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Mathematics

Strongly nonlinear parabolic initial-boundary value problems

(Dirichlet boundary conditions/compactness theorems/approximation theorems with convex side conditions)

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ABSTRACT An existence and uniqueness result is presented for the solution of a parabolic initial-boundary value problem under Dirichlet null boundary conditions for a general parabolic equation of order $2m$ with a strongly nonlinear zeroth-order perturbation. This is the parabolic generalization of a class of elliptic results considered earlier by the writers and others and is based upon a new compactness theorem.

Let Ω be a bounded open set in R^n, $(n \geq 1)$, Q the cylinder $\Omega \times [0,T]$ for a given $T > 0$. Consider the quasilinear parabolic partial differential equation of order $2m$ on Q, $(m \geq 1)$, of the form

$$\frac{\partial u}{\partial t} + A_t(u) + g(x,t,u) = f(x,t) \qquad [1]$$

with the initial-boundary conditions

$$u(x,0) = 0 \text{ for } x \text{ in } \Omega; \frac{\partial^j u}{\partial N^j}(x,t) = 0 \text{ for } x \text{ in bdry}(\Omega), t > 0, 0 \leq j \leq m-1 \qquad [2]$$

(N being the normal derivative). Using the conventional notation (as described, for example, in ref. 1), A_t for each t in $[0,T]$ is an elliptic operator of order $2m$ in the generalized divergence form

$$A_t(u) = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta A_\beta(x,t,u,\ldots,D^m u) \qquad [3]$$

with the coefficient functions $A_\beta(x,t,\xi)$ of x in Ω, t in $[0,T]$, and $\xi = \{\xi_\alpha : |\alpha| \leq m\}$ continuous in ξ and measurable in (x,t).

In a preceding paper (1), the writers studied the Dirichlet problem for the elliptic equation $A(u) + g(x,u) = f(x)$. Here, we consider the corresponding parabolic problem under the assumption that A_t is a regular elliptic operator in the Sobolev space $W^{m,p}(\Omega)$ for a given exponent $p \geq 2$—i.e., satisfies the following three conditions:

(*i*) There exists $c_o \geq 0$, h_0 in $L^{p'}(Q)$, $(p' = p/(p-1))$, such that

$$|A_t(x,t,\xi)| \leq c_o\{|\xi|^{p-1} + h_0(x,t)\}$$

for all (x,t,ξ).

(*ii*) For (x,t) outside of a null set, all lower-order jets η, and $\zeta \neq \zeta\#$,

$$\sum_{|\beta| = m} [A_\beta(x,t,\eta,\zeta) - A_\beta(x,t,\eta,\zeta\#)](\zeta_\beta - \zeta_\beta\#) > 0.$$

(*iii*) There exists $c_1 > 0$, h_1 in $L^1(Q)$ such that for all (x,t,ξ)

$$\sum_{|\beta| \leq m} A_\beta(x,t,\xi)\xi_\beta \geq c_1|\xi|^p - h_1(x,t).$$

For the strongly nonlinear perturbing term $g(x,t,u)$, we assume no *a priori* growth restriction, but aside from the usual condition that $g(x,t,u)$ is measurable in (x,t), continuous in u, we impose the following set of conditions:

(*iv*) There exists a continuous nondecreasing function $h:R^1 \to R^1$ with $h(0) = 0$ such that for all (x,t) in Q, r in R^1, and a fixed C

$$rg(x,t,r) \geq 0; \ |g(x,t,r)| \leq |h(r)|; \ |h(r)| \leq C\{|g(x,t,r)| + |r|^{p-1} + 1\}.$$

The following two theorems, for the first of which we sketch the most important steps in the proof, are our basic result for this parabolic case:

THEOREM 1. *Let Ω be a bounded open subset of R^n whose boundary satisfies the mild smoothness condition* (s) *of* Definition 1 *below, and consider a parabolic equation* 1 *satisfying the conditions i, ii, iii, and iv for a given $p \geq 2$. Let f be a distribution in $L^{p'}(0,T:W^{-m,p'}(\Omega))$.*

Then: There exists u in $L^p(0,T:W_0^{m,p}(\Omega)) \cap C(0,T:L^2(\Omega))$ with $u(0) = 0$ such that g(u) and ug(u) lie in $L^1(Q)$ that satisfies the equation 1 *with the additional condition:*
For $0 \leq t \leq T$,

$$\frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_0^t \langle A_s(u(s)),u(s)\rangle ds + \int_{Q_t} ug(u) = \int_0^t \langle f(s),u(s)\rangle ds \qquad [4]$$

(*in which* $Q_t = \Omega \times [0,t]$).

THEOREM 2. *If, in addition,* g(x,t,r) *is nondecreasing in* r *and each* A_s *is monotone, the solution* u *of* Theorem 1 *is uniquely determined by* f.

The most important new ingredient in the proof of the parabolic result is the following compactness theorem:

PROPOSITION 1. *Let Ω be a bounded open set in R^n, $\{u_k\}$ a bounded sequence in $L^p(0,T:W_0^{m,p}(\Omega))$ such that $\partial u_k/\partial t = w_k + z_k$ where $\{w_k\}$ is a bounded sequence in $L^{p'}(0,T:W^{-m,p'}(\Omega))$ and $\{z_k\}$ is sequentially weakly compact in $L^1(Q)$.*

Then: $\{u_k\}$ is strongly compact in $L^p(Q)$.

Proposition 1 is a special case of a more general result, which is of great interest in its own right:

THEOREM 3. *Let X_0, X_1, X_2 be three Banach spaces with X_0 having a compact linear embedding in X_1, X_1 a continuous linear embedding in X_2. Let $\{u_k\}$ be a bounded sequence in $L^p(0,T:X_0)$ for $p \geq 1$ with du_k/dt lying in $L^1(0,T:X_2)$. Suppose that there exists a function $\gamma:R^+ \to R^+$ with $\gamma(r) \to 0$ as $r \to 0$ such that for any pair (s,t) in [0,T] with s < t and all k,*

$$\int_s^t \left\|\frac{du_k}{dt}(r)\right\|_{X_2} dr \leq \gamma(t-s).$$

Then: $\{u_k\}$ is strongly compact in $L^p(0,T:X_1)$.

We obtain *Proposition 1* from *Theorem 3* by the following specialization: We set $X_0 = W_0^{m,p}(\Omega)$, $X_1 = L^p(\Omega)$, and $X_2 = W^{-j,p}(\Omega)$ with $j = n + m$. Then X_0 is compactly embedded in X_1 by the boundedness of the domain Ω and the compactness part of the Sobolev embedding theorem, and X_1 is continuously embedded in X_2. The hypothesis of *Theorem 3* is satisfied on the derivatives, for the $\{w_k\}$ by Holder's inequality and for the $\{z_k\}$ by the Dunford–Pettis theorem (because for a suitable

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function γ with $\gamma(r) \to 0$ as $r \to 0$, $\int_s^t \|z_k(s)\|_{L^1(\Omega)}\, ds \le \gamma(t - s)$) because both $W^{-m,p'}(\Omega)$ and $L^1(\Omega)$ are continuously embedded in X_2.

Proof of Theorem 3: If we multiply the functions $u_k(t)$ by $\xi(t)$ with ξ in $C^1(R^1)$ such that $\xi(t) = 1$ for $t \le \frac{1}{2}T$, $\xi(t) = 0$ for $t \ge \frac{3}{4}T$, and note that both $\{\xi u_k\}$ and $\{(1-\xi)u_k\}$ satisfy the same hypotheses as $\{u_k\}$, it suffices to assume that for all k, $u_k(t)$ is defined for all $t \ge 0$ and has its support in $[0,T)$.

Let j be a nonnegative function in $\mathcal{D}(R^1)$ with support in $[0,1]$ such that $\int_0^\infty j(s)\, ds = 1$. For each $\delta > 0$, we set $j_\delta(s) = \delta^{-1} j(\delta^{-1}s)$. For each k and δ, we define

$$v_{k,\delta}(t) = \int_0^\infty j_\delta(s) u_k(t+s)\, ds.$$

Because $\{u_k\}$ is a bounded sequence in $L^p(0,T{:}X_0)$, it follows that $\{v_{k,\delta}\}$ is bounded in $L^p(0,T{:}X_0)$ for all k and all $\delta > 0$. For each fixed $\delta > 0$, $\{v_{k,\delta}\}$ is a bounded sequence in $C^1(0,T{:}X_0)$ and by the compactness of the embedding of X_0 into X_1, $\{v_{k,\delta}\}$ for fixed δ is strongly compact in $L^p(0,T{:}X_1)$. Hence, it suffices to show that $\int_0^T \|u_k(t) - v_{k,\delta}(t)\|_{X_1}^p\, dt \to 0$ as $\delta \to 0$, uniformly in k.

Because X_0 is compactly embedded in X_1 and X_1 is continuously embedded in X_2, for each $\epsilon > 0$ there exists K_ϵ such that for all u in X_0, $\|u\|_{X_1}^p \le \epsilon \|u\|_{X_0}^p + K_\epsilon \|u\|_{X_2}^p$. Hence

$$\int_0^T \|u_k(t) - v_{k,\delta}(t)\|_{X_1}^p \le c\epsilon \left(\int_0^T \{\|u_k(t)\|_{X_0}^p + \|v_{k,\delta}(t)\|_{X_0}^p\} dt \right) + K \int_0^T \|u_k(t) - v_{k,\delta}(t)\|_{X_2}^p\, dt.$$

The first term is bounded by ϵM. On the other hand, for t in $[0,T]$

$$\|u_k(t) - v_{k,\delta}(t)\|_{X_2} \le \sup_{0\le s\le \delta} \|u_k(t) - u_k(t+s)\|_{X_2} \le \int_t^{k+\delta} \left\| \frac{du_k}{dt}(r) \right\|_{X_2} dr \le \gamma(\delta).$$

Hence, choosing $\epsilon > 0$ sufficiently small and then $\delta > 0$ small, the desired conclusion follows. q.e.d.

Definition 1: Let Ω be an open subset of R^n. For each $\delta > 0$, let $\Omega_\delta = \{x \mid x \in \Omega, \operatorname{dist}(x, \operatorname{bdry}(\Omega)) < \delta\}$. Then Ω is said to satisfy (s) if there exists $C > 0$, $\delta_0 > 0$, such that for $0 < \delta < \delta_0$ and all φ in $\mathcal{D}(\Omega)$,

$$\int_{\Omega_\delta} |\varphi|^p dx \le C\delta^p \int_{\Omega_{C\delta}} |\nabla\varphi|^p dx.$$

We use the following approximation result in the proof of *Theorems 1* and *2*:

PROPOSITION 2. *Let* Ω *be an open subset of* R^n *that satisfies* (s), *and let* H *be a continuous, nonnegative, convex function on the reals with* $H(0) = 0$. *Let* u *be an element of* $L^p(0,T{:}W_0^{m,p}(\Omega))$ *for some* $p \ge 1$ *with* $H(u)$ *lying in* $L^1(Q)$.

Then there exists a sequence $\{v_j\}$ *in* $C(0,T,\mathcal{D}(\Omega))$ *with* $\partial v_j/\partial t \in L^2$, $v_j(0) = 0$ *for each* j *such that* v_j *converges strongly to* u *in* $L^p(0,T{:}W_0^{m,p}(\Omega))$, v_j *converges a.e. to* u *in* Q, $H(v_j)$ *converges strongly to* $H(u)$ *in* $L^1(Q)$, *and*

$$\overline{\lim} \int_0^t \left(\frac{dv_j}{dt}(s), v_j(s) - u(s) \right) ds \le 0$$

for all t *in* [0,T].

We apply *Proposition 2* to the convex function $H(r) = \int_0^r h(s)ds$, in which $h(r)$ is the nondecreasing continuous function of condition (*iv*) with $h(0) = 0$. Then H satisfies the conditions of *Proposition 2*, and $H' = h$. Suppose that u is an element of $L^p(0,T{:}W_0^{m,p}(\Omega))$ with $ug(u)$ in $L^1(Q)$. Because

$$0 \le H(u) \le uh(u) \le Cug(u) + C|u|^p + C|u|,$$

it follows that $uh(u)$ and $H(u)$ lie in $L^1(Q)$. If we consider the sequence $\{v_j\}$ described by the conclusions of *Proposition 2*, then $(g(u)v_j)^+$ converges a.e. on Q to $ug(u)$. On the other hand, the subgradient relation $H(r) - H(s) \ge h(s)(r-s)$ for all r and s implies that $(h(s)r)^+ \le H(r) + sh(s)$. Hence

$$(g(u)v_j)^+ \le (h(u)v_j)^+ \le H(v_j) + uh(u)$$

where the bounding sequence is strongly convergent in $L^1(Q)$. Hence for any t in $[0,T]$,

$$\int_{Q_t} g(u)v_j \le \int_{Q_t} (g(u)v_j)^+ \to \int_{Q_t} ug(u).$$

Proof of Theorem 1: Let g_k be the truncation of g at level k. By the corresponding existence theorem for regular parabolic problems, for each k there exists u_k in $L^p(0,T{:}W_0^{m,p}(\Omega)) \cap C(0,T{:}L^2(\Omega))$, $u(0) = 0$, such that

$$\frac{\partial u_k}{\partial t} + A(u_k) + g_k(u_k) = f.$$

Moreover, for each v in $L^p(0,T{:}W_0^{m,p}(\Omega)) \cap C^1(0,T{:}L^2(\Omega)) \cap L^\infty(Q)$ with $v(0) = 0$, we have for all t in $[0,T]$,

$$\tfrac{1}{2}\|u_k(t) - v(t)\|_{L^2(\Omega)}^2 + \int_0^t \langle A_s(u_k(s)), u_k(s) - v(s)\rangle ds + \int_{Q_t} g_k(u_k)(u_k - v) = \int_0^t \langle f(s), u_k(s) - v(s)\rangle ds + \int_0^t \left(\frac{dv}{dt}(s), v(s) - u_k(s) \right) ds. \quad [5]$$

In particular, if we set $v(t) \equiv 0$, it follows as in the elliptic case that $\{u_k\}$ is a bounded sequence in $L^p(0,T{:}W_0^{m,p}(\Omega))$ and in $L^\infty(0,T{:}L^2(\Omega))$ and that $\{u_k g_k(u_k)\}$ is bounded in $L^1(Q)$. In particular, $\{g_k(u_k)\}$ is sequentially weakly compact in $L^1(Q)$.

We may now apply the compactness result of *Proposition 1* to extract an infinite subsequence (again denoted by $\{u_k\}$) such that u_k converges weakly to u in $L^p(0,T{:}W_0^{m,p}(\Omega))$ and strongly to u in $L^p(Q)$. We may also assume that u_k converges to u a.e. in Q, $g_k(u_k)$ converges to $g(u)$ strongly in $L^1(Q)$, and, for t outside of a null set N, $u_k(t)$ converges strongly to $u(t)$ in $L^2(\Omega)$. The limit function u lies in $L^p(0,T{:}W_0^{m,p}(\Omega)) \cap L^\infty(0,T{:}L^2(\Omega))$, $g(u)$ lies in $L^1(Q)$, and, by Fatou's Lemma, $ug(u)$ lies in $L^1(Q)$. We may also assume that $A(u_k)$ converges weakly in $L^{p'}(0,T{:}W^{-m,p'}(\Omega))$ to some w. In the sense of distributions on Q,

$$\frac{\partial u}{\partial t} + w + g(u) = f.$$

Hence, it suffices to prove that $w = A(u)$ and that Eq. 4 holds.

If we transform Eq. 5 above, we see that

$$\int_0^t \langle A_s(u_k(s)), u_k(s) - u(s)\rangle ds + \int_{Q_t} \{g_k(u_k)u_k - g(u)u\} = J_k(v) + R_k(v)$$

with

$$J_k(v) = \int_0^t \left\{ \langle f(s), u_k(s) - v(s)\rangle - \langle A_s(u_k(s)), u(s) - v(s)\rangle + \left(\frac{dv}{dt}(s), v(s) - u_k(s) \right) \right\} ds - \tfrac{1}{2}\|u_k(t) - v(t)\|_{L^2}^2,$$

$$R_k(v) = \int_{Q_t} \{g_k(u_k)v - g(u)u\}.$$

For t outside N, $J_k(v)$ converges to $J(v)$, in which

$$J(v) = \int_0^t \{ \langle f(s) - w(s), u(s) - v(s)\rangle$$

$$+ \left\{\frac{dv}{dt}(s),v(s) - u(s)\right\}\Big\}ds - \tfrac{1}{2}\|u(t) - v(t)\|^2_{L^2}.$$

Moreover, $R_k(v)$ converges to $R(v)$ given by

$$R(v) = \int_{Q_t} \{g(u)v - g(u)u\}.$$

Consider the sequence $\{v_j\}$ corresponding to u in the sense of *Proposition 2* with respect to the convex function H. Then

$$R(v_j) \le \int_{Q_t} \{(g(u)v_j)^+ - ug(u)\} \to 0.$$

Furthermore

$$\overline{\lim}\, J(v_j) \le 0.$$

Hence for all t outside of N,

$$\overline{\lim} \int_0^t < A_s(u_k(s)),u_k(s) - u(s) > ds + \int_{Q_t} \{g_k(u_k)u_k - g(u)u\} \le 0.$$

Because

$$\overline{\lim} \int_{Q_t} \{g_k(u_k)u_k - g(u)u\} \ge 0,$$

it follows that

$$\overline{\lim} \int_0^t < A_s(u_k(s)),u_k(s) - u(s) > ds \le 0.$$

Applying a slight variant of the pseudomonotonicity argument of ref. 2, it follows that $A(u_k)$ converges weakly to $A(u)$ in $L^{p'}(0,T:W^{-m,p'}(\Omega))$. Moreover,

$$\int_0^t <A_s(u_k(s)),u_k(s) - u(s) > ds \to 0$$

and

$$\int_0^t < A_k(u_k(s)),u_k(s) > ds \to \int_0^t < A(u(s)),u(s) > ds.$$

In particular, it follows that

$$\overline{\lim} \int_{Q_t} g_k(u_k)u_k \le \int_{Q_t} g(u)u,$$

so that

$$\int_{Q_t} g_k(u_k)u_k \to \int_{Q_t} g(u)u.$$

Finally, taking the limit of Eq. **5** with $v = 0$, we find that, for t outside of N,

$$\tfrac{1}{2}\|u(t)\|^2 + \int_0^t < A_s(u(s)),u(s) > ds + \int_{Q_t} ug(u) = \int_0^t < f(s),u(s) > ds.$$

Hence, $\|u(t)\|_{L_2}$ is identical with a continuous function for $t \notin N$. It follows immediately that if we redefine u on N, the resulting function lies in $C(0,T:L^2(\Omega))$ and Eq. **4** holds for all t in $[0,T]$. q.e.d.

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