nabla u|^2 - 2\frac{\varphi_u}{\eta^2}\nabla u \cdot \nabla\eta + 2\frac{\varphi}{\eta^3}|\nabla\eta|^2.$$

Now, suppose (x,t) is an internal point of this domain at which Z first becomes non-negative.

At an internal spatial maximum of Z, $\nabla Z = 0$ so

$$\begin{aligned} |\nabla v|^2 &= \left|\nabla\left(\frac{\varphi}{\eta}\right)\right|^2 \\ &= \frac{\varphi_u^2}{\eta^2}\,|\nabla u|^2 - 2\frac{\varphi\varphi_u}{\eta^3}\nabla u\cdot\nabla\eta + \frac{\varphi^2}{\eta^4}\,|\nabla\eta|^2\,. \end{aligned}$$

Use this to replace the $v^{-1}\,|\nabla v|^2$ term in the evolution equation, so that at this point

$$\begin{aligned} \left(\frac{d}{dt} - \Delta\right) Z &= -|A|^2 v - 2\frac{\eta}{\varphi}\left(\frac{\varphi_u^2}{\eta^2}\,|\nabla u|^2 - 2\frac{\varphi\varphi_u}{\eta^3}\nabla u\cdot\nabla\eta + \frac{\varphi^2}{\eta^4}\,|\nabla\eta|^2\right) \\ &\quad - \frac{1}{\eta}\left(\varphi_t - \varphi_{uu}|\nabla u|^2\right) - 2\frac{\varphi}{\eta^2}\,|\nabla u|^2 - 2\frac{\varphi_u}{\eta^2}\nabla u\cdot\nabla\eta + 2\frac{\varphi}{\eta^3}|\nabla\eta|^2 \\ &= -|A|^2 v - \frac{1}{\eta}\left(\varphi_t - \varphi_{uu}|\nabla u|^2 + 2\,|\nabla u|^2\,\frac{\varphi_u^2}{\varphi}\right) \\ &\quad - 2\,|\nabla u|^2\,\frac{\varphi}{\eta^2} + 2\frac{\varphi_u}{\eta^2}\nabla u\cdot\nabla\eta. \end{aligned}$$

We can bound the $\nabla u \cdot \nabla\eta$ term by a v^{-1} term, since (using a local orthonormal frame $\{e_i\}$)

$$\begin{aligned}\nabla u \cdot \nabla \eta &= \nabla \langle \mathbf{x}, \omega \rangle \cdot \nabla \left(-\langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{x}, \omega \rangle^2 \right) \\ &= \langle \nabla_i \mathbf{x}, \omega \rangle e_i \cdot (-2\langle \mathbf{x}, \nabla_j \mathbf{x} \rangle e_j + 2\langle \mathbf{x}, \omega \rangle \langle \nabla_j \mathbf{x}, \omega \rangle e_j) \\ &= \langle e_i, \omega \rangle e_i \cdot (-2\langle \mathbf{x}, e_j \rangle e_j + 2\langle \mathbf{x}, \omega \rangle \langle e_j, \omega \rangle e_j) \\ &= g^{ij} \langle e_i, \omega \rangle \left(-2\langle \mathbf{x}, e_j \rangle + 2\langle \mathbf{x}, \omega \rangle \langle e_j, \omega \rangle \right),\end{aligned}$$

and

$$g^{ij} = \delta^{ij} - \frac{D_i u D_j u}{1 + |Du|^2}$$
$$\langle e_i, \omega \rangle = D_i u$$
$$\langle e_i, \mathbf{x} \rangle = \mathbf{x}_i + u D_i u$$

so that (writing x for $\mathbf{x} - u\omega$, the position in the hyperplane)

$$\begin{aligned}\nabla u \cdot \nabla \eta &= -2\langle Du, x \rangle + 2\frac{\langle Du, x \rangle |Du|^2}{1 + |Du|^2} \\ &= -2\frac{\langle Du, x \rangle}{1 + |Du|^2} \\ &\leq 2\frac{|x|}{\sqrt{1 + |Du|^2}} \\ &\leq 2\frac{\sqrt{R^2 - 2nt}}{v}.\end{aligned}$$

With this we estimate the term in the evolution equation—

$$2\frac{\varphi_u}{\eta^2} \nabla u \cdot \nabla \eta \leq 4\frac{|\varphi_u|}{\varphi \eta} R. \tag{9.4}$$

The evolution equation itself becomes

$$\begin{aligned}\left(\frac{d}{dt} - \Delta \right) Z \leq -|A|^2 v - 2\,|\nabla u|^2\, \frac{\varphi}{\eta^2} \\ - \frac{1}{\eta} \left(\varphi_t - \varphi_{uu} |\nabla u|^2 + 2\,|\nabla u|^2\, \frac{\varphi_u^2}{\varphi} - 4R\frac{|\varphi_u|}{\varphi} \right).\end{aligned}$$

This time, we choose

$$\varphi(u, t) = \Phi(u, t)^{-q}$$

for some $q > 1$, where Φ still satisfies the heat equation $\Phi_t = c\Phi''$. Then

$$\varphi' = -q\Phi^{-q-1}\Phi',$$
$$\varphi'' = q(q+1)\Phi^{-q-2}(\Phi')^2 - q\Phi^{-q-1}\Phi'',$$
$$\varphi_t = -q\Phi^{-q-1}\Phi_t,$$

so that the equation satisfied by φ is

$$\varphi_t = c\varphi'' - c\left(1 + \frac{1}{q}\right)\frac{\varphi'^2}{\varphi}.$$

The final term in the evolution equation is

$$\begin{aligned}
\varphi_t - \varphi_{uu}|\nabla u|^2 + 2\,|\nabla u|^2\,\frac{\varphi_u^2}{\varphi} - 4R\frac{|\varphi_u|}{\varphi} \\
&= q\frac{\Phi''}{\Phi^{q+1}}\left(-c + |\nabla u|^2\right) + q\frac{\Phi'^2}{\Phi^{q+2}}\,|\nabla u|^2\,(q-1) - 4Rq\frac{|\Phi'|}{\Phi} \\
&= q\frac{\Phi''}{\Phi^{q+1}}\left(-c + 1 - \Phi^{2q}\right) + q\frac{|\Phi'|}{\Phi}\left((q-1)(1-\Phi^{2q})\frac{|\Phi'|}{\Phi^{q+1}} - 4R\right),
\end{aligned}$$

where we have used $|\nabla u|^2 = 1 - v^{-2} = 1 - \Phi^{2q}$. The first term above is positive if, as in the previous case, $\Phi'' \geq 0$ and $c - 1 + \Phi^{2q} \leq 0$. The second term is positive if q satisfies

$$q \geq 1 + \frac{4R\Phi^{q+1}}{|\Phi'|(1-\Phi^2)}.$$

Choose Φ to be a fundamental solution of the heat equation, as in the previous proof —

$$\Phi(u,t) = \frac{1}{\sqrt{t}}\exp\left(-c\frac{(u \pm 2M)^2}{4t}\right).$$

Now we can choose some $c < 1$ and T' small so that $\Phi'' > 0$ and $\Phi \leq (1-c) < 1$ for $t < T'$. Here, T' is dependent on M and c only. We can also find q dependent on T', M, c and R, satisfying

$$1 + \frac{4R\Phi^{q+1}}{|\Phi'|(1-\Phi^2)} \leq 1 + \frac{8RT'}{cM(1-(1-c)^2)} \leq q.$$

At an internal maximum of Z, $Z_t \leq 0$ and the result follows. □

9.2 Gradient estimates for anisotropic mean curvature flows

A more general case of curvature flows is that of *anisotropic mean curvature flow*. This has been specifically studied by Almgren, Taylor and Wang [1], Gurtin and Angenent [9], and Andrews [3, 4], among others. The anisotropic surface energy arises in applications from materials science, such as crystalline growth and phase changes; it also arises in Finsler geometry [11] (on a Finsler manifold, at each point only a normed space is defined, rather than an inner product space as on a Riemannian manifold).

In this section, we use the framework and notation of [4].

As before, we consider surfaces that can be written (either entirely or locally) as graphs, so that $M_t = \{(x^1, \ldots, x^n, u(x^1, \ldots, x^n, t))\} =$ graph $u(x,t)$.

The equation for motion of the graph by anisotropic mean curvature is derived in [4]; in the present work, we set the homogeneous degree zero "mobility factor" m to be identically 1, and so

$$u_t = F(Du)D^{ij}F\big|_{Du}\, u_{ij}, \tag{9.5}$$

where $u_{ij} = D^2u(e_i, e_j)$, with respect to some basis for the tangent space $\{e_1, \dots, e_n\}$.

The function $F : \mathbb{R}^n \to \mathbb{R}$ is defined by $F(p_1, \dots, p_n) := \bar{F}(p_i\phi^i - \phi^0)$, where $\{\phi^0, \phi^1, \dots, \phi^n\}$ is a basis for the cotangent space V^*, with dual basis for $V = \mathbb{R}^{n+1}$ given by $\{e_0, e_1, \dots, e_n\}$, and where $\bar{F} : V^* \to \mathbb{R}$ is a positive convex function that is homogeneous of degree one, $\bar{F}(\lambda\omega) = \lambda\bar{F}(\omega)$ for $\lambda > 0$. The *unit ball* of $\bar{F}$

$$\bar{F}^{-1}(1) := \{\omega \in V^* : \bar{F}(\omega) = 1\}$$

must be strictly convex. We also require that $\bar{F}$ is at least C^3.

Differences between the isotropic and anisotropic cases

The introduction of the unspecified anisotropic $\bar{F}$ into the flow has the effect of highlighting the special nature of the *isotropic* case, when $\bar{F}(p_i\phi^i + p_0\phi^0) = \left(\sum_{i=0}^n p_i^2\right)^{1/2}$.

One immediately notices that in the isotropic case, the term with third derivatives arising in the evolution equation is zero. In (9.21), this is the term

$$D\bar{F}\big|_z(\phi^k)\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(\phi^m, \hat{\phi}^i, \hat{\phi}^j)u_{mk}u_{ij}.$$

This absence of third derivatives is apparent in the third identity of Lemma 9.1 —

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2v^{-1}|\nabla v|^2;$$

the left-hand side involves second derivatives of the gradient so we might expect to see some derivatives of curvature in the right hand side — instead we see only curvature terms and first derivatives of the gradient function.

The second difference is that there is no estimate of the form

$$\nabla u \cdot \nabla \eta \le \frac{c}{v},$$

as in (9.4) for the isotropic case. An equivalent estimate in the anisotropic context would be: given $q = \sum_{i=1}^n q_i\phi^i$, find $c = c(q)$ so that

$$\bar{F}(p - \phi^0)\; \bar{F}D^2\bar{F}\big|_{p-\phi^0}(p, q) \le c(q)$$

for all $p = \sum_{i=1}^n p_i\phi^i$. This is certainly true if we restrict p to the unit ball, $\bar{F}(p) = 1$. If we replace p by sp (s is a scalar), the putative estimate would be

$$\bar{F}(s\bar{p} - \phi^0)\; \bar{F}D^2\bar{F}\big|_{s\bar{p}-\phi^0}(s\bar{p}, q) \le c(q).$$

Rewriting the left-hand side using homogeneity gives us

$$s\bar{F}(\bar{p} - \phi^0/s)\; \bar{F}D^2\bar{F}\big|_{\bar{p}-\phi^0/s}(\phi^0, q) \le c(q).$$

As s increases, the left-hand side is converging to a constant defined on the unit ball, $F(\bar{p})\; \bar{F}D^2\bar{F}\big|_{\bar{p}}(\phi^0, q)$, multiplied by s. Unless $F(\bar{p})\; \bar{F}D^2\bar{F}\big|_{\bar{p}}(\phi^0, q)$ is zero , the left-hand side will not remain bounded by the right-hand side as $s \to \infty$. The estimate will not hold without further restrictions on $\bar{F}$.

Calculating with the homogeneous function $\bar{F}$

We make some observations about properties arising directly from the homogeneity and convexity of $\bar{F}$, and introduce some notation.

Let $\{\phi^0, \ldots, \phi^n\}$ be a basis for the cotangent space V^* dual to $\{e_0, \ldots, e_n\}$, the basis for the tangent space V. Both V^* and V are copies of $\mathbb{R}^{n+1}$.

For $p = p_i\phi^i$ (all repeated indices are summed from 1 to n unless indicated otherwise) we will write

$$z := p - \phi^0.$$

In general we will prefer to write all derivatives of $\bar{F}$ in a form that is homogeneous of degree zero, that is, as $D\bar{F}$, $\bar{F}D^2\bar{F}$, or $\bar{F}^2D^3\bar{F}$. This means that we can evaluate them on the unit ball, or scale as we wish — for example, we can use $\bar{F}D^2\bar{F}\big|_{p-\phi^0/t}$ instead of $\bar{F}D^2\bar{F}\big|_{tp-\phi^0}$.

Homogeneity also means that some derivatives in the radial direction disappear — for all $\omega \in V^*$,

$$D\bar{F}\big|_{\omega}(\omega) = \bar{F}(\omega) \tag{9.6}$$

$$\bar{F}D^2\bar{F}\big|_{\omega}(\omega, \cdot) = \bar{F}D^2\bar{F}\big|_{\omega}(\cdot, \omega) = 0 \tag{9.7}$$

$$D(\bar{F}D^2\bar{F})\big|_{\omega}(\omega, \cdot, \cdot) = 0. \tag{9.8}$$

The strict convexity of the unit ball of $\bar{F}$ means that for all $\omega, r \in V^*$ on the unit ball, with $r \neq \pm\omega$,

$$D^2\bar{F}\big|_{\omega}(r, r) > 0.$$

As $\bar{F}$ is homogeneous, all the level sets of $\bar{F}$ are also strictly convex, so this holds for all non-zero ω.

We denote by $\widehat{\ }$ the removal of a component in the direction of z from ϕ^k, $k = 0, \ldots, n$,

$$\widehat{\phi}^k := \phi^k - c^k z,$$

where c^k is such that $\widehat{\phi}^k$ is tangent to the unit ball of $\bar{F}$,

$$\begin{aligned} 0 &= D\bar{F}\big|_z(\widehat{\phi}^k) \\ &= D\bar{F}\big|_z(\phi^k - c^k z) \\ &= D\bar{F}\big|_z(\phi^k) - c^k\, D\bar{F}\big|_z(z) \\ &= D\bar{F}\big|_z(\phi^k) - c^k\bar{F}(z). \end{aligned}$$

We have used (9.6) in the last line. It follows that

$$c^k = \frac{D\bar{F}\big|_z(\phi^k)}{\bar{F}(z)}.$$

In the next two lemmas, we show that the coefficients of the evolution operator satisfy a condition similar to the control on degeneracy that we required with condition (5.9) of Chapter 5.

Lemma 9.4. *Let $\bar{F} : V^* \to \mathbb{R}$ be a C^2, positive, homogeneous degree one function with a strictly convex unit ball $\bar{F}^{-1}(1) = \{\omega : \bar{F}(\omega) = 1\}$.*

Let $\phi^0, \dots, \phi^n$ be a basis for V^*.

Then there exist positive constants A and P so that

$$\bar{F} D^2 \bar{F}\big|_{p-\phi^0}(p,p) \geq A$$

for all $p = \sum_{i=1}^n p_i \phi^i$ with $\bar{F}(p) \geq P$.

Proof: Write

$$B(p) := \bar{F} D^2 \bar{F}\big|_{p-\phi^0}(p,p) = \bar{F} D^2 \bar{F}\big|_{p-\phi^0}(\widehat{p},\widehat{p}),$$

where

$$\widehat{p} = p - \frac{D\bar{F}\big|_{p-\phi^0}(p)}{F(p-\phi^0)}\left(p - \phi^0\right)$$

is non-zero whenever $p \neq 0$. As $\widehat{p}$ is a non-zero tangent covector, the strict convexity of the unit ball means that $B(p) > 0$ if $p \neq 0$.

Fix $p = p_i \phi^i$ on the unit ball of $\bar{F}$, $\bar{F}(p) = 1$.

Consider $B(sp)$. We want to show that we can find some P_p and some strictly positive A_p so that for all $s \geq P_p$, $B(sp) \geq A_p$. If this is not possible, then we can find a sequence $s_k \to \infty$ with

$$\lim_{k\to\infty} B(s_k p) = 0.$$

Since $\bar{F}$ is C^2, $\bar{F} D^2 \bar{F}\big|_z$ is continuous in z, and we have that

$$\begin{aligned}
\lim_{k\to\infty} \bar{F} D^2 \bar{F}\big|_{s_k p - \phi^0}(s_k p, s_k p) &= \lim_{k\to\infty} \bar{F} D^2 \bar{F}\big|_{s_k p - \phi^0}(\phi^0,\phi^0) \\
&= \lim_{k\to\infty} \bar{F} D^2 \bar{F}\big|_{p - \phi^0/s_k}(\phi^0,\phi^0) \\
&= \bar{F} D^2 \bar{F}\big|_{\lim_{k\to\infty} p - \phi^0/s_k}(\phi^0,\phi^0).
\end{aligned}$$

Clearly,

$$\lim_{k\to\infty} p - \phi^0/s_k = p,$$

so

$$\lim_{k\to\infty} B(s_k p) = \bar{F} D^2 \bar{F}\big|_p\left(\phi^0,\phi^0\right) = \bar{F} D^2 \bar{F}\big|_p\left(\widehat{\phi}^0,\widehat{\phi}^0\right) > 0,$$

by the strict convexity of the unit ball, as at p,

$$\widehat{\phi}^0 = \phi^0 - \frac{D\bar{F}\big|_p(\phi^0)}{\bar{F}(p)} p$$

is a non-zero tangent covector. The contradiction implies that we can indeed find such P_p and A_p.

We can find such P_q and A_q for every $q = q_i\phi^i$. Let

$$A := \inf_{q:\bar{F}(q)=1} A_q, \qquad P := \sup_{q:\bar{F}(q)=1} P_q.$$

As we are optimizing over a compact space, $A > 0$ and $P < \infty$.

The result follows directly, for given any $p = p_i\phi^i$ with $\bar{F}(p) \geq P$,

$$B(p) = B(\bar{F}(p)\bar{p}) \geq A_{\bar{p}} \geq A,$$

where $\bar{p} = p/\bar{F}(p)$ is on the unit ball. □

We use this to show that the anisotropic mean curvature flow satisfies (5.9), the condition controlling the degeneracy of the parabolic operator in Chapter 5.

Lemma 9.5. *For all non-zero $v = v_i\phi^i$ and $p = p_i\phi^i$, we can find positive constants P and A_0 such that*

$$\frac{|p|^4}{(v\cdot p)^2}\, \bar{F}D^2\bar{F}\Big|_{p-\phi^0}(v,v) \geq A_0 \tag{9.9}$$

whenever $\bar{F}(p) \geq P$. Here, $(v\cdot p)^2 = \sum(v_ip_i)^2$ and $|p|^2 = p\cdot p$.

Proof: Set $B(p,v) = \frac{|p|^4}{(v\cdot p)^2}\, \bar{F}D^2\bar{F}\Big|_{p-\phi^0}(v,v)$.

Since $B(\cdot,v)$ is invariant under $v \mapsto sv$, we need only to consider v in the unit ball.

Suppose that p is in the unit ball. Since $\widehat{v}$ is a non-zero tangent covector at $p-\phi^0$, $\bar{F}D^2\bar{F}\Big|_{p-\phi^0}(v,v) > 0$ by the strict convexity of the unit ball. By compactness,

$$\inf_{p\in\bar{F}^{-1}(1)}\ \inf_{v\in\bar{F}^{-1}(1)} \bar{F}D^2\bar{F}\Big|_{p-\phi^0}(v,v) \geq c_1 > 0.$$

Also, as neither p nor v are zero,

$$\frac{|p|^4}{(v\cdot p)^2} \geq c_2 > 0,$$

and so $\inf_{p,v\in F^{-1}(1)} B(p,v) \geq c_1c_2 > 0$.

Suppose, in order to obtain a contradiction, that there is a pair (v,p) in the unit ball for which there are no such constants A_0 and P. That is,

$$\lim_{s\to\infty} B(sp,v) = \lim_{s\to\infty} \frac{s^2|p|^4}{(v\cdot p)^2}\, \bar{F}D^2\bar{F}\Big|_{sp-\phi^0}(v,v) = 0.$$

There are two possibilities here: $v \neq p$ or $v = p$. In the first case, we must have

$$\begin{aligned} 0 &= \lim_{s\to\infty} \bar{F}D^2\bar{F}\Big|_{sp-\phi^0}(v,v) \\ &= \lim_{s\to\infty} \bar{F}D^2\bar{F}\Big|_{p-\phi^0/s}(v,v) \\ &= \bar{F}D^2\bar{F}\Big|_{p}(v,v). \end{aligned}$$

However, since $v \neq p$, $\bar{F}D^2\bar{F}\Big|_{p}(v,v) > 0$ which is a contradiction.

On the other hand, if $v = p$, then

$$\begin{aligned}\lim_{s\to\infty} B(sp,p) &= \lim_{s\to\infty} s^2\ \bar{F}D^2\bar{F}\big|_{sp-\phi^0}(p,p)\\ &= \lim_{s\to\infty} \bar{F}D^2\bar{F}\big|_{sp-\phi^0}(sp,sp)\\ &\geq A\end{aligned}$$

by Lemma 9.4, which is again a contradiction.

Therefore for every pair of covectors (v,p), there is a pair of positive constants A_0 and P such that $B(p,v) \geq A_0$ whenever $F(p) \geq P$. To get bounds for all (v,p) we take the infimum of the A_0 and the supremum of the P. □

We will consider two different restrictions on $\bar{F}$. The first is that third derivatives are small; the second is a symmetry in the distinguished direction ϕ^0.

In order to define the first condition, consider the tensor

$$Q(p,q,r) := \bar{F}^2(z)\ D^3\bar{F}\big|_z(p,q,r), \tag{9.10}$$

for p,q,r covectors tangent to the unit ball of $\bar{F}$ at z, so that $D\bar{F}\big|_z(p) = 0$ (and similarly for q and r).

(This is the *Cartan tensor* of Bao, Chern and Shen [10], or the tensor Q of [4] restricted to the tangent space of the unit ball.)

The *smallness-of-third-derivatives condition* is that Q satisfies

$$Q(p,q,r) \leq C_1\left[\bar{F}(z)^3\ D^2\bar{F}\big|_z(p,p)\ D^2\bar{F}\big|_z(q,q)\ D^2\bar{F}\big|_z(r,r)\right]^{1/2}, \tag{9.11}$$

for all p,q,r tangent to the unit ball of $\bar{F}$, where C_1 is a positive constant dependent on n.

The *symmetry condition* is that

$$\bar{F}(p+\phi^0) = \bar{F}(p-\phi^0) \text{ for all } p = \sum_{i=1}^{n} p_i\phi^i. \tag{9.12}$$

Lemma 9.6. *If $\bar{F}$ satisfies* (9.12)*, then*

$$D\bar{F}\big|_p(\phi^0) = 0 \tag{9.13}$$
$$D^2\bar{F}\big|_p(\phi^0,\phi^j) = 0$$
$$D^3\bar{F}\big|_p(\phi^0,\phi^j,\phi^k) = 0 \tag{9.14}$$
$$D^3\bar{F}\big|_p(\phi^0,\phi^0,\phi^0) = 0, \tag{9.15}$$

for all $p = \sum_{i=1}^{n} p_i\phi^i$ and all $j,k \neq 0$.

Proof: This is a direct consequence of homogeneity. □

We can show that the symmetry condition (9.12) can be used in a similar way to the smallness-of-third-derivatives condition (9.11).

Lemma 9.7. *Suppose the symmetry condition* (9.12) *holds. Then a constant depen-*

dent only on $\bar{F}$ bounds

$$\left| \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{p-\phi^0}(p,\widehat{q},\widehat{q})}{G(p,p)^{1/2}G(q,q)} \right|, \tag{9.16}$$

for all $p = \sum_{i=1}^n p_i\phi^i$ and $q = \sum_{i=1}^n q_i\phi^i$, where $G = \bar{F}D^2\bar{F}\big|_{p-\phi^0}$. Furthermore, for all $\epsilon > 0$ we can find S_ϵ so that when $\bar{F}(p) \geq S_\epsilon$,

$$\left| \bar{F}D(\bar{F}D^2\bar{F})\big|_{p-\phi^0}(p,\widehat{q},\widehat{q}) \right| \leq \epsilon G(p,p)^{1/2}G(q,q).$$

Proof: Consider p and q on the unit ball and set

$$C := \sup_{\substack{p,q\in\bar{F}^{-1}(1) \\ p=p_i\phi^i,\, q=q_i\phi^i}} \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{p-\phi^0}(p,\widehat{q},\widehat{q})}{G(p,p)^{1/2}G(q,q)}$$

When we project p and q onto the tangent plane at $p-\phi^0$ they give non-zero tangent covectors $\widehat{p}$ and $\widehat{q}$,

$$\widehat{p} = p - c^p(p-\phi^0), \qquad \widehat{q} = q - c^q(p-\phi^0),$$

where

$$c^p = \frac{D\bar{F}\big|_{p-\phi^0}(p)}{\bar{F}(p-\phi^0)}, \qquad c^q = \frac{D\bar{F}\big|_{p-\phi^0}(q)}{\bar{F}(p-\phi^0)},$$

so the terms in the denominator of (9.16), $G(p,p)$ and $G(q,q)$, are strictly positive, and hence bounded below when we take the supremum over p and q in the unit ball.

Also, $\bar{F}D(\bar{F}D^2\bar{F})$ is a homogeneous degree zero tensor, and so bounded above on the unit ball.

It follows that C is finite.

The constant C is unchanged if we scale $q \mapsto sq$ so we only need to consider the behaviour of (9.16) as $\bar{F}(p)$ becomes large: that is,

$$\lim_{s\to\infty} \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{sp-\phi^0}(sp,\widehat{q},\widehat{q})}{G(sp,sp)^{1/2}G(q,q)}, \tag{9.17}$$

where $G = \bar{F}D^2\bar{F}\big|_{sp-\phi^0}$.

Firstly, consider the case that q is parallel to p. Let p be on the unit ball, and without loss of generality, let $q = +p$.

We note that the $G(q,q) = G(p,p)$ term in the denominator converges to zero,

$$\begin{aligned} \lim_{s\to\infty} G(p,p) &= \lim_{s\to\infty} \bar{F}D^2\bar{F}\big|_{p-\phi^0/s}(p,p) \\ &= \bar{F}D^2\bar{F}\big|_p(p,p) = 0, \end{aligned}$$

so to deal with this we will multiply both top and bottom by s^2:

$$\begin{aligned}
&\lim_{s\to\infty} \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{sp-\phi^0}(sp,\widehat{p},\widehat{p})}{G(sp,sp)^{1/2}G(p,p)} \\
&\quad= \lim_{s\to\infty} \frac{s^2\ \bar{F}D(\bar{F}D^2\bar{F})\big|_{sp-\phi^0}\left(\phi^0, p-c^p(sp-\phi^0), p-c^p(sp-\phi^0)\right)}{s^2G(sp,sp)^{1/2}G(p,p)} \\
&\quad= \lim_{s\to\infty} \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{sp-\phi^0}\left(\phi^0, s(1-sc^p)p+sc^p\phi^0, s(1-sc^p)p+sc^p\phi^0\right)}{G(sp,sp)^{3/2}}. \quad (9.18)
\end{aligned}$$

Now, the denominator is strictly positive, and by Lemma 9.4 bounded below —

$$\lim_{s\to\infty} G(sp,sp) = \lim_{s\to\infty} \bar{F}D^2\bar{F}\big|_{sp-\phi^0}(sp,sp) \geq A > 0.$$

The limiting value of the coefficient of ϕ^0 in (9.18) is

$$\begin{aligned}
\lim_{s\to\infty} sc^p &= \lim_{s\to\infty} s\frac{D\bar{F}\big|_{p-\phi^0/s}(p)}{\bar{F}(sp-\phi^0)} \\
&= \lim_{s\to\infty} \frac{D\bar{F}\big|_{p-\phi^0/s}(p)}{\bar{F}(p-\phi^0/s)} \\
&= \frac{D\bar{F}\big|_{p}(p)}{\bar{F}(p)} \\
&= 1,
\end{aligned}$$

using (9.6).

The limiting value of the coefficient of p is

$$\begin{aligned}
\lim_{s\to\infty} s(1-sc^p) &= \lim_{s\to\infty} s\left(1 - s\frac{D\bar{F}\big|_{p-\phi^0/s}(p)}{\bar{F}(sp-\phi^0)}\right) \\
&= \lim_{r\to 0} \frac{1}{r}\left(1 - \frac{D\bar{F}\big|_{p-r\phi^0}(p)}{\bar{F}(p-r\phi^0)}\right) \\
&= \lim_{r\to 0} \frac{1}{r}\left(\frac{D\bar{F}\big|_{p}(p)}{\bar{F}(p)} - \frac{D\bar{F}\big|_{p-r\phi^0}(p)}{\bar{F}(p-r\phi^0)}\right)
\end{aligned}$$

where we use that $D\bar{F}\big|_p(p) = \bar{F}(p)$. The above term is a derivative, so we have

$$\begin{aligned}
\lim_{s\to\infty} s(1-sc^p) &= \frac{d}{dr}\left(\frac{D\bar{F}\big|_{p+r\phi^0}(p)}{\bar{F}(p+r\phi^0)}\right)\Bigg|_{r=0} \\
&= \frac{D^2\bar{F}\big|_p(\phi^0,p)}{\bar{F}(p)} - \frac{D\bar{F}\big|_p(p)\ D\bar{F}\big|_p(\phi^0)}{\bar{F}(p)^2} \\
&= 0,
\end{aligned}$$

where the first term of the second last line is zero due to (9.7) while the second term is zero as consequence (9.13) of the symmetry condition.

The equation (9.18) is then

$$\lim_{s\to\infty} \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{sp-\phi^0}\left(\phi^0, s(1-sc^p)p+sc^p\phi^0, s(1-sc^p)p+sc^p\phi^0\right)}{G(sp,sp)^{1/2}G(sp,sp)}$$
$$= \frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_p\left(\phi^0,\phi^0,\phi^0\right)}{\bar{F}D^2\bar{F}\big|_p(\phi^0,\phi^0)^{3/2}}$$
$$= 0,$$

where symmetry has been used again in the form of (9.15).

Now consider the case that q is *not* parallel to p. In this case, the denominator of (9.17) is non-zero:

$$\begin{aligned}\lim_{s\to\infty} G(sp,sp)^{1/2}G(q,q) &= \lim_{s\to\infty} \bar{F}D^2\bar{F}\big|_{p-\phi^0/s}(\phi^0,\phi^0)\ \bar{F}D^2\bar{F}\big|_{p-\phi^0/s}(q,q)\\ &= \bar{F}D^2\bar{F}\big|_p(\phi^0,\phi^0)\ \bar{F}D^2\bar{F}\big|_p(q,q)\\ &> 0,\end{aligned}$$

since $\lim_{s\to\infty}\widehat{q}$ is a non-zero tangent vector at p,

$$\lim_{s\to\infty}\widehat{q} = \lim_{s\to\infty} q - \frac{D\bar{F}\big|_{sp-\phi^0}(q)}{\bar{F}(sp-\phi^0)}(sp-\phi^0) = q - \frac{D\bar{F}\big|_p(q)}{\bar{F}(p)}p.$$

This cannot be zero as q is not parallel to p.

Then

$$\begin{aligned}&\lim_{s\to\infty}\frac{\bar{F}D(\bar{F}D^2\bar{F})\big|_{sp-\phi^0}(sp,\widehat{q},\widehat{q})}{G(sp,sp)^{1/2}G(q,q)}\\ &= \lim_{s\to\infty}\left(G(sp,sp)^{1/2}G(q,q)\right)^{-1}\lim_{s\to\infty}\bar{F}D(\bar{F}D^2\bar{F})\big|_{p-\phi^0/s}(\phi^0,\widehat{q},\widehat{q})\\ &= 0,\end{aligned}$$

where the second limit is zero by (9.14), since $\lim_{s\to\infty}\widehat{q}$ has no component in the direction of ϕ^0.

We have shown that for a fixed p on the unit ball, the quantity (9.16) is bounded above (by C), and that as p is scaled outwards this decreases to zero, so for fixed p the quantity is bounded above.

By compactness of the unit ball, it is bounded for all p. □

Estimates for periodic, anisotropic mean curvature flows

Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a H_2, bounded

$$|u(x,t)| \leq M,$$

periodic

$$u(x+L,t) = u(x,t) \text{ for some lattice } L,$$

solution to the anisotropic curvature flow equation (9.5), where $\bar{F}$ is a positive convex function, homogeneous degree one, with a strictly convex unit ball.

Remark: We have one estimate in the case that $\bar{F}$ satisfies the smallness-of-third-derivatives condition (Theorem 9.8) and another in the case that $\bar{F}$ satisfies the symmetry condition (Theorem 9.9). However, in Theorem 5.1 and Corollary 5.2 we found an estimate for periodic anisotropic flows without the need to impose either of these conditions. In that case, the fact that we were estimating on the difference quotient, rather than the first derivative itself, avoided the need to take derivatives of the flow coefficients. Strictly speaking, Theorems 9.8 and 9.9 are redundant, but they are a good introduction for the interior estimate of Theorem 9.12.

Theorem 9.8. *If the tensor Q given by* (9.10) *satisfies* (9.11) *with*

$$C_1{}^2 < \frac{4}{\sqrt{n}}, \tag{9.19}$$

then

$$F(Du) \leq \max\left\{t^{q/2}\exp\left(\frac{Aq(|u|-2M)^2}{4t}\right), P\right\}$$

for $0 < t \leq T'$, where T' depends on M, A and P (both given by Lemma 9.4), and $1 < q = (1 - C_1^2\sqrt{n}/4)^{-1}$.

Theorem 9.9. *If $\bar{F}$ satisfies the symmetry condition* (9.12)*, then*

$$F(Du) \leq \max\left\{t\exp\left(\frac{A(|u|-2M)^2}{2t}\right), P, S_{(2/n)^{1/2}}\right\}$$

for $0 < t \leq T'$, where T' depends on M, A and P (both given by Lemma 9.4), and S is given by Lemma 9.7.

Proof of Theorem 9.8: As in the previous sections, define the quantity

$$Z := F(Du) - \varphi(u,t)$$

where P is given by Lemma 9.4, and where φ is a positive function chosen so that $\varphi \to \infty$ as $t \to 0$. Suppose that we are at the first point where Z is no longer negative, and let us assume that at this point, $\varphi \geq P$. This point will be a spatial maximum of Z, due to the periodicity of Z.

Then at this point, $F(Du) = \varphi$ and the first derivative condition is

$$0 = D_k Z = D\bar{F}\,|_z(\phi^m)u_{mk} - \varphi_u u_k,$$

where $z = Du - \phi^0$. That is, for all vectors $v \in$ span$\{e_1, \ldots, e_n\}$,

$$D^2u\left(D\bar{F}\big|_z(\phi^m)e_m, v\right) = \varphi_u Du\,(v)\,. \tag{9.20}$$

Using (9.7), we can rewrite the evolution equation for u in terms of the purely tangential directions $\widehat{\phi}^i$,

$$u_t = \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)D^2u(e_i,e_j) = \bar{F}D^2\bar{F}\big|_z(\widehat{\phi}^i,\widehat{\phi}^j)D^2u(e_i,e_j).$$

We make use of this in finding an evolution equation for F

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= D\bar{F}\big|_z(\phi^k)u_{kt} \\
&= D\bar{F}\big|_z(\phi^k)\left[\bar{F}D^2\bar{F}\big|_z(\widehat{\phi}^i,\widehat{\phi}^j)u_{ij}\right]_k \\
&= D\bar{F}\big|_z(\phi^k)\Big[\, D\left(\bar{F}D^2\bar{F}\right)\big|_z(D_k z,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} + \bar{F}D^2\bar{F}\big|_z(D_k\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} \\
&\qquad\qquad + \bar{F}D^2\bar{F}\big|_z(\widehat{\phi}^i,D_k\widehat{\phi}^j)u_{ij} + \bar{F}D^2\bar{F}\big|_z(\widehat{\phi}^i,\widehat{\phi}^j)u_{ijk}\Big] \\
&\qquad\qquad + \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)D_{ij}F \\
&\qquad\qquad - \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)\left[D^2\bar{F}\big|_z(\phi^m,\phi^l)u_{mi}u_{lj} + D\bar{F}\big|_z(\phi^m)u_{mij}\right] \\
&= D\bar{F}\big|_z(\phi^k)\Big[\, D\left(\bar{F}D^2\bar{F}\right)\big|_z(D_k z,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} + \bar{F}D^2\bar{F}\big|_z(D_k\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} \\
&\qquad\qquad + \bar{F}D^2\bar{F}\big|_z(\widehat{\phi}^i,D_k\widehat{\phi}^j)u_{ij}\Big] \\
&\qquad + \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)D_{ij}F - \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)\; D^2\bar{F}\big|_z(\phi^m,\phi^l)u_{mi}u_{lj},
\end{aligned}
$$

where in the third step we have added and subtracted second derivatives of F.

Derivatives of z are $D_k z = u_{mk}\phi^m$. We this to simplify those terms with derivatives of $\widehat{\phi}^i$—

$$
\begin{aligned}
D^2\bar{F}\big|_z(D_k\widehat{\phi}^i,\widehat{\phi}^j) &= D^2\bar{F}\big|_z(D_k(-c^{\phi^i}z),\widehat{\phi}^j) \\
&= D^2\bar{F}\big|_z(-D_k(c^{\phi^i})z - c^{\phi^i}u_{mk}\phi^m,\widehat{\phi}^j),
\end{aligned}
$$

and remembering that $D^2\bar{F}\big|_z(z,\cdot) = 0$, this is

$$
\begin{aligned}
&= -c^{\phi^i}\; D^2\bar{F}\big|_z(u_{mk}\phi^m,\widehat{\phi}^j) \\
&= -\frac{D\bar{F}\big|_z(\phi^i)}{\bar{F}(z)}\; D^2\bar{F}\big|_z(u_{mk}\phi^m,\widehat{\phi}^j).
\end{aligned}
$$

The evolution equation is now

$$
\begin{aligned}
\frac{\partial F}{\partial t} = &\; D\bar{F}\big|_z(\phi^k)\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(\phi^m,\widehat{\phi}^i,\widehat{\phi}^j)u_{mk}u_{ij} \\
&- D\bar{F}\big|_z(\phi^k)\left[D\bar{F}\big|_z(\phi^i)\; D^2\bar{F}\big|_z(\phi^m,\widehat{\phi}^j) + D\bar{F}\big|_z(\phi^j)\; D^2\bar{F}\big|_z(\phi^m,\widehat{\phi}^i)\right]u_{mk}u_{ij} \\
&+ \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)D_{ij}F - \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)\; D^2\bar{F}\big|_z(\phi^m,\phi^l)u_{mi}u_{lj}. \qquad (9.21)
\end{aligned}
$$

When we are at a critical point of Z, we can use the first derivative condition (9.20) to simplify further. The first term of (9.21) becomes

$$
\begin{aligned}
D^2u(\,D\bar{F}\big|_z(\phi^k)e_k,e_m)\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(\phi^m,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} \\
= \varphi_u Du(e_m)\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(\phi^m,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} \\
= \varphi_u\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(Du,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij},
\end{aligned}
$$

while the second becomes

$$
\begin{aligned}
- D\bar F\big|_z(\phi^k)\, D\bar F\big|_z(\phi^i)\; D^2\bar F\big|_z(\phi^m,\widehat{\phi}^j)u_{mk}u_{ij}& \\
&= -\,D^2\bar F\big|_z(\phi^m,\widehat{\phi}^j) D^2u\left(D\bar F\big|_z(\phi^k)e_k, e_m\right) D^2u\left(D\bar F\big|_z(\phi^i)\, e_i, e_j\right)\\
&= -\varphi_u^2\; D^2\bar F\big|_z(Du(e_m)\phi^m, Du(e_j)\widehat{\phi}^j)\\
&= -\varphi_u^2\; D^2\bar F\big|_z(Du, Du),
\end{aligned}
$$

as does the third.

The evolution equation for F, at the local maximum of Z, is now

$$
\begin{aligned}
\frac{\partial F}{\partial t} = \frac{\varphi_u}{\varphi}\; \bar F D\left(\bar F D^2\bar F\right)\big|_z(Du,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} - 2\frac{\varphi_u^2}{\varphi}\;\bar F D^2\bar F\big|_z(Du, Du)& \\
+ \bar F D^2\bar F\big|_z(\phi^i,\phi^j)D_{ij}F - \frac{1}{\varphi}\;\bar F D^2\bar F\big|_z(\phi^i,\phi^j)\;\bar F D^2\bar F\big|_z(\phi^m,\phi^l)u_{mi}u_{lj},&
\end{aligned}
$$

where we have multiplied some terms through by $1 = \bar F/\varphi$ (since we assume that $Z = 0$ here) in order that derivatives of $\bar F$ appear as homogeneous degree zero terms.

Derivatives of φ are given by

$$
\begin{aligned}
D\varphi &= \varphi_u Du\\
D_{ij}\varphi &= \varphi_{uu}u_i u_j + \varphi_u u_{ij}\\
\frac{d\varphi}{dt} &= \varphi_u u_t + \varphi_t
\end{aligned}
$$

for $i, j \neq 0$, so an evolution equation for φ is

$$
\begin{aligned}
\frac{d\varphi}{dt} &= \varphi_u u_t + \varphi_t + \bar F D^2\bar F\big|_z(\phi^i,\phi^j)\left(D_{ij}\varphi - \varphi_{uu}u_i u_j - \varphi_u u_{ij}\right)\\
&= \varphi_t + \bar F D^2\bar F\big|_z(\phi^i,\phi^j)\left(D_{ij}\varphi - \varphi_{uu}u_i u_j\right),
\end{aligned}
$$

and the entire evolution equation for Z, at a local maximum, is

$$
\begin{aligned}
\frac{dZ}{dt} = \bar F D^2\bar F\big|_z(\phi^i,\phi^j)D_{ij}Z + \frac{\varphi_u}{\varphi}\;\bar F D\left(\bar F D^2\bar F\right)\big|_z(Du,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij}& \\
- 2\frac{\varphi_u^2}{\varphi}\;\bar F D^2\bar F\big|_z(Du, Du) - \frac{1}{\varphi}\;\bar F D^2\bar F\big|_z(\phi^i,\phi^j)\;\bar F D^2\bar F\big|_z(\phi^m,\phi^l)u_{mi}u_{lj}& \\
- \varphi_t + \bar F D^2\bar F\big|_z(Du, Du)\varphi_{uu}.&
\end{aligned}
$$

Notice that all the covectors ϕ^i, Du appear in places where they may be replaced by $\widehat{\phi}^i$, $\widehat{Du}$ respectively (using (9.7) and (9.8)). That is, we are working exclusively on the tangent space to the unit ball.

Restricted to the tangent space, $D^2\bar F$ is positive definite so we can define a Riemannian metric on the tangent space $G^{\alpha\beta} := \bar F D^2\bar F\big|_z(\widehat{\phi}^\alpha,\widehat{\phi}^\beta)$, $\alpha,\beta \neq 0$. We can choose the basis $\{\phi^1,\ldots,\phi^n\}$ so that G is the identity at our maximum point,

$G^{\alpha\beta} = \delta^{\alpha\beta}$. The evolution equation for Z is now

$$\frac{dZ}{dt} = G^{ij} D_{ij} Z + \frac{\varphi_u}{\varphi}\ \bar{F} D\left(\bar{F} D^2 \bar{F}\right)\Big|_z (\widehat{Du}, \widehat{\phi}^i, \widehat{\phi}^j) u_{ij} - 2\frac{\varphi_u^2}{\varphi} G(Du, Du) - \frac{1}{\varphi} G^{ij} G^{ml} u_{mi} u_{lj} - \varphi_t + G(Du, Du)\varphi_{uu}. \tag{9.22}$$

Recall the Cauchy-Schwarz inequality for positive matrices: If A is a positive semi-definite $n \times n$ matrix, then for $v, w \in \mathbb{R}^n$,

$$0 \le \left((2\epsilon)^{1/2} v - (2\epsilon)^{-1/2} w \right)^T A \left((2\epsilon)^{1/2} v - (2\epsilon)^{-1/2} w \right)$$

and so

$$v^T A w \le \epsilon v^T A v + \frac{1}{4\epsilon} w^T A w.$$

If A is positive definite, then we can replace w by $A^{-1} w$ to find that

$$v^T w \le \epsilon v^T A v + \frac{1}{4\epsilon} w^T A^{-1} w.$$

As we assume that u is smooth, G is positive definite and we can use the above inequality to estimate the second term of (9.22) :

$$\begin{aligned}
&\frac{\varphi_u}{\varphi}\ \bar{F} D\left(\bar{F} D^2 \bar{F}\right)\Big|_z (\widehat{Du}, \widehat{\phi}^i, \widehat{\phi}^j) u_{ij} \\
&\qquad = \frac{\varphi_u}{\varphi}\left[D\bar{F}\Big|_z (\widehat{Du})\ \bar{F} D^2 \bar{F}\Big|_z (\widehat{\phi}^i, \widehat{\phi}^j) + \bar{F}^2 D^3 \bar{F}\Big|_z (\widehat{Du}, \widehat{\phi}^i, \widehat{\phi}^j) \right] u_{ij} \\
&\qquad = \frac{\varphi_u}{\varphi} u_k Q^{kij} u_{ij} \\
&\qquad \le \epsilon \frac{{\varphi_u}^2}{\varphi} G(Du, Du) + \frac{1}{4\epsilon\varphi} G_{\alpha\beta} Q^{\alpha ij} u_{ij} Q^{\beta kl} u_{kl},
\end{aligned} \tag{9.23}$$

where the first term of the second line is zero because $\widehat{Du}$ is tangent to the unit ball, so $D\bar{F}\big|_z (\widehat{Du}) = 0$. In the last line, we have used the notation for the inverse $G_{\alpha\beta} = (G^{-1})^{\alpha\beta}$. We will choose $\epsilon < 1$ later.

We can use (9.11), the smallness-of-third-derivatives condition, to estimate the second term in this inequality:

$$\begin{aligned}
\frac{1}{4\epsilon\varphi} G_{\alpha\beta} Q^{\alpha ij} u_{ij} Q^{\beta kl} u_{kl} &= \frac{1}{4\epsilon\varphi} Q\left(G_{\alpha\beta} \widehat{\phi}^\alpha, u_{ij} \widehat{\phi}^i, \widehat{\phi}^j \right) Q\left(\widehat{\phi}^\beta, u_{kl} \widehat{\phi}^k, \widehat{\phi}^l \right) \\
&\le \frac{{C_1}^2}{4\epsilon\varphi} \Big(G(G_{\alpha\beta}\widehat{\phi}^\alpha, G_{\gamma\beta}\widehat{\phi}^\gamma) G(u_{ij}\widehat{\phi}^i, u_{mj}\widehat{\phi}^m) G(\widehat{\phi}^j, \widehat{\phi}^j) \\
&\qquad \times G(\widehat{\phi}^\beta, \widehat{\phi}^\beta) G(u_{kl}\widehat{\phi}^k, u_{pl}\widehat{\phi}^p) G(\widehat{\phi}^l, \widehat{\phi}^l) \Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{{C_1}^2}{4\epsilon\varphi}\left(G_{\alpha\beta}G^{\alpha\gamma}G_{\gamma\beta}u_{ij}u_{mj}G^{im}G^{jj}G^{\beta\beta}G^{kp}u_{kl}u_{pl}G^{ll}\right)^{1/2}\\
&= \frac{{C_1}^2}{4\epsilon\varphi}\left(G_{\beta\beta}u_{ij}u_{ij}G^{jj}G^{\beta\beta}u_{kl}u_{kl}G^{ll}\right)^{1/2}\\
&= \frac{{C_1}^2}{4\epsilon\varphi}\sqrt{n}\left(G^{ij}G^{kl}u_{ik}u_{jl}\right).
\end{aligned}
$$

Now we can estimate (9.22) from above —

$$
\begin{aligned}
\frac{dZ}{dt} &\leq G^{ij}D_{ij}Z + \frac{{C_1}^2}{4\epsilon\varphi}\sqrt{n}G^{ij}G^{kl}u_{ik}u_{jl} + \epsilon\frac{{\varphi_u}^2}{\varphi}G(Du,Du)\\
&\qquad - 2\frac{\varphi_u^2}{\varphi}G(Du,Du) - \frac{1}{\varphi}G^{ij}G^{ml}u_{mi}u_{lj} - \varphi_t + \varphi_{uu}G(Du,Du)\\
&= G^{ij}D_{ij}Z + \frac{1}{\varphi}\left(\frac{{C_1}^2}{4\epsilon}\sqrt{n} - 1\right)G^{ij}G^{kl}u_{ik}u_{jl}\\
&\qquad + \frac{{\varphi_u}^2}{\varphi}(\epsilon-2)\,G(Du,Du) - \varphi_t + \varphi_{uu}G(Du,Du).
\end{aligned}
$$

The second term is zero if we choose

$$\epsilon \;=\; {C_1}^2\sqrt{n}/4 \;<\; 1;$$

the inequality here is a consequence of (9.19).

As in the proof of Theorem 9.3, choose $\varphi = \Phi^{-q}$ for some $q > 1$, with Φ given by (9.3). This satisfies the heat equation $\Phi_t = c\Phi''$. We will choose $c = A$, where A is given by Lemma (9.4). Substituting Φ and its derivatives for φ and its derivatives (see page 88) we find that

$$
\begin{aligned}
&\frac{{\varphi_u}^2}{\varphi}(\epsilon-2)\,G(Du,Du) - \varphi_t + \varphi_{uu}G(Du,Du)\\
&\qquad = q^2\Phi^{-q-2}\Phi'^2(\epsilon-2)G(Du,Du)\\
&\qquad\quad + \left[q(q+1)\Phi^{-q-2}\Phi'^2 - q\Phi^{-q-1}\Phi''\right]G(Du,Du) + Aq\Phi^{-q-1}\Phi''\\
&\qquad = q\Phi^{-q-2}\Phi'^2\left[q(\epsilon-1)+1\right]G(Du,Du) + q\Phi^{-q-1}\Phi''\left[A - G(Du,Du)\right].
\end{aligned}
$$

Now, the first term is zero if we choose

$$q = \frac{1}{1-\epsilon} = \frac{1}{1-{C_1}^2\sqrt{n}/4}.$$

As we assumed at the beginning that $\bar{F}(Du) \geq P$, Lemma 9.4 ensures that

$$G(Du,Du) \geq A.$$

Consequently, if we choose $T' > 0$ small, Φ'' is positive and then the final term will be negative.

On the other hand, if we consider the possibility that $\varphi < P$ at this local maximum, we could replace φ by $\sup\{\varphi, P\}$ in the definition of Z. In that case, the first maximum of Z occurs at a point where the barrier is flat, and so the first variation is

$$0 = D_k Z = D\bar{F}\big|_z (\phi^k) u_{mk},$$

and the evolution equation for Z at the local maximum is

$$\begin{aligned}\frac{dZ}{dt} &= \bar{F}D^2\bar{F}\big|_z (\phi^i, \phi^j) D_{ij} Z - D^2\bar{F}\big|_z (\phi^i, \phi^j)\ \bar{F}D^2\bar{F}\big|_z (\phi^m, \phi^l) u_{mi} u_{lj} \\ &\le 0.\end{aligned}$$

Since $Z_t \le 0$ at the first point where $Z = 0$, $Z \le 0$ for all $t < T'$ and the conclusion follows. □

Proof of Theorem 9.9: We begin by assuming that at a local maximum point $F(Du) \ge \max\{P, S_{(2/n)^{1/2}}\}$, and follow the proof of Theorem 9.8 up to equation (9.22), the evolution equation for Z at a local maximum:

$$\begin{aligned}\frac{dZ}{dt} = G^{ij} D_{ij} Z &+ \frac{\varphi_u}{\varphi}\ \bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z (\widehat{Du}, \widehat{\phi}^i, \widehat{\phi}^j) u_{ij} - 2\frac{\varphi_u^2}{\varphi} G(Du, Du) \\ &- \frac{1}{\varphi} G^{ij} G^{ml} u_{mi} u_{lj} - \varphi_t + G(Du, Du)\varphi_{uu}.\end{aligned}$$

This time we do not choose coordinates to make G the identity.

We use Cauchy-Schwarz (with $\epsilon = 1/2$) to estimate the second term —

$$\begin{aligned}&\frac{\varphi_u}{\varphi}\ \bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z (Du, \widehat{\phi}^i, \widehat{\phi}^j) u_{ij} \\ &\quad \le \frac{{\varphi_u}^2}{2\varphi} G(Du, Du) + \frac{1}{2\varphi} G(Du, Du)^{-1} \left[\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z (Du, \widehat{\phi}^i, \widehat{\phi}^j) u_{ij}\right]^2.\end{aligned}$$

Choose the basis $\{\phi^1, \dots, \phi^n\}$ at this point so that $D^2 u$ is diagonal. Also, let I be the $n \times n$ diagonal matrix with $+1$ or -1 as its diagonal entries, chosen so that $I^{ii} u_{ii} = |u_{ii}|$. As I^2 is the identity matrix, $u_{ij} = I^{ik} |u_{kj}|$.

In these coordinates,

$$\begin{aligned}&\frac{1}{2\varphi} G(Du, Du)^{-1} \left[\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z (Du, \widehat{\phi}^i, \widehat{\phi}^j) u_{ij}\right]^2 \\ &\qquad = \frac{1}{2\varphi}\frac{1}{G(Du, Du)}\left[\sum_{i=1}^n \bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z (Du, \widehat{\phi}^i, \widehat{\phi}^i) I^{ii} |u_{ii}|\right]^2 \\ &\qquad \le \frac{1}{2\varphi}\frac{1}{G(Du, Du)}\left[\sum_{i=1}^n \left|\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z (Du, \widehat{\phi}^i, \widehat{\phi}^i)\right| |u_{ii}|\right]^2 \\ &\qquad \le \frac{1}{2\varphi}\frac{1}{G(Du, Du)}\left[\sum_{i=1}^n \left(\frac{2}{n}\right)^{1/2} G(Du, Du)^{1/2} G^{ii} |u_{ii}|\right]^2 \\ &\qquad = \frac{1}{n\varphi}\left[\sum_{i=1}^n G^{ii} |u_{ii}|\right]^2,\end{aligned}$$

where the term $(2/n)^{1/2}$ comes from the estimation of $\left| \bar{F}D\left(\bar{F}D^2\bar{F}\right)\Big|_z (Du, \widehat{\phi}^i, \widehat{\phi}^i)\right|$ in Lemma 9.7, under the assumption that $\bar{F}(Du) \geq S_{(2/n)^{1/2}}$.

This term is dominated by the fourth term of the evolution equation, since (in the same coordinates)

$$\begin{aligned}
\frac{1}{\varphi}G^{ij}G^{ml}u_{mi}u_{lj} &= \frac{1}{\varphi}G^{ij}G^{ml}I^{m\alpha}|u_{\alpha i}|I^{l\beta}|u_{\beta j}| \\
&= \frac{1}{\varphi}G^{ij}\left[I^{\alpha m}G^{ml}I^{l\beta}\right]|u_{\alpha i}||u_{\beta j}| \\
&= \frac{1}{\varphi}G^{ij}G^{\alpha\beta}|u_{\alpha i}||u_{\beta j}| \\
&\geq \frac{1}{\varphi}\frac{1}{n}\left[\sum_{i,j}G^{ij}|u_{ji}|\right]^2
\end{aligned}$$

where we use the trace inequality $(\operatorname{trace} A)^2 \leq n \operatorname{trace}(A^2)$

$$= \frac{1}{\varphi}\frac{1}{n}\left[\sum_{i}G^{ii}|u_{ii}|\right]^2.$$

What is left of the evolution equation is

$$\frac{dZ}{dt} \leq G^{ij}D_{ij}Z - \frac{3}{2}\frac{{\varphi_u}^2}{\varphi}G(Du, Du) - \varphi_t + G(Du, Du)\varphi_{uu}.$$

This is negative at a local maximum if we make the same choice of barrier as before — $\varphi = \Phi^{-q}$ for $q = 2$ with Φ given by (9.3).

If our assumption that $F(Du) \geq \max\{P, S_{(2/n)^{1/2}}\}$ does not hold, then we can replace φ by $\max\{P, S_{(2/n)^{1/2}}, \varphi\}$. At the local maximum, $Z_t \leq 0$ and so the conclusion follows. □

Remark: In the last theorem, we have chosen $q = 2$ somewhat arbitrarily; in fact q needs only to be strictly greater than 1, since we can set $q = (1 - n\epsilon^2/4)^{-1}$, for ϵ given by Lemma 9.7. However, a smaller ϵ may entail a larger S_ϵ, so the optimal choice would depend on the exact form of $\bar{F}$.

Interior estimate for anisotropic mean curvature flow

We begin by showing that when we have the symmetry condition, an estimate analogous to (9.4) in the isotropic case is possible.

Lemma 9.10. *Suppose that $\bar{F}$ satisfies the symmetry condition* (9.12)*. Then there exists a constant C_2 depending only on $\bar{F}$ such that*

$$\bar{F}D^2\bar{F}\Big|_{p-\phi^0}(p, q) \leq C_2\frac{\bar{F}(q)}{\bar{F}(p - \phi^0)}$$

for all $p = p_i\phi^i$ and $q = q_i\phi^i$.

Proof: This estimate is unchanged under $q \mapsto sq$, so we need only to show that this holds for q on the unit ball. Let $q = q_i\phi^i$ be a fixed point on the unit ball.

By compactness, the estimate holds for all p on the unit ball.

Let p be a fixed point on the unit ball and consider

$$\begin{aligned}\lim_{s\to\infty} \bar{F}(sp-\phi^0)\, \bar{F}D^2\bar{F}\big|_{sp-\phi^0}(sp,q) \\ &= \lim_{s\to 0} \bar{F}(p-s\phi^0)\frac{1}{s}\,\bar{F}D^2\bar{F}\big|_{p-s\phi^0}(\phi^0,q) \\ &= \lim_{s\to 0} \bar{F}(p-s\phi^0)\frac{1}{s}\left[\bar{F}D^2\bar{F}\big|_{p-s\phi^0}(\phi^0,q) - \bar{F}D^2\bar{F}\big|_{p}(\phi^0,q)\right]\end{aligned}$$

where we have added zero in the form of $\bar{F}D^2\bar{F}\big|_{p}(\phi^0,q)$

$$= \bar{F}(p)\left[-D\left(\bar{F}D^2\bar{F}\right)\big|_{p}(\phi^0,\phi^0,q)\right].$$

So, either the supremum of $\bar{F}(sp-\phi^0)\;\bar{F}D^2\bar{F}\big|_{sp-\phi^0}(sp,q)$ over $s\in[0,\infty)$ is the limit above, or else it is attained at some finite s. In either case,

$$C_p(q) = \sup_{s\geq 0} \bar{F}(sp-\phi^0)\;\bar{F}D^2\bar{F}\big|_{sp-\phi^0}(sp,q)$$

is finite, and we can set

$$C_2 = \sup_{\{q:\bar{F}(q)=1\}}\;\sup_{\{p:\bar{F}(p)=1\}} C_p(q)$$

to complete the lemma. □

The next lemma shows that the trace of $\bar{F}D^2\bar{F}$ is bounded below.

Lemma 9.11. *Let $\{\phi^0,\phi^1,\ldots,\phi^n\}$ be a basis for V^*, where $n>1$. Then there is a constant $k>0$ so that for all $p=\sum_{i=1}^n p_i\phi^i$,*

$$\sum_{i=1}^n G\big|_{p-\phi^0}(\phi^i,\phi^i) \geq k,$$

where $G=\bar{F}D^2\bar{F}$.

Proof: By compactness and strict convexity of the unit ball,

$$\sup_{\{p:\bar{F}(p)\leq 1\}} G\big|_{p-\phi^0}(\phi^i,\phi^i) = \sup_{\{p:\bar{F}(p)\leq 1\}} G\big|_{p-\phi^0}(\widehat{\phi^i},\widehat{\phi^i}) > 0,$$

since $\widehat{\phi^i}$ is a non-zero tangent covector.

Now consider

$$\lim_{t\to\infty}\sum_{i=1}^n G\big|_{tp-\phi^0}(\phi^i,\phi^i) = \sum_{i=1}^n G\big|_{p}(\widehat{\phi^i},\widehat{\phi^i}),$$

where, in the limit, $\widehat{\phi^i} = \phi^i - \frac{D\bar{F}\big|p(\phi^i)}{\bar{F}(p)}p$.

At most one of the ϕ^i may be parallel to p — suppose it is ϕ^1, in which case $\lim_{t\to\infty} G|_{tp-\phi^0}(\phi^1,\phi^1) = 0$. For the remaining $(n-1)$ basis covectors, $\widehat{\phi}^i$ are non-zero tangent covectors (to the unit ball at p) and so $G(\widehat{\phi}^i,\widehat{\phi}^i)$ is again bounded below for $i \neq 1$.

It follows that

$$\lim_{t\to\infty}\sum_{i=1}^{n} G|_{tp-\phi^0}(\phi^i,\phi^i) \geq \sum_{i=2,n} G|_p(\widehat{\phi}^i,\widehat{\phi}^i) > 0.$$

If we take the infimum of all such lower bounds, over all p in the unit ball, then the conclusion follows. □

Let $u : B_R(0)\times[0,T]\to\mathbb{R}$ be a H_2, bounded

$$|u(x,t)| \leq M$$

solution on the ball of radius R to the anisotropic curvature flow equation (9.5), where $\bar{F}$ is a positive, convex homogeneous degree one function, with a strictly convex unit ball.

Theorem 9.12 (Interior estimate for anisotropic mean curvature flow). *If $\bar{F}$ satisfies both the smallness of third derivatives condition* (9.11) *with some constant*

$$C_1^2 < 2/\sqrt{n},$$

and the symmetry condition (9.12)*, then*

$$F(Du) \leq t^{q/2}\exp\left(\frac{Aq(|u|-2M)^2}{4t}\right)\left(R^2-2kt-|x|^2\right)^{-r}$$

for $0 < t \leq T'$.

Here, A is given by Lemma 9.4 and depends on $\bar{F}$; k is given by Lemma 9.11 and depends on $\bar{F}$; and $T' > 0$, $q > 1$, and $r > 1$ depend on M, A, P (which is also given by Lemma 9.4) and k.

Proof: We introduce the localising term η into our definition of Z, now restricted to the shrinking ball:

$$Z := F(Du) - \frac{\varphi}{\eta}$$

for $(x,t)\in B_{\sqrt{R^2-2kt}}\times[0,T]$, where k is the constant given by Lemma 9.11, $\varphi = \varphi(u,t)$ is a smooth strictly positive function chosen so that $Z < 0$ at the initial time, and η is a smooth positive function chosen so that $\eta\to 0$ on the boundary of the shrinking ball.

Assume that at the first interior point where $Z = 0$, $F(Du) \geq P$.

Then $F(Du) = \varphi/\eta$ and as this is a spatial maximum (since the choice of η ensures that there are no boundary maxima) we have a first derivative condition

$$0 = D_k Z = D\bar{F}\,|_z(\phi^m)u_{mk} - D_k(\varphi/\eta)\,. \tag{9.24}$$

An evolution equation for φ/η is:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\varphi}{\eta}\right) &= \frac{1}{\eta}\left(\varphi_u u_t + \varphi_t\right) - \frac{\varphi}{\eta^2}\frac{d\eta}{dt} + \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)D_{ij}\left(\frac{\varphi}{\eta}\right) \\ &\quad - \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)\Bigg[\frac{1}{\eta}\left(\varphi_{uu}u_iu_j + \varphi_u u_{ij}\right) - \frac{\varphi_u}{\eta^2}\left(u_jD^i\eta + u_iD^j\eta\right) \\ &\qquad + 2\frac{\varphi}{\eta^3}D^i\eta D^j\eta - \frac{\varphi}{\eta^2}D_{ij}\eta\Bigg] \\ &= G^{ij}D_{ij}\left(\frac{\varphi}{\eta}\right) + \frac{1}{\eta}\left[\varphi_t - G(Du,Du)\varphi_{uu}\right] - \frac{\varphi}{\eta^2}\left(\frac{d}{dt} - G^{ij}D_{ij}\right)\eta \\ &\quad + 2\frac{\varphi_u}{\eta^2}G(Du,D\eta) - 2\frac{\varphi}{\eta^3}G(D\eta,D\eta),\end{aligned}$$

where in the last line we have used the notation $G^{ij} = \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)$.

We can incorporate the first derivative condition (9.24) into (9.21), the evolution equation for F:

$$\begin{aligned}\frac{dF}{dt} &= \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)D_{ij}F + D\bar{F}\big|_z(\phi^k)\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(\phi^m,\widehat{\phi}^i,\widehat{\phi}^j)u_{mk}u_{ij} \\ &\quad - D\bar{F}\big|_z(\phi^k)\left[D\bar{F}\big|_z(\phi^i)\; D^2\bar{F}\big|_z(\phi^m,\widehat{\phi}^j) + D\bar{F}\big|_z(\phi^j)\; D^2\bar{F}\big|_z(\phi^m,\widehat{\phi}^i)\right]u_{mk}u_{ij} \\ &\qquad - \bar{F}D^2\bar{F}\big|_z(\phi^i,\phi^j)\; D^2\bar{F}\big|_z(\phi^m,\phi^l)u_{mi}u_{lj} \\ &= G^{ij}D_{ij}F + D^m(\varphi/\eta)\; D\left(\bar{F}D^2\bar{F}\right)\big|_z(\phi^m,\widehat{\phi}^i,\widehat{\phi}^j)u_{ij} \\ &\qquad - 2\frac{\eta}{\varphi}G\left(D(\varphi/\eta),D(\varphi/\eta)\right) - \frac{\eta}{\varphi}G^{ij}G^{ml}u_{mi}u_{lj} \\ &= G^{ij}D_{ij}F + \frac{\eta}{\varphi}\;\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(D(\varphi/\eta),\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij} \\ &\quad - 2\frac{\eta}{\varphi}\left[\frac{\varphi_u^2}{\eta^2}G(Du,Du) - 2\frac{\varphi\varphi_u}{\eta^3}G(Du,D\eta) + \frac{\varphi^2}{\eta^4}G\left(D\eta,D\eta\right)\right] \\ &\qquad - \frac{\eta}{\varphi}G^{ij}G^{ml}u_{mi}u_{lj}.\end{aligned}$$

Putting the last two steps together gives an evolution equation for Z at a local maximum:

$$\begin{aligned}\frac{dZ}{dt} &= \frac{dF}{dt} - \frac{d}{dt}\left(\frac{\varphi}{\eta}\right) \\ &= G^{ij}D_{ij}Z + \frac{\eta}{\varphi}\;\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(D(\varphi/\eta),\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij} - \frac{\eta}{\varphi}G^{ij}G^{ml}u_{mi}u_{lj} \\ &\qquad - \frac{1}{\eta}\left[\varphi_t - G(Du,Du)\varphi_{uu} + 2\frac{\varphi_u^2}{\varphi}G(Du,Du)\right] \\ &\qquad + \frac{\varphi}{\eta^2}\left(\frac{d}{dt} - G^{ij}D_{ij}\right)\eta + 2\frac{\varphi_u}{\eta^2}G(Du,D\eta).\end{aligned}$$

The second term here may be split up into a part with $D\varphi$ and a part with $D\eta$:

$$\begin{aligned}
&\frac{\eta}{\varphi}\,\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(\frac{\varphi_u}{\eta}Du-\frac{\varphi}{\eta^2}D\eta,\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij}\\
&\quad=\frac{\varphi_u}{\varphi}\,\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(Du,\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij}-\frac{1}{\eta}\,\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(D\eta,\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij}.
\end{aligned}$$

These may be individually estimated using the Cauchy-Schwarz inequality and the smallness-of-third-derivatives condition, as described in the proof of Theorem 9.8 on page 101 —

$$\begin{aligned}
&\frac{\varphi_u}{\varphi}\,\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(Du,\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij}\\
&\qquad\qquad\leq\mu_1\frac{\varphi_u^2}{\varphi\eta}G(Du,Du)+\frac{1}{4\mu_1}\frac{\eta}{\varphi}C_1^2\sqrt{n}\left(G^{ij}G^{ml}u_{mi}u_{lj}\right),\\
&-\frac{1}{\eta}\,\bar{F}D\left(\bar{F}D^2\bar{F}\right)\big|_z\left(D\eta,\widehat{\phi}^i,\widehat{\phi}^j\right)u_{ij}\\
&\qquad\qquad\leq\mu_2\frac{\varphi}{\eta^3}G(D\eta,D\eta)+\frac{1}{4\mu_2}\frac{\eta}{\varphi}C_1^2\sqrt{n}\left(G^{ij}G^{kl}u_{ik}u_{jl}\right),
\end{aligned}$$

for some $0<\mu_1,\mu_2<1$.

We choose the localising term to be $\eta:=\tilde{\eta}^r$ for some $r>1$, $\tilde{\eta}=R^2-2kt-|x|^2$, and k given in Lemma 9.11. Then

$$\begin{aligned}
D_i\eta&=r\tilde{\eta}^{r-1}D_i\tilde{\eta}\\
D_{ij}\eta&=r\tilde{\eta}^{r-1}D_{ij}\tilde{\eta}+r(r-1)\tilde{\eta}^{r-2}D_i\tilde{\eta}D_j\tilde{\eta},
\end{aligned}$$

and the second-last term of the evolution equation is

$$\begin{aligned}
\frac{\varphi}{\eta^2}\left(\frac{d}{dt}-G^{ij}D_{ij}\right)\eta&=\frac{\varphi}{\eta^2}r\tilde{\eta}^{r-1}\left[2k-2\operatorname{trace}G-(r-1)\tilde{\eta}^{-1}G(D\tilde{\eta},D\tilde{\eta})\right]\\
&\leq\frac{\varphi}{\eta^2}r\tilde{\eta}^{r-2}(1-r)G(D\tilde{\eta},D\tilde{\eta}).
\end{aligned}$$

As $\bar{F}$ satisfies the symmetry condition (9.12), we may use Lemma 9.10 to estimate the final term of the evolution equation:

$$\begin{aligned}
2\frac{\varphi_u}{\eta^2}G(Du,D\eta)&=2\frac{\varphi_u}{\eta^2}\,\bar{F}D^2\bar{F}\big|_{Du-\phi^0}(Du,D\eta)\\
&\leq2\frac{\varphi_u}{\eta^2}\frac{C_2\bar{F}(D\eta)}{\bar{F}(Du-\phi^0)}\\
&=2C_2\bar{F}(D\eta)\frac{\varphi_u}{\varphi\eta}.
\end{aligned}$$

The evolution equation can now be estimated from above —

$$\frac{dZ}{dt}\leq G^{ij}D_{ij}Z+\frac{\eta}{\varphi}\left(\frac{1}{4\mu_1}C_1^2\sqrt{n}+\frac{1}{4\mu_2}C_1^2\sqrt{n}-1\right)G^{ij}G^{ml}u_{mi}u_{lj}$$

$$- \frac{1}{\eta}\left[\varphi_t - G(Du,Du)\varphi_{uu} + (2-\mu_1)\frac{\varphi_u^2}{\varphi}G(Du,Du) - 2C_2\bar{F}(D\eta)\frac{\varphi_u}{\varphi}\right] + \frac{\varphi}{\eta^2} r\tilde{\eta}^{r-2}\left(1 - r + r\mu_2\right)G(D\tilde{\eta}, D\tilde{\eta}). \tag{9.25}$$

Since $C_1^2\sqrt{n}/4 < 1/2$, we can choose $\mu_1 < 1$ and $\mu_2 < 1$ such that

$$\frac{C_1^2\sqrt{n}}{4}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) \leq 1.$$

With such choices, the second term of the evolution inequality (9.25) will be negative. We can also set $r = (1-\mu_2)^{-1} > 1$, so the coefficient of $\tilde{\eta}^{-1}G(D\tilde{\eta}, D\tilde{\eta})$ is zero.

As in the previous cases we can set $\varphi = \Phi^{-q}$ where Φ is given by (9.3) with $c = A$, where A is given by Lemma 9.4.

The bracketted part of the second line of the evolution equation is then

$$\begin{aligned}
&\varphi_t - G(Du,Du)\varphi_{uu} + (2-\mu_1)\frac{\varphi_u^2}{\varphi}G(Du,Du) - 2C_2\bar{F}(D\eta)\frac{\varphi_u}{\varphi} \\
&\quad = -q\Phi^{-q-1}\left(\Phi_t - G(Du,Du)\Phi''\right) \\
&\qquad + G(Du,Du)q{\Phi'}^2\Phi^{-q-2}\left[-1 + (1-\mu_1)q - 2\frac{C_2\bar{F}(D\eta)}{G(Du,Du)}\frac{\Phi^{q+1}}{|\Phi'|}\right]
\end{aligned} \tag{9.26}$$

If we choose T' small enough that $\Phi'' \geq 0$, then the term $\Phi_t - G(Du,Du)\Phi'' = (A - G(Du,Du))\,\Phi''$ is negative.

As $r > 1$, $\bar{F}(D\eta) = r\tilde{\eta}^{r-1}\bar{F}(D\tilde{\eta}) \leq C_3 r R^{2r-1}$, where $C_3 > 0$ depends only on $\bar{F}$.

With this choice of Φ,

$$\begin{aligned}
2\frac{C_2\bar{F}(D\eta)}{G(Du,Du)}\frac{\Phi^{q+1}}{|\Phi'|} &\leq 2\frac{C_2C_3rR^{2r-1}}{A}\frac{\Phi^{q+1}}{|\Phi'|} \\
&\leq 4\frac{C_2C_3rR^{2r-1}\Phi^q t}{A^2M} \\
&\leq 4\frac{C_2C_3rR^{2r-1}t}{A^2M}
\end{aligned}$$

if we choose T' small enough that $\Phi \leq 1$.

In order to ensure that the last part of (9.26) is positive, we choose q so that

$$q \geq \frac{1}{1-\mu_1}\left(1 + 4\frac{C_2C_3rR^{2r-1}T'}{A^2M}\right).$$

So, at such maxima, $Z_t \leq 0$.

At local maxima where $F(Du) < P$, then in the definition of Z we replace φ/η by $\max\{\varphi/\eta, P\}$, in which case the barrier is flat at the local maxima, and we again find that $Z_t \leq 0$.

In either case, the maximum principle ensures that Z is never greater than zero and the conclusion follows. ☐

Appendix A

Function spaces and regularity estimates for parabolic equations

Here, we define relevant function spaces, and survey some regularity results used in the existence theorems of Chapters 4, 6 and 7. This treatment follows the books of Krylov [23] and Lieberman [25].

A.1 Function spaces

On the space of continuous functions $u : \Omega \to \mathbb{R}$, we have the supremum norm

$$|u|_{0;\Omega} := \sup_{x \in \Omega} |u(x)|.$$

Define the Hölder semi-norm with exponent $\alpha \in (0,1]$ by

$$[u]_{\alpha;\Omega} := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}.$$

For an integer $k \geq 0$, we define the Hölder $(k+\alpha)$-norm by

$$|u|_{k+\alpha;\Omega} := \sum_{|\beta| \leq k} |D^\beta u|_{0;\Omega} + \sum_{|\beta| = k} [D^\beta u]_{\alpha;\Omega},$$

where β is a multi-index — an n-tuple of non-negative integers with $|\beta| = \sum \beta_i$, and where $D^\beta u := \dfrac{\partial^{|\beta|} u}{\partial(x^1)^{\beta_1} \ldots \partial(x^n)^{\beta_n}}$.

The Banach space associated with this norm is $C^{k+\alpha}(\Omega) = \{u : |u|_{k+\alpha} < \infty\}$.

The parabolic Hölder spaces H_α

With parabolic equations, it is useful to weight the space and time variables differently- that is, two space derivatives to one time derivative. Following Lieberman, we will denote parabolic Hölder spaces by H rather than C. For points $z = (x,t)$ in a domain

$Q \subset \mathbb{R}^n \times \mathbb{R}$, define the parabolic Hölder semi-norm by

$$[u]_{\alpha,\alpha/2;Q} := \sup_{\substack{z_1,z_2\in Q\\ z_1\neq z_2}} \frac{|u(z_1)-u(z_2)|}{(|x_1-x_2|+|t_1-t_2|^{\frac{1}{2}})^{\alpha}},$$

where $\alpha \in (0,1]$. The parabolic norm is

$$|u|_{\alpha,\alpha/2;Q} := |u|_{0;Q} + [u]_{\alpha,\frac{\alpha}{2};Q}.$$

Higher spatial derivatives and derivatives in time are bounded by $k+\alpha$ norms, where $k \geq 0$ is an integer:

$$|u|_{k+\alpha,\alpha/2;Q} = \sum_{|\beta|+2j\leq k} \sup |D_x^{\beta} D_t^{j} u| + \sum_{|\beta|+2j=k} [D_x^{\beta} D_t^{j} u]_{\alpha,\alpha/2}.$$

The Banach space associated with the $|\cdot|_{k+\alpha,\alpha/2}$ norm is

$$H_{k+\alpha}(Q) = \{u : |u|_{k+\alpha,\alpha/2} < \infty\}.$$

When the region Q is a cylinder, in the sense that $Q = \Omega \times [0,T]$, the parabolic boundary is given by

$$\mathcal{P}\,(\Omega \times [0,T]) = \Omega \times \{0\} \cup \partial\Omega \times [0,T].$$

On the boundary, we can define parabolic norms $H_{k+\alpha}(\mathcal{P})$ exactly as above.

A.2 Regularity estimates

In the following, $\boldsymbol{P}$ is a quasilinear parabolic operator

$$\boldsymbol{P}u = -u_t + a^{ij}(Du,x,t)D_{ij}u + b(Du,u,x,t)$$

with positive constants λ_K and Λ_K such that

$$a^{ij}(x,p)\xi_i\xi_j \geq \lambda_K|\xi|^2 \text{ for } \xi \in \mathbb{R}^n$$
$$|a^{ij}(x,p)| \leq \Lambda_K,$$

whenever $|p| \leq K$. We work on a domain $\Omega \times [0,T]$ for some smoothly bounded $\Omega \subseteq \mathbb{R}^n$.

Here Q_r is the intersection of the region and a cylinder:

$$Q_R(x_0,t_0) := \{(x,t) \in \Omega \times [0,T] : |x-x_0| < R, t_0 - R^2 < t < t_0\}.$$

We begin with an oscillation estimate for the gradient of a solution for a Neumann problem near a flat boundary:

Theorem A.1. *Suppose that $a^{ij} = a^{ij}(Du,x)$, $b = b(x)$, and that inside Q_R, the boundary of Ω is $x^n = 0$. Let $u \in C^{2,1} \cap H_1(Q_R)$ be a solution of $\boldsymbol{P}u = 0$ when $x^n > 0$ and $D_n u = 0$, when $x^n = 0$.*

Suppose there are positive constants b_0 and λ_0 such that

$$|a^{ij}{}_p| \leq \lambda_0, \quad |b(x)| \leq b_0$$

for all $(x,p) \in Q_R \times \mathbb{R}^n$ with $|p| \leq K$.

If $|Du| \leq K$, then there are positive constants θ and σ determined only by K, n, λ, Λ and λ_0 such that

$$\operatorname*{osc}_{\Omega\cap Q_R} a^{ij}(\cdot,p) \leq \sigma \text{ for all } |p| \leq K$$

implies

$$\operatorname*{osc}_{\Omega\cap Q_r} Du \leq C(K,n,\lambda,\Lambda,\lambda_0)\left(\frac{r}{R}\right)^{\theta}[\operatorname*{osc}_{\Omega\cap Q_R} Du + b_0 R].$$

Similarly, we can find an estimate near the boundary for problems with Dirichlet boundary conditions:

Theorem A.2. *Suppose that a^{ij} and b are uniformly continuous, that a^{ij} is differentiable with respect to (p,q,x), and where, if $|q|+|p| \leq K$, we can find a positive constant μ_K such that*

$$K[|a^{ij}{}_x| + |a^{ij}{}_q||p|] + |b| \leq \mu_K. \tag{A.1}$$

If $u \in C^{2,1} \cap H_1(\Omega \times [0,T])$ satisfies $\boldsymbol{P}u = 0$ on $\Omega \times [0,T]$ and $u = 0$ on $\partial\Omega \times (0,T)$, then there are positive constants C and α depending on n, λ_K, Λ_K and $\mu_K R/K$ such that for any $(x_0,t_0) \in \partial\Omega \times (0,T)$

$$\operatorname*{osc}_{Q_r(x_0,t_0)} Du \leq C\left[\left(\frac{r}{R}\right)^{\alpha} K + +\mu_K r\right]$$

for $0 < r < R < t_0^{1/2}$.

On the interior of the domain, one can also find a Hölder bound for the gradient:

Theorem A.3. *Suppose that a^{ij} and a are continuous; a^{ij} is differentiable with respect to (p,q,x,t); and where if $|q|+|p| \leq K$, we can find a positive constant μ_K satisfying* (A.1)*.*

If $u \in C^{2,1}$ satisfies $\boldsymbol{P}u = 0$ and $|Du|+|u| \leq K$ in $\Omega \times [0,T]$, then there is a positive constant α determined by $n, \lambda_K, \Lambda_K, \sup|a^{ij}{}_p|K,$ and $\mu_K R/K$ such that for interior sets $\Omega' \times [t_1,t_2] \subset\subset \Omega \times [0,T]$ we have

$$[Du]_{\alpha;\Omega'\times[t_1,t_2]} \leq C\left(n,K,\lambda_K,\Lambda_K,\mu_K,\operatorname{diam}(\Omega\times[0,T])\right)d^{-\alpha},$$

where $d = \operatorname{dist}(\Omega' \times [t_1,t_2], \mathcal{P}\Omega \times [0,T])$.

If Q_r is a cylinder in the interior of the domain, we also have

$$\operatorname*{osc}_{Q_r(x_0,t_0)} Du \leq C(n,K,\lambda_K,\Lambda_K)\left(\frac{r}{R}\right)^{\alpha}\left(\operatorname*{osc}_{Q_R(x_0,t_0)} Du + \mu_K R\right)$$

as long as $0 < r \leq R \leq d(x_0,t_0)$, where $d(x_0,t_0)$ is the distance from (x_0,t_0) to the parabolic boundary $\mathcal{P}(\Omega \times [0,t_0])$.

The following interior estimate is a $H_{2+\alpha}$ bound for u when the coefficients of $\boldsymbol{P}$ are smooth:

Theorem A.4. *Suppose that* $a^{ij} = a^{ij}(x,t)$, *and* $b(Du,u,x,t) = b^i(x,t)D_iu + c(x,t)u$, *and that there is a constant* K *such that*

$$|a,b,c|_{\alpha,\alpha/2} \leq K,$$

for $\alpha \in (0,1)$.

Then for any $R > 0$ *there is a constant* C *dependent on* R, λ, α, K *and* n *such that if* $u \in H_{2+\alpha}(Q_{3R})$, *then*

$$|u|_{2+\alpha,1+\alpha/2;Q_R} \leq C\left(|\boldsymbol{P}u|_{\alpha,\alpha/2;Q_{2R}} + |u|_{0;Q_{2R}}\right).$$

In a similar vein, there are higher regularity estimates on the interior of a domain:

Theorem A.5. *If we have* $|D^\alpha a|_{\alpha,\alpha/2;Q_{2R}}$, $|D^\alpha b|_{\alpha,\alpha/2;Q_{2R}}$, $|D^\alpha c|_{\alpha,\alpha/2;Q_{2R}} \leq K$ *for any* $|\alpha| \leq k$, *and if* $u \in H_{2+\alpha}(Q_{2R})$ *and* $D^\alpha(\boldsymbol{P}u) \in H_\alpha(Q_{2R})$, *then* $D^\alpha u \in H_\alpha(Q_R)$ *for* $|\alpha| \leq k$, *and there is a constant* $C = C(R,K,\alpha,k,d,\lambda)$ *such that*

$$\sum_{|\beta|\leq k} |D^\beta u|_{2+\alpha,1+\alpha/2;Q_R} \leq \sum_{|\beta|\leq k} |D^\beta|_{2+\alpha,1+\alpha/2;Q_{2R}} + C|u|_{0;Q_{2R}}.$$

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