# A NONLINEAR HEAT EQUATION WITH SINGULAR INITIAL DATA

By

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## 1. Introduction

We consider the question of local existence, for the model problem

$$(1) \qquad \begin{cases} u_t - \Delta u = |u|^{p-1}u & \text{in } (0,T) \times \Omega, \\ u = 0 \quad \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x) \quad \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain and $p > 1$. It is well-known that if $u_0 \in L^\infty(\Omega)$, there is a unique solution defined on a maximal interval $[0, T_{\max})$. This solution satisfies:

$$(2) \qquad \bullet \quad u \text{ is a classical solution of (1) on } (0,T_{\max}) \times \overline{\Omega}$$
$$(u \text{ is } C^1 \text{ in } t \in (0,T_{\max}) \text{ and } C^2 \text{ in } x \in \overline{\Omega}),$$

$$(3) \qquad \bullet \quad u \in L^\infty((0,T) \times \Omega) \quad \text{for all } T < T_{\max},$$

$$(4) \qquad \bullet \quad \|u(t) - T(t)u_0\|_{L^\infty} \to 0 \quad \text{as } t \downarrow 0,$$

where $(T(t))_{t\geq 0}$ is the linear heat flow (see e.g. [6]).

We are concerned with the question of what happens if $u_0 \notin L^\infty(\Omega)$. More precisely, we assume that $u_0 \in L^q(\Omega)$ for some $1 \leq q < \infty$. This type of problem was first considered by F. B. Weissler [17, 18] who has obtained interesting existence and uniqueness results. Further results have been obtained by Y. Giga [12] and by W.-M. Ni and P. Sacks [16]. Our purpose is to shed some new light on the problem; in particular, we prove uniqueness in a larger, more natural, class (see the precise statements in Theorems 1 and 4 below).

The value

$$q = \frac{N(p-1)}{2}$$

plays a critical role, and one has to distinguish two cases:

**Case 1:** $q \geq N(p-1)/2$.

**Case 2:** $q < N(p-1)/2$.

Roughly speaking, in case 1 we obtain the existence and uniqueness of a local solution for any $u_0 \in L^q(\Omega)$. In case 2, it seems that there exists no local solution in any reasonable sense for some initial conditions $u_0 \in L^q(\Omega)$ (see Weissler [18, 19] and Section 7.6 below).

Our main existence and uniqueness result is the following.

**Theorem 1** *Assume $q > N(p-1)/2$ (resp. $q = N(p-1)/2$) and $q \geq 1$ (resp. $q > 1$), $N \geq 1$. Given any $u_0 \in L^q(\Omega)$, there exist a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], L^q(\Omega))$ with $u(0) = u_0$, which is a classical solution of* (1) *on $(0,T) \times \overline{\Omega}$ (in the sense of* (2)*).*

*Moreover, we have:*

(i) *Smoothing effect and continuous dependence, namely*

$$\|u(t) - v(t)\|_{L^q} + t^{N/2q}\|u(t) - v(t)\|_{L^\infty} \leq C\|u_0 - v_0\|_{L^q}, \tag{5}$$

*for all $t \in (0,T]$ where $T = \min\{T(u_0), T(v_0)\}$ and $C$ can be estimated in terms of $\|u_0\|_{L^q}$ and $\|v_0\|_{L^q}$.*

(ii) $\lim_{t \downarrow 0} t^{N/2q}\|u(t)\|_{L^\infty} = 0$.

(iii) *If $u_0 \geq 0$, then $u(t) \geq 0$ for all $t \in [0, T(u_0)]$.*

*Furthermore, for any bounded set (resp. compact set) $\mathcal{K}$ in $L^q(\Omega)$, there is a (uniform) time $T = T(\mathcal{K})$ such that for any $u_0 \in \mathcal{K}$ the solution of* (1) *exists on $[0,T]$.*

The uniqueness part in Theorem 1 seems totally new. Previously, many people have established uniqueness results for nonlinear evolution equations with singular initial conditions, in particular the Navier-Stokes and the Euler equations (see e.g. Kato and Fujita [15], Kato [14], Ben-Artzi [4], Weissler [18]). In all these works it is assumed that $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ and **also** that $\lim_{t\downarrow 0} t^\alpha \|u(t)\|_{L^\infty} = 0$ for some appropriate $\alpha > 0$. Our main point is that such an assumption is redundant. A similar observation was first made in [5].

**Remark 2** Since $u$ is a classical solution on $(0,T) \times \overline{\Omega}$, the usual blow up alternative holds: either $T_{\max} = +\infty$ or else $T_{\max} < \infty$ and $\lim_{t\uparrow T_{\max}} \|u(t)\|_{L^\infty} = +\infty$.

**Remark 3** The "doubly critical" case, $q = N(p-1)/2$ and $q = 1$, in Theorem 1 is delicate and widely open (see Section 7.5). For example, when $N = 1$, the very simple equation

$$u_t - u_{xx} = u^3,$$

with an initial condition $u_0 \in L^1(\Omega)$, enters in this category.

It seems that for some $u_0 \in L^1(\Omega)$ there is not even a local solution. See Open Problem 2. We are, at least, able to find some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that equation (1) has no **nonnegative** solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. See Section 7.5 below.

When $q \geq p$, it makes sense to talk about weak solutions $u \in C([0,T], L^q(\Omega))$ in the integral sense, i.e.

$$u(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s)\,ds, \tag{6}$$

for all $t \in [0,T]$. Uniqueness holds in that class:

**Theorem 4** *Assume $q > N(p-1)/2$ (resp. $q = N(p-1)/2$) and $q \geq p$ (resp. $q > p$), $N \geq 1$. Then uniqueness for* (6) *holds in the class $C([0,T], L^q(\Omega))$.*

The conclusion of Theorem 4 (at least when $q > N(p-1)/2$ and $q > p$) is essentially contained in a work of Weissler [17].

**Remark 5** In the "doubly critical" case $q = N(p-1)/2$ and $q = p$, i.e. $q = p = N/(N-2)$ with $N \geq 3$, the conclusion of Theorem 4 fails, i.e. uniqueness fails in the class $C([0,T], L^q(\Omega))$. See Ni and Sacks [16] and Section 7.4 below.

**Remark 6** The solution $u$ of (1) given in Theorem 1 also satisfies (6); here, there is no restriction about $q$ except for the assumptions of Theorem 1. This is not completely obvious since the integral on the right hand side of (6) need not be well-defined. To establish the convergence of this integral, we rely on the smoothing effect (5). Clearly we have

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\,d\sigma, \tag{7}$$

for all $0 < s < t < T$. We let $s \downarrow 0$ in (7); to justify this passage to the limit it suffices to check that

$$\int_0^t \|T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\|_{L^q}\,d\sigma < \infty.$$

The only difficulty is when $\sigma$ is near 0. But

$$\|T(t-\sigma)|u(\sigma)|^{p-1}u(\sigma)\|_{L^q} \leq (t-\sigma)^{-\frac{N}{2}\left(\frac{q-1}{q}\right)}\|u(\sigma)\|^p_{L^p}.$$

(See e.g. Lemma 7 below.) We may always assume that $p > q$ (the case $q \geq p$ has been handled above); and so,

$$\|u(\sigma)\|^p_{L^p} \leq \|u(\sigma)\|^q_{L^q}\|u(\sigma)\|^{p-q}_{L^\infty} \leq C\sigma^{-\frac{N(p-q)}{2q}}.$$

The result follows, since $\frac{N(p-q)}{2q} = 1 - \frac{1}{q}\left(q - \frac{N(p-1)}{2}\right) - \frac{N}{2q}(q-1) < 1$.

In several places, it is convenient to view the nonlinear equation (1) as a linear problem

$$u_t - \Delta u = au,$$

and we have collected in the Appendix some useful facts about this linear heat equation with a potential.

The plan of the paper is the following. In Section 2, we prove Theorem 4. In Sections 3 and 4, we prove the existence part of Theorem 1 in the super-critical and critical cases. In Section 5 we prove the uniqueness part of Theorem 1, and in Section 6 we prove the stability and continuous dependence (properties (i)–(iii)). Finally, Section 7 is devoted to additional results and open problems.

## 2. Proof of Theorem 4

We will frequently use the standard smoothing effect of $(T(t))_{t\geq 0}$:

**Lemma 7** *Let* $1 \leq \beta \leq \gamma \leq \infty$. *Then*

$$\|T(t)\varphi\|_{L^\gamma(\Omega)} \leq \frac{1}{t^{\frac{N}{2}\left(\frac{1}{\beta}-\frac{1}{\gamma}\right)}}\|\varphi\|_{L^\beta(\Omega)}, \tag{8}$$

*for all* $t > 0$ *and all* $\varphi \in L^\beta(\Omega)$.

We consider separately two cases:

**Case A:** $q > N(p-1)/2$ and $q \geq p$,

**Case B:** $q = N(p-1)/2$ and $q > p$.

**Case A** Let $u$ and $v$ be two solutions, $u, v \in C([0,T], L^q(\Omega))$. We have

$$u(t) - v(t) = \int_0^t T(t-s)\left(|u(s)|^{p-1}u(s) - |v(s)|^{p-1}v(s)\right) ds. \tag{9}$$

Thus, by the smoothing effect of $T(t) : L^{q/p}(\Omega) \to L^q(\Omega)$,

$$\begin{aligned}\|u(t) - v(t)\|_{L^q} &\leq C\int_0^t (t-s)^{-\alpha}\|\,(|u|^{p-1} + |v|^{p-1})|u-v|\,\|_{L^{q/p}}\,ds\\ &\leq C\int_0^t (t-s)^{-\alpha}(\|u\|_{L^q}^{p-1} + \|v\|_{L^q}^{p-1})\|u-v\|_{L^q}\,ds,\end{aligned}$$

where $\alpha = N(p-1)/2q < 1$ since we are in case A. Let

$$M = \sup_{0\leq t\leq T}\|u(t)\|_{L^q} + \|v(t)\|_{L^q} \quad \text{and} \quad \psi(t) = \sup_{0\leq s\leq t}\|u(t) - v(t)\|_{L^q},$$

for $t \in [0,T]$. We deduce that

$$\psi(t) \le CM^{p-1}\frac{T^{1-\alpha}}{1-\alpha}\psi(t).$$

Hence $\psi(t) = 0$ for $t$ sufficiently small. Repeating the same argument, we see that $\psi(t) = 0$ for $t \in [0,T]$.

**Case B** Note that $q = N(p-1)/2 > p$, thus $N \ge 3$. Let $u, v$ be two solutions and let $w = u - v$. We set

$$a(t,x) = \begin{cases} \dfrac{|u|^{p-1}u - |v|^{p-1}v}{u-v} & \text{if } u \ne v, \\ p|u|^{p-1} & \text{if } u = v, \end{cases} \tag{10}$$

so that

$$w(t) = \int_0^t T(t-s)a(s)w(s)\,ds.$$

We claim that

$$a \in C([0,T], L^{N/2}(\Omega)). \tag{11}$$

We may then apply Theorem A2 (in the Appendix at the end of this paper) to conclude that $w \equiv 0$. Note that (since we are in case B) $q > N/(N-2)$. □

**Proof of (11)** We have $|a| \le p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0,T), L^{N/2}(\Omega))$. We now establish (11) by contradiction. Otherwise, there exist $\varepsilon > 0$, $t \in [0,T]$ and a sequence $(t_n)_{n\ge 0} \in [0,T]$ such that $t_n \to t$ and $\|a(t_n,\cdot) - a(t,\cdot)\|_{L^{N/2}} \ge \varepsilon$. On the other hand, by possibly extracting a subsequence, we may assume that $u(t_n) \to u(t)$ and $v(t_n) \to v(t)$ in $L^q(\Omega)$ and almost everywhere, and that there exists $\varphi \in L^q(\Omega)$ such that $|u(t_n)| + |v(t_n)| \le \varphi$ almost everywhere. It follows easily that $a(t_n) \to a(t)$ almost everywhere and that $|a(t_n)| \le C|\varphi|^{p-1} \in L^{N/2}(\Omega)$. By dominated convergence, we deduce $a(t_n) \to a(t)$ in $L^{N/2}(\Omega)$, which is absurd. □

## 3. Proof of the existence part in Theorem 1 when $q > N(p-1)/2$ and $q \ge 1$

We first establish the existence of a solution

$$u \in L^\infty((0,T), L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^{pq}(\Omega)).$$

We use the contraction mapping principle in a somewhat unusual space (this idea is due to F. B. Weissler [17]). Fix $M \ge \|u_0\|_{L^q}$ and let

$$E = L^\infty((0,T), L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^{pq}(\Omega)),$$

and

$$K = K(T) = \{u \in E;\ \|u(t)\|_{L^q} \le M+1 \text{ and } t^\alpha \|u(t)\|_{L^{pq}} \le M+1 \text{ for } t \in (0,T)\},$$

with $\alpha = N(p-1)/2pq < 1/p < 1$. We equip $K$ with the distance

$$d(u,v) = \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^{pq}},$$

so that $(K,d)$ is a nonempty complete metric space. Given $u \in K$, we set

$$\Phi(u)(t) = T(t)u_0 + \int_0^t T(t-s)|u(s)|^{p-1}u(s)\,ds.$$

For $u \in K$, we have

$$\begin{aligned}
\|\Phi(u)(t)\|_{L^q} &\le \|u_0\|_{L^q} + \int_0^t \|u(s)\|_{L^{pq}}^p\,ds \\
&\le \|u_0\|_{L^q} + \Big(\sup_{0<t<T} t^\alpha \|u(t)\|_{L^{pq}}\Big)^p \int_0^t s^{-p\alpha}\,ds \\
&\le \|u_0\|_{L^q} + \frac{T^{1-p\alpha}}{1-p\alpha}(M+1)^p.
\end{aligned}$$

Next,

$$\begin{aligned}
t^\alpha \|\Phi(u)(t)\|_{L^{pq}} &\le \|u_0\|_{L^q} + t^\alpha \int_0^t (t-s)^{-\alpha} \|u(s)\|_{L^{pq}}^p\,ds \\
&\le \|u_0\|_{L^q} + t^\alpha (M+1)^p \int_0^t (t-s)^{-\alpha} s^{-p\alpha}\,ds \\
&\le \|u_0\|_{L^q} + T^{1-p\alpha}(M+1)^p \int_0^1 (1-\sigma)^{-\alpha}\sigma^{-p\alpha}\,d\sigma.
\end{aligned}$$

Similarly, one shows that for $u, v \in K$,

$$t^\alpha \|\Phi(u)(t) - \Phi(v)(t)\|_{L^{pq}} \le CT^{1-p\alpha}(M+1)^{p-1} d(u,v).$$

It follows from the above estimates that if $T$ is small enough (depending on $M$), then $\Phi : K \to K$ is a strict contraction. Thus $\Phi$ has a unique fixed point in $K$.

To complete the argument, it suffices to show that $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ (once $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$, it must be a classical solution on $(0,T) \times \overline{\Omega}$). Since $u \in K$ and $p\alpha < 1$, we have $|u|^{p-1}u \in L^1((0,T), L^q(\Omega))$. This implies that $u \in C([0,T], L^q(\Omega))$. (Recall that, in a general setting, if $f \in L^1((0,T), X)$ and $u(t) = \int_0^t T(t-s)f(s)\,ds$, then $u \in C([0,T], X)$.)

Next, we prove that $u \in L^{\infty}_{\rm loc}((0,T),L^{\infty}(\Omega))$. Indeed, we have $u \in L^{\infty}_{\rm loc}((0,T),L^{pq}(\Omega))$. Therefore, we may apply Theorem A1 in the Appendix (with $r = pq$ and $\sigma = pq/(p-1)$) on every interval $(\varepsilon, T-\varepsilon)$, with $a = |u|^{p-1}$.

Note that the choice of $T$ depends only on $M$. This establishes the last assertion in Theorem 1. □

## 4. Proof of the existence part in Theorem 1 when $q = N(p-1)/2$ and $q > 1$

We will use the following lemma.

**Lemma 8** *Given a compact set $\mathcal{K} \subset L^q(\Omega)$ and $q < r \leq \infty$, there exists a function $\gamma : (0,1] \to (0,\infty)$ with*

$$\lim_{t \downarrow 0} \gamma(t) = 0,$$

*such that*

$$t^{\alpha}\|T(t)u_0\|_{L^r} \leq \gamma(t),$$

*for all $t \in (0,1)$ and all $u_0 \in \mathcal{K}$, where* $\alpha = \dfrac{N}{2}\left(\dfrac{1}{q} - \dfrac{1}{r}\right)$.

**Proof** If $\mathcal{K}$ is reduced to a single point $u_0$, the result is clear. Indeed, for any $v_0 \in L^{\infty}(\Omega)$

$$\begin{aligned} t^{\alpha}\|T(t)u_0\|_{L^r} &\leq t^{\alpha}\|T(t)(u_0 - v_0)\|_{L^r} + t^{\alpha}\|T(t)v_0\|_{L^r} \\ &\leq \|u_0 - v_0\|_{L^q} + Ct^{\alpha}\|v_0\|_{L^{\infty}}; \end{aligned}$$

and so

$$\limsup_{t \downarrow 0} t^{\alpha}\|T(t)u_0\|_{L^r} \leq \|u_0 - v_0\|_{L^q}.$$

The assertion follows since $v_0$ is arbitrary.

In the general case, given any $\rho > 0$, there is a finite covering of $\mathcal{K}$ by balls $B(u_i, \rho)$ in $L^q(\Omega)$. Any $u_0 \in \mathcal{K}$ belongs to some $B(u_i, \rho)$, and we then write

$$\begin{aligned} t^{\alpha}\|T(t)u_0\|_{L^r} &\leq t^{\alpha}\|T(t)(u_0 - u_i)\|_{L^r} + t^{\alpha}\|T(t)u_i\|_{L^r} \\ &\leq \|u_0 - u_i\|_{L^q} + t^{\alpha}\|T(t)u_i\|_{L^r} \\ &\leq \rho + t^{\alpha}\|T(t)u_i\|_{L^r}. \end{aligned}$$

The conclusion of the lemma then follows from the first assertion. □

We now return to the proof of the theorem. The strategy is the same as in Section 3 with some minor technical differences. Fix any $r \in (q,pq)$, $r \geq p$, and set

$$\widetilde{E} = L^\infty((0,T),L^q(\Omega)) \cap \{u \in L^\infty_{\mathrm{loc}}((0,T),L^r(\Omega));\ t^\alpha u \in L^\infty((0,T),L^r(\Omega))\},$$

and

$$E = L^\infty((0,T),L^q(\Omega)) \cap \{u \in L^\infty_{\mathrm{loc}}((0,T),L^r(\Omega));\ t^\alpha u \in C_0([0,T],L^r(\Omega))\},$$

with $\alpha = \dfrac{N}{2}\left(\dfrac{1}{q} - \dfrac{1}{r}\right) < \dfrac{1}{p} < 1$ (since $r < pq$). Here $C_0$ means that we consider functions which vanish at $t = 0$. Fix $M \geq \|u_0\|_{L^q}$. Given $\delta > 0$ to be chosen later, let

$$\widetilde{K} = \widetilde{K}(T) = \{u \in \widetilde{E};\ \|u(t)\|_{L^q} \leq M+1 \text{ and } t^\alpha\|u(t)\|_{L^r} \leq \delta \text{ for } t \in (0,T)\},$$

and

$$K = K(T) = \widetilde{K} \cap E.$$

We equip $\widetilde{K}$ with the distance

$$d(u,v) = \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^r},$$

so that $(\widetilde{K},d)$ and $(K,d)$ are nonempty complete metric spaces. Consider the same mapping $\Phi$ as in Section 3. Let $a = \dfrac{N}{2}\left(\dfrac{p}{r} - \dfrac{1}{q}\right)$. For $u \in \widetilde{K}$, we have by using the smoothing effect $L^{r/p} \to L^q$ (note that $r < pq$, so that $r/p < q$)

$$\begin{aligned} \|\Phi(u)(t)\|_{L^q} &\leq \|u_0\|_{L^q} + \int_0^t (t-s)^{-a}\|u(s)\|^p_{L^r}\,ds \\ &\leq \|u_0\|_{L^q} + \left(\sup_{0<t<T} t^\alpha\|u(t)\|_{L^r}\right)^p \int_0^t (t-s)^{-a}s^{-p\alpha}\,ds \\ &\leq \|u_0\|_{L^q} + C_1\delta^p, \end{aligned} \tag{12}$$

since $a + p\alpha = 1$. Here the constant $C_1$ (and the constants $C_2, C_3$ below) depends only on $p,q,r,N$. Therefore,

$$\|\Phi(u)(t)\|_{L^q} \leq M+1,$$

provided

$$C_1\delta^p \leq 1. \tag{13}$$

Next, using the smoothing effect $L^{r/p} \to L^r$, we have

$$
\begin{aligned}
t^{\alpha}\|\Phi(u)(t)\|_{L^r} &\le \sup_{0<t<T} t^{\alpha}\|T(t)u_0\|_{L^r} + t^{\alpha}\int_0^t (t-s)^{-N(p-1)/2r}\|u(s)\|_{L^r}^p\,ds \\
&\le \sup_{0<t<T} t^{\alpha}\|T(t)u_0\|_{L^r} \\
&\quad + \left(\sup_{0<t<T} t^{\alpha}\|u(t)\|_{L^r}\right)^p t^{\alpha}\int_0^t (t-s)^{-N(p-1)/2r}s^{-p\alpha}\,ds \\
&\le \sup_{0<t<T} t^{\alpha}\|T(t)u_0\|_{L^r} + C_2\delta^p,
\end{aligned}
\tag{14}
$$

since $p\alpha + N(p-1)/2r = \alpha + 1$. Therefore,

$$
\sup_{0<t<T} t^{\alpha}\|\Phi(u)(t)\|_{L^r} \le \sup_{0<t<T} t^{\alpha}\|T(t)u_0\|_{L^r} + \delta/2, \tag{15}
$$

provided

$$
C_2\delta^{p-1} \le \tfrac{1}{2}. \tag{16}
$$

Similarly, one shows that for $u, v \in \widetilde{K}$,

$$
\sup_{0<t<T} t^{\alpha}\|\Phi(u)(t) - \Phi(v)(t)\|_{L^r} \le C_3\delta^{p-1}d(u,v) \le \tfrac{1}{2}d(u,v), \tag{17}
$$

provided

$$
C_3\delta^{p-1} \le \tfrac{1}{2}, \tag{18}
$$

for some constant $C_3$. It follows from the above estimates that $\Phi : \widetilde{K} \to \widetilde{E}$.

We fix any $\delta > 0$ small enough so that (13), (16) and (18) are satisfied. The choice of $\delta$ depends only on $N, p, q, r$.

Next, we fix $T > 0$ such that

$$
\sup_{0<t<T} t^{\alpha}\|T(t)u_0\|_{L^r} \le \delta/2. \tag{19}
$$

In view of Lemma 8, the choice of $T$ depends only on the compact set $\mathcal{K} \subset L^q(\Omega)$. This establishes the last assertion in Theorem 1.

By (17), (15) and (19), $\Phi : \widetilde{K} \to \widetilde{K}$ is a strict contraction, and thus has a unique fixed point in $\widetilde{K}$.

Next, we claim that this fixed point belongs to $K$. For this purpose, it suffices to verify that $\Phi : K \to K$. We have to check that $\Phi(u) \in C((0,T], L^r(\Omega))$ and that $\lim_{t\downarrow 0} t^{\alpha}\Phi(u)(t) = 0$ in $L^r(\Omega)$. Since by Lemma 8, $T(t)u_0$ satisfies the above requirements, we may always assume that $u_0 = 0$. It is clear that $\Phi(u) \in K$ when $u \in C([0,T], L^\infty(\Omega))$. Since $K \cap C([0,T], L^\infty(\Omega))$ is dense in $K$ equipped with the metric $d$, the result follows from (17).

We now show that $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$. Indeed, we have $u \in L^\infty_{\text{loc}}((0,T), L^r(\Omega))$. Therefore, we can apply Theorem A1 (with $\sigma = r/(p-1)$) on every interval $(\varepsilon, T-\varepsilon)$, with $a = |u|^{p-1}$. Indeed, $r \geq p > p-1$, so that $\sigma > 1$; $r > q = N(p-1)/2$, so that $\sigma > N/2$; and $r \geq p$, so that $r \geq \sigma'$.

Finally, we show that $u \in C([0,T], L^q(\Omega))$. Indeed, we have $u \in K$, so that in particular $u \in C((0,T], L^r(\Omega)) \subset C((0,T], L^q(\Omega))$. Therefore, it remains to show that $u(t) - T(t)u_0 \underset{t\downarrow 0}{\longrightarrow} 0$ in $L^q(\Omega)$. As in (12) we have

$$\|u(t) - T(t)u_0\|_{L^q} \leq C_1 \sup_{0<s<t} (s^\alpha \|u(s)\|_{L^r})^p \underset{t\downarrow 0}{\longrightarrow} 0,$$

since $u \in E$. □

## 5. Proof of the uniqueness in Theorem 1

For every $u_0 \in L^q(\Omega)$, we denote by $U(t)u_0$ **the** solution constructed via the above contraction argument (in Section 3 or 4) on some interval $[0, T(u_0)]$. We shall need the following lemma.

**Lemma 9** *Let $u_0 \in L^\infty(\Omega)$ and consider the classical solution $\tilde{u}$ of* (1) *defined on the maximal interval $[0, T_{\max}(u_0))$ and given by* (2)–(4). *Then $T(u_0) < T_{\max}(u_0)$ and $\tilde{u}(t) = U(t)u_0$ for all $t \in [0, T(u_0)]$.*

**Proof** It is clear that $\tilde{u} \in K(\tau)$ for some $0 < \tau \leq T(u_0)$ sufficiently small. By uniqueness in $K(\tau)$ we have

$$\tilde{u}(t) = U(t)u_0, \quad \text{for } 0 \leq t \leq \tau.$$

After time $\tau$, both $\tilde{u}(t)$ and $U(t)u_0$ are classical solutions. Hence the result. □

We now return to the proof of uniqueness. Here we use the same idea as in [5]. We give the proof only in the critical case $q = N(p-1)/2$ and $q > 1$; the other case is simpler. Let $v \in C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ be a solution of (1) with $v(0) = u_0$. Recall that $v$ is a classical solution of (1) on $(0,T) \times \overline{\Omega}$. We are going to prove that $v(t) = U(t)u_0$ on some interval $[0, T')$. Then, $v(t) = U(t)u_0$ as long as both exist, by standard uniqueness in $L^\infty(\Omega)$.

Set

$$\mathcal{K} = v([0,T]),$$

and

$$M = \sup_{0\leq t\leq T} \|v(t)\|_{L^q}.$$

Since $\mathcal{K}$ is a compact set in $L^q(\Omega)$, there is a uniform $T_1 > 0$ such that $U(t)v_0$ is well defined for all $v_0 \in \mathcal{K}$ and all $t \in [0, T_1]$. Moreover, since $U(t)v(s) \in K(T_1)$ (considered as a function of $t$), we have

$$\begin{aligned} &\|U(t)v(s)\|_{L^q} \leq M + 1, \\ &t^\alpha \|U(t)v(s)\|_{L^r} \leq \delta, \end{aligned} \tag{20}$$

for all $s \in (0, T)$ and all $t \in (0, T_1)$.

Fix any $0 < s < T$. It follows from Lemma 9 that

$$v(t+s) = U(t)v(s) \quad \text{for } 0 \leq t \leq \min\{T - s, T_1\}. \tag{21}$$

Combining (20) and (21) we obtain

$$\begin{aligned} &\|v(t+s)\|_{L^q} \leq M + 1, \\ &t^\alpha \|v(t+s)\|_{L^r} \leq \delta, \end{aligned}$$

for $t + s < T$ and $t < T_1$. Passing to the limit as $s \downarrow 0$, we deduce that

$$\begin{aligned} &\|v(t)\|_{L^q} \leq M + 1, \\ &t^\alpha \|v(t)\|_{L^r} \leq \delta, \end{aligned}$$

for $0 < t < \min\{T, T_1\}$. Therefore, $v(t) \in \widetilde{K}(T')$ where $T' = \min\{T, T_1\}$. We may now argue as in Remark 6 to assert that

$$v(t) = T(t)u_0 + \int_0^t T(t-s)|v(s)|^{p-1}v(s)\,ds, \tag{22}$$

i.e. $v = \Phi(v)$. By (17) we deduce $v(t) = U(t)u_0$ on $[0, T']$. □

## 6. Smoothing effect and stability: proof of (i), (ii) and (iii) in Theorem 1

To prove (5) we consider three cases. The methods are essentially the same in all three cases with some minor technical changes.

**Case a:** $q > N(p-1)/2$, $q \geq p - 1$ and $q \geq 1$;

**Case b:** $q > N(p-1)/2$ and $1 \leq q < p - 1$;

**Case c:** $q = N(p-1)/2$ and $q > 1$;

**Case a:** $q > N(p-1)/2$ **and** $q \geq p-1$. We apply Theorem A1 with $a$ given by (10). We have $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((0,T), L^\sigma(\Omega))$ with $\sigma = q/(p-1) > N/2$, $\sigma \geq 1$. By (A3) we have

$$\|u(t) - v(t)\|_{L^\infty} \leq Ce^{Ct\|a\|^\alpha_{L^\infty((0,T),L^\sigma)}}(t^{-N/2q} + 1)\|u_0 - v_0\|_{L^q},$$

with $\alpha = N(p-1)/2pq$. By construction, $u, v \in K$ where $M$ is chosen such that $M \geq \|u_0\|_{L^q}$ and $M \geq \|u_0\|_{L^q}$; and thus, the $L^\infty$ estimate of (5) follows.

On the other hand, we have

$$\|u(t) - v(t)\|_{L^q} \leq \|u_0 - v_0\|_{L^q} + C\int_0^t (\|u\|^{p-1}_{L^{pq}} + \|v\|^{p-1}_{L^{pq}})\|u - v\|_{L^{pq}}.$$

Since $u, v \in K$, we have

$$\|u(s)\|^{p-1}_{L^{pq}} + \|v(s)\|^{p-1}_{L^{pq}} \leq \frac{C}{s^{\alpha(p-1)}}(M+1)^{p-1}.$$

Therefore,

$$\sup_{0<t<T} \|u(t) - v(t)\|_{L^q} \leq \|u_0 - v_0\|_{L^q} + C(M+1)^{p-1} \sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^{pq}}. \tag{23}$$

Here, we use the fact that $\alpha p < 1$. Furthermore, by using the $L^q(\Omega) \to L^{pq}(\Omega)$ smoothing effect, we have

$$\|u(t) - v(t)\|_{L^{pq}} \leq t^{-\alpha}\|u_0 - v_0\|_{L^q} + CA\int_0^t (t-s)^{-\alpha} s^{-\alpha(p-1)}\|u - v\|_{L^{pq}},$$

where $A = \sup_{0<s<T} s^{\alpha(p-1)}(\|u(s)\|^{p-1}_{L^{pq}} + \|v(s)\|^{p-1}_{L^{pq}})$. From the singular Gronwall lemma below, we deduce

$$\sup_{0<t<T} t^\alpha \|u(t) - v(t)\|_{L^{pq}} \leq C\|u_0 - v_0\|_{L^q}. \tag{24}$$

Combining (23) and (24) we obtain the $L^q$ estimate of (5).

**A singular Gronwall lemma** *Let $T > 0$, $A \geq 0$, $0 \leq \alpha, \beta \leq 1$ and let $f$ be a nonnegative function with $f \in L^p(0,T)$ for some $p > 1$ such that $p' \max\{\alpha, \beta\} < 1$. Consider a nonnegative function $\varphi \in L^\infty(0,T)$ such that*

$$\varphi(t) \leq A\, t^{-\alpha} + \int_0^t (t-s)^{-\beta} f(s)\varphi(s)\, ds, \quad \textit{for almost all } t \in [0,T].$$

*Then there exists $C$, depending only on $T$, $\alpha$, $\beta$, $p$ and $\|f\|_{L^p}$, such that*

$$\varphi(t) \leq A\, C\, t^{-\alpha},$$

*for almost all $t \in [0, T]$.*

For the proof, see e.g. [6].

**Case b.** Here, we cannot apply Theorem A1 since $\sigma = q/(p-1) < 1$. However, the second part of the proof is unchanged and we deduce as above that

$$\|u(t) - v(t)\|_{L^q} + t^{\alpha}\|u(t) - v(t)\|_{L^{pq}} \leq C\|u_0 - v_0\|_{L^q},$$

for all $t \in [0, T]$. This establishes the $L^q$ estimate of (5).

We now turn to the proof of the $L^\infty$ estimate of (5). First note that

$$\|u(t/2) - v(t/2)\|_{L^{pq}} \leq Ct^{-\alpha}\|u_0 - v_0\|_{L^q}.$$

We now apply Theorem A1 on the interval $(t/2, t)$ with $r = pq$ and $\sigma = pq/(p-1) > 1$ and with $a$ given by (10). Since $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, we have that $a \in L^\infty((t/2, t), L^\sigma(\Omega))$; and $\|a\|_{L^\infty((t/2,t),L^\sigma)} \leq C(M+1)^{p-1}t^{-\alpha(p-1)}$. It follows that

$$\begin{aligned}\|u(t) - v(t)\|_{L^\infty} \leq & C\exp\left(Ct(M+1)^{\frac{2\sigma(p-1)}{2\sigma-N}}(t^{-\alpha(p-1)})^{\frac{2\sigma}{2\sigma-N}}\right)\\ & \times (1 + t^{-\frac{N}{2pq}})\|u(t/2) - v(t/2)\|_{L^{pq}};\end{aligned}$$

and so

$$\|u(t) - v(t)\|_{L^\infty} \leq C\exp\left(C(M+1)^{\frac{2\sigma(p-1)}{2\sigma-N}}t^{\gamma}\right)(1 + t^{-\frac{N}{2pq}})\|u(t/2) - v(t/2)\|_{L^{pq}},$$

with $\gamma = 1 - \dfrac{2\sigma\alpha(p-1)}{2\sigma - N}$. Since $\gamma > 0$, we obtain

$$\|u(t) - v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2pq}}\|u(t/2) - v(t/2)\|_{L^{pq}}.$$

Therefore,

$$\|u(t) - v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2pq}}t^{-\alpha}\|u_0 - v_0\|_{L^q} = Ct^{-\frac{N}{2q}}\|u_0 - v_0\|_{L^q},$$

which is the $L^\infty$ estimate of (5).

**Case c: $q = N(p-1)/2$ and $q > 1$.** Since $u - v = T(t)(u_0 - v_0) + \Phi(u) - \Phi(v)$, it follows from (17) that

$$\sup_{0<t<T} t^{\alpha}\|u(t) - v(t)\|_{L^r} \leq \sup_{0<t<T} t^{\alpha}\|T(t)(u_0 - v_0)\|_{L^r} + \tfrac{1}{2}\sup\nolimits_{0<t<T} t^{\alpha}\|u(t) - v(t)\|_{L^r},$$

with $\alpha = N(r-q)/2qr$; and so

$$\sup_{0<t<T} t^{\alpha}\|u(t) - v(t)\|_{L^r} \leq 2\sup_{0<t<T} t^{\alpha}\|T(t)(u_0 - v_0)\|_{L^r} \leq 2\|u_0 - v_0\|_{L^q}. \tag{25}$$

Furthermore (as in (12)) we have

$$\begin{aligned}\|u(t) - v(t)\|_{L^q} \leq &\|u_0 - v_0\|_{L^q} \\ &+ C\int_0^t (t-s)^{-a}(\|u(s)\|_{L^r}^{p-1} + \|v(s)\|_{L^r}^{p-1})\|u(s) - v(s)\|_{L^r}\, ds \\ \leq &\|u_0 - v_0\|_{L^q} + C\delta^{p-1}\sup_{0<s<t} t^{\alpha}\|u(s) - v(s)\|_{L^r}\int_0^t (t-s)^{-a}s^{-\alpha p}\, ds \\ \leq &C\|u_0 - v_0\|_{L^q},\end{aligned}$$

by (25). This establishes the $L^q$ estimate of (5).

To prove the $L^\infty$ estimate, we apply Theorem A1 on the interval $(t/2, t)$ with $\sigma = r/(p-1) > N/2$, $\sigma > 1$, and with $a$ given by (10). We have $|a| \leq p(|u|^{p-1} + |v|^{p-1})$, so that $a \in L^\infty((t/2,t), L^\sigma(\Omega))$; and since $u, v \in K$, $\|a\|_{L^\infty((t/2,t),L^\sigma)} \leq Ct^{-\alpha(p-1)}$. It follows that

$$\|u(t) - v(t)\|_{L^\infty} \leq C\exp\left(Ct(t^{-\alpha(p-1)})^{\frac{2\sigma}{2\sigma-N}}\right)(1 + t^{-\frac{N}{2r}})\|u(t/2) - v(t/2)\|_{L^r}.$$

But $1 - \dfrac{2\sigma\alpha(p-1)}{2\sigma - N} = 0$; and so

$$\|u(t) - v(t)\|_{L^\infty} \leq C(1 + t^{-\frac{N}{2r}})\|u(t/2) - v(t/2)\|_{L^r}. \tag{26}$$

Combining (25) and (26), we obtain

$$\|u(t) - v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2r}}t^{-\alpha}\|u_0 - v_0\|_{L^q} = Ct^{-\frac{N}{2q}}\|u_0 - v_0\|_{L^q},$$

which is the desired estimate. In fact, in this case the constant $C$ in (5) is independent of $\|u_0\|_{L^q}$ and $\|u_0\|_{L^q}$.

Finally, (ii) and (iii) are clearly true when $u_0 \in L^\infty(\Omega)$, and the general case follows by continuous dependence (5).

## 7. Additional results and open problems

**7.1.** The above results hold for more general nonlinearities with similar proofs. More precisely, one can replace $|u|^{p-1}u$ by $g(u)$ where $g : \mathbf{R} \to \mathbf{R}$ verifies $|g(x) - g(y)| \leq C(1 + |x|^{p-1} + |y|^{p-1})|x - y|$.

**7.2.** Even when $q < p$, uniqueness holds in a class larger than $C([0,T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$. More precisely, uniqueness holds in $C([0,T], L^q(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^p(\Omega))$. Indeed, note first that the equation makes sense in that class (since $|u|^{p-1}u \in L^\infty_{\mathrm{loc}}((0,T), L^1(\Omega))$). Furthermore, if $u \in L^\infty_{\mathrm{loc}}((0,T), L^p(\Omega))$, then $|u|^{p-1} \in L^\infty_{\mathrm{loc}}((0,T), L^{p/(p-1)}(\Omega))$. We have $p > q = N(p-1)/2$, so that $p/(p-1) > N/2$. Therefore, it follows from Theorem A1 that $u \in L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$.

**7.3. Open problem 1** Recall that in the critical case $q = N(p-1)/2$ and $q > 1$, the time of existence $T(u_0)$ depends (in our proof of Theorem 1) on $u_0$ and not only on $\|u_0\|_{L^q}$. It would be interesting to clarify this point. In particular, is it possible to construct a sequence of initial conditions $(u_0^n)_{n\geq 0}$ which is bounded in $L^q(\Omega)$ and such that $T_{\max}(u_0^n) \underset{n\to\infty}{\longrightarrow} 0$ ?

**7.4. The "doubly critical" case in Theorem 4** As we have already mentioned in Remark 5, if $q = N(p-1)/2$ and $q = p$, i.e. $q = p = N/(N-2)$ ($N \geq 3$), then the conclusion of Theorem 4 fails, i.e. uniqueness fails in the class $C([0,T], L^q(\Omega))$. This is a result of Ni and Sacks [16], and we sketch their argument. First, a simple lemma.

**Lemma 10** *Let $\varphi, f \in L^1(\Omega)$ satisfy the equation*

$$\begin{cases} -\Delta\varphi = f & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

*in the sense that*

$$-\int_\Omega \varphi\Delta\zeta = \int_\Omega f\zeta, \tag{27}$$

*for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. Then*

$$\varphi = T(t)\varphi + \int_0^t T(s)f\,ds, \tag{28}$$

*for all $t \geq 0$.*

**Proof** The conclusion is trivial if $\varphi$ is smooth. In the general case, let $(f_n)_{n\geq 0} \subset \mathcal{D}(\Omega)$ be such that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $L^1(\Omega)$ and let $\varphi_n$ be the corresponding solution of (27).

Then $\varphi_n \to \varphi$ in $L^1(\Omega)$ (see e.g. Lemma 1 in [7]) and one passes to the limit in (28). □

In the case $\Omega =$ the unit ball of $\mathbf{R}^N$, Ni and Sacks [16] have constructed a radial function $\psi \in C^2(\overline{\Omega}\setminus\{0\})$, $\psi > 0$ in $\Omega$, $\psi = 0$ on $\partial\overline{\Omega}$, $\psi \in L^p(\Omega)$, $\lim_{x\to 0}\psi(x) = +\infty$, satisfying the equation

$$\begin{cases} -\Delta\psi = \psi^p & \text{in } \Omega \quad \text{with } p = N/(N-2),\\ \psi = 0 & \text{in } \partial\Omega, \end{cases}$$

in the sense that

$$-\int_\Omega \psi\Delta\zeta = \int_\Omega \psi^p\zeta,$$

for all $\zeta \in C^2(\overline{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. In view of Lemma 10, $v(t) \equiv \psi$ is a solution of (22) in $C([0,\infty), L^p(\Omega))$. On the other hand, the solution $u$ of (1) (with initial condition $\psi$) given by Theorem 1 has a smoothing effect. Hence, the two solutions are distinct.

**7.5. The "doubly critical" case in Theorem 1** If $q = N(p-1)/2$ and $q = 1$, i.e. $p = (N+2)/N$ $(N \geq 1)$ Theorem 1 does not apply and we suspect that the conclusions fail. (This concerns, for example, the simple case $N = 1, p = 3$, $q = 1$.) More precisely:

**Open problem 2** Is there some $u_0 \in L^1(\Omega)$ for which there is no (local) solution of (1) ? This means that for every $T > 0$ there is no function $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ satisfying (1) in the sense of Theorem 1.

**Open problem 3** Is there some $u_0 \in L^1(\Omega)$ for which uniqueness fails in the class $C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ for some $T > 0$ ?

**Open problem 4** Could there be failure of the maximum principle? More precisely, is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ and a solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ for some $T > 0$ which does not preserve the positivity ?

**Open problem 5** Is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that problem (1) with the "truncated" initial condition

$$u_0^n = \min\{u_0, n\}$$

has a (classical) solution $u^n$ on some maximal interval $[0, T_{\max}(u_0^n))$ satisfying $T_{\max}(u_0^n) \underset{n\to\infty}{\longrightarrow} 0$ ?

Alternatively, consider the "truncated" problem

$$\begin{cases} u_t^n - \Delta u^n = g^n(u^n) & \text{in } (0,\infty)\times\Omega, \\ u^n = 0 & \text{on } (0,\infty)\times\partial\Omega, \\ u^n(0) = u_0 & \text{in } \Omega, \end{cases}$$

where $g^n(t) = \min\{|t|^p, n\}\,\mathrm{sign}\, t$. Is there some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that $u^n(t,x) \underset{n\to\infty}{\longrightarrow} +\infty$ for all $x \in \Omega$ and all $t > 0$ ?

Here is some evidence suggesting that the answers to the above questions might be positive.

**Theorem 11** *Assume again* $p = (N+2)/N$, $q = 1$. *There is some* $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ *such that for every* $T > 0$ *problem* (1) *has no nonnegative solution* $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T), L^\infty(\Omega))$.

Here $N \geq 1$ and $\Omega$ can be arbitrary.

**Proof** Fix any open ball $\omega \subset \Omega$ with $\overline{\omega} \subset \Omega$.

**Claim** There is some $u_0 \in L^1(\Omega)$, $u_0 \geq 0$ such that $v(t) = T(t)u_0$ satisfies

$$\int_0^1 \int_\omega v^p(t,x)\,dx\,dt = +\infty. \tag{29}$$

(Note that $v \geq 0$ by the maximum principle.)

**Proof of claim** We argue by contradiction. Suppose that for every $u_0 \in L^1(\Omega)$, $u_0 \geq 0$,

$$\int_0^1 \int_\omega v^p(t,x)\,dx\,dt < \infty.$$

It follows that for every $u_0 \in L^1(\Omega)$,

$$\int_0^1 \int_\omega |v|^p(t,x)\,dx\,dt < \infty.$$

Applying the closed graph theorem to the operator $u_0 \mapsto v_{|(0,1)\times\omega}$, we see that there is a constant $C$ such that

$$\int_0^1 \int_\omega |v|^p(t,x)\,dx\,dt \leq C\|u_0\|_{L^1}^p, \tag{30}$$

for every $u_0 \in L^1(\Omega)$. Without loss of generality, we may assume that $0 \in \omega$ and we consider a sequence $(u_0^n)_{n\geq 0} \in \mathcal{D}(\omega)$ such that $\|u_0^n\|_{L^1} \leq 1$ and $u_0^n \underset{n\to\infty}{\longrightarrow} \delta$ (= the Dirac mass at 0) in the weak* topology of measures. The corresponding solutions

$v^n(t,x)$ converge to $v = T(t)\delta$. From (30) applied to $u_0^n$, we deduce (using Fatou's lemma) that

$$\int_0^1 \int_\omega |T(t)\delta|^p(x)\,dx\,dt < \infty.$$

On the other hand, $|T(t)\delta - E(t,\cdot)| \le C$ on $(0,1) \times \omega$, where $E$ is the fundamental solution of the heat equation in $\mathbf{R}^N$. In particular,

$$\int_0^1 \int_\omega |E(t,x)|^p(t,x)\,dx\,dt < \infty. \tag{31}$$

Since $E(t,x) = (4\pi t)^{-N/2}e^{-|x|^2/4t}$, a direct computation shows that (31) does not hold. □

We now return to the proof of Theorem 11 and we choose $u_0$ as in the claim. Assume by contradiction that for some $T > 0$ there is a nonnegative solution $u \in C([0,T], L^1(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ of (1). We have

$$u(t+s) \ge T(t)u(s),$$

for all $t \ge 0$, $s > 0$, $t+s < T$. As $s \downarrow 0$ we find

$$u(t) \ge T(t)u_0 = v(t), \tag{32}$$

for all $t \in [0,T]$. Since $u$ is a classical solution of (1) for $t \in (0,T)$, we may multiply (1) by $\zeta \in \mathcal{D}(\Omega)$, $\zeta \ge 0$ on $\Omega$, $\zeta \ge 1$ on $\omega$ and we obtain

$$\frac{d}{dt}\int_\Omega u\zeta + \int_\Omega u(-\Delta\zeta) = \int_\Omega u^p\zeta \ge \int_\omega u^p.$$

Integrating on $(\varepsilon, T)$ and letting $\varepsilon \downarrow 0$ (since $u \in C([0,T], L^1(\Omega))$), we deduce that

$$\int_0^T \int_\omega u^p < \infty,$$

which contradicts (29) and (32). □

### 7.6. The "subcritical" case $1 \le q < N(p-1)/2$

**Open problem 6** Is there some $u_0 \in L^q(\Omega)$ for which there is no (local) solution of (1)? This means that given any $T > 0$ (as small as we please) there is no function $u \in C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ satisfying (1).

Here is a suggestion how to construct such a $u_0$. Let $\Omega$ be the unit ball in $\mathbf{R}^N$, and let $\varphi = \varphi(r)$ with $r = |x|$, $\varphi \in C^1(\overline{\Omega})$, $\varphi > 0$ in $\Omega$, $\varphi = 0$ on $\partial\Omega$, $\varphi'(r) < 0$ for

$r \in (0,1)$, $\varphi''(0) < 0$ and $\Delta\varphi + \varphi^p \geq 0$ in $\Omega$ be such that the solution $v$ of (1) with the initial condition $v(0) = \varphi$ blows up in finite time $T_{\max}$. (It is well known that such a $\varphi$ exists.) By Theorem 2.4 of Friedman and McLeod [11],

$$\sup_{0 \leq t < T_{\max}} \|v(t)\|_{L^q} < \infty \quad \text{for all } 1 \leq q < \frac{N(p-1)}{2}.$$

Set

$$u_0 = \lim_{t \uparrow T_{\max}} v(t).$$

This $u_0$ belongs to $L^q(\Omega)$ for all $1 \leq q < N(p-1)/2$. We suspect that for such an initial condition $u_0$, there exists no local solution of (1) in any reasonable sense. That there is no nonnegative solution follows from Baras and Cohen [2]. Indeed, suppose there is a nonnegative solution $u$ of (1) with the initial condition $u(0) = u_0$ on $[0, T]$ for some $T > 0$. Set

$$w(t) = \begin{cases} v(t) & \text{for } 0 < t \leq T_{\max}, \\ u(t - T_{\max}) & \text{for } T_{\max} \leq t \leq T_{\max} + T. \end{cases}$$

This is an integral solution of (1) in the sense of Baras–Cohen [2] and Baras–Pierre [3] which blows up at $t = T_{\max}$. From [2], one knows that the only way to continue a solution beyond blow up time is by $+\infty$ everywhere.

**Remark 12** Baras [1] has given examples showing that uniqueness for problem (1) fails in the class $C([0,T], L^q(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ for $1 \leq q < N(p-1)/2$. Here, the initial condition can be any smooth function $u_0$, for example $u_0 = 0$. Such a phenomenon had been observed earlier by Haraux and Weissler [13] when $\Omega = \mathbf{R}^N$.

**7.7. Blow up and $L^q$ norms** Let $u_0 \in L^\infty(\Omega)$ and let $u$ be the corresponding solution of (1). Assume that $T_{\max} < \infty$ (it is well known that $T_{\max} < \infty$ if $u_0$ is "large" enough). By the blow up alternative, we know that $u$ blows up in $L^\infty(\Omega)$, i.e.

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^\infty} = +\infty.$$

It is natural to ask whether $\|u(t)\|_{L^q}$ also blows up as $t \uparrow T_{\max}$ for some $q < \infty$. It turns out that

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^q} = +\infty,$$

for $q \geq 1$, $q > N(p-1)/2$ (see Weissler [18, 19]). We can also derive this result as a simple consequence of Theorem 1 (see below). In fact, F. Weissler [20] (see also [6]) has a rate of blow up:

$$\liminf_{t \uparrow T_m} (T_m - t)^\delta \|u(t)\|_{L^q} > 0,$$

with $\delta = \dfrac{1}{p-1} - \dfrac{N}{2q}$.

**Corollary 13** *Let $u_0 \in L^\infty(\Omega)$, let $u$ be the corresponding solution of* (1) *and assume that $T_{\max} < \infty$. Then*

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^q} = +\infty, \tag{33}$$

*for any $q \geq 1$, $q > N(p-1)/2$.*

*Suppose in addition that $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and that*

$$\Delta u_0 + |u_0|^{p-1} u_0 \geq 0, \tag{34}$$

*a.e. in $\Omega$. If*

$$\frac{N(p-1)}{2} > 1, \tag{35}$$

*then*

$$\lim_{t \uparrow T_{\max}} \|u(t)\|_{L^{N(p-1)/2}} = +\infty. \tag{36}$$

**Proof of (33)** Suppose by contradiction that $\liminf_{t \uparrow T_{\max}} \|u(t)\|_{L^q} < \infty$ and let $(t_n)_{n \geq 0}$ be a sequence such that $t_n \uparrow T_{\max}$ as $n \to \infty$ and $\sup_{n \geq 0} \|u(t_n)\|_{L^q} < \infty$. Applying Theorem 1 with $u(t_n)$ as initial condition, we obtain a uniform $T > 0$. Thus $T_{\max} \geq t_n + T$. This is impossible as $n \to \infty$. □

**Proof of (36)** We argue as above, except that since we are in a critical case, we must know that $u(t_n)$ is contained in a compact set of $L^{N(p-1)/2}(\Omega)$. This is indeed the case, since, by the maximum principle, $u_t(t,x) \geq 0$ on $(0, T_{\max}) \times \Omega$. And thus $(u(t_n))_{n \geq 0}$ is a nonincreasing sequence and has a limit in $L^{N(p-1)/2}(\Omega)$. □

**Open problem 7** We do not know if the assumptions (34) and (35) in Corollary 13 are essential. In other words, if $N(p-1)/2 \geq 1$, does $\|u(t)\|_{L^{N(p-1)/2}}$ blow up as $t \uparrow T_{\max}$ for any initial condition $u_0 \in L^\infty(\Omega)$?

There is much evidence in favor of a positive answer:

- Assuming $N \geq 3$, $N(p-1)/2 = 1$ and (34), then (36) holds (see [6]).
- Assuming $N(p-1)/2 = 2$, i.e. $p = 1 + 4/N$, then (36) holds without having to assume (34) (see [6]).

**Remark 14** The second assertion in Corollary 13 was already known (see Weissler [19]) under further restrictions on $p$: $p > 1 + 4/N$ and $(N-2)p < N+2$.

Since we are discussing the blow up of $L^q$ norms as $t \uparrow T_{\max}$, we want to call attention to an interesting open problem connected with the result of Friedman and McLeod [11] mentioned above.

**Open problem 8** Does $\|u(t)\|_{L^q}$ remain bounded as $t \uparrow T_{\max}$, for **any** $u_0 \in L^\infty(\Omega)$ with $T_{\max} < \infty$ and any $q$, $1 \le q < N(p-1)/2$ ?

**Appendix: The linear heat equation with a potential**

Here we consider the equation

$$\text{(A1)} \qquad \begin{cases} u_t - \Delta u - a(t,x)u = 0, \\ u_{|\partial\Omega} = 0, \\ u(0) = u_0, \end{cases}$$

under various assumptions on the potential $a$. We write the equation (A1) in the form

$$\text{(A2)} \qquad u(t) = T(t)u_0 + \int_0^t T(t-s)a(s)u(s)\,ds.$$

**Theorem A1** *Let $0 < T < \infty$, let $\sigma > N/2$, $\sigma \ge 1$, and let $a \in L^\infty((0,T), L^\sigma(\Omega))$. Given $u_0 \in L^r(\Omega)$, $1 \le r < \infty$, there exists a unique solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ of equation* (A1). *Moreover, there is a constant $C$ depending only on $N, \sigma, r, |\Omega|$ such that $u$ satisfies*

$$\text{(A3)} \qquad \|u(t)\|_{L^\infty} \le C e^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}} t^{-N/2r} \|u_0\|_{L^r},$$

*for all $t \in (0,T]$, with $\beta = 2\sigma/(2\sigma - N)$.*

*Uniqueness also holds in the class $L^\infty((0,T), L^r(\Omega))$ provided $r \ge \sigma'$ (without having to assume $u \in L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$).*

**Proof** By a solution $u \in C([0,T], L^r(\Omega)) \cap L^\infty_{\text{loc}}((0,T), L^\infty(\Omega))$ of equation (A1), we mean that

(A4)
$$\begin{cases} u(t) = T(t-\varepsilon)u(\varepsilon) + \displaystyle\int_\varepsilon^t T(t-s)(a(s)u(s) + f(s))\,ds \quad \text{for } 0 < \varepsilon \le t \le T, \\ u(t) \underset{t \downarrow 0}{\longrightarrow} u_0 \quad \text{in } L^r(\Omega). \end{cases}$$

If $r \geq \sigma'$ and $u \in L^\infty((0,T),L^r(\Omega))$, then $au \in L^\infty((0,T),L^1(\Omega))$, so that the equation (A2) makes sense in $L^1(\Omega)$ and is equivalent to (A4). We now proceed in seven steps.

**Step 1** Given $u_0 \in L^\infty(\Omega)$, there exists a unique solution $u \in L^\infty((0,T),L^\infty(\Omega))$ of (A2) on $(0,T)$, and it satisfies

$$\|u(t)\|_{L^\infty} \leq 2e^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}}\|u_0\|_{L^\infty}, \tag{A5}$$

for all $t \in (0,T)$.

We apply the contraction mapping principle to the map $\Psi : L^\infty((0,T),L^\infty(\Omega)) \to L^\infty((0,T),L^\infty(\Omega))$ defined by

$$\Psi(u)(t) = T(t)u_0 + \int_0^t T(t-s)a(s)u(s)\,ds,$$

for $t \in (0,T)$. Note that

$$\begin{aligned}\|\Psi(u)(t) - \Psi(v)(t)\|_{L^\infty} &\leq \int_0^t (t-s)^{-\frac{N}{2\sigma}}\|a(s)\|_{L^\sigma}\|u(s)-v(s)\|_{L^\infty}\,ds\\ &\leq CT^{1-\frac{N}{2\sigma}}\|a\|_{L^\infty((0,T),L^\sigma)}\|u-v\|_{L^\infty((0,T),L^\infty)}.\end{aligned}$$

Hence, $\Psi$ is a strict contraction for example if

$$C\|a\|_{L^\infty((0,T),L^\sigma)}T^{1-\frac{N}{2\sigma}} \leq \frac{1}{2}.$$

Therefore, $\Psi$ has a fixed point, which is a solution of (A2). In this case, the conclusion of the lemma follows with $\|u(t)\|_{L^\infty} \leq 2\|u_0\|_{L^\infty}$. The general case follows by a standard iteration argument.

**Step 2** There exists $C$ such that for every $r \in [1,\infty]$ and every $u_0 \in L^\infty(\Omega)$, the corresponding solution $u$ of (A2) given by Step 1 verifies

$$\|u(t)\|_{L^r} \leq 2e^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}}\|u_0\|_{L^r}, \tag{A6}$$

for all $t \in (0,T)$.

By duality, we deduce from (A5) that

$$\|u(t)\|_{L^1} \leq 2e^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}}\|u_0\|_{L^1}. \tag{A7}$$

The general case $1 < r < \infty$ now follows from (A5), (A7) and Riesz–Thorin's interpolation theorem.

**Step 3** There exists $C$ such that for every $u_0 \in L^\infty(\Omega)$, the corresponding solution $u$ of (A2) given by Step 1 verifies

$$\|u(t)\|_{L^2} \le Ce^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}} t^{-N/4} \|u_0\|_{L^1}, \tag{A8}$$

for all $t \in (0,T)$.

We assume $u_0 \neq 0$. Multiplying the equation (A1) by $u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 = \int_\Omega au^2. \tag{A9}$$

By Hölder's and Gagliardo–Nirenberg's inequalities, we have

$$\int_\Omega au^2 \le \|a\|_{L^\sigma}\|u\|^2_{L^{2\sigma'}} \le C\|a\|_{L^\sigma}\left(\int_\Omega |\nabla u|^2\right)^{\frac{N}{2\sigma}}\left(\int_\Omega u^2\right)^{\frac{2\sigma-N}{2\sigma}}.$$

By Young's inequality,

$$\begin{aligned}
C\|a\|_{L^\sigma}\left(\int_\Omega |\nabla u|^2\right)^{\frac{N}{2\sigma}}\left(\int_\Omega u^2\right)^{\frac{2\sigma-N}{2\sigma}} &= C\left(\int_\Omega |\nabla u|^2\right)^{\frac{N}{2\sigma}}\left(\|a\|^\beta_{L^\sigma}\int_\Omega u^2\right)^{\frac{2\sigma-N}{2\sigma}} \\
&\le \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{C}{2}\|a\|^\beta_{L^\sigma}\int_\Omega u^2.
\end{aligned}$$

Therefore, we deduce from (A9) that

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega |\nabla u|^2 \le C\|a\|^\beta_{L^\sigma}\int_\Omega u^2. \tag{A10}$$

On the other hand, by Gagliardo–Nirenberg's inequality and (A7) we have

(A11)

$$\left(\int_\Omega u^2\right)^{\frac{N+2}{N}} \le C\int_\Omega |\nabla u|^2\left(\int_\Omega |u|\right)^{\frac{4}{N}} \le C\int_\Omega |\nabla u|^2\left(Ce^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}}\|u_0\|_{L^1}\right)^{\frac{4}{N}}.$$

It now follows from (A10) and (A11) that the function

$$f(t) = \int_\Omega u^2(t,x)\,dx$$

satisfies the differential inequality

$$f'(t) + Af(t)^{1+\frac{2}{N}} \le Bf(t),$$

with

$$A = \left(Ce^{Ct\|a\|^\beta_{L^\infty((0,t),L^\sigma)}}\|u_0\|_{L^1}\right)^{-\frac{4}{N}},$$

and

$$B = C\|a\|^{\beta}_{L^\infty((0,t),L^\sigma)}.$$

This yields

$$f(t) \le \left(\frac{2}{NAt}\right)^{\frac{N}{2}} e^{Bt},$$

which is the desired estimate.

**Step 4** If $u_0 \in L^\infty(\Omega)$, then for every $1 \le r \le \infty$ the estimate (A3) holds. By duality, we deduce from (A8) that

$$\|u(t)\|_{L^\infty} \le Ce^{Ct\|a\|^{\beta}_{L^\infty((0,t),L^\sigma)}} t^{-N/4}\|u_0\|_{L^2}. \tag{A12}$$

Combining (A8) and (A12), we deduce that

$$\|u(t)\|_{L^\infty} \le Ce^{Ct\|a\|^{\beta}_{L^\infty((0,t),L^\sigma)}} t^{-N/2}\|u_0\|_{L^1}. \tag{A13}$$

(A3) now follows from (A5), (A13) and Riesz–Thorin's interpolation theorem. (The arguments used in Steps 3 and 4 use an idea of Fabes and Stroock [10].)

**Step 5** Existence in the class $C([0,T],L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T),L^\infty(\Omega))$. Let $u_0 \in L^r(\Omega)$, and let $(u_0^n)_{n\ge 0} \subset L^\infty(\Omega)$ be such that $u_0^n \underset{n\to\infty}{\longrightarrow} u_0$ in $L^r(\Omega)$. Let $u^n$ be the corresponding solutions of (A2). It follows from (A6) and (A3) that $u^n$ converges to a limit $u$ in $C([0,T],L^r(\Omega))$ and in $C([\varepsilon,T],L^\infty(\Omega))$ for every $0<\varepsilon<T$. Therefore, $u$ solves the equation (A4) and satisfies the estimate (A3).

**Step 6** Uniqueness in the class $C([0,T],L^r(\Omega)) \cap L^\infty_{\mathrm{loc}}((0,T),L^\infty(\Omega))$. Suppose $u_0 = 0$. It follows from Step 4 that

$$\|u(t+\varepsilon)\|_{L^\infty} \le Ce^{Ct\|a\|^{\beta}_{L^\infty((0,T),L^\sigma)}} t^{-N/2r}\|u(\varepsilon)\|_{L^r},$$

for all $t \in (0, T-\varepsilon)$. Letting $\varepsilon \downarrow 0$, we obtain $u(t) = 0$ for all $t \in (0,T)$.

**Step 7** Uniqueness in the class $L^\infty((0,T),L^{\sigma'}(\Omega))$. If $u$ and $v$ are two solutions, we have

$$\begin{aligned}
\|u(t)-v(t)\|_{L^{\sigma'}} &\le \int_0^t (t-s)^{-\frac{N}{2}(1-\frac{1}{\sigma'})}\|a(u-v)\|_{L^1}\,ds \\
&\le \int_0^t (t-s)^{-\frac{N}{2\sigma}}\|a\|_{L^\sigma}\|u-v\|_{L^{\sigma'}}\,ds \\
&\le \|a\|_{L^\infty((0,T),L^\sigma)} \int_0^t (t-s)^{-\frac{N}{2\sigma}}\|u-v\|_{L^{\sigma'}}\,ds,
\end{aligned}$$

and it follows from the singular Gronwall lemma that $u = v$. This completes the proof. □

Finally, we present a uniqueness result when $a \in C([0,T], L^{N/2}(\Omega))$.

**Theorem A2** *Assume $N \geq 3$. Let $T > 0$ and $a \in C([0,T], L^{N/2}(\Omega))$. If $u \in L^\infty((0,T), L^q(\Omega))$ with $q > N/(N-2)$ satisfies*

(A14) $$u(t) = \int_0^t T(t-s)a(s)u(s)\,ds,$$

*for all $t \in [0,T]$, then $u(t) \equiv 0$.*

**Proof** We have $au \in L^\infty((0,T), L^{r_0}(\Omega))$, with $1/r_0 = 1/q + 2/N$. In particular, $1 < r_0 < \infty$, so that by maximal regularity (see [9]) we have $u \in L^p((0,T), W^{2,r_0}(\Omega) \cap W_0^{1,r_0}(\Omega))$ for every $p < \infty$, and $u$ satisfies

(A15) $$u_t - \Delta u = au,$$

in $L^{r_0}\Omega)$ for almost all $t \in (0,T)$.

We now use a duality argument. Fix $t_0 \in (0,T)$, and $\psi \in \mathcal{D}(\Omega)$. Let $a_n = \min\{n, \max\{a, -n\}\}$. We have $(a_n)_{n\geq 0} \subset C([0,T], L^{N/2}(\Omega)) \cap L^\infty((0,T) \times \Omega)$. Moreover (as in the proof of (11)) $a_n \to a$ in $C([0,T], L^{N/2}(\Omega))$ as $n \to \infty$.

Let $v_n$ be the solution of

$$\begin{cases} -(v_n)_t - \Delta v_n = a_n v_n, & \text{in } (0,t_0) \times \Omega, \\ v_n = 0 \quad \text{on } (0,t_0) \times \partial\Omega, \\ v_n(t_0) = \psi \quad \text{in } \Omega. \end{cases}$$

We now multiply the equation (A15) by $v_n$ and integrate on $(0,t_0) \times \Omega$. We obtain

$$\left[\int_\Omega u v_n\right]_0^{t_0} = \int_0^{t_0}\int_\Omega (u(v_n)_t + u_t v_n) = \int_0^{t_0}\int_\Omega (u(-\Delta v_n - a_n v_n) + v_n(\Delta u + au))$$
$$= \int_0^{t_0}\int_\Omega (a - a_n) u v_n.$$

Therefore,

$$\int_\Omega u(t_0)\psi = \int_0^{t_0}\int_\Omega (a - a_n) u v_n.$$

Hence

(A16) $$\left|\int_\Omega u(t_0)\psi\right| \leq t_0 \|a - a_n\|_{C([0,t_0],L^{N/2})} \|u\|_{L^\infty(0,t_0),L^q)} \|v_n\|_{L^\infty((0,t_0),L^\theta)},$$

with $1/\theta = 1 - 1/q - 2/N > 0$. In particular, we have $\theta < \infty$. We claim that for every $2 \le r < \infty$ there exists a constant $C$ ($C$ depends on $r$) such that

$$\sup_{n\ge 0} \|v_n\|_{L^\infty((0,t_0),L^{r'})} \le C\|\psi\|_{L^{r'}}. \tag{A17}$$

Assuming the claim, we let $n \to \infty$ in (A16) and we obtain

$$\int_\Omega u(t_0)\psi = 0.$$

Since $t_0 \in (0,T)$ and $\psi \in \mathcal{D}(\Omega)$ are arbitrary, we deduce that $u \equiv 0$. □

**Proof of Claim (A17)** We use the same method as in Brezis and Kato [8]. It is convenient to introduce $w_n(t) = v_n(t_0 - t)$ so that $w_n$ satisfies

$$\begin{cases} (w_n)_t - \Delta w_n = b_n w_n & \text{in } (0,t_0)\times\Omega, \\ w_n = 0 \quad \text{in } (0,t_0)\times\partial\Omega, \\ w_n(0) = \psi \quad \text{in } \Omega, \end{cases} \tag{A18}$$

with $b_n(s) = a_n(t_0 - s)$. We multiply the equation (A18) by $|w_n|^{r-2}w_n$ to obtain

$$\frac{1}{r}\frac{d}{dt}\int_\Omega |w_n(t)|^r + \frac{4(r-1)}{r^2}\int_\Omega |\nabla|w_n|^{r/2}|^2 \le \int_\Omega |b_n|\,|w_n|^r \le \int_\Omega |b|\,|w_n|^r, \tag{A19}$$

where $b(t) = a(t_0 - t)$ for $0 \le t \le t_0$. Given $j \ge 0$ to be chosen large enough, we write $b = b - b_j + b_j$, and we estimate

$$\begin{aligned} \int_\Omega |b|\,|w_n|^r &\le \int_\Omega |b - b_j|\,|w_n|^r + \int_\Omega |b_j|\,|w_n|^r \\ &\le \|b - b_j\|_{L^{N/2}}\|w_n\|^r_{L^{Nr/(N-2)}} + j\|w_n\|^r_{L^r} \\ &\le C\|b - b_j\|_{L^{N/2}}\int_\Omega |\nabla|w_n|^{r/2}|^2 + j\|w_n\|^r_{L^r}, \end{aligned} \tag{A20}$$

where the last inequality follows from Sobolev's inequality. We now choose $j$ large enough (independent of $n$) so that

$$C\|b - b_j\|_{L^{N/2}} \le \frac{4(r-1)}{r^2}.$$

(Recall that $b_j \underset{j\to\infty}{\longrightarrow} b$ in $C([0,t_0],L^{N/2}(\Omega))$. It is here that we use the assumption $a \in C([0,T],L^{N/2}(\Omega))$; $a \in L^\infty((0,T),L^{N/2}(\Omega))$ would not be sufficient.) It now follows from (A19) and (A20) that

$$\frac{1}{r}\frac{d}{dt}\int_\Omega |w_n(t)|^r \le j\|w_n\|^r_{L^r},$$

from which we deduce $\|w_n(t)\|_{L^r}^r \le \|\psi\|_{L^r}^r e^{jrt}$. □

**Remark A3** The conclusion of Theorem A2 fails if $q = N/(N-2)$. To construct such an example, we use the technique of Ni and Sacks [16]. Let $\psi$ be as in Section 7.4 and let $v$ be the solution of (1) with the initial condition $v(0) = \psi$. Set $u = v - \psi$ and

$$a = \begin{cases} \dfrac{v^p - \psi^p}{v - \psi} & \text{if } v \neq \psi, \\ v^{p-1} & \text{if } v = \psi. \end{cases}$$

$u$ satisfies (A14) but $u \neq 0$.

**Open problem 9** Can one replace the assumption $a \in C([0,T], L^{N/2}(\Omega))$ by $a \in L^\infty((0,T), L^{N/2}(\Omega))$ in Theorem A2 ? The problem is open even under the additional assumption $u \in C_c^\infty((0,T) \times \Omega)$.

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## REFERENCES

[1] P. Baras, *Non unicité des solutions d'une équation d'évolution non linéaire*, Ann. Fac. Sci. Toulouse **5** (1983), 287–302.

[2] P. Baras and L. Cohen, *Complete blow-up after $T_{max}$ for the solution of a semilinear heat equation*, J. Functional Analysis **71** (1987), 142–174.

[3] P. Baras and M. Pierre, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. Henri Poincaré, Analyse Non Linéaire **2** (1985), 185–212.

[4] M. Ben-Artzi, *Global solutions of two-dimensional Navier-Stokes and Euler equations*, Arch. Rational Mech. Anal. **128** (1994), 330–358.

[5] H. Brezis, *Remarks on the preceding paper by M. Ben-Artzi "Global solutions of two-dimensional Navier-Stokes and Euler equations"*, Archive Rat. Mech. Anal. **128** (1994), 359–360.

[6] H. Brezis and T. Cazenave, *Nonlinear evolution equations*, in preparation.

[7] H. Brezis, T. Cazenave, Y. Martel and A. Ramiandrisoa, *Blow up for $u_t - \triangle u = g(u)$ revisited*, Advances in Differential Equations **1** (1996), 73–90.

[8] H. Brezis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pures Appl. **58** (1979), 137–151.

[9] T. Coulhon and D. Lamberton, *Régularité $L^p$ pour les équations d'évolution*, Publ. Math. Univ. Paris VII, Sem. Analyse Fonct., 1984–1985, pp. 155–165.

[10] E. B. Fabes and D. W. Stroock, *A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash*, Arch. Rational Mech. Anal. **96** (1986), 327–338.

[11] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J. **34** (1985), 425–447.

[12] Y. Giga, *Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system*, J. Differential Equations **62** (1986), 186–212.

[13] A. Haraux and F. B. Weissler, *Non uniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. **31** (1982), 167–189.

[14] T. Kato, *Strong $L^p$-solutions of the Navier-Stokes equation in $\mathbb{R}^N$*, Math. Z. **187** (1984), 471–480.

[15] T. Kato and H. Fujita, *On the nonstationary Navier-Stokes system*, Rend. Sem. Math. Univ. Padova **32** (1962), 243–260.

[16] W.-M. Ni and P. Sacks, *Singular behavior in nonlinear parabolic equations*, Trans. Amer. Math. Soc. **287** (1985), 657–671.

[17] F. B. Weissler, *Semilinear evolution equations in Banach spaces*, J. Funct. Anal. **32** (1979), 277–296.

[18] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in $L^p$*, Indiana Univ. Math. J. **29** (1980), 79–102.

[19] F. B. Weissler, *$L^p$ energy and blowup for a semilinear heat equation*, Proc. Symp. Pure Math. **45**, part 2 (1986), 545–552.

[20] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. **38** (1981), 29–40.

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