On the Uniqueness of Weak Solutions of Navier-Stokes Equations: Remarks on a Clay Institute Prize Problem

Uniqueness of Weak Solutions of Navier-Stokes Equations

J. Hoffman and C. Johnson

Abstract

We consider the Clay Institute Prize Problem asking for a mathematical analytical proof of existence, smoothness and uniqueness (or a converse) of solutions to the incompressible Navier-Stokes equations. We argue that the present formulation of the Prize Problem asking for a strong solution is not reasonable in the case of turbulent flow always occuring for higher Reynolds numbers, and we propose to focus instead on weak solutions. Since weak solutions are known to exist by a basic result by J. Leray from 1934, only the uniqueness of weak solutions remains as an open problem. To seek to give some answer we propose to reformulate this problem in computational form as follows: For a given flow what quantity of interest can be computed to what tolerance to what cost? We give computational evidence that quantities of interest (or output quantitites) such as the mean value in time of the drag force of a bluff body subject to a turbulent high Reynolds number flow, is computable on a PC up to a tolerance of a few percent. We also give evidence that the drag force at a specific point in time is uncomputable even on a very high performance computer. We couple this evidence to the question of uniqueness of weak solutions to the Navier-stokes equations, and thus give computational evidence of both uniqueness and non-uniqueness in outputs of weak solutions. The basic tool of investigation is a representation of the output error in terms of the residual of a computed solution and the solution of an associated linear dual problem acting as a weight. By computing the dual solution coupled to a certain output and measuring the energy-norm of the dual velocity, we get quantitiative information of computability of different outputs, and thus information on output uniqueness of weak solutions.

Key words: Clay Institute Prize Problem, uniqueness of weak solutions, Navier-Stokes equations, computability, a posteriori error estimate, turbulence
PACS: 47.27.Eq, 47.11.+j, 02.70.Dh, 02.30.Jr

1 Introduction

One of the Clay Institute $\$1$ million Prize Problems concerns the existence, uniqueness and smoothness of solutions to the Navier-Stokes equations for incompressible fluid flow. The Navier-Stokes equations take the form of an initial value problem for a set of partial differential equations expressing conservation of momentum and mass. The existence of at least one *weak solution* for a given set of data, was proved by J. Leray 1934 [7]. A weak solution satisfies the Navier-Stokes partial differential equations in an average sense, while a *strong solution* or *smooth* solution is required to satisfy the equations in a pointwise sense. A strong solution is also a weak solution, but a weak solution may not be a strong solution.

Leray also referred to the weak solution he proved to exist as a *turbulent solution*. Leray could not prove the uniqueness of a weak (turbulent) solution, neither could he prove existence of a strong solution. Despite heavy efforts by many excellent mathematicians, little improvements on Leray's result have been made. Today the following problems are open:

- (PS) Is there a unique strong (smooth) solution to the Navier-Stokes equations?
- (PW) Is a weak solution of the Navier-Stokes equations unique?

The Clay Prize Problem concerns the mathematical proof of existence (or non-existence) of strong solutions, which we may summarize in (PS), as formulated by C. Fefferman in [2]. We remark that uniqueness of a strong (smooth) solution is considered easy to prove mathematically, so (PS) may be reduced to the question of existence (or non-existence) of strong solutions.

The purpose of this note is to propose an approach to the Clay Prize Problem using modern computational methods. The idea is thus to give input to the question of existence and uniqueness of solutions to the Navier-Stokes equations by computing approximate solutions and studying their uniqueness by computational means. We believe that computational methods may indeed offer some new perspectives. Using a computational approach we may study the question of existence and uniqueness for a set of specific cases with given data, which may be representative for a wider selection of data, but we will not be able to give one answer for all possible data, as the ideal analytical mathematical proof would give. We are thus restricted to a case by case study, and a reformulation of the Prize Problem as a set of 10^3 Prizes each of $\$10^3$, would seem more natural.

Before further scrutinizing the two problems formulated above, we recall that the *Reynolds number* $Re = \frac{UL}{\nu}$, where U is a characteristic flow velocity, L a characteristic length scale, and $\nu > 0$ the *viscosity* of the fluid, may be used to characterize different flow regimes. If Re is relatively small ($Re \leq 10 - 100$), then the flow is viscous and the flow field is ordered and smooth or *laminar*, while for larger Re, the flow will at least partly be *turbulent* with time-dependent non-ordered features on a range of length scales down to a smallest scale (which may be estimated to be of size $Re^{-3/4}$, assuming $L = 1$). We may expect a laminar flow to be determined pointwise in space-time, while in a turbulent flow, because of its rapid fluctuations, we can only expect various mean values to be uniquely determined. In many applications of scientific and industrial importance Re is very large, of the order 10^6 or larger, and the flow shows a combination of laminar and turbulent features.

An example, to which we will return below, is the flow of air around our car when we are traveling at say 60 mph, which is an example of the flow around a *bluff body*, that is a body which is not very streamlined, at $Re \approx 10^6$. We know from observation that there is a large volume behind the car (the "wake") where the air flow is very irregular (turbulent) and seemingly unpredictive in a pointwise sense. A corresponding solution to the Navier-Stokes equations would be *non-smooth* signifying that derivatives of the solution would be very large corresponding to a rapidly fluctuating solution.

Below we shall give also computational evidence of the existence of turbulent solutions. So even if we cannot analytically construct turbulent solutions to the Navier-Stokes equations, we can observe turbulent flow in real life and we can also compute approximate solutions which are turbulent. Of course it is natural to expect that computed solutions approximate the weak (turbulent) solutions proved to exist by Leray.

In this note we now focus on flows at moderate to large Reynolds numbers, where we thus expect to meet both laminar and turbulent flow features. Normalizing the flow velocity U and the lenght scale L both to one, we thus focus on flows with *small vicosity* ν, say typically $\nu \leq 10^{-6}$. We then can argue that (PS) does not seem to give a reasonable formulation of the Clay Prize Problem, because the answer is either trivial or impossible to give. The reason is of course that a turbulent flow is non-smooth and it would seem impossible to uniquely define the exact value of the velocity at a specific point in space-time, as would be required for a strong solution. Thus, because turbulent fluid flow is observed to exist both experimentally and computationally and it appears that the Navier-Stokes equations describe fluid flow, we seem to have clear evidence that strong (smooth) solutions to the Navier-Stokes do not exist in general. So the answer to (PS) would simply be that smooth solutions cannot exist in general, and (PS) would then be solved almost without effort (in the negative sense).

At this point we may have to remark that there may be some (pure) mathematicians who would insist that a turbulent solution could be viewed as a smooth solution with possibly very large derivatives, which indeed would satsify the Navier-Stokes equations in a strong sense. The formulation of the Clay Prize Problem given in [2] indicates that indeed this may be the standpoint. However, we believe that this point of view is not scientifically reasonable because of the extreme sensitivity of pointwise values of a turbulent flow to small perturbations, which effectively makes it impossible to determine a unique velocity and pressure at a specific point in space-time. This couples to the suggested (very simple) proof of uniqueness of smooth solutions which would use a standard Gronwall-type argument involving a constant of the form e^{KT} where K would measure the size of the first derivatives of the velocity. There is evidence that in a turbulent flow we typically have $K \sim Re^{1/2}$, and thus with $Re = 10^6$ and $T = 1$, we would have to deal with amplification factors of size e^{1000}, which is a number beyond any comprehension, and a corresponding uniqueness proof would have no scientific meaning. We shall below also give computational evidence of strong pointwise sensitivity in turbulent flows. Altogether, we believe that from a scientific point of view it is not reasonable to maintain that a turbulent solution may mathematically be viewed as a smooth solution with large derivatives and very strong sensitive to perturbations. We thus believe, following Leray, that a turbulent solution has to be viewed as a weak solution of the Navier-Stokes equations. In mathematical terms we may express the strong pointwise sensitivity in turbulent flow, as follows: The Navier-Stokes equations are not well-posed in a strong sense in the case of turbulent flow.

Having now discarded (PS) as trivial and thus not correctly posed as a Clay Prize Problem, we now focus on (PW) instead: Is a weak solution unique? To attempt to give some (partial) answer, we have to make the uniqueness question more precise. This is because from our experience of turbulent flow, we cannot hope a flow to be uniquely determined in a pointwise sense in space-time, and we must therefore seek some less precise way of measuring uniqueness. As already indicated, it is then natural to consider instead of pointwise quantities some more or less local *mean values* in space-time. More precisely, we choose as a *quantity of interest* or *output* a certain mean value. In the case of the car it may be a meanvalue in time of the total *drag force* $D(t)$ at time t acting on the car in the direction opposite to the motion of the car. The consumption of fuel of a car is directly related to the mean value in time of the drag force $D(t)$, which suitably normalized is referred to as the c_D-coefficient, or *drag coefficient*. Some car manufacturers like to present the c_D of a certain car as an indication of fuel economy (for example $c_D < 0.3$). For a jumbo-jet a decrease in drag with one percent could save $\$400$ million in fuel cost over a 25 year life span.

So we may ask, for example, if the c_D of a car would be uniquely determined? Or in the setting of weak solutions: Will two weak solutions give the same c_D? The corresponding normalized mean value in time of the total force perpendicular to the direction of motion is referred as the *lift coefficient* c_L, which is crucial for flying vehicles (or sailing boats and also very fast cars).

We will approach this type of problem by computational methods, and it is then natural to rephraze the problem as a problem of *computability*. We then specify an output, an *error tolerance* TOL, a certain amount of computational work W (or computational cost), and we ask if we can compute the output up to the tolerance TOL with the available work W. For example, we may ask if we can compute the

c_D-coefficient of a specific car up to a tolerance of 5% on our PC within 1 hour?

More generally we propose the following formulation of the Clay Prize Problem:

- (PC) For a given flow, what output can be computed to what tolerance to what cost?

We may view (PC) as a computational version of a variant of (PW) of the form:

- (PWO) Is the output of a weak solution unique?

(PWO) can alternatively be phrazed as a question of well-posedness in a weak sense. We refer to (PWO) as a question of *weak uniqueness* with respect to a given output. Below we will approach the questions of weak uniqueness of the mean value c_D and the momentary value $D(t)$ of the drag force. Of course, (PWO) couples to the concept *observable quantities* of basic relevance in physic. It may seem that only uniqueness of observable quantities could be the subject of scientific investigation. This couples to questions of classical vs quantum mechanics, e.g. the question if an electron can be located at a specific point in time and space.

We will now address (PC) using the technique of adaptive finite element methods with a posteriori error estimation based on duality developed in [5,6]. The a posteriori error estimate results from an *error representation* expressing the output error as a space-time integral of the *residual* of a computed solution multiplied with *weights* which related to derivatives of the solution of an associated *dual problem*. The weights express *sensitivity* of a certain output with respect to the residual of a computed solution, and their size determine the degree of computability of a certain output: The larger the weights are, the smaller the residual has to be and the more work is required. In general the weights increase as the size of the mean value in the output decreases, indicating increasing computational cost for more local quantities. We give computational evidence in a bluff body problem that a mean value in time of the drag (the c_D) is computable to a reasonable tolerance at a reasonable computational cost, while the value of the drag at a specific point in time appears to be uncomputable even at a very high computational cost.

We can rephrase this result for (PC) as the following result for (PWO): Two weak solutions of a bluff body problem give the same c_D. At least we have then given computational evidence of a certain output uniqueness of weak solutions.

As a general remark on approximate solutions obtained using the finite element method, we recall that a finite element solution is set up to be an approximate weak solution, and thus there is a strong connection between finite element solutions and weak solutions.

Lerays proof of existence of weak solutions is based on a basic energy estimate for approximate solutions of the Navier-Stokes equations, which could be finite element solutions. Using the the basic energy estimate one may extract a weakly convergent subsequence of approximate solutions as the mesh size tends to zero, and this way obtain a proof of existence of a weak solution. Even if the finite element solution on each given mesh is unique, a weak limit of a sequence of finite element solutions does not have to be unique, and thus the Leray solution is not necessarily unique. Of course, with this perspective the questions (PWO) and (PC) become closely coupled: (PWO) is close to the question of output uniqueness of a weak limit of a sequence of finite element solutions, which is close to the output computability (PC).

2 The Navier-Stokes equations

The Navier-Stokes equations for an incompressible fluid with constant kinematic viscosity $\nu > 0$ occupying a volume Ω in $\mathbb{R}^3$ with boundary Γ, take the form:

$$\begin{aligned} \dot{u} + (u \cdot \nabla)u - \nu\Delta u + \nabla p &= f \qquad && \text{in } \Omega \times I, \\ \nabla \cdot u &= 0 && \text{in } \Omega \times I, \\ u &= 0 && \text{on } \Gamma \times I, \\ u(\cdot, 0) &= u^0 && \text{in } \Omega, \end{aligned} \tag{1}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the *velocity* and $p(x,t)$ the *pressure* of the fluid at $(x,t) = (x_1, x_2, x_3, t)$, and $f(x,t)$, $u^0(x)$, $I = (0,T)$, is a given driving force, initial data and time interval, respectively. For simplicity and definiteness we assume homogeneous Dirichlet boundary conditions for the velocity.

The first equation in (1) expresses conservation of momentum (Newton's Second Law) and the second equation expresses conservation of mass in the form of incompressibility.

The Navier-Stokes equations formulated 1821-45 appear to give an accurate description of fluid flow including both laminar and turbulent flow features. *Computational Fluid Dynamics CFD* concerns the computational simulation of fluid flow by solving the Navier-Stokes equations numerically. To computationally resolve all the features of a flow in a *Direct Numerical Simulation DNS* seems to require of the order Re^3 mesh points in space time, so already a flow at $Re = 10^6$ would require $Re^3 = 10^{18}$ mesh points in space-time, and thus would seem to be impossible to solve on any forseeable computer.

The computational challenge is to compute high Reynolds number flows (e.g $Re = 10^6$) using less computational effort than in a DNS. We shall see that for certain mean value outputs such as the c_D or c_L coefficients, this indeed appears to be possible: We give evidence that the c_D and c_L of a surface mounted cube may be computed on a PC up to a tolerance of a few percent (but not less).

3 The Basic Energy Estimate for the Navier-Stokes Equations

We now derive a basic stability estimate of energy type for the velocity u of the Navier-Stokes equations (1), assuming for simplicity that $f = 0$. This is about the only analytical a priori estimate which is known for the Navier-Stokes equations.

Scalar multiplication of the momentum equation by u and integration with respect to x gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,dx + \nu\sum_{i=1}^{3}\int_\Omega |\nabla u_i|^2\,dx = 0,$$

because by partial integration (with boundary terms vanishing),

$$\int_\Omega \nabla p\cdot u\,dx = -\int_\Omega p\nabla\cdot u\,dx = 0$$

and

$$\int_\Omega (u\cdot\nabla)u\cdot u\,dx = -\int_\Omega (u\cdot\nabla)u\cdot u\,dx - \int_\Omega \nabla\cdot u|u|^2\,dx$$

so that

$$\int_\Omega (u\cdot\nabla)u\cdot u\,dx = 0.$$

Integrating next with respect to time, we thus obtain the following basic a priori stability estimate for $T > 0$:

$$\begin{aligned} \|u(\cdot,T)\|^2 + D_\nu(u,T) &= \|u^0\|^2, \\ D_\nu(u,T) &= \nu\sum_{i=1}^{3}\int_0^T \|\nabla u_i\|^2\,dt, \end{aligned} \tag{2}$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$-norm. This estimate gives a bound on the kinetic energy of the velocity with $D_\nu(u,T)$ representing the total *dissipation* from the viscosity of the fluid over the time interval $[0,T]$. We see that the growth of this term with time corresponds to a decrease of the velocity (momentum) of the flow (with $f = 0$).

The characteristic feature of a turbulent flow is that $D_\nu(u,T)$ is comparatively large, while in a laminar flow with ν small, $D_\nu(u,T)$ is small. With $D_\nu(u,T) \sim 1$ in a turbulent flow and $|\nabla u|$ uniformly distributed, we may expect to have pointwise

$$|\nabla u_i| \sim \nu^{-1/2}. \tag{3}$$

4 Weak solutions

From the basic energy estimate, Leray derived the existence of a weak solution $(u, p) \in V \times Q$ of the Navier-Stokes equations defined by:

$$\begin{aligned} R_\nu(u, p; v, q) \equiv & ((\dot{u}, v)) + ((u \cdot \nabla u, v)) - ((\nabla \cdot v, p)) + ((\nabla \cdot u, q)) \\ & + ((\nu \nabla u, \nabla v)) - ((f, v)) = 0 \quad \forall (v, q) \in V \times Q, \end{aligned} \tag{4}$$

assuming $u(0) = u^0 \in L_2(\Omega)^3$ and $f \in L_2(I; H^{-1}(\Omega)^3)$, where

$$\begin{aligned} V &= \{v : v \in L_2(I; H_0^1(\Omega)^3), \dot{v} \in L_2(I; H^{-1}(\Omega)^3)\}, \\ Q &= L_2(I; L_2(\Omega)), \end{aligned}$$

where $H_0^1(\Omega)^3$ is the usual Sobolev space of vector functions being square integrable together with their first derivatives over Ω, with dual $H^{-1}(\Omega)^3$, and $((\cdot, \cdot))$ denoting the corresponding $L_2(I; L_2(\Omega))$ inner product or pairing. As usual, $L_2(I; X)$ with X a Hilbert space denotes the set of functions $v : I \rightarrow X$ which are square integrable. Below we write $L_2(X)$ instead of $L_2(I; X)$ and $L_2(H^1)$ and $L_2(H^{-1})$ instead of $L_2(H_0^1(\Omega)^3)$ and $L_2(H^{-1}(\Omega)^3)$. Note that the term $((u \cdot \nabla u, v))$ is interpreted as $-\sum_{i,j}((u_i u_j, v_{j,i}))$, where $v_{j,i} = \partial v_j / \partial x_i$.

5 Computational solution

We now consider a computational solution of the Navier-Stokes equations. In [5,6] we have developed stabilized Galerkin methods for solving the Navier-Stokes equations based on the weak formulation (4). Without here going into details of the construction of these methods, which we refer to as Generalized Galerkin or G^2, we can describe these methods as producing an approximate solution $(u_h, p_h) \in V_h \times Q_h$, where $V_h \times Q_h$ is a piecewise polynomial finite element subspace of $V \times Q$ defined on space-time meshes with h representing the mesh size in space-time, defined by the following discrete analog of (4)

$$R_h(u_h, p_h; v, q) = 0 \quad \text{for all } (v, q) \in V_h \times Q_h, \tag{5}$$

expressing that the discrete residual $R_h(u_h, p_h) = R_h(u_h, p_h : \cdot, \cdot)$ is orthogonal to $V_h \times Q_h$. Note that in the finite element method (5) we use an *artificial viscosity* of size h instead of the physical viscosity ν assuming $h > \nu$. There are other more sophisticated ways of introducing a (necessary) artificial viscosity coupled to weighted least squares stabiliztion in G^2, but here we consider the simplest form of stabilization.

The finite element solution satisfies an energy estimate analogous to (2) of the form

$$\|\sqrt{h} \nabla u_h\| \leq C, \tag{6}$$

where $\|\cdot\|$ denotes the $L_2(L_2)$-norm, which follows by choosing $(v, q) = (u_h, p_h)$ in (5). Here and below, C is a positive constant of unit size.

We will see below that to estimate an output error, we will have to estimate $R_\nu(u_h, p_h; \varphi_h, \theta_h)$, where (φ_h, θ_h) is the solution of a certain linear dual problem with data connected to the output. In general, (φ_h, θ_h) will not belong to the finite element subspace, and we will thus need to estimate $R_\nu(u_h, p_h; \varphi_h, \theta_h)$. The basic estimate for this quantity takes the form

$$|R_\nu(u_h, p_h; \varphi_h, \theta_h)| \leq C\sqrt{h}\|\varphi_h\|_{L_2(H^1)}, \tag{7}$$

if we omit the relevant θ_h-term assuming exact incompressibility, and C denotes a constant of moderate size. To motivate this estimate, we observe that estimating separately the dissipative term in G^2 (with viscosity h) using the energy estimate (6), we get by Cauchy's inequality

$$|((h\nabla u_h, \nabla\varphi))| \leq \|\sqrt{h}\nabla u_h\|\|\sqrt{h}\nabla\varphi\| \leq C\sqrt{h}\|\varphi\|_{L_2(H^1)}.$$

One can now argue that the remaining part of the residual can be estimated similarly, which leads to (7). We conclude that we expect the residual of (u_h, p_h) to be small (of size $h^{1/2}$) in a weak norm. However, we cannot expect the residual to be small in a strong sense: We would except the residual in an L_2-sense to be of size $h^{-1/2}$ reflecting the basic energy estimate (6), which suggests that $|\nabla u_h| \sim h^{-1/2}$ paralleling (3).

6 Output error representation

We now proceed to estimate the error in certain mean value outputs of a computed finite element solution (u_h, p_h) as compared to the output of a weak solution (u, p). We then consider an output of the form

$$M(u) = ((u, \psi))$$

where $\psi \in L_2(L_2)$ is a given (smooth) function. The output $M(u)$ then corresponds to a mean value in space and time of the velocity u with the function ψ appearing as a weight. We then establish an *error representation* in terms of the residual of the computed solution and the solution (φ_h, θ_h) of a certain linear dual problem (with coefficients depending on both u and u_h) to be specified below, of the form

$$M(u) - M(u_h) = R_\nu(u_h, p_h; \varphi_h, \theta_h). \tag{8}$$

We can then attempt to estimate the output error by using (7) to get

$$|M(u) - M(u_h)| \leq C\sqrt{h}\|\varphi_h\|_{L_2(H^1)}, \tag{9}$$

and the crucial question will thus concern the size of $\|\varphi_h\|_{L_2(H^1)}$. More precisely, we compute (an approximation of) the dual solution (φ_h, θ_h) and directly evaluate $R_\nu(u_h, p_h; \varphi_h, \theta_h)$, but we may use (9) to get a rough idea on the dependence of $M(u) - M(u_h)$ on the mesh size h. We will then obtain convergence in output if, roughly speaking, $\|\varphi_h\|_{L_2(H^1)}$ grows slower than $h^{-1/2}$.

We need here to make the role of h vs ν more precise. We assume that ν is quite small, say $\nu \leq 10^{-6}$ so that it is inconceivable that in computation we could reach $h \leq \nu$; we would rather have $10^{-4} \leq h \leq 10^{-2}$. In the finite element method we use an artificial viscosity of size h instead of the physical viscosity ν and thus computing on a sequence of meshes with decreasing h, could be seen as computing a sequence of solutions to problems with decreasing effective viscosity of size h. We would then be interested in the "limit" with $h = \nu$, and we would by observing the convergence (or divergence) for $h > \nu$ seek to draw a conclusion concerning the case $h = \nu$. So, in the computational examples to be presented we compute on a sequence of successively refined meshes with decreasing h and we evaluate the quantity $R_\nu(u_h, p_h; \varphi_h, \theta_h)$ to seek to determine convergence (or divergence) for a specific output.

7 The dual problem

The dual problem takes the following form, starting from a finite element solution (u_h, p_h) and a weak solution (u, p) with ψ a given (smooth) function: Find (φ_h, θ_h) with $\varphi_h = 0$ on Γ, such that

$$
\begin{aligned}
-\dot{\varphi}_h - (u \cdot \nabla)\varphi_h + \nabla u_h \cdot \varphi_h - \nu\Delta\varphi_h + \nabla\theta_h &= \psi \qquad && \text{in } \Omega \times I, \\
\operatorname{div} \varphi_h &= 0 && \text{in } \Omega \times I, \\
\varphi_h(\cdot, T) &= 0 && \text{in } \Omega,
\end{aligned}
\tag{10}
$$

where $(\nabla u_h \cdot \varphi_h)_j = (u_h)_{,j} \cdot \varphi_h$. This is a linear convection-diffusion-reaction problem, where the time variable runs "backwards" in time with initial value ($= 0$) given at final time T. The reaction coefficient ∇u_h is large and highly fluctuating, and the convection velocity u is of unit size and is also fluctuating. A standard Gronwall type estimate of the solution (φ_h, θ_h) in terms of the data ψ would bring in an exponential factor e^{KT} with K a pointwise bound of $|\nabla u_h|$ which would be enormous, as indicated above. When we compute the solution (φ_h, θ_h) we note that (φ_h, θ_h) does not seem to explode exponentially at all, as would be indicated by Gronwall. Intuitively, by cancellation in the reaction term, with roughly as much production as consumption, (φ_h, θ_h) grows very slowly with deceasing h, and as we have said, the crucial question will be the growth of the quantity $\|\varphi_h\|_{L_2(H^1)}$.

To establish the error representation (8) we multiply (10) by $u - u_h$, integrate by parts, and use the fact that

$$(u \cdot \nabla)u - (u_h \cdot \nabla)u_h = (u \cdot \nabla)e + \nabla u_h \cdot e$$

where $e = u - u_h$.

In the computation, we have to replace the convection velocity u by the computed velocity u_h. We don't expect u_h to necessarily be close pointwise to u, so we have to deal with the effect of a large perturbation in the dual linear problem. In the computations we get evidence that the effect on a crucial quantity like $\|\varphi_h\|_{L_2(H^1)}$ may be rather small, if the output is c_D or c_L. More precisely, our computations show in these cases a quite slow logarithmic growth of $\|\varphi_h\|_{L_2(H^1)}$ in terms of $1/h$, which indicates that the large perturbation in u indeed has little influence on the error representation for c_D and c_L.

The net result is that we get evidence of output uniqueness of weak solutions in the case the output is c_D or c_L. We contrast this with computational evidence that an output of the momentary drag $D(t)$ for a given specific point in time t, is not uniquely determined by a weak solution.

8 Output uniqueness of weak solutions

Suppose we have two weak solutions (u, p) and $(\hat{u}, \hat{p})$ of the Navier Stokes equations with the same data. Let (φ_h, θ_h) be a corresponding dual solution defined by the dual equation (10) with u_h replaced by $\hat{u}$ and a given output (given by the function ψ). Output uniqueness will then hold if $\|\varphi_h\|_{L_2(H^1)} < \infty$.

In practice, we will seek to compute $\|\varphi_h\|_{L_2(H^1)}$ approximatively, replacing both u and $\hat{u}$ as coefficients in the dual problem by a computed solution u_h, thus obtaining an approximate dual velocity φ_h. We then study $\|\varphi_h\|_{L_2(H^1)}$ as h decreases and we extrapolate to $h = \nu$. If the extrapolated value $\|\varphi_\nu\|_{L_2(H^1)} < \infty$, or rather is not too large, then we have evidence of output uniqueness. If the extrapolated value is very large, we get indication of output non-uniqueness. As a crude test of largeness of $\|\varphi_\nu\|_{L_2(H^1)}$, it appears natural to use $\|\varphi_\nu\|_{L_2(H^1)} >> \nu^{-1/2}$.

We may further use a slow growth of $\|\varphi_h\|_{L_2(H^1)}$ as evidence that it is possible to replace both u and $\hat{u}$ by u_h in the computation of the solution of the dual problem: a near constancy indicates a desired robustness to (possibly large) perturbations of the coefficients u and $\hat{u}$.

We now proceed to give computational evidence.

9 Computational results: Uniqueness of c_D and c_L

The computational example is a bluff body benchmark problem at the *CDE-Forum* [1], and is described in detail in [3,4].

We compute the mean value in time of drag and lift forces on a surface mounted cube in a rectangular channel from an incompressible fluid governed by the Navier-Stokes equations (1), at $Re = 40.000$ based on the cube side length and the bulk inflow velocity. We compute the mean values over a time interval of a length corresponding to 40 cube side lengths, which we take as approximations of c_D and c_L defined as mean values over very long time.

The incoming flow is laminar time-independent with a laminar boundary layer on the front surface of the body, which separates and develops a turbulent time-dependent wake attaching to the rear of the body. The flow is thus very complex with a combination of laminar and turbulent features including boundary layers and a large turbulent wake, see Figure 2.

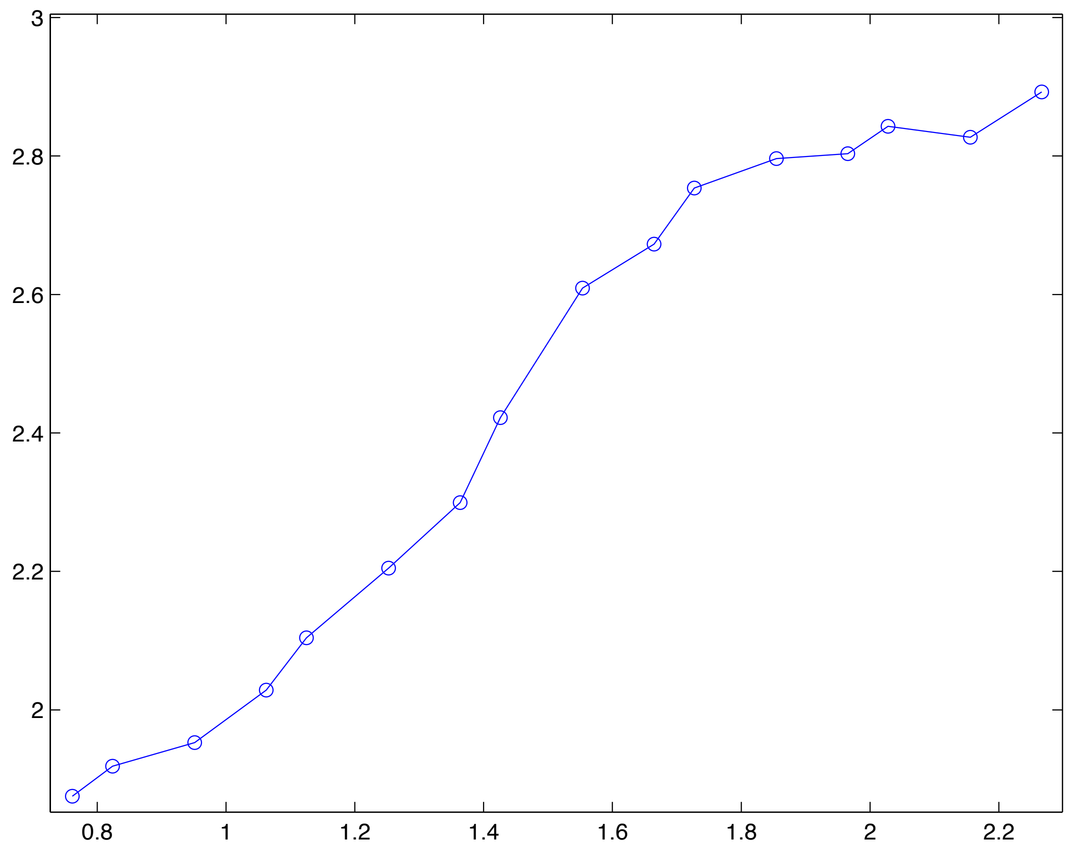

Figure 1. $\log_{10}$-$\log_{10}$-plot of $\|\varphi_h\|_{L_2(H^1)}$ as a function of $1/h$.

The dual problem corresponding to c_D has boundary data of unit size for φ_h on the cube in the direction of the main flow, acting on the time interval underlying

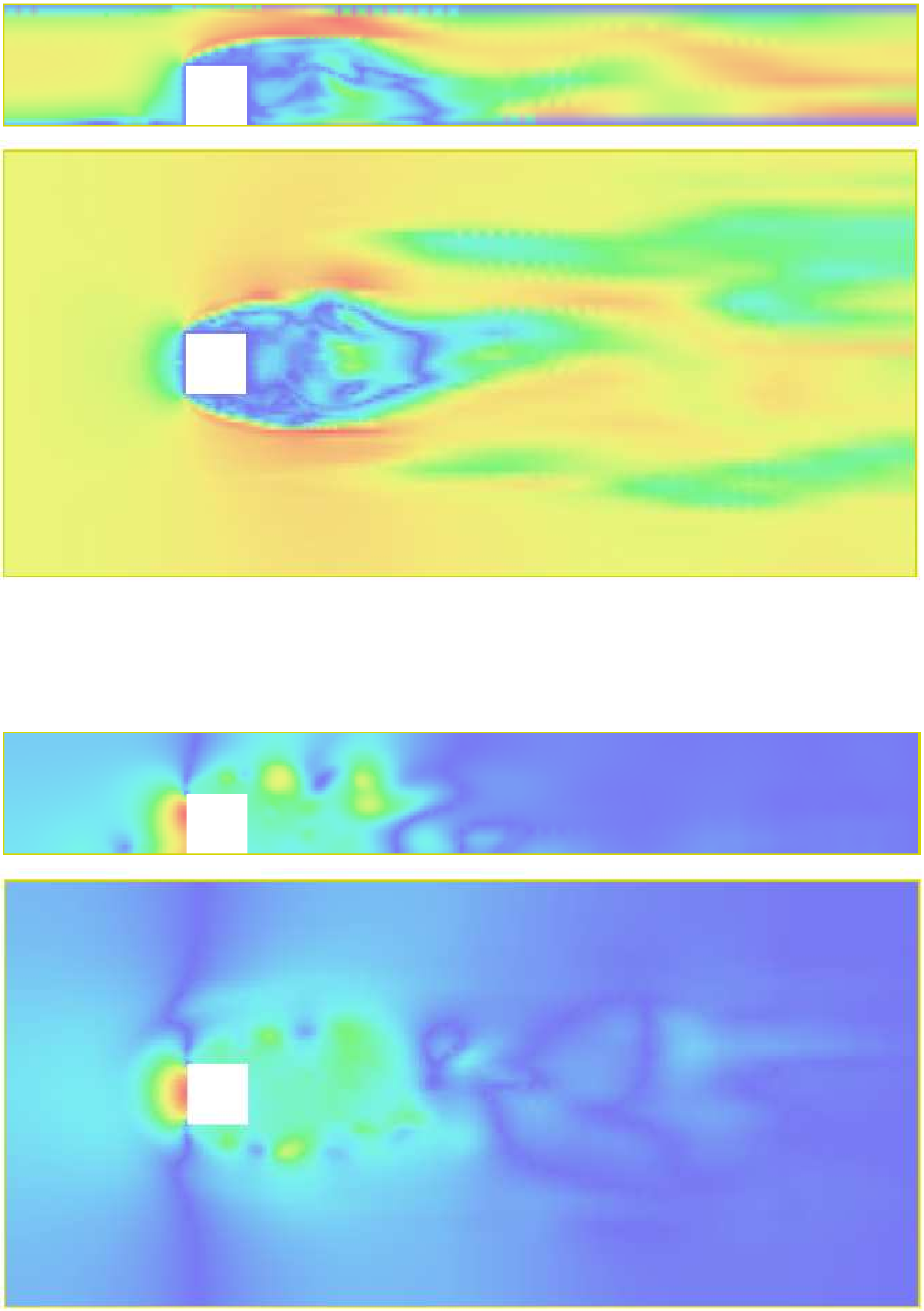

Figure 2. Velocity $|u_h|$ (upper) and pressure $|p_h|$ (lower), from side and top respectively.

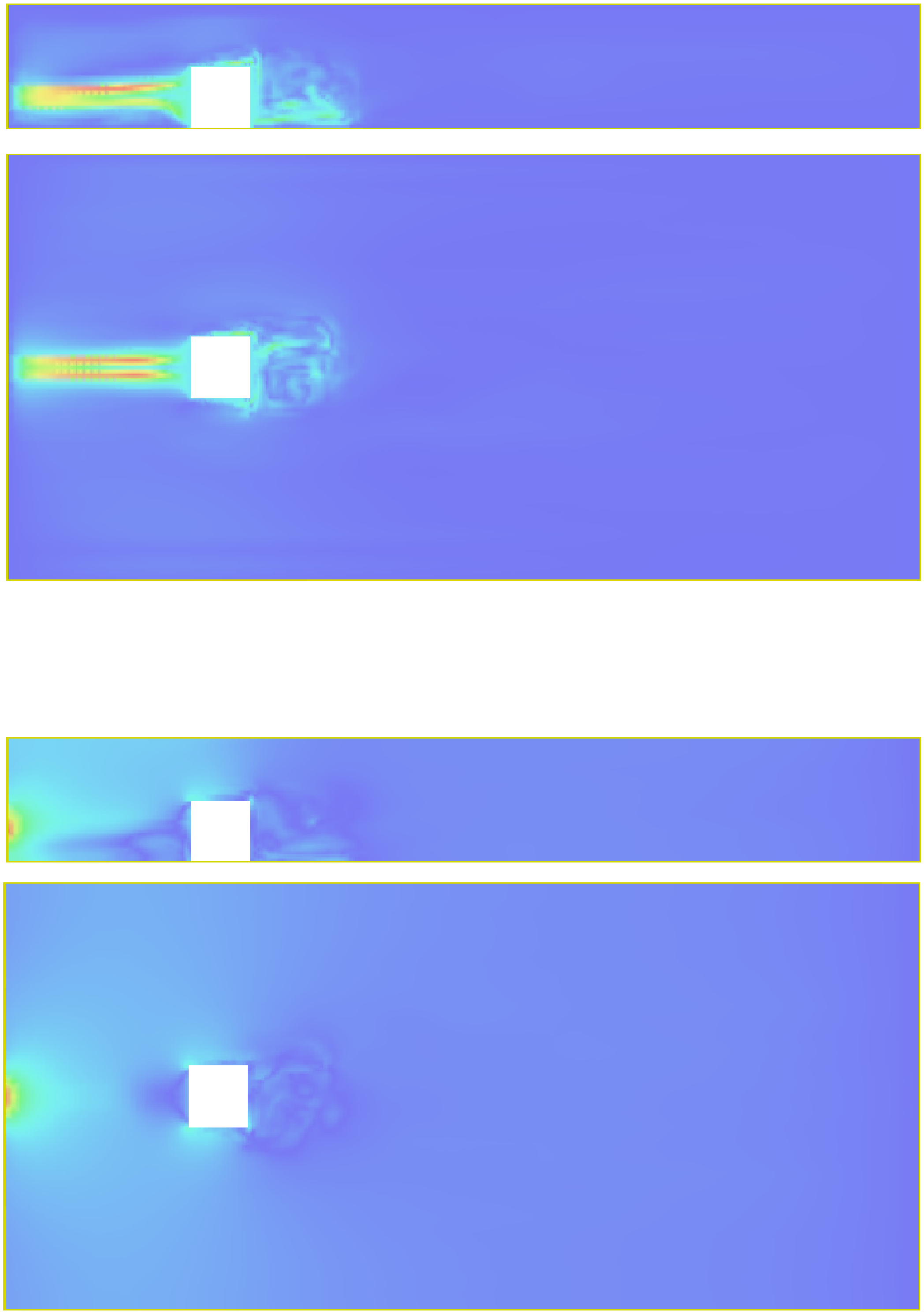

Figure 3. Dual velocity $|\varphi_h|$ (upper) and dual pressure $|\theta_h|$ (lower), with respect to the computation of mean drag c_D, from side and top respectively.

the mean value, and zero boundary data elsewhere. A snapshot of the dual solution corresponding to c_D is shown in Figure 3, and in Figure 1 we plot $\|\varphi_h\|_{L_2(H^1)}$ as a function of h^{-1}, with h the smallest element diameter in the computational mesh.

We find that $\|\varphi_h\|_{L_2(H^1)}$ shows a slow logarithmic growth, and extrapolating we find that $\|\varphi_\nu\|_{L_2(H^1)} \sim \nu^{-1/2}$. We take this as evidence of computability and weak uniqueness of c_D. We obtain similar results for the lift coefficient c_L.

10 Computational results: Non-uniqueness of $D(t)$

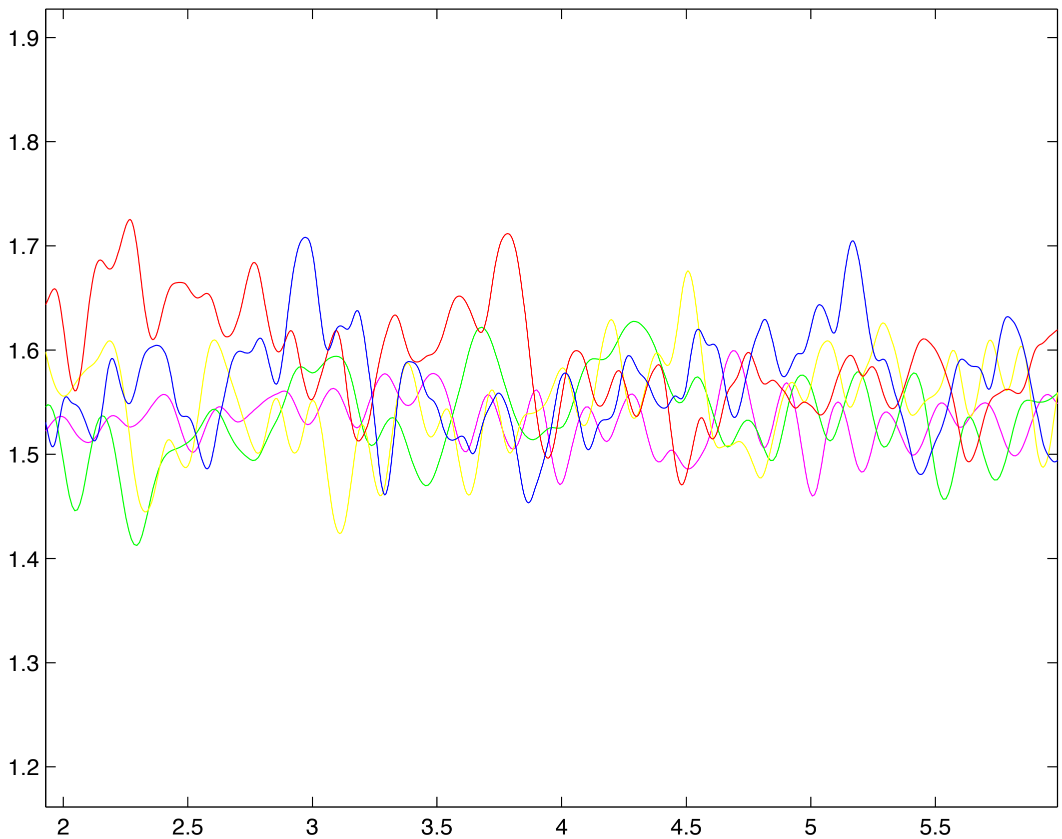

Figure 4. $D(t)$ (normalized) as a function of time, for the 5 finest computational meshes.

We now investigate the computability and weak uniqueness of the total drag force $D(t)$ at a specific time t. In Figure 4 we show the variation in time of $D(t)$ computed on different meshes, and we notice that $D(t)$ for a given t appears to converge very slowly or not at all with decreasing h.

We now choose one of the finer meshes corresponding to $h^{-1} \approx 150$, and we compute the dual solution over a time interval $[0, T]$ with data corresponding to a mean value of $D(t)$ over a time interval $[T_0, T]$, where we let $T_0 \rightarrow T$. We thus seek to compute $D(T)$.

In Figure 5 we find a growth of $\|\varphi_h\|_{L_2(H^1)}$ similar to $|T - T_0|^{-1/2}$. The results show that for $|T - T_0| = 1/16$ we have $\|\varphi_h\|_{L_2(H^1)} \approx 10\nu^{-1}$, and extrapolation of the computational results indicate further growth of $\|\varphi_h\|_{L_2(H^1)}$ as $T_0 \to T$ and $h \to \nu$. We take this as evidence of non-computability and weak non-uniqueness of $D(T)$.

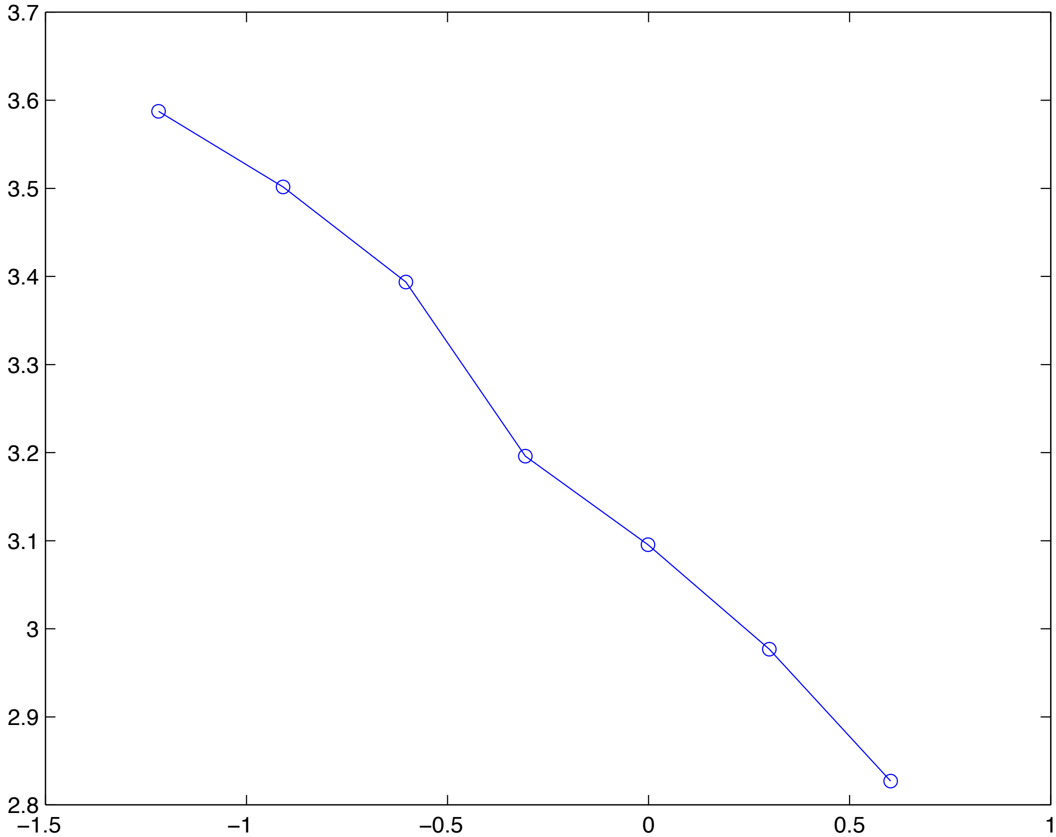

Figure 5. $\|\varphi_h\|_{L_2(H^1)}$ corresponding to computation of the mean drag force (normalized) over a time interval $[T_0, T]$, as a function of the interval length $|T - T_0|$ ($\log_{10}$-$\log_{10}$-plot).

11 Conclusion

We have given computational evidence of weak uniqueness of mean values such as c_D and c_L and weak non-uniqueness of a momentary value $D(t)$ of the total drag. In the computations we observe this phenomenon as a continuous degradation of computability (increasing error tolerance) as the length of the mean value decreases to zero. When the error tolerance is larger than one, then we have effectively lost computability, since the oscillation of $D(t)$ is of unit size. We compute c_D and c_L as mean values of finite length (of size 10), and thus we expect some variation also in these values, but on a smaller scale than for $D(t)$, maybe of size $= 0.1$ with 0.01 as a possible lower limit with present computers. Thus the distinction between computability (or weak uniqueness) and non-computability (weak non-uniqueness) may in practice be just one or two orders of magnitude in output error, rather than a difference between 0 and ∞.

Of course, this is what you may expect in a quantified computational world, as compared to an ideal mathematical world. In particular, we are led to a point of view where we measure residuals of approximate weak solutions, rather than working with exact weak solutions with zero residuals. A such quantified mathematical world is in fact richer than an ideal zero residual world, and thus possibly more accessible.

Acknowledgments

The first author would like to aknowledge the support by DOE grant DE-FG02-88ER25053.

References

[1] *Cde-forum*, `http://www.phi.chalmers.se/cdeforum/`.

[2] C. L. Fefferman, *Official clay prize problem description: Existence and smoothness of the navier-stokes equation*, http://www.claymath.org/millennium/, (2000).

[3] J. Hoffman, *Computation of the mean drag of a surface mounted cube using adaptive dns/les*, submitted to SIAM J. Sci. Comput., (2004).

[4] ——, *Computation of the mean lift of a surface mounted cube using adaptive dns/les*, in preparation, (2004).

[5] J. Hoffman and C. Johnson, *Computability and Adaptivity in CFD*, Encyclopedia of Computational Mechanics (Ed. E. Stein, R. de Borst and T.J.R. Hughes), John Wiley and Sons, 2004.

[6] ——, *A new approach to computational turbulence modeling*, to appear in Comput. Methods Appl. Mech. Engrg., (2004).

[7] J. Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace.*, Acta Mathematica, 63 (1934), pp. 193–248.