# Antisymmetric Hamiltonians: Variational Resolutions for Navier-Stokes and Other Nonlinear Evolutions

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**Abstract**

The theory of anti-self-dual (ASD) Lagrangians, introduced in [6], is developed further to allow for a variational resolution of nonlinear PDEs of the form $\Lambda u + Au + \partial\varphi(u) + f = 0$ where $\varphi$ is a convex lower-semicontinuous function on a reflexive Banach space $X$, $f \in X^*$, $A : D(A) \subset X \to X^*$ is a positive linear operator, and $\Lambda : D(\Lambda) \subset X \to X^*$ is a nonlinear operator that satisfies suitable continuity and antisymmetry properties. ASD Lagrangians on path spaces also yield variational resolutions for nonlinear evolution equations of the form $\dot{u}(t) + \Lambda u(t) + Au(t) + \partial\varphi(u(t)) + f = 0$ starting at $u(0) = u_0$. In both stationary and dynamic cases, the equations associated to the proposed variational principles are not derived from the fact that they are critical points of the action functional, but because they are also zeroes of the Lagrangian itself. For that we establish a general nonlinear variational principle that has many applications, in particular to Navier-Stokes-type equations, to generalized Choquard-Pekar Schrödinger equations with nonlocal terms, as well as to complex Ginsburg-Landau-type initial-value problems. The case of Navier-Stokes evolutions is more involved and will be dealt with in [9]. The general theory of antisymmetric Hamiltonians and its applications is developed in detail in an upcoming monograph [7]. © 2007 Wiley Periodicals, Inc.

## 1 Introduction

We consider the question of solving variationally various nonlinear elliptic equations of the type

$$\begin{cases} -\Delta u + f(x, u, \nabla u) = 0 & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a given nonlinearity or general equations such as

$$\begin{cases} -\Delta u + \Lambda(x, u) = 0 & \text{for } x \in \Omega, \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Lambda(x, \cdot)$ can be a general nonlocal operator as well as the corresponding evolution equations. It is well-known that a variational resolution of (1.1) is possible whenever $f$ is—for example—free of the gradient term, since (weak) solutions can then be obtained as critical points of the functional $I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2\,dx +$

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$F(x,u)dx$ where $F(x,\cdot)$ is a primitive of $f(x,\cdot)$ and that minimization of $I$ suffices as soon as $s \to F(x,s)$ is convex. This method, however, breaks down as soon as $f(x,u,\nabla u)$ comprises terms such as the *first-order* transport operator $\mathbf{a}\cdot\nabla u$ where $\mathbf{a}$ is a given vector field on $\Omega$, or the *nonlinear* Stokes operator $u\cdot\nabla u$, or when we deal with *nonlocal* convolution operators such as $\Lambda u = (w\star g(u))h(u)$ for different functions $g$ and $h$. In such cases, equations (1.1) and (1.2) are not Euler-Lagrange equations associated to an appropriate energy functional, and our main premise in this paper is that they can still, in many important cases, be resolved via an appropriately designed variational principle.

Motivated by our proof in [10] of a conjecture of Brezis and Ekeland [3, 4] (see also Auchmuty [1, 2], such a variational framework was developed in [6] for various semilinear PDEs and dissipative evolutions, which are not normally of Euler-Lagrange type, but whose solutions can still be obtained as minima of functionals of the form $I(u) = L(u, Au)$ or, in the case of evolution equations,

$$I(u) = \int_0^T L(t, u(t), \dot{u}(t) + Au(t))dt + \ell(u(0), u(T)). \tag{1.3}$$

The Lagrangians $L$ and $\ell$ must obey certain "self-duality" conditions, while the operators $A$ are essentially linear and skew-adjoint. For such "anti-self-dual" (ASD) Lagrangians, defined below, the minimal value of these functionals is always zero and, just like the self (and anti-self) dual equations of quantum field theory (e.g., Yang-Mills and others), the equations associated to such minima are not derived from the fact they are critical points of the action functional, but because an associated nonnegative Lagrangian, obtained by completing the square, is zero as soon as the functional attains its natural infimum.

The most basic *anti-self-dual Lagrangians* on phase space $X \times X^*$, where $X$ is a reflexive Banach space, is of the form $L(x,p) = \varphi(x) + \varphi^*(-p)$ where $\varphi$ is a convex, lower-semicontinuous function on $X$ and $\varphi^*$ is its Fenchel-Legendre transform. They yield variational resolution for differential inclusions and parabolic evolutions of the form

$$-Au \in \partial\varphi(u) \quad \text{and} \quad -\dot{u}(t) - Au(t) \in \partial\varphi(u(t)) \quad \text{with } u(0) = u_0. \tag{1.4}$$

The self-dual functionals in these cases are $I(u) = \varphi(u) + \varphi^*(-Au)$ and

$$\begin{aligned} I(u) = &\int_0^T \{\varphi(u(t)) + \varphi^*(-Au(t) - \dot{u}(t))\}\,dt \\ &+ \frac{1}{2}(|u(0)|^2 + |u(T)|^2) - 2\langle u(0), u_0\rangle + |u_0|^2, \end{aligned}$$

respectively. Note that here the operators $A$ are linear but not necessarily self-adjoint and hence cannot be treated via standard Euler-Lagrange theory. Our goal in this paper is to develop the theory further to be able to include some of the most basic nonlinear operators in this new variational framework. For that, we need to work with the dual framework of *antisymmetric Hamiltonians*, which is a substantial enlargement of the class of Hamiltonians that are obtained from self-dual Lagrangians by taking their Legendre transforms in one of the variables.

To illustrate our approach on equation (1.1), we use that the Laplacian $-\Delta$ is the differential of the Dirichlet energy $\varphi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx$, which is a convex continuous function on the space $H_0^1(\Omega)$, and consider the Legendre-Fenchel dual functional of $\varphi$ defined on $H^{-1}(\Omega)$ by

$$\varphi^*(v) = \sup\{\langle v, u\rangle - \varphi(u) : u \in H_0^1(\Omega)\} = \frac{1}{2}\int_\Omega |(-\Delta)^{-1/2}v|^2\,dx,$$

We then consider the following functional on $H_0^1(\Omega)$:

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |(-\Delta)^{-1/2} f(x,u,\nabla u)|^2\,dx + \int_\Omega u f(x,u,\nabla u)dx \\ &= \varphi(u) + \varphi^*(-f(\,\cdot\,,u,\nabla u) + \langle u, f(\,\cdot\,,u,\nabla u)\rangle. \end{aligned}$$

Legendre-Fenchel duality yields that $J(u) \geq 0$ on $H_0^1(\Omega)$, and the limiting case of this inequality reduces the problem of solving (1.1) to proving that the infimum of the functional $J$ is 0, in addition to being attained.

Unlike the linear case, where the problem of attainability and the more important issue of identifying the value of the infimum were dealt with via duality theory in convex optimization, we approach the nonlinear case via a version of the Ky Fan min-max theorem [5]. Indeed, we can write for $u \in H_0^1(\Omega)$,

$$\begin{aligned} J(u) &:= \varphi(u) + \varphi^*(-f(\,\cdot\,,u,\nabla u)) + \langle u, f(\,\cdot\,,u,\nabla u)\rangle \\ &= \sup\{\varphi(u) - \varphi(w) + \langle f(\,\cdot\,,u,\nabla u), u-w\rangle : w \in H_0^1(\Omega)\} \\ &:= \sup\{H(u,w) : w \in H_0^1(\Omega)\} \end{aligned}$$

The problem is then to show that the infimum

$$\inf_{u\in H_0^1} J(u) = \inf_{u\in H_0^1}\sup_{w\in H_0^1} H(u,w)$$

is equal to 0, in addition to being achieved. Note that for every $u \in H_0^1(\Omega)$, $H(u,u) = 0$ and $H(u,w)$ is concave in $w$, and so in order to use the Ky-Fan theorem, what is left is to impose conditions on $f$ to insure coercivity (i.e., $J(u) \to +\infty$ if $|u|_{H_0^1} \to \infty$) and some compactness by ensuring that $u \to H(u,w)$ is weakly lower-semicontinuous. One can then deduce by the min-max principle mentioned above that there exists $\bar{u}$ such that $\sup_{w\in H_0^1(\Omega)} H(\bar{u},w) \leq 0$, which is then a solution for (1.1). Now this simple idea becomes quite powerful once one notices that it applies to a much larger class of Lagrangians than those of the form $\varphi(x) + \varphi^*(-p)$. Indeed, it is also shown in [6] that the class of anti-self-dual Lagrangians is rich enough to contain:[1]

[1] More recently the author actually established that a self-dual Lagrangian can be associated to any equation involving a maximal monotone vector field [8].

- the superposition of skew-adjoint operators with the gradients of convex functions,
- the addition of appropriate boundary Lagrangians in order to solve problems with boundary constraints, and
- the lifting of anti-self-dual Lagrangians to path spaces yielding variational resolutions to dissipative initial-value problems.

The above variational approach applied to general anti-self-dual Lagrangians allows us to resolve variationally a large class of PDEs, and in particular nonlinear Lax-Milgram problems of type

$$\Lambda u + Au + f \in -\partial\varphi(u) \tag{1.5}$$

as well as parabolic evolution equations of the form

$$\begin{cases} \dot{u}(t) + \Lambda u(t) + Au(t) + f(t) \in -\partial\varphi(t, u(t)) & \text{a.e. } t \in [0, T], \\ u(0) = u_0, \end{cases} \tag{1.6}$$

where $u_0$ is a given initial value. Here $\varphi$ is a convex lower-semicontinuous functional, $\Lambda$ is a nonlinear "conservative" operator, and $A$ is a linear and not necessarily bounded positive operator.

As applications to the method, we provide a variational resolution to equations involving nonlinear operators such as the Navier-Stokes equation for a fluid driven by its boundary:

$$\begin{cases} (u \cdot \nabla)u + f = \nu\Delta u - \nabla p & \text{on } \Omega, \\ \operatorname{div} u = 0 & \text{on } \Omega, \\ u = u^0 & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where $u^0 \in H^{3/2}(\partial\Omega)$ is such that $\int_{\partial\Omega} u^0 \cdot \mathbf{n}\, d\sigma = 0$, $\nu > 0$, and $f \in L^p(\Omega; \mathbb{R}^3)$. We can also deal with the superposition of such nonlinear operators with non-self-adjoint first-order operators such as

$$\begin{cases} (u \cdot \nabla)u + a \cdot \nabla u + a_0 u + |u|^{m-2}u + f = \nu\Delta u - \nabla p & \text{on } \Omega, \\ \operatorname{div} u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where $a \in C^\infty(\bar{\Omega})$ is a smooth vector field and $a_0 \in L^\infty$ are such that $a_0 - \frac{1}{2}\operatorname{div} a \geq 0$. The method is also applicable to nonlinear equations involving nonlocal terms such as the following generalized Choquard-Pekar Schrödinger equation:

$$-\Delta u + V(x)u = (w * f(u))g(u) \tag{1.9}$$

where $V$ and $w$ are real functions, $V(x) \geq \delta > 0$ for $x \in \mathbb{R}^N$, and $w * f(u)$ denotes the convolution of $f(u)$ and $w$.

The methods extend to the dynamic case where typically we give a variational resolution to the complex Ginzburg-Landau initial-value problem on $\Omega \subseteq \mathbb{R}^N$

$$\begin{cases} \dot{u}(t) - (\kappa + i\alpha)\triangle u + (\gamma + i\beta)|u|^{q-1}u - wu = 0, \\ u(x, 0) = u_0, \end{cases} \tag{1.10}$$

where $\kappa \geq 0$, $\gamma \geq 0$, $q \geq 1$, and $\alpha, \beta \in \mathbb{R}$.

The paper, though sufficiently self-contained, is better read in conjunction with [6]. It is organized as follows: In Section 2, we give the main nonlinear self-dual variational principle, while Section 3 contains its first applications to the variational resolution of various nonlinear equations. In Section 4 we deal with the dynamic case where we provide a general principle for the variational resolution of nonlinear parabolic initial-value problems. This is illustrated in Section 5 by an application to the complex Ginsburg-Landau initial-value problem with various parameters. Further applications to other models in hydrodynamics and magnetohydrodynamics will follow in a forthcoming paper [9]. The general theory of antisymmetric Hamiltonians is detailed in the upcoming monograph [7].

## 2 A Nonlinear Self-Dual Variational Principle

Let $X$ be a reflexive Banach space and let $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a convex lower-semicontinuous function that is not identically equal to $+\infty$. Its Legendre-Fenchel dual (in both variables) is defined on $X^* \times X$ as

$$L^*(q, y) = \sup\{\langle q, x\rangle + \langle p, y\rangle - L(x, p) : x \in X, p \in X^*\}.$$

The (partial) domain of a Lagrangian $L$ is defined as

$$\mathrm{dom}_1(L) = \{x \in X : L(x, p) < +\infty \text{ for some } p \in X^*\}.$$

To each Lagrangian $L$ on $X \times X^*$, we associate its Hamiltonian $H_L : X \times X \to \bar{\mathbb{R}}$ by

$$H_L(x, y) = \sup\{\langle y, p\rangle - L(x, p) : p \in X^*\},$$

which is the Legendre transform in the second variable. The following class of Lagrangians plays a significant role in our proposed variational formulation.

DEFINITION 2.1 Let $L : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ be a convex lower-semicontinuous function. Then $L$ is an *anti-self-dual Lagrangian* (*ASD*) on $X \times X^*$ if

$$L^*(p, x) = L(-x, -p) \quad \text{for all } (p, x) \in X^* \times X. \tag{2.1}$$

We shall frequently use the following basic properties of an ASD Lagrangian:

$$L(x, p) + \langle x, p\rangle \geq 0 \quad \text{for every } (x, p) \in X \times X^* \tag{2.2}$$

and

$$L(x, p) + \langle x, p\rangle = 0 \quad \text{if and only if} \quad (-p, -x) \in \partial L(x, p). \tag{2.3}$$

Basic examples of ASD Lagrangians are $L(x, p) = \varphi(x) + \varphi^*(-p)$ where $\varphi$ is convex lower-semicontinuous on $X$. But as shown in [6], the class of ASD Lagrangians is much richer. For example, if $A : X \to X^*$ is a skew-adjoint operator, then $L(x, p) = \varphi(x) + \varphi^*(-Ax - p)$ is also an ASD-Lagrangian, and as shown in Lemma 2.6 below, this property still holds for a class of unbounded skew-adjoint operators.

Consider now the Hamiltonian $H = H_L$ associated to an ASD Lagrangian $L$ on $X \times X^*$. It is easy to check that they can be characterized as those functions $H : X \times X \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ such that:

- for each $y \in X$, the function $H_y : x \to -H(x, y)$ from $X$ to $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ is convex and
- the function $x \to H(-y, -x)$ is the convex lower-semicontinuous envelope of $H_y$.

It readily follows that for such a Hamiltonian, the function $y \to H(x, y)$ is convex and lower-semicontinuous for each $x \in X$, and that the following inequality holds for every $(x, y) \in X \times X$:

$$H(-y, -x) \leq -H(x, y). \tag{2.4}$$

In particular, we have for every $x \in X$,

$$H(x, -x) \leq 0. \tag{2.5}$$

Note that $H_L$ is always concave in the first variable; however, it is not necessarily upper-semicontinuous in the first variable. This leads to the following notion:

DEFINITION 2.2 A Lagrangian $L \in \mathcal{L}(X)$ will be called *tempered* if for each $y \in \operatorname{dom}_1(L)$, the map $x \to H_L(x, -y)$ from $X$ to $\mathbb{R} \cup \{-\infty\}$ is upper-semicontinuous.

If $L$ is a tempered anti-self-dual Lagrangian, then its corresponding Hamiltonian satisfies

$$H_L(y, x) = -H_L(-x, -y) \quad \text{for all } (x, y) \in X \times \operatorname{dom}_1(L) \tag{2.6}$$

and therefore

$$H_L(x, -x) = 0 \quad \text{for all } x \in \operatorname{dom}_1(L). \tag{2.7}$$

It is also easy to see that if $L$ is tempered, then $\operatorname{dom}_1(L)$ is closed and convex. A typical tempered Lagrangian (respectively, tempered ASD-Lagrangian) is $L(x, p) = \varphi(x) + \psi^*(p)$ (respectively, $L(x, p) = \varphi(x) + \varphi^*(-p)$) where $\varphi$ and $\psi$ are convex and lower-semicontinuous on $X$.

We now introduce the following notion, which extends considerably the class of Hamiltonians associated to self-dual Lagrangians.

DEFINITION 2.3 Let $E$ be a convex subset of a reflexive Banach space $X$. A functional $M : E \times E \to \mathbb{R}$ is said to be an *antisymmetric Hamiltonian* on $E \times E$ if it satisfies the following conditions:

(i) For every $x \in E$, the function $y \to M(x, y)$ is concave on $E$.

(ii) For every $y \in E$, the function $x \to M(x, y)$ is weakly lower-semicontinuous.

(iii) $M(x, x) \leq 0$ for every $x \in E$.

The class of *antisymmetric Hamiltonians* on a given convex set $E$, denoted $\mathcal{H}_B^{\mathrm{asym}}(E)$, is an interesting class of its own. It contains the Maxwellian Hamiltonians $M(x, y) = \varphi(y) - \varphi(x) + \langle Ay, x\rangle$, where $\varphi$ is convex and $A$ is skew-adjoint. More generally:

(1) If $L$ is an anti-self-dual Lagrangian on a Banach space $X$, then the Hamiltonian $M(x, y) = H_L(y, -x)$ is in $\mathcal{H}^{\mathrm{asym}}(X)$.

(2) If $\Lambda : D(\Lambda) \subset X \to X^*$ is a (not necessarily) linear operator that is continuous on its domain for the weak topologies of $X$ and $X^*$, and if the function $x \to \langle \Lambda x, x\rangle$ is weakly lower-semicontinuous on a convex subset $E \subset D(\Lambda)$, then the Hamiltonian $H(x, y) = \langle x - y, \Lambda x\rangle$ is in $\mathcal{H}^{\mathrm{asym}}(E)$.

Examples of such operators are of course the linear positive operators. On the other hand, they also include some linear but not necessarily positive operators such as $\Lambda u = J\dot{u}$, which is weakly continuous on the Sobolev space $H^1_{\mathrm{per}}[0, T]$ of $\mathbb{R}^{2N}$-valued periodic functions on $[0, T]$, where $J$ is the symplectic matrix. They also include the nonlinear Navier-Stokes operator (see below).

Since $\mathcal{H}^{\mathrm{asym}}(X)$ is obviously a convex cone, we can therefore superpose certain nonlinear operators with anti-self-dual Lagrangians, via their corresponding antisymmetric Hamiltonians, to obtain a remarkably rich family that generates nonconvex self-dual functionals as follows:

DEFINITION 2.4 Say that a functional $I : X \to \mathbb{R}^+ \cup \{+\infty\}$ is *self-dual on a convex set* $E \subset X$ if there exists an antisymmetric Hamiltonian $M : E \times E \to \mathbb{R}$ such that

$$I(x) = \sup_{y \in E} M(x, y) \quad \text{for every } x \in E.$$

A key aspect of our variational approach is that solutions of many nonlinear PDEs can be obtained by minimizing self-dual functionals in such a way that the infimum is actually 0, i.e., $I(u) = \inf_{v \in E} I(v) = 0$. This theory of self-dual functionals will be developed in full in the upcoming monograph [7]. In this paper, we shall concentrate on their first and most basic applications.

## 2.1 Unbounded Skew-Adjoint Operators and Regular Maps

Let now $A$ be a linear, not necessarily bounded map from its domain $D(A) \subset X$ into $X^*$ such that $D(A)$ is dense in $X$; we consider the domain of its adjoint $A^*$ defined as:

$$D(A^*) = \{x \in X : \sup\{\langle x, Ay\rangle : y \in D(A), \|y\|_X \leq 1\} < +\infty\}.$$

DEFINITION 2.5 Say that

- $A$ is *antisymmetric* if $D(A) \subset D(A^*)$ and if $A^* = -A$ on $D(A)$ and
- $A$ is *skew-adjoint* if it is antisymmetric and if $D(A) = D(A^*)$.

LEMMA 2.6 *Let $L : X \times X^* \to \mathbb{R}$ be an ASD Lagrangian on a reflexive Banach space $X$ and let $A$ be a linear map from its domain $D(A) \subset X$ into $X^*$. The Lagrangian $L_A$ defined by*

$$L_A(x, p) = \begin{cases} L(x, Ax + p) & \text{if } x \in D(A), \\ +\infty & \text{if } x \notin D(A), \end{cases}$$

*is then itself anti-self-dual on $X \times X^*$ provided one of the following conditions hold*:

(i) $D(A) = X$ (*i.e., $A$ is a bounded skew-adjoint operator*).
(ii) *$A$ is antisymmetric, $L$ is tempered, and* $0 \in \mathrm{dom}_1(L) \subset D(A)$, *or*
(iii) *$A$ is skew-symmetric and the function $x \to L(x, 0)$ is bounded on the unit ball of $X$.*

PROOF: For $(q, y) \in X^* \times D(A)$, set $r = Ax + p$ and write:

$$\begin{aligned} L_A^*(q, y) &= \sup\{\langle q, x\rangle + \langle y, p\rangle - L(x, Ax + p); (x, p) \in X \times X^*\} \\ &= \sup\{\langle q, x\rangle + \langle y, r - Ax\rangle - L(x, r); (x, r) \in D(A) \times X^*\} \\ &= \sup\{\langle q + Ay, x\rangle + \langle y, r\rangle - L(x, r); (x, r) \in X \times X^*\} \\ &= L^*(q + Ay, y) = L(-y, -q - Ay) \\ &= L_A(-y, -q). \end{aligned}$$

(i) If $D(A) = X$, we are done.

Now assume $y \notin D(A)$; therefore $-y \notin D(A)$, and we distinguish the two remaining cases.

(ii) In the antisymmetric case, we have $-y \notin \mathrm{dom}_1(L)$; hence $-H_L(-y, 0) = +\infty$. Since $L$ is tempered, and $0 \in \mathrm{dom}_1(L)$, we get from (2.6) that $H_L(0, y) = -H_L(-y, 0) = +\infty$. It follows that

$$\begin{aligned} L_A^*(q, y) &= \sup_{\substack{x \in \mathrm{dom}_1(L) \\ r \in X^*}} \{\langle y, r - Ax\rangle + \langle x, q\rangle - L(x, r)\} \\ &\geq \sup_{r \in X^*}\{\langle y, r\rangle - L(0, r)\} - C \\ &= H_L(0, y) - C = -H_L(-y, 0) - C = +\infty = L_A(-y, -q). \end{aligned}$$

(iii) In the skew-adjoint case, write

$$\begin{aligned} L_A^*(q,y) &= \sup_{\substack{x\in D(A)\\ r\in X^*}} \{\langle y, r-Ax\rangle + \langle x,q\rangle - L(x,r)\} \\ &\ge \sup_{\substack{x\in D(A)\\ \|x\|_X<1}} \{\langle -y, Ax\rangle + \langle x,q\rangle - L(x,0)\} \end{aligned}$$

Since by assumption $L(x,0) \le K$ whenever $\|x\|_X \le 1$, we obtain since $y \notin D(A) = D(A^*)$ as soon as $-y \notin D(A)$, that

$$\begin{aligned} L_{A,\ell}^*(q,y) &\ge \sup_{\substack{x\in D(A)\\ \|x\|_X\le 1}} \{\langle -y, Ax\rangle - \|q\| - K\} \\ &= +\infty \\ &= L_A(-y,-q) \end{aligned}$$

Therefore $L_A^*(q,y) = L_A(-y,-q)$ for all $(y,q) \in X \times X^*$ and $L_A$ is an anti-self-dual Lagrangian. □

DEFINITION 2.7

(i) Say that a (not necessarily) linear map $\Lambda : D(\Lambda) \subset X \to X^*$ is a *regular map* if

(2.8) $$u \to \Lambda u \text{ is weak-to-weak continuous}$$

and

(2.9) $$u \to \langle \Lambda u, u\rangle \text{ is weakly lower-semicontinuous on } D(\Lambda).$$

(ii) Say that $\Lambda : D(\Lambda) \subset X \to X^*$ is a *regular conservative map* if it satisfies (2.8) and

(2.10) $$\langle \Lambda u, u\rangle = 0 \text{ for all } u \text{ in its domain } D(\Lambda).$$

It is clear that positive, bounded linear operators are necessarily *regular maps* and that regular conservative maps (which include skew-symmetric bounded linear operators) are also *regular* maps. However, there are also plenty of nonlinear regular maps, many of them appearing in the basic equations of hydrodynamics and magnetohydrodynamics (see below and [15]).

## 2.2 A Nonlinear Self-Dual Variational Principle

If now $L$ is an anti-self-dual Lagrangian on $X \times X^*$, then for any map $\Lambda : D(\Lambda) \subset X \to X^*$ we have the following inequality:

(2.11) $$L(x,\Lambda x) + \langle x, \Lambda x\rangle \ge 0 \quad \text{for all } x \in D(\Lambda).$$

What is remarkable is that, just as in the case of linear skew-adjoint operators [6], the infimum will often be 0 as long as $\Lambda$ is a regular map, a fact that will allow us to derive variationally several nonlinear PDEs without using Euler-Lagrange theory.

THEOREM 2.8 *Let $L$ be an anti-self-dual Lagrangian on a reflexive Banach space $X \times X^*$ such that* $\mathrm{dom}_1(L)$ *is closed, and let $H_L$ be the corresponding Hamiltonian. Let $\Lambda : D(\Lambda) \subset X \to X^*$ be a regular map such that*

$$\mathrm{dom}_1(L) \subset D(\Lambda) \quad \textit{and} \quad \lim_{\|x\| \to +\infty} H_L(0, x) + \langle \Lambda x, x \rangle = +\infty. \tag{2.12}$$

*Then, the functional $I(x) = L(x, \Lambda x) + \langle \Lambda x, x \rangle$ is self-dual on $D := \mathrm{dom}_1(L)$, and consequently there exists $\bar{x} \in D(\Lambda)$ such that*

$$I(\bar{x}) = \inf_{x \in X} I(x) = 0, \tag{2.13}$$

$$(-\Lambda \bar{x}, -\bar{x}) \in \partial L(\bar{x}, \Lambda \bar{x}). \tag{2.14}$$

Theorem 2.8 is a nice application of the following Ky-Fan-type min-max theorem, which is essentially due to Brezis, Nirenberg, and Stampacchia [5].

LEMMA 2.9 *Let $D$ be a nonbounded convex and closed subset of a reflexive Banach space $X$, and let $M(x, y)$ be a real-valued function on $D \times D \subset X \times X$ that satisfies the following conditions*:

(i) *$M(x, x) \leq 0$ for every $x \in D$.*
(ii) *For each $x \in D$, the function $y \to M(x, y)$ is concave.*
(iii) *For each $y \in D$, the function $x \to M(x, y)$ is weakly lower-semicontinuous.*
(iv) *The set $D_0 = \{x \in D : M(x, 0) \leq 0\}$ is bounded in $X$.*

*Then there exists $x_0 \in D$ such that* $\sup_{y \in D} M(x_0, y) \leq 0$.

PROOF OF THEOREM 2.8: Under assumption (2.12) we can write for each $x \in D := \mathrm{dom}_1(L)$, since the Lagrangian $L$ is anti-self-dual,

$$\begin{aligned}
I(x) &= L(x, \Lambda x) + \langle \Lambda x, x \rangle \\
&= L^*(-\Lambda x, -x) + \langle \Lambda x, x \rangle \\
&= \sup\{\langle y, -\Lambda x \rangle + \langle p, -x \rangle - L(y, p) : y \in X, p \in X^*\} + \langle \Lambda x, x \rangle \\
&= \sup\{\langle y, -\Lambda x \rangle + \langle p, -x \rangle - L(y, p) : y \in D, p \in X^*\} + \langle \Lambda x, x \rangle \\
&= \sup\Big\{\langle y, -\Lambda x \rangle + \sup\{\langle p, -x \rangle - L(y, p) : p \in X^*\} : y \in D\Big\} + \langle \Lambda x, x \rangle \\
&= \sup\{\langle x - y, \Lambda x \rangle + H_L(y, -x) : y \in D\} \\
&= \sup_{y \in D} M(x, y)
\end{aligned}$$

where $M(x, y) = \langle x - y, \Lambda x \rangle + H_L(y, -x)$, and where $H_L$ is the Hamiltonian associated to $L$. We now claim that $M$ satisfies all the properties of the Ky-Fan min-max lemma above. Indeed:

(i) For each $x \in D$, we have $y \to M(x, y)$ is concave since $y \to \langle x - y, \Lambda x \rangle$ is clearly linear, and $y \to H_L(y, x)$ is concave.
(ii) For each $y \in D$, the function $x \to M(x, y)$ is weakly lower-semicontinuous since $x \to \langle x - y, \Lambda x \rangle$ is weakly continuous by hypothesis while $x \to H_L(y, -x)$ is clearly the supremum of continuous affine functions.

(iii) To show that $M(x,x) \le 0$ for each $x \in D$, use the fact that $H_L$ is an ASD Hamiltonian; hence $H_L(x,-x) \le 0$.
(iv) The set $D_0 = \{x \in D : M(x,0) \le 0\}$ is bounded in $X$ since $M(x,0) = H_L(0,-x) + \langle \Lambda x, x\rangle$ and the latter goes to infinity with $\|x\|$.

It follows from Lemma 2.9 that there exists $\bar{x} \in D$ with $I(\bar{x}) = \sup_{y\in D} M(\bar{x}, y) \le 0$. On the other hand, by (2.2) we have for any $x \in X$ that $I(x) = L(x, \Lambda x) + \langle \Lambda x, x\rangle \ge 0$. It follows that $L(\bar{x}, \Lambda\bar{x}) + \langle \Lambda\bar{x}, \bar{x}\rangle = I(\bar{x}) = 0 = \inf_{x\in X} I(x)$. Claim (2.14) now follows from (2.3). □

*Remark* 2.10. Weaker hypotheses on $M$ are sufficient to obtain the same conclusion as in the Ky-Fan min-max theorem above. For our purpose, this translates to only assuming that the operator $\Lambda$ is *pseudoregular* in the sense that it only needs to satisfy the following property:

$$\text{If } x_n \rightharpoonup x \text{ in } X \text{ and } \limsup_{n\to+\infty}\langle \Lambda x_n, x_n - x\rangle \le 0, \quad \text{then } \liminf_{n\to+\infty}\langle \Lambda x_n, x_n - y\rangle \ge \langle \Lambda x, x - y\rangle. \tag{2.15}$$

The same conclusion as in Theorem 2.8 will still hold (see [5]). This weakening will be useful in the application to the complex Ginsburg-Landau evolution in Section 5.

COROLLARY 2.11 *Let $L$ be an anti-self-dual Lagrangian on a reflexive space $X \times X^*$ such that* $\mathrm{dom}_1(L)$ *is closed and consider $\Lambda : D(\Lambda) \subset X \to X^*$ to be a regular map and $B : D(B) \subset X \to X^*$ to be a linear operator satisfying*

$$\lim_{\|x\|\to+\infty} H_L(0,x) + \langle x, Bx + \Lambda x\rangle = +\infty. \tag{2.16}$$

*Suppose one of the following two conditions holds*:

(i) *$B$ is a positive operator such that* $\mathrm{dom}_1(L) \subset D(B) \cap D(\Lambda)$ *or*
(ii) *$B$ is skew-adjoint and $x \to L(x,0)$ is bounded on the unit ball of $X$ while* $D(B) \cap \mathrm{dom}_1(L) \subset D(\Lambda)$.

*Then the functional $I(x) = L(x, Bx + \Lambda x) + \langle x, Bx + \Lambda x\rangle$ is self-dual on* $\mathrm{dom}_1(L_B) = \mathrm{dom}_1(L) \cap D(B)$, *and there exists $\bar{x} \in$* $\mathrm{dom}_1(L) \cap D(B) \cap D(\Lambda)$ *such that*

$$I(\bar{x}) = \inf_{x\in X} I(x) = 0, \tag{2.17}$$

$$(-\Lambda\bar{x} - B\bar{x}, -\bar{x}) \in \partial L(\bar{x}, B\bar{x} + \Lambda\bar{x}). \tag{2.18}$$

PROOF: In case (i), since $\mathrm{dom}_1(L)$ is closed, we apply Theorem 2.8 to the regular operator $\tilde{\Lambda} = \Lambda + B$, which also satisfies $\mathrm{dom}_1(L) \subset D(B) \cap D(\Lambda) \subset D(\tilde{\Lambda})$.

In case (ii), that is, if the domain of the linear operator $B$ is not large enough but $B$ is skew-symmetric, we can use the fact that $L_B$, defined in Lemma 2.6, is then an anti-self-dual Lagrangian and apply Theorem 2.8 to $L_B$ and the regular operator $\Lambda$. □

We now apply the above results to the most basic ASD Lagrangians of the form $L(x, p) = \varphi(x) + \varphi^*(-Bx - p)$ where $\varphi$ is a convex function and $B$ is a linear antisymmetric but not necessarily bounded operator. The domain of the nonlinear operator $\Lambda$ needs to be large, but we have much more flexibility with the linear operator $B$. The applications differ because they will depend on the "size" and "position" of the domain of $B$ vis-à-vis the domains of $\varphi$ and the domain of $\Lambda$. Roughly speaking, $B$ can be any positive operator if its domain is large enough to contain the domains of $\varphi$ (and the domain of the regular operator $\Lambda$ if any), while if $B$ has a smaller domain than $\varphi$, then one can still conclude provided it is skew-adjoint.

COROLLARY 2.12 *Let $\varphi$ be a proper, convex, lower-semicontinuous function on a reflexive Banach space $X$, let $\Lambda : D(\Lambda) \subset X \to X^*$ be a regular operator, and consider a linear operator $B : D(B) \subset X \to X^*$ such that*

$$\lim_{\|x\|\to+\infty} \frac{\varphi(x) + \langle \Lambda x + Bx, x\rangle}{\|x\|} = +\infty.$$

*Suppose one of the following two conditions holds*:

(i) *$B$ is positive on $X$ and* $\mathrm{dom}(\varphi) \subset D(B) \cap D(\Lambda)$.
(ii) *$B$ is skew-adjoint on $X$, $\varphi$ is bounded on the unit ball of $X$, and* $\mathrm{dom}(\varphi) \cap D(B) \subset D(\Lambda)$.

*Then, for every $f \in X^*$ there exists $\bar{x} \in \mathrm{dom}(\varphi) \cap D(B) \cap D(\Lambda)$ that solves*

$$0 \in f + \Lambda x + Bx + \partial\varphi(x). \tag{2.19}$$

*It is obtained as a minimizer of the self-dual functional*

$$I(x) = \varphi(x) + \langle f, x\rangle + \varphi^*(-\Lambda x - Bx - f) + \langle x, \Lambda x + Bx\rangle, \tag{2.20}$$

*whose infimum on $X$ is equal to* 0.

PROOF: This corollary is an immediate consequence of Corollary 2.11 applied to the Lagrangian $L(x, p) = \psi(x) + \psi^*(-p)$ where $\psi(x) = \varphi(x) + \langle f, x\rangle$. Note that its Hamiltonian is now $H(x, y) = \varphi(-y) - \varphi(x) - \langle f, x + y\rangle$, which means that the coercivity hypothesis implies that $H(0, y) + \langle y, \Lambda y + By\rangle \to +\infty$ with $\|y\|$. Corollary 2.11 then applies with the Lagrangian $L$ and the regular operator $\Lambda$ to obtain that the minimum in (2.20) is attained at some $\bar{x} \in X$. We then get

$$\varphi(\bar{x}) + \varphi^*(-B\bar{x} - \Lambda\bar{x} - f) = \langle -B\bar{x} - \Lambda\bar{x} - f, \bar{x}\rangle,$$

which yields, in view of Legendre-Fenchel duality, that $-B\bar{x} - \Lambda\bar{x} - f \in \partial\varphi(\bar{x})$. □

An immediate application is the case where the linear operator component is bounded, which already covers many interesting applications.

COROLLARY 2.13 *Let $\varphi$ be a function on a reflexive Banach space $X$ and let $B : X \to X^*$ be a bounded linear operator such that the function $\psi(x) := \varphi(x) + \frac{1}{2}\langle Bx, x\rangle$ is proper, convex, lower-semicontinuous, and bounded below on $X$. Let $\Lambda : X \to X^*$ be any regular operator such that $\lim_{\|x\|\to\infty} \|x\|^{-1}(\psi(x) + \langle x, \Lambda x\rangle) = +\infty$. Then for any $f \in X^*$, there exists a solution $\bar{x} \in X$ to the equation*

$$0 \in f + \Lambda x + Bx + \partial\varphi(x). \tag{2.21}$$

*It can be obtained as a minimizer of the self-dual functional*

$$I(x) = \psi(x) + \langle f, x\rangle + \psi^*(-\Lambda x - B^a x - f) + \langle x, \Lambda x\rangle, \tag{2.22}$$

*where $B^a$ is the antisymmetric part of $B$.*

PROOF: Apply the above corollary to $\psi(x) + \langle f, x\rangle$ and to $B^a = \frac{1}{2}(B - B^*)$, the antisymmetric part of $B$. We then get $\bar{x} \in X$ such that $-B^a\bar{x} - \Lambda\bar{x} - f \in \partial\psi(\bar{x}) = B^s\bar{x} + \partial\varphi(\bar{x})$; hence $\bar{x}$ satisfies (2.21). □

We can also give a variational resolution for certain nonlinear systems.

COROLLARY 2.14 *Let $\varphi$ be proper and convex lower-semicontinuous on $X \times Y$, let $A : X \to Y^*$ be any bounded linear operator, and let $B_1 : X \to X^*$ (respectively, $B_2 : Y \to Y^*$) be two positive bounded linear operators such that*

$$\lim_{\|x\|+\|y\|\to\infty} \frac{\varphi(x, y) + \frac{1}{2}\langle B_1 x, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle}{\|x\| + \|y\|} = +\infty.$$

*If $\Lambda := (\Lambda_1, \Lambda_2) : X \times Y \to X^* \times Y^*$ is a regular conservative operator, then for any $(f, g) \in X^* \times Y^*$, there exists $(\bar{x}, \bar{y}) \in X \times Y$, which solves the following system:*

$$\begin{cases} \Lambda_1(x, y) - A^*y - B_1 x + f \in \partial_1\varphi(x, y), \\ \Lambda_2(x, y) + Ax - B_2 y + g \in \partial_2\varphi(x, y). \end{cases} \tag{2.23}$$

*The solution is obtained as a minimizer on $X \times Y$ of the self-dual functional*

$$I(x, y) = \psi(x, y) + \psi^*(-A^*y - B_1^a x + \Lambda_1(x, y), Ax - B_2^a y + \Lambda_2(x, y)),$$

*where*

$$\psi(x, y) = \varphi(x, y) + \frac{1}{2}\langle B_1 x, x\rangle + \frac{1}{2}\langle B_2 y, y\rangle - \langle f, x\rangle - \langle g, y\rangle,$$

*and where $B_1^a$ (respectively, $B_2^a$) are the skew-symmetric parts of $B_1$ and $B_2$.*

PROOF: Consider the following ASD Lagrangian (see [6])

$$L((x, y), (p, q)) = \psi(x, y) + \psi^*(-A^*y - B_1^a x + p, Ax - B_2^a y + q).$$

Theorem 2.8 yields that $I(x,y) = L((x,y),\Lambda(x,y))$ attains its minimum at some point $(\bar{x},\bar{y}) \in X \times Y$ and that the minimum is 0. In other words,

$$\begin{aligned}0 &= I(\bar{x},\bar{y})\\ &= \psi(\bar{x},\bar{y}) + \psi^*(-A^*\bar{y} - B_1^a\bar{x} + \Lambda_1(\bar{x},\bar{y}), A\bar{x} - B_2^a\bar{y} + \Lambda_2(\bar{x},\bar{y}))\\ &= \psi(\bar{x},\bar{y}) + \psi^*(-A^*\bar{y} - B_1^a\bar{x} + \Lambda_1(\bar{x},\bar{y}), A\bar{x} - B_2^a\bar{y} + \Lambda_2(\bar{x},\bar{y}))\\ &\quad - \langle(\bar{x},\bar{y}), (-A^*\bar{y} - B_1^a\bar{x} + \Lambda_1(\bar{x},\bar{y}), A\bar{x} - B_2^a\bar{y} + \Lambda_2(\bar{x},\bar{y}))\rangle\end{aligned}$$

from which follows that

$$\begin{cases}-A^*y - B_1^a x + \Lambda_1(x,y) \in \partial_1\varphi(x,y) + B_1^s(x) - f,\\ Ax - B_2^a y + \Lambda_1(x,y) \in \partial_2\varphi(x,y) + B_2^s(y) - g.\end{cases} \tag{2.24}$$

□

## 3 Applications to Stationary Navier-Stokes and Other Nonlinear Systems

### Example 1: Variational Resolution for Stationary Navier-Stokes Equations

Consider the incompressible stationary Navier-Stokes equation on a smooth bounded domain $\Omega$ of $\mathbb{R}^3$

$$\begin{cases}(u\cdot\nabla)u + f = \nu\Delta u - \nabla p & \text{on } \Omega,\\ \operatorname{div} u = 0 & \text{on } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{3.1}$$

where $\nu > 0$ and $f \in L^p(\Omega;\mathbb{R}^3)$. Let

$$\Phi(u) = \frac{\nu}{2}\int_\Omega \sum_{j,k=1}^{3}\left(\frac{\partial u_j}{\partial x_k}\right)^2 dx \tag{3.2}$$

be the convex and coercive function on $X = \{u \in H_0^1(\Omega;\mathbf{R}^3) : \operatorname{div} v = 0\}$. Its Legendre transform $\Phi^*$ on $X^*$ can be characterized as $\Phi^*(v) = \langle Sv, v\rangle$ where $S : X^* \to X$ is the bounded linear operator that associates to $v \in X^*$ the solution $\hat{v} = Sv$ of the Stokes' problem

$$\begin{cases}\nu\Delta\hat{v} + \nabla p = -v & \text{on } \Omega,\\ \operatorname{div}\hat{v} = 0 & \text{on } \Omega,\\ \hat{v} = 0 & \text{on } \partial\Omega.\end{cases} \tag{3.3}$$

It is easy to see that (3.1) can be reformulated as

$$\begin{cases}(u\cdot\nabla)u + f \in -\partial\Phi(u) = \nu\Delta u - \nabla p,\\ u \in X.\end{cases} \tag{3.4}$$

Consider now the nonlinear operator $\Lambda : X \to X^*$ defined as

$$\langle \Lambda u, v\rangle = \int_\Omega \sum_{j,k=1}^{3} u_k \frac{\partial u_j}{\partial x_k} v_j \, dx = \langle (u\cdot\nabla)u, v\rangle.$$

We can deduce the following:

THEOREM 3.1 *Assume $\Omega$ is a bounded domain in $\mathbb{R}^3$ and consider $f \in L^p(\Omega;\mathbb{R}^3)$ for $p > \frac{6}{5}$. Then the infimum of the self-dual functional*

$$I(u) = \Phi(u) + \Phi^*(-(u\cdot\nabla)u + f) - \int_\Omega \sum_{j=1}^{3} f_j u_j$$

*on $X$ is equal to* 0 *and is attained at a solution of equation* (3.1).

PROOF: To apply Theorem 2.8, it remains to show that $\Lambda$ is a regular conservative operator. It is standard to show that $\langle \Lambda u, u\rangle = 0$ on $X$. For the weak-to-weak continuity, assume that $u^n \to u$ weakly in $H^1(\Omega)$ and fix $v \in V$. We have that

$$\langle \Lambda u^n, v\rangle = \int_\Omega \sum_{j,k=1}^{3} u_k^n \frac{\partial u_j^n}{\partial x_k} v_j \, dx = -\int_\Omega \sum_{j,k=1}^{3} u_k^n \frac{\partial v_j}{\partial x_k} u_j^n \, dx$$

converges to $\langle \Lambda u, v\rangle = \int_\Omega \sum_{j,k=1}^{3} u_k \frac{\partial v_j}{\partial x_k} u_j \, dx$. Indeed, the Sobolev embedding in dimension 3 implies that $(u^n)$ converges strongly in $L^p(\Omega;\mathbb{R}^3)$ for $1 \le p < 6$. On the other hand, $\frac{\partial u_j}{\partial x_k}$ is in $L^2(\Omega)$, and the result follows from an application of Hölder's inequality. $\square$

### Example 2: Variational Resolution for a Fluid Driven by Its Boundary

The full strength of Corollary 2.13 comes out when one deals with the Navier-Stokes equation with a boundary moving with a prescribed velocity:

$$\begin{cases} (u\cdot\nabla)u + f = \nu\Delta u - \nabla p & \text{on } \Omega,\\ \operatorname{div} u = 0 & \text{on } \Omega,\\ u = u^0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $\int_{\partial\Omega} u^0 \cdot \mathbf{n}\, d\sigma = 0$, $\nu > 0$, and $f \in L^p(\Omega;\mathbb{R}^3)$. Assuming that $u^0 \in H^{3/2}(\partial\Omega)$ and that $\partial\Omega$ is connected, a classical result of Hopf then yields for each $\epsilon > 0$, the existence of $v^0 \in H^2(\Omega)$ such that

$$v^0 = u^0 \text{on } \partial\Omega, \quad \operatorname{div} v^0 = 0,$$

$$\int_\Omega \sum_{j,k=1}^{3} u_k \frac{\partial v_j^0}{\partial x_k} u_j \, dx \le \epsilon \|u\|_X^2 \quad \text{for all } u \in X. \tag{3.6}$$

By setting $v = u + v^0$, solving (3.5) reduces to finding a solution for

$$\begin{cases} (u \cdot \nabla)u + (v^0 \cdot \nabla)u + (u \cdot \nabla)v^0 + f - \nu\Delta v^0 + (v^0 \cdot \nabla)v^0 \\ \quad = \nu\Delta u - \nabla p & \text{on } \Omega, \\ \operatorname{div} u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This can be reformulated as the following equation in the space $X$:

$$(u \cdot \nabla)u + (v^0 \cdot \nabla)u + (u \cdot \nabla)v^0 + g \in -\partial\Phi(u) \tag{3.7}$$

where $\Phi$ is again as in (3.2) and $g := f - \nu\Delta v^0 + (v^0 \cdot \nabla)v^0 \in X^*$. In other words, this is an equation of the form

$$\Lambda u + Bu + g \in -\partial\Phi(u) \tag{3.8}$$

with $\Lambda u = (u \cdot \nabla)u$ a regular conservative operator and $Bu = (v^0 \cdot \nabla)u + (u \cdot \nabla)v^0$ is a bounded linear operator. Note that the component $B^1 u := (v^0 \cdot \nabla)u$ is skew-symmetric, which means that Hopf's result yields the required coercivity condition:

$$\Psi(u) := \Phi(u) + \frac{1}{2}\langle Bu, u\rangle \geq \frac{1}{2}(\nu - \epsilon)\|u\|^2 \quad \text{for all } u \in X.$$

$\Psi$ is then convex and coercive and therefore we can apply Corollary 2.13 to deduce the following:

THEOREM 3.2 *Under the above hypothesis, with $A^a$ denoting the antisymmetric part of the operator $Au = (u \cdot \nabla)v^0$, the self-dual functional*

$$I(u) = \Psi(u) + \Psi^*(-(u \cdot \nabla)u - (v^0 \cdot \nabla)u - A^a u + g) - \int_\Omega \sum_{j=1}^{3} g_j u_j$$

*has* 0 *for an infimum on the space $X$, which is attained at a solution $\bar{u}$ for* (3.7).

### Example 3: Variational Resolution for a Fluid Driven by Transport

Let $\mathbf{a} \in C^\infty(\bar{\Omega}, \mathbb{R}^3)$ be a smooth vector field on a neighborhood of a $C^\infty$ bounded open set $\Omega \subset \mathbb{R}^3$, let $a_0 \in L^\infty(\Omega)$, and consider again the space $X$ defined above. The transport operator $B : u \mapsto (\mathbf{a} \cdot \nabla)u + \frac{1}{2}\operatorname{div}(\mathbf{a})u$ with domain $D(B) = \{u \in X : \mathbf{a} \cdot \nabla u + \frac{1}{2}\operatorname{div}(\mathbf{a})u \in X^*\}$ into $X^*$ is clearly a skew-adjoint on the space $X$ (see [11]).

Consider now the following equation on the domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} (u \cdot \nabla)u + (\mathbf{a} \cdot \nabla)u + a_0 u + |u|^{m-2}u + f = \nu\Delta u - \nabla p & \text{on } \Omega, \\ \operatorname{div} u = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\nu > 0$, $6 \geq m \geq 1$, and $f \in L^q(\Omega; \mathbb{R}^3)$ for $q \geq \frac{6}{5}$. Suppose

$$\frac{1}{2}\operatorname{div}(\mathbf{a}) - a_0 \geq 0 \quad \text{on } \Omega, \tag{3.10}$$

and consider the functional

$$\Psi(u) = \frac{\nu}{2}\int_\Omega \sum_{j,k=1}^{3}\left(\frac{\partial u_j}{\partial x_k}\right)^2 dx + \frac{1}{4}\int_\Omega (\operatorname{div}\mathbf{a} - 2a_0)|u|^2\,dx + \frac{1}{m}\int_\Omega |u|^m\,dx + \int_\Omega uf\,dx,$$

which is convex and a coercive function on $X$. Corollary 2.13 then applies to yield the following:

THEOREM 3.3 *Under the above hypothesis, the self-dual functional*

$$I(u) = \Psi(u) + \Psi^*\left(-(u\cdot\nabla)u - \mathbf{a}\cdot\nabla u - \frac{1}{2}\operatorname{div}(\mathbf{a})u\right)$$

*has* 0 *for an infimum and the latter is attained at a solution* $\bar{u}$ *for* (3.9).

## Example 4: Nonlinear Transport Equations

Let again $\mathbf{a}\in C^\infty(\bar{\Omega},\mathbb{R}^n)$ be a smooth vector field on a neighborhood of a $C^\infty$ bounded open set $\Omega\subset\mathbb{R}^N$, and consider the equation

$$\begin{cases}\mathbf{a}\cdot\nabla u - \Delta u = |u|^{p-1}u - |u|^{q-1}u + f(x) & \text{on } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{3.11}$$

where $f\in H^{-1}$ and $1 < p < q < (N+2)/(N-2)$. Consider the functional

$$\psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{q+1}\int_\Omega |u|^{q+1} - \int_\Omega f(x)u,$$

which is convex, lower-semicontinuous, and coercive on $H_0^1(\Omega)$. Corollary 2.12 then applies to yield the following:

THEOREM 3.4 *Under the above hypothesis, the self-dual functional*

$$I(u) = \psi(u) + \psi^*(-\mathbf{a}\cdot\nabla u - |u|^{p-1}u) + \frac{1}{2}\int_\Omega \operatorname{div}(\mathbf{a})|u|^2\,dx - \int_\Omega |u|^{p+1}\,dx$$

*has an infimum of* 0*, which is attained at a solution* $\bar{u}$ *for* (3.11).

PROOF: It suffices to check that the nonlinear operator $\Lambda u = -\mathbf{a}\cdot\nabla u - |u|^{p-1}u$ is regular. Indeed, it is weak-to-weak continuous from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ since the Sobolev embedding and Hölder's inequality imply that $|u_n|^{p-1}u_n \to |u|^{p-1}u$ strongly in $L^{2N/(N+2)}$ as soon as $u_n\to u$ weakly in $H_0^1(\Omega)$. On the other hand,

$$u \to \langle\Lambda u, u\rangle = \int_\Omega (-\mathbf{a}\cdot\nabla u - |u|^{p-1}u)u\,dx = \frac{1}{2}\int_\Omega \operatorname{div}(\mathbf{a})u^2\,dx - \int_\Omega |u|^{p+1}\,dx$$

is also weakly lower-semicontinuous on $H_0^1(\Omega)$ while the functional

$$\psi(u) + \langle \Lambda u, u\rangle = \\ \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{q}\int_\Omega |u|^{q+1}\,dx + \frac{1}{2}\int_\Omega \operatorname{div}(\mathbf{a})u^2\,dx - \int_\Omega |u|^{p+1}\,dx$$

is coercive since $q > p > 1$. Now we can apply Corollary 2.12. □

## Example 5: A Variational Resolution for Nonlinear Coupled Equations

Let $\mathbf{b_1} : \Omega \to \mathbb{R}^n$ and $\mathbf{b_2} : \Omega \to \mathbb{R}^n$ be two smooth vector fields on the neighborhood of a bounded domain $\Omega$ of $\mathbb{R}^n$, and let $B_1 v = \mathbf{b_1}\cdot\nabla v$ and $B_2 v = \mathbf{b_2}\cdot\nabla v$ be the corresponding first-order linear operators. Consider the Dirichlet problem:

$$\begin{cases} \Delta(v+u) + \mathbf{b_1}\cdot\nabla u = |u|^{p-2}u + u^{m-1}v^m + f & \text{on } \Omega, \\ \Delta(v-u) + \mathbf{b_2}\cdot\nabla v = |v|^{q-2}q - u^m v^{m-1} + g & \text{on } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

We can use Corollary 2.14 to get the following:

THEOREM 3.5 *Assume* $\operatorname{div}(\mathbf{b_1}) \geq 0$ *and* $\operatorname{div}(\mathbf{b_2}) \geq 0$ *on* $\Omega$, $2 < p, q \leq 2n/(n-2)$, *and* $1 < m < (n+2)/(n-2)$, *and consider on* $H_0^1(\Omega)\times H_0^1(\Omega)$ *the self-dual functional*

$$I(u,v) = \Psi(u) + \Psi^*\left(\mathbf{b_1}.\nabla u + \frac{1}{2}\operatorname{div}(\mathbf{b_1})\,u + \Delta v - u^{m-1}v^m\right) \\ + \Phi(v) + \Phi^*\left(\mathbf{b_2}.\nabla v + \frac{1}{2}\operatorname{div}(\mathbf{b_2})\,v - \Delta u + u^m v^{m-1}\right)$$

*where*

$$\Psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{p}\int_\Omega |u|^p\,dx + \int_\Omega fu\,dx + \frac{1}{4}\int_\Omega \operatorname{div}(\mathbf{b_1})\,|u|^2\,dx,$$

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx + \frac{1}{q}\int_\Omega |v|^q\,dx + \int_\Omega gv\,dx + \frac{1}{4}\int_\Omega \operatorname{div}(\mathbf{b_2})\,|v|^2\,dx.$$

*Then there exists* $(\bar u, \bar v) \in H_0^1(\Omega)\times H_0^1(\Omega)$ *such that*

$$I(\bar u,\bar v) = \inf\{I(u,v) : (u,v) \in H_0^1(\Omega)\times H_0^1(\Omega)\} = 0,$$

*and* $(\bar u, \bar v)$ *is a solution of* (3.12).

PROOF: Let $A = \Delta$ on $H_0^1$, $B_1 = \mathbf{b_1}.\nabla$, and $B_2 = \mathbf{b_2}.\nabla$. Let $X = H_0^1 \times H_0^1$, and consider on $X \times X^*$ the ASD Lagrangian

$$L((u,v),(r,s))) = \Psi(u) + \Psi^*\Big(\mathbf{b_1}.\nabla u + \frac{1}{2}\operatorname{div}(\mathbf{b_1})u + \Delta v + r\Big) + \Phi(v) + \Phi^*\Big(\mathbf{b_2}.\nabla v + \frac{1}{2}\operatorname{div}(\mathbf{b_2})v - \Delta u + s\Big).$$

It is also easy to verify that the nonlinear operator $\Lambda : H_0^1 \times H_0^1 \to H^{-1} \times H^{-1}$ defined by $\Lambda(u,v) = (-u^{m-1}v^m, u^m v^{m-1})$ is regular and conservative. □

## Example 6: Nonlinear Schrödinger Equation with a Nonlocal Term

Consider the generalized Choquard-Pekar equation

$$-\Delta u + V(x)u = (w * f(u))g(u) \tag{3.13}$$

where $V$ and $w$ are real functions. We consider the case where $f(u) = |u|^p$ and $g(u) = |u|^{q-2}u$, and note that if $p = q$ the problem can then be solved by the usual variational method since weak solutions are critical points of the energy function, $\Phi(u) = \psi(u) - \varphi(u)$, where

$$\psi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2}\int_\Omega V(x)u^2 \quad \text{and} \quad \varphi(u) = \int_\Omega \big(w * f(u)\big) f(u)\, dx.$$

However, as soon as $p \neq q$, (3.13) ceases to be an Euler-Lagrange equation. Nevertheless, we can proceed in the following way:

THEOREM 3.6 *Consider $w \in L^1(\mathbb{R}^N)$ and $V$ such that $V(x) \geq \delta > 0$ for $x \in \mathbb{R}^N$. Assume that either $V$ and $w$ are both radial or that $\lim_{\|x\|\to\infty} V(x) = +\infty$. If, moreover, one of the following conditions holds*:

(i) $1 \leq p < \frac{2^*}{2}$, $1 < q < \frac{2^*}{2}$, *and* $w(x) \leq 0$ *on* $\mathbb{R}^N$, *or*
(ii) $1 \leq p < \frac{2^*}{2}$, $1 < q < \frac{2^*}{2}$, *and* $1 \leq pq < 2$,

*then the self-dual functional*

$$I(u) = \psi(u) + \psi^*((w * |u|^p)|u|^{q-1}u) - \int_\Omega (w * |u|^p)|u|^q\, dx$$

*has an infimum of* 0 *on* $H^1(\mathbb{R}^N)$*, which is attained at a solution of* (3.13).

PROOF: It uses the following standard facts:

- Let $w \in L^r(\mathbb{R}^N)$, $r \geq 1$, and $s = 2r/(2r-1)$. The bilinear map $(u,v) \to (w * u)v$ is then well-defined and continuous from $L^s \times L^s$ into $L^1$ and satisfies $|(w * u)v|_{L^1(\Omega)} \leq \|w\|_r \|u\|_s \|v\|_s$. Moreover, if $(v_n)$ and $(u_n) \subseteq L^s(\mathbb{R}^N)$ are bounded and if either $u_n \to u$ in $L^s(\mathbb{R}^N)$ and $v_n \to v$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ or vice versa $u_n \to u$ in $L^s_{\text{loc}}(\mathbb{R}^N)$ and $v_n \to v$ in $L^s(\mathbb{R}^N)$, then $(w * u_n)v_n \to (w * u)v$ in $L^1$.

- If $\limsup_{|x|\to+\infty} V(x) = +\infty$, then the space $X = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty\}$ embeds compactly in $L^k(\mathbb{R}^N)$ provided $2 \le k < 2^*$.
- The space $H_r^1(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u \text{ is radial}\}$ also embeds compactly in $L^k(\mathbb{R}^N)$ for $2 \le k < 2^*$.

□

We now show that the operator $\Lambda : X \to X^*$ defined by $\Lambda u = -(w * |u|^p)|u|^{q-1}u$ is regular when $X$ is either $H_r^1(\mathbb{R}^N)$ for the radial case or when $X = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\,dx < \infty\}$ for the case when $\lim_{|x|\to+\infty} V(x) = +\infty$.

First note that $\Lambda : X \to X^*$ is well-defined since by Young's inequality and then by Hölder's we have

$$\begin{aligned}
|\langle \Lambda u, v\rangle| &= \left| -\int_{\mathbb{R}^N} (w * |u|^p)|u|^{q-2}uv\,dx \right| \\
&\le \|w\|_1\, \||u|^p\|_2\, \||u|^{q-1}|v|\|_2 \\
&\le \|w\|_1\, \|u\|_{2p}^p\, \|u\|_{2q}^{q-1}\, \|v\|_{2q} < \infty.
\end{aligned}$$

To show that $\Lambda$ is weak-to-weak continuous, let $u_n \rightharpoonup u$ weakly in $X$ so that $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$ for $2 \le r < 2^*$. It follows that $|u_n|^p \to |u|^p$ strongly in $L^2(\mathbb{R}^N)$, and $|u_n|^{q-2}u \to |u|^{q-2}u$ strongly in $L^{2q/(q-1)}(\mathbb{R}^N)$. For every $v \in L^{2q}$, the sequence $|u_n|^{q-2}uv$ then converges strongly to $|u|^{q-2}uv$ in $L^2(\mathbb{R}^N)$. Therefore by Young's inequality, we get that $\langle \Lambda u_n, v\rangle \to \langle \Lambda u, v\rangle$, and consequently $\Lambda$ is weak-to weak continuous. On the other hand, in case (i) we have

$$\langle \Lambda u, u\rangle = -\int_{\mathbb{R}^N} (w * |u|^p)|u|^{q+1}\,dx \ge 0,$$

so that the functional $\psi(u) + \langle \Lambda u, u\rangle$ is coercive. For the second case (ii), even though $\langle \Lambda u, u\rangle$ may be nonpositive, the functional $\varphi(u) + \langle \Lambda u, u\rangle$ does not lose its coercivity since $1 < pq < 2$. Corollary 2.13 then applies to yield the claimed result.

## 4 Self-Dual Variational Principles for Nonlinear Evolution Equations

Consider now an evolution triple $X \subset H \subset X^*$; that is, $H$ is a Hilbert space with $\langle \cdot, \cdot\rangle$ as a scalar product, and $X$ is a dense vector subspace of $H$ that is a reflexive Banach space once equipped with its own norm $\|\cdot\|$. Assuming the canonical injection $X \to H$, continuous, we identify the Hilbert space $H$ with its dual $H^*$ and we "inject" $H$ in $X^*$ in such a way that

$$\langle h, u\rangle_{X^*,X} = \langle h, u\rangle_H \quad \text{for all } h \in H \text{ and all } u \in X.$$

This injection is continuous and one-to-one, and $H$ is also dense in $X^*$. In other words, the dual $X^*$ of $X$ is represented as the completion of the Hilbert space $H$ for the dual norm $\|h\| = \sup\{\langle h, u\rangle_H : \|u\|_X \le 1\}$.

Let $[0, T]$ be a fixed real interval and consider the following Banach spaces:

- the space $L^2_X$ of Bochner integrable functions from $[0, T]$ into $X$ with norm
$$\|u\|^2_{L_2(X)} = \left(\int_0^T \|u(t)\|^2_X \, dt\right)^{1/2}$$
- the space $\mathcal{X}_2$ of all functions in $L^2_X$ such that $\dot{u} \in L^2_{X^*}$, equipped with the norm
$$\|u\|_{\mathcal{X}} = (\|u\|^2_{L_2(X)} + \|\dot{u}\|^2_{L_2(X^*)})^{1/2}.$$

Note that this last space is different from the Sobolev space $A^2_X = \{u : [0, T] \to X;\ \dot{u} \in L^2_X\}$, unless $X = H$, and that we actually have $A^2_X \subset \mathcal{X}_2 \subset A^2_{X^*}$.

DEFINITION 4.1 A *time-dependent Lagrangian* on $[0, T] \times X \times X^*$ is any function $L : [0, T] \times X \times X^* \to \mathbb{R} \cup \{+\infty\}$ that is measurable with respect to the $\sigma$-field generated by the products of Lebesgue sets in $[0, T]$ and Borel sets in $H \times H$. The Hamiltonian $H_L$ of $L$ is the function defined on $[0, T] \times X \times X$ by
$$H(t, x, y) = \sup\{\langle y, p\rangle - L(t, x, p) : p \in X^*\}.$$
We say that $L$ is an *anti-self-dual Lagrangian* (ASD) on $[0, T] \times X \times X^*$ if for any $t \in [0, T]$, the map $L_t : (x, p) \to L(t, x, p)$ is ASD on $X \times X^*$, that is, if
$$L^*(t, p, x) = L(t, -x, -p) \quad \text{for all } (x, p) \in X \times X^*,$$
where here $L^*$ is the Legendre transform in the last two variables.

The most basic time-dependent ASD Lagrangians are again of the form
$$L(t, x, p) = \varphi(t, x) + \varphi^*(t, -p)$$
where for each $t$, the function $x \to \varphi(t, x)$ is convex and lower-semicontinuous on $X$. We now show how this property naturally "lifts" to the path space. For that, we associate to each time-dependent Lagrangian $L$ on $[0, T] \times X \times X^*$ the corresponding Lagrangian $\mathcal{L}$ on the path space $L^2_X \times L^2_{X^*}$ defined by
$$\mathcal{L}(u, p) := \int_0^T L(t, u(t), p(t))dt.$$
Define the dual of $\mathcal{L}$ in both variables as
$$\mathcal{L}^*(q, v) = \sup\left\{\int_0^T \big(\langle q(t), u(t)\rangle + \langle p(t), v(t)\rangle - L(t, u(t), p(t))\big)dt : (u, p) \in L^2_X \times L^2_{X^*}\right\}$$
and denote the associated Hamiltonian on path space by
$$H_{\mathcal{L}}(u, v) = \sup\left\{\int_0^T \big(\langle p(t), v(t)\rangle - L(t, u(t), p(t))\big)dt : p \in L^2_{X^*}\right\}$$
The following is standard (see [6]).

PROPOSITION 4.2 *Suppose that $L$ is a Lagrangian on $[0,T]\times X\times X^*$ such that the corresponding Lagrangian $\mathcal{L}$ is proper on the path space $L^2_X\times L^2_{X^*}$. Then*

(i) $\mathcal{L}^*(p,u)=\int_0^T L^*(t,p(t),u(t))dt$.
(ii) $H_{\mathcal{L}}(u,v)=\int_0^T H_L(t,u(t),v(t))dt$.
(iii) *If $L$ is an anti-self-dual Lagrangian on $[0,T]\times X\times X^*$, then $\mathcal{L}$ is anti-self-dual on $L^2_X$.*

We also consider *boundary Lagrangians* $\ell : H\times H\to\mathbb{R}\cup\{+\infty\}$, which are also proper convex and lower-semicontinuous, and their Legendre transform in both variables $\ell^*$.

DEFINITION 4.3 Say that $\ell$ is a *compatible boundary Lagrangian* if

$$\ell^*(-r,s)=\ell(r,s)\quad\text{for all }(r,s)\in H\times H. \tag{4.1}$$

It is easy to see that such a boundary Lagrangian satisfies the inequality

$$\ell(r,s)\geq\frac{1}{2}(\|s\|^2-\|r\|^2)\quad\text{for all }(r,s)\in H\times H, \tag{4.2}$$

with equality holding if and only if

$$(-r,s)\in\partial\ell(r,s). \tag{4.3}$$

The basic example of a compatible boundary Lagrangian is given by a function $\ell$ on $H\times H$ of the form $\ell(r,s)=\psi_1(r)+\psi_2(s)$, with $\psi_1^*(r)=\psi_1(-r)$ and $\psi_2^*(s)=\psi_2(s)$. Here the choices for $\psi_1$ and $\psi_2$ are rather limited and the typical sample is

$$\psi_1(r)=\frac{1}{2}\|r\|^2-2\langle a,r\rangle+\|a\|^2\quad\text{and}\quad\psi_2(s)=\frac{1}{2}\|s\|^2$$

where $a$ is given in $H$.

The following shows how anti-self-dual Lagrangians "lift" to appropriate path spaces.

PROPOSITION 4.4 *Let $X\subset H\subset X^*$ be an evolution pair and consider an anti-self-dual Lagrangian on $[0,T]\times X\times X^*$ such that*

(4.4) *for each $p\in L^2_{X^*}$, the map $u\to\int_0^T L(t,u(t),p(t))dt$ is continuous on $L^2_X$, and*

(4.5) *the map $u\to\int_0^T L(t,u(t),0)dt$ is bounded on the unit ball of $L^2_X$.*

*Let $\ell$ be a compatible boundary Lagrangian on $H\times H$ such that*

$$\ell(a,b)\leq C(1+\|a\|^2_H+\|b\|^2_H)\quad\textit{for all }(a,b)\in H\times H. \tag{4.6}$$

*Then the Lagrangian*

$$\mathcal{M}_L(u,p)=\begin{cases}\int_0^T L(t,u(t),p(t)+\dot{u}(t))dt+\ell(u(0),u(T)) & \textit{if } u\in\mathcal{X}_2,\\ +\infty & \textit{otherwise},\end{cases}$$

*is anti-self-dual on $L^2_X\times L^2_{X^*}$.*

PROOF: For $(q, v) \in L^2_X \times \mathcal{X}_2$, note first that

$$\mathcal{M}_L^*(q,v) = \sup_{u\in\mathcal{X}_2}\sup_{p\in L^2_{X^*}}\left\{\int_0^T \big(\langle u(t), q(t)\rangle + \langle v(t), p(t)\rangle - L(t, u(t), p(t)+\dot u(t))\big)dt - \ell(u(0), u(T))\right\}.$$

Make a substitution $p(t) + \dot u(t) = r(t) \in L^2_{X^*}$. Since $u$ and $v$ are in $\mathcal{X}_2$, we have

$$\int_0^T \langle v, \dot u\rangle = -\int_0^T \langle \dot v, u\rangle + \langle v(T), u(T)\rangle - \langle v(0), u(0)\rangle,$$

and since the subspace $\mathcal{X}_{2,0} = \{u \in \mathcal{X}_2 : u(0) = u(T) = 0\}$ is dense in $L^2_X$, we obtain

$$\begin{aligned}
\mathcal{M}_L^*(q,v) &= \sup_{u\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\left\{\int_0^T \big(\langle u(t), q(t)\rangle + \langle v(t), r(t)-\dot u(t)\rangle - L(t,u(t),r(t))\big)dt - \ell(u(0),u(T))\right\}\\
&= \sup_{u\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\left\{\int_0^T \big(\langle u(t), q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,u(t),r(t))\big)dt\right.\\
&\qquad\left. - \langle v(T), u(T)\rangle + \langle v(0), u(0)\rangle - \ell(u(0),u(T))\right\}\\
&= \sup_{u\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\sup_{u_0\in\mathcal{X}_{2,0}}\left\{\int_0^T \big(\langle u(t), q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,u,r)\big)dt\right.\\
&\qquad - \langle v(T), (u+u_0)(T)\rangle + \langle v(0), (u+u_0)(0)\rangle\\
&\qquad\left. - \ell((u+u_0)(0), (u+u_0)(T))\right\}\\
&= \sup_{w\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\sup_{u_0\in\mathcal{X}_{2,0}}\left\{\int_0^T \big(\langle w(t)-u_0(t), q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle\right.\\
&\qquad - L(t, w(t)-u_0(t), r(t))\big)dt\\
&\qquad\left. - \langle v(T), w(T)\rangle + \langle v(0), w(0)\rangle - \ell(w(0), w(T))\right\}
\end{aligned}$$

$$= \sup_{w\in\mathcal{X}_2}\sup_{r\in L^2_{X^*}}\sup_{x\in L^2_X}\Big\{\int_0^T \big(\langle x(t), q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,x,r)\big)dt - \langle v(T), w(T)\rangle + \langle v(0), w(0)\rangle - \ell(w(0), w(T))\Big\}$$

Here we have used the fact that $\mathcal{X}_{2,0}$ is dense in $L^2_X$ and the continuity of the map $u \to \int_0^T L(t,u(t),p(t))dt$ on $L^2_X$ for each $p$.

Now, for each $(a,b) \in X\times X$, there is $w\in\mathcal{X}_2$ such that $w(0)=a$ and $w(T)=b$, namely, the linear path

$$w(t) = \frac{(T-t)}{T}a + \frac{t}{T}b.$$

Since $X$ is also dense in $H$ and $\ell$ is continuous on $H$, we finally obtain that

$$\begin{aligned}
\mathcal{M}^*_L(q,v) &= \sup_{(a,b)\in X^2}\sup_{r\in L^2_{X^*}}\sup_{x\in L^2_X}\Big\{\int_0^T \big(\langle x(t), q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,x,r)\big)dt \\
&\qquad - \langle v(T), b\rangle + \langle v(0), a\rangle - \ell(a,b)\Big\} \\
&= \sup_{x\in L^2_X}\sup_{r\in L^2_{X^*}}\Big\{\int_0^T (\langle x(t), q(t)+\dot v(t)\rangle + \langle v(t), r(t)\rangle - L(t,x(t),r(t)))dt \\
&\qquad + \sup_{a\in H}\sup_{b\in H}\{-\langle v(T), b\rangle + \langle v(0), a\rangle - \ell(a,b)\Big\} \\
&= \int_0^T L^*(t, q(t)+\dot v(t), v(t))dt + \ell^*(v(0), -v(T)) \\
&= \int_0^T L(t, -v(t), -\dot v(t) - q(t))dt + \ell(-v(0), -v(T)) \\
&= M(-v,-q).
\end{aligned}$$

If now $(q,v)\in L^2_{X^*}\times L^2_X\setminus\mathcal{X}_2$, then we use the fact that $u\to\int_0^T L(t,u(t),0)dt$ is bounded on the unit ball of $\mathcal{X}_2$ and the growth condition on $\ell$ to deduce

$$\begin{aligned}
&\mathcal{M}^*_L(q,v) \\
&\quad\geq \sup_{u\in\mathcal{X}_2}\sup_{r\in\mathcal{X}_2}\Big\{\int_0^T \big(\langle u(t), q(t)\rangle + \langle v(t), r(t)\rangle - \langle v(t), \dot u(t)\rangle \\
&\qquad - L(t,u(t),r(t))\big)dt - \ell(u(0), u(T))\Big\}
\end{aligned}$$

$$\geq \sup_{u\in\mathcal{X}_2}\sup_{r\in\mathcal{X}_2}\Big\{-\|u\|_{L^2_X}\|q\|_{L^2_{X^*}}-\|v\|_{L^2_X}\|r\|_{L^2_{X^*}}$$
$$-\int_0^T\big(\langle v(t),\dot u(t)\rangle+L(t,u(t),r(t))\big)dt-\ell(u(0),u(T))\Big\}$$
$$\geq \sup_{\|u\|_{\mathcal{X}_2}\leq 1}\Big\{-\|q\|_2+\int_0^T(\langle -v(t),\dot u(t)\rangle-L(t,u(t),0))dt-\ell(u(0),u(T))\Big\}$$
$$\geq \sup_{\|u\|_{\mathcal{X}_2}\leq 1}\Big\{C+\int_0^T(\langle -v(t),\dot u(t)\rangle-L(t,u(t),0))dt$$
$$-\frac{1}{2}(\|u(0)\|^2+\|u(T)\|^2)\Big\}$$
$$\geq \sup_{\|u\|_{\mathcal{X}_2}\leq 1}\Big\{D+\int_0^T\langle -v(t),\dot u(t)\rangle dt-\frac{1}{2}(\|u(0)\|_X^2+\|u(T)\|_X^2)\Big\}.$$

Since now $v$ does not belong to $\mathcal{X}_2$, we have that

$$\sup_{\|u\|_{\mathcal{X}_2}\leq 1}\int_0^T(\langle v(t),\dot u(t)\rangle dt+\frac{1}{2}(\|u(0)\|_X^2+\|u(T)\|_X^2)=+\infty,$$

which means that $M^*(q,v)=+\infty=M(-v,-q)$. □

Now we can prove the following:

THEOREM 4.5 *Let $X\subset H\subset X^*$ be an evolution pair, and consider an anti-self-dual Lagrangian $L$ on $[0,T]\times X\times X^*$ and a compatible boundary Lagrangian $\ell$ on $H\times H$. Assume the following conditions:*

(4.7) $u\to\int_0^T L(t,u(t),p(t))dt$ *is bounded on the balls of* $L^2_X$ *for each* $p\in L^2_{X^*}$

(4.8) $$\lim_{\|v\|_{L^2(X)}\to+\infty}\int_0^T H_L(t,0,v(t))dt=+\infty,$$

(4.9) $\ell(a,b)\leq C(1+\|a\|_H^2+\|b\|_H^2)$ *for all* $(a,b)\in H\times H$.

(i) *For any regular map $\Lambda: D(\Lambda)\subset L^2_X\to L^2_{X^*}$ such that $\mathcal{X}_2\subset D(\Lambda)$, the functional*

$$I_{\ell,L,\Lambda}(u)=\int_0^T\big\{L(t,u(t),\Lambda u(t)+\dot u(t))+\langle\Lambda u(t),u(t)\rangle\big\}dt+\ell(u(0),u(T))$$

*is self-dual on $\mathcal{X}_2$ and has 0 for an infimum. Moreover, there exists $v\in\mathcal{X}_2$ such that*

(4.10) $$(v(t),\Lambda v(t)+\dot v(t))\in\operatorname{dom}(L)\quad\textit{for almost all } t\in[0,T],$$

(4.11) $$I_{\ell,L,\Lambda}(v)=\inf_{u\in\mathcal{X}_2}I_{\ell,L,\Lambda}(u)=0,$$

$$(-v(0), v(T)) \in \partial\ell(v(0), v(T)), \tag{4.12}$$

$$(-\dot{v}(t) - \Lambda v(t), -v(t)) \in \partial L(t, v(t), \dot{v}(t) + \Lambda v(t)). \tag{4.13}$$

(ii) *In particular, for every* $v_0 \in H$ *the self-dual functional*

$$I_{v_0,L,\Lambda}(u) = \int_0^T \big\{L(t, u(t), \Lambda u(t) + \dot{u}(t)) + \langle \Lambda u(t), u(t)\rangle\big\}dt \\ + \frac{1}{2}\|u(0)\|^2 - 2\langle v_0, u(0)\rangle + \|v_0\|^2 + \frac{1}{2}\|u(T)\|^2$$

*has* 0 *for an infimum on* $\mathcal{X}_2$. *It is attained at a unique path* $v$ *such that* $v(0) = v_0$ *and satisfying* (4.10)–(4.13). *In particular, we have the following "conservation of energy type" formula: For every* $t \in [0, T]$,

$$\|v(t)\|_H^2 = \|v_0\|^2 - 2\int_0^t \big\{L(s, v(s), \Lambda v(s) + \dot{v}(s)) + \langle v(s), \Lambda v(s)\rangle\big\}ds. \tag{4.14}$$

PROOF: (i) We first apply Proposition 4.4 to get that the Lagrangian

$$\mathcal{M}_L(u, p) = \begin{cases} \int_0^T L(t, u(t), p(t) + \dot{u}(t))dt + \ell(u(0), u(T)) & \text{if } u \in \mathcal{X}_2, \\ +\infty & \text{otherwise,} \end{cases}$$

is anti-self-dual on $L^2_X$. We then apply Theorem 2.8 with the space $L^2_X$, since $\mathrm{dom}_1(\mathcal{M}) \subset \mathcal{X}_2 \subset D(\Lambda)$ to conclude that the infimum of $\mathcal{M}_L(u, \Lambda u)$ on $\mathcal{X}_2$ is equal to 0 and is achieved. This yields claim (4.10) and (4.11).

Since by (2.2), we have $L(t, v(t), \Lambda v(t) + \dot{v}(t)) + \langle v(t), \Lambda v(t) + \dot{v}(t)\rangle \geq 0$ for all $t \in [0, T]$, and by (4.2) we have $\ell(v(0), v(T)) \geq \frac{1}{2}(\|v(T)\|_H^2 - \|v(0)\|_H^2)$, claims (4.15) and (4.16) follow from the following identity:

$$0 = I_{\ell,L,\Lambda}(v) = \int_0^T \big\{L(t, v(t), \Lambda v(t) + \dot{v}(t) + \langle v(t), \Lambda v(t) + \dot{v}(t)\rangle\big\}dt \\ -\frac{1}{2}(\|v(T)\|_H^2 - \|v(0)\|_H^2) + \ell(v(0), v(T)).$$

It follows that

$$L(t, v(t), \Lambda v(t) + \dot{v}(t)) + \langle v(t), \Lambda u(t) + \dot{v}(t)\rangle = 0, \quad t \in [0, T], \tag{4.15}$$

and

$$\ell(v(0), v(T)) = \frac{1}{2}(\|v(T)\|_H^2 - \|v(0)\|_H^2), \tag{4.16}$$

which imply (4.13) and (4.12), respectively.

(ii) It suffices to apply the first part with the boundary Lagrangian

$$\ell(r, s) = \frac{1}{2}\|r\|^2 - 2\langle v_0, r\rangle + \|v_0\|^2 + \frac{1}{2}\|s\|^2,$$

which clearly satisfies the conditions in Proposition 4.4. We then get

$$I_{\ell,L,\Lambda}(u) = \int_0^T [L(t, u(t), \Lambda u(t) + \dot{u}(t)) + \langle u(t), \Lambda u(t) + \dot{u}(t)\rangle]dt + \|u(0) - v_0\|^2. \tag{4.17}$$

Note also that (4.15) yields

$$\frac{d(|v(s)|^2)}{ds} = -2[L(s, v(s), \Lambda v(s) + \dot{v}(s)) + \langle \Lambda v(s), v(s)\rangle],$$

which readily implies (4.14). □

COROLLARY 4.6 *Let $X \subset H \subset X^*$ be an evolution triple, and consider for each $t \in [0, T]$ a bounded linear operator $A_t : X \to X^*$ and $\varphi : [0, T] \times X \to \mathbb{R}$ such that for each $t$ the functional $\psi(t, x) := \varphi(t, x) + \frac{1}{2}\langle A_t x, x\rangle$ is convex and lower-semicontinuous and satisfies for some $C > 0$, $m, n > 1$, the following growth condition: For $x \in L^2_X$,*

$$\frac{1}{C}(\|x\|^m_{L^2_X} - 1) \le \int_0^T \left\{\varphi(t, x(t)) + \frac{1}{2}\langle A_t x(t), x(t)\rangle\right\}dt \le C(\|x\|^n_{L^2_X} + 1). \tag{4.18}$$

*If $\Lambda : D(\Lambda) \subset L^2_X \to L^2_{X^*}$ is a regular map such that $\mathcal{X}_2 \subset D(\Lambda)$, we consider for any $v_0 \in X$ the following self-dual functional on $\mathcal{X}_2$:*

$$I(x) = \int_0^T \{\psi(t, x(t)) + \psi^*(t, -\Lambda x(t) - A^a_t x(t) - \dot{x}(t)) + \langle \Lambda x(t), x(t)\rangle\}\, dt + \frac{1}{2}(|x(0)|^2 + |x(T)|^2) - 2\langle x(0), v_0\rangle + |v_0|^2,$$

*where for each $t \in [0, T]$, $A^a_t$ is the antisymmetric part of the operator $A_t$. Then there exists a path $v \in \mathcal{X}_2$ such that*

$$I(v) = \inf_{x\in\mathcal{X}_2} I(x) = 0, \tag{4.19}$$

$$\begin{cases} -\dot{v}(t) - A_t v(t) - \Lambda v(t) \in \partial\varphi(t, v(t)) & \textit{for a.e. } t \in [0, T], \\ v(0) = v_0. \end{cases} \tag{4.20}$$

PROOF: The Lagrangian $L(t, x, p) := \psi(t, x) + \psi^*(t, -A^a x - p)$ is an ASD Lagrangian on $X \times X^*$ by Proposition 2.6. Consider $\ell$ on $H \times H$ to be $\ell(r, s) = \frac{1}{2}(|r|^2 + |s|^2) - 2\langle r, v_0\rangle + |v_0|^2$. It is easy to check that all the conditions of Theorem 4.5 are satisfied by $L$, $\ell$, and $\Lambda$; hence there exists $v \in \mathcal{X}_2$ such that $I(v) = 0$. We obtain

$$0 = \int_0^T \big(\psi(t, v(t)) + \psi^*(t, -\Lambda v(t) - A^a_t v(t) - \dot{v}(t)) + \langle v(t), \Lambda v(t) + A^a_t v(t) + \dot{v}(t)\rangle\big)dt + \frac{1}{2}\|v(0) - v_0\|^2_H$$

which, since the integrand is nonnegative for each $t$ and since we are now in the limiting case of Legendre-Fenchel duality, yields that

$$\begin{cases} -\dot{v}(t) - A_t^a v(t) - \Lambda v(t) \in \partial\varphi(t, v(t)) + A_t^s v(t) & \text{for a.e. } t \in [0, T], \\ v(0) = v_0. \end{cases} \tag{4.21}$$

□

## 5 Application to Complex Ginsburg-Landau Evolutions

We consider the initial boundary value problem for the complex Ginzburg-Landau equation in $\Omega \subseteq \mathbb{R}^N$

$$\begin{cases} \dot{u}(t) - (\kappa + i\alpha)\triangle u + (\gamma + i\beta)|u|^{q-1}u - wu = 0, \\ u(x, 0) = u_0, \end{cases} \tag{5.1}$$

where $\kappa \geq 0$, $\gamma \geq 0$, $q \geq 1$, and $\alpha, \beta \in \mathbb{R}$. Note first that $A := -i\alpha\triangle$ is a skew-adjoint operator in $L^2(\mathbb{R}^N)$ and in $H_0^1(\Omega)$. We shall distinguish two cases:

- when $\kappa > 0$, in which case Corollary 4.6 above will apply, and
- when $\kappa = 0$, in which case we shall need that either $\beta = 0$ or $q = 1$, so as to apply the "semilinear" theory developed in [6, 11].

### Example 7: Ginsburg-Landau Evolution in the Presence of Diffusion

THEOREM 5.1 *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $\kappa > 0$, $\gamma \geq 0$, $\beta \in \mathbb{R}$, and $q > 1$. Let $H := L^2(\Omega)$, $X := H_0^1(\Omega)$, $V_1 := L_X^2$, $V_2 := L^{q+1}(0, T; L^{q+1}(\Omega))$, and $V := V_1 \cap V_2$. Then for every $u_0 \in X$ there exists $u \in V$ with $\dot{u} \in V^*$ satisfying equation* (5.1).

We would like to apply Corollary 4.6 with the nonlinear operator

$$\Lambda u := -i\triangle u + i\beta|u|^{q-1}u - wu$$

and the convex functional

$$\Phi(x) := \frac{\kappa}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\gamma}{q+1}\int_\Omega |u|^{q+1}\,dx$$

with $X = H_0^1(\Omega)$ and $H := L^2(\Omega)$. However, the operator $\Lambda$ is not regular from its domain in $L_X^2$ into $L_{X^*}^2$. We shall therefore replace $\Lambda$ by the "pseudomonotone" operator

$$\Lambda_\lambda(u) := -i\triangle u + i\beta\partial\psi_\lambda(u) - wu,$$

where $\psi_\lambda$ is the $\lambda$-regularization of $\psi(u) = \gamma/(q+1)\int |u|^{q+1}\,dx$ on $H := L^2(\Omega)$. In this case, $\Phi$ needs to be replaced by

$$\Phi_\lambda(x) := \frac{\kappa}{2}\int_\Omega |\nabla u|^2\,dx + \psi_\lambda(u).$$

We shall first prove the following:

PROPOSITION 5.2 *Suppose $\kappa, \gamma, w > 0$ and $u_0 \in X$. For every $0 < \lambda < 1/(2w)$, there exists a solution $u_\lambda \in \mathcal{X}_2$ of the $\lambda$-regularized problem*

$$\begin{cases} \dot u(t) - (\kappa + i\alpha)\Delta u + (\gamma + i\beta)\partial\psi_\lambda(u) - wu = 0 \quad \text{on } \Omega, \\ u(x,0) = u_0. \end{cases} \tag{5.2}$$

PROOF: In order to apply Corollary 4.6, we need to show that $\Lambda_\lambda$ is pseudoregular on $L^2_X$ and that the functional

$$\Phi_\lambda(u) + \langle \Lambda_\lambda u, u\rangle = \frac{\kappa}{2}\int_\Omega |\nabla u|^2\,dx + \psi_\lambda(u) - w\|u\|_H^2$$

is coercive on $H$. For $\Lambda_\lambda$, we first note that the operator $-i\Delta u - wu$ is bounded linear and so clearly "lifts" to a regular operator from $L^2_X \to L^2_{X^*}$ since $-i\Delta$ is skew-adjoint and that $u \to -wu$ is compact from $L^2_X \to L^2_H$. So we only need to verify that $B_\lambda(u) := i\partial\psi_\lambda u : L^2_X \to L^2_{X^*}$ is pseudoregular. For that, suppose that $x_n \rightharpoonup x$ weakly in $L^2_X$. Since $B_\lambda$ is Lipschitz-continuous on $L^2_H$, we can assume that $B_\lambda x_n \rightharpoonup y$ weakly in $L^2_{X^*}$. Since $\langle u, B_\lambda(u)\rangle = 0$ for every $u \in X$, it therefore suffices to show that $y = B_\lambda x$ as long as $0 \le \langle x, y\rangle$.

Now by monotonicity of $\partial\psi_\lambda$ we have $\langle B_\lambda x_n - B_\lambda u, x_n - u\rangle \ge 0$ for every $u \in L^2_X$. It follows that

$$\begin{aligned} \langle y - B_\lambda u, x - u\rangle &\ge \langle y, -u\rangle + \langle -B_\lambda u, x - u\rangle \\ &\ge \lim_n \langle B_\lambda x_n, -u\rangle + \lim_n \langle -B_\lambda u, x_n - u\rangle \\ &= \lim_n \langle B_\lambda x_n - B_\lambda u, x_n - u\rangle \\ &\ge 0. \end{aligned}$$

Hence $\langle y - B_\lambda u, x - u\rangle \ge 0$ for all $u \in L^2_X$. For $w \in L^2_X$, set $u = x - tw$ with $t > 0$ in such a way that

$$0 \le \frac{1}{t}\langle y - B_\lambda u, x - u\rangle = \langle y - B_\lambda(x - tw), w\rangle. \tag{5.3}$$

Since $B_\lambda$ is Lipschitz-continuous on $L^2_H$, we have that $\lim_{t\to 0}\langle B_\lambda(x - tw), w\rangle = \langle B_\lambda x, w\rangle$, which yields that $\langle y - B_\lambda x, w\rangle \ge 0$ for every $w \in L^2_X$ and therefore $y = B_\lambda x$.

For the coercivity, it suffices to show that for every $w > 0$ with $0 < \lambda < 1/(2w)$ the functional $\psi_\lambda(u) - w\|u\|_H^2$ is coercive on $H$. For that, write for $u \in H$,

$$\begin{aligned} \psi_\lambda(u) - w\|u\|_H^2 &= \inf_{v\in H}\left\{\psi(v) + \frac{\|u - v\|_H^2}{2\lambda}\right\} - w\|u\|_H^2 \\ &= \inf_{v\in H}\left\{\psi(v) + \frac{\|u\|_H^2}{2\lambda} + \frac{\|v\|_H^2}{2\lambda} - \frac{1}{\lambda}\langle u, v\rangle - w\|u\|_H^2\right\} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{1}{2\lambda} - w\right)\|u\|_H^2 + \inf_{v \in H}\left\{\psi(v) + \frac{\|v\|_H^2}{2\lambda} - \frac{1}{\lambda}\langle u, v\rangle\right\} \\
&= \left(\frac{1}{2\lambda} - w\right)\|u\|_H^2 - \sup_{v \in H}\left\{\frac{1}{\lambda}\langle u, v\rangle - \psi(v) - \frac{\|v\|_H^2}{2\lambda}\right\} \\
&\geq \left(\frac{1}{2\lambda} - w\right)\|u\|_H^2 - \sup_{v \in H}\left\{\frac{1}{\lambda}\langle u, v\rangle - \psi(v)\right\} \\
&\geq \left(\frac{1}{2\lambda} - w\right)\|u\|_H^2 - \psi^*\left(\frac{1}{\lambda}u\right) \\
&= \left(\frac{1}{2\lambda} - w\right)\|u\|_H^2 - \frac{\gamma^{-1/q}}{p\lambda^p}\int_\Omega |u|^p\,dx,
\end{aligned}
$$

where $1/p + 1/(q+1) = 1$. Since $q + 1 > 2$ we have $p < 2$, which implies the required coercivity of $\psi_\lambda(u) - w\|u\|_H^2$ on $H$.

All conditions of Corollary 4.6 are therefore satisfied and there exists then a solution $u_\lambda \in \mathcal{X}_2$ of (5.2).

In order to complete the proof of Theorem 5.1, we need some estimates for $u_\lambda$. For that, we multiply equation (5.2) with $u_\lambda$ to get for all $t \in [0, T]$

$$
\frac{1}{2}\frac{d}{dt}\|u_\lambda\|_H^2 + \kappa\int_\Omega |\nabla u_\lambda|^2\,dx + \psi_\lambda(u_\lambda) - w\|u_\lambda\|_H^2 \leq 0. \tag{5.4}
$$

Using Gronwall's inequality, we obtain that $\|u_\lambda\|_H$ is bounded and that consequently $u_\lambda$ is bounded in $L_X^2$. It also follows from the above inequality that the family $\int_0^T \psi_\lambda(u_\lambda)dt$ is bounded. On the other hand, the regularization process gives for every $\lambda > 0$ a unique $j_\lambda u_\lambda \in L_H^2$ such that for some constant $C > 0$,

$$
\int_0^T \psi_\lambda(u_\lambda)dt = \int_0^T \left\{\psi(j_\lambda u_\lambda) + \frac{\|u_\lambda - j_\lambda u_\lambda\|_H^2}{2\lambda}\right\} dt \leq C. \tag{5.5}
$$

We now claim that there exists $u \in V$ such that

$$
u_\lambda \rightharpoonup u \quad \text{weakly in } L_H^2, \tag{5.6}
$$

$$
u_\lambda \to u \quad \text{a.e. in } [0, T] \times \Omega. \tag{5.7}
$$

Indeed, it follows from (5.4) and (5.5) that $u_\lambda$ and $j_\lambda u_\lambda$ are bounded in $V_1$ and $V_2$, respectively. Since $\partial\psi$ and $-\Delta$ are duality maps, $-\Delta u_\lambda$ and $\partial\psi_\lambda(u_\lambda) = \partial\psi(j_\lambda u_\lambda)$ are bounded in $V_1^*$ and $V_2^*$, respectively. Let $m \in \mathbb{N}$ with $m > N/2$ in such a way that $V_0 = W_0^{m,2}(\Omega)$ continuously embeds in $X \cap L^{q+1}(\Omega)$. It follows from equation (5.2) that $\{u_\lambda\}$ is bounded in the space

$$
\mathcal{Y} := \{u \in L_X^2;\, \dot{u} \in L^p(0, T : V_0^*)\} \quad \text{where } p = \frac{q+1}{q}.
$$

Since $X \subseteq H \subseteq V_0^*$ where $X \subseteq H$ is compact and $H \subseteq V_0^*$ is continuous, the injection $\mathcal{Y} \subseteq L^2_H$ is compact, and therefore there exists $u \in \mathcal{Y}$ such that $u_\lambda \to u$ in $L^2_H$ and $u_\lambda \to u$ a.e. in $[0, T] \times \Omega$.

It follows that, up to a subsequence, we have

$$u_\lambda \rightharpoonup u \quad \text{weakly in } L^2_X, \tag{5.8}$$

$$j_\lambda u_\lambda \rightharpoonup u \quad \text{weakly in } V_2, \tag{5.9}$$

$$\partial\psi_\lambda(u_\lambda) \rightharpoonup \partial\psi(u) \quad \text{weakly in } V_2^*, \tag{5.10}$$

$$u_\lambda(T) \rightharpoonup a \quad \text{weakly in } H \text{ for some } a \in H. \tag{5.11}$$

Indeed, since $(u_\lambda)$ is bounded in $L^2_X$, and since $u_\lambda \to u$ a.e. in $[0, T] \times \Omega$, we easily get (5.8), that $j_\lambda u_\lambda \to u$, and that $\partial\psi_\lambda(u_\lambda) \to \partial\psi(u)$ a.e. in $[0, T] \times \Omega$, which together with the fact that $\partial\psi_\lambda(u_\lambda)$ is bounded in $V_2^*$, implies (5.9) and (5.10). To prove (5.11), it suffices to note that $u_\lambda(T)$ is bounded in $H$; hence $u_\lambda(T) \rightharpoonup a$ for some $a \in H$.

To complete the proof of Theorem 5.1, let $v \in C^1([0, T]; V)$ and deduce from equation (5.2) that

$$\begin{aligned} 0 = &\int_0^T \langle -(\kappa + i\alpha)\triangle u_\lambda + (\gamma + i\beta)\partial\psi_\lambda(u_\lambda) - wu_\lambda, v(t)\rangle\, dt \\ &- \int_0^T \langle \dot{v}(t), u_\lambda(t)\rangle + \langle u_\lambda(T), v(T)\rangle - \langle u_0, v(0)\rangle. \end{aligned}$$

Letting $\lambda$ go to 0, it follows from (5.8)–(5.11) that

$$\begin{aligned} 0 = &\int_0^T \langle -(\kappa + i\alpha)\triangle u + (\gamma + i\beta)\partial\psi(u) - wu, v(t)\rangle dt \\ &- \int_0^T \langle \dot{v}(t), u(t)\rangle + \langle a, v(T)\rangle - \langle u_0, v(0)\rangle. \end{aligned}$$

Therefore $u \in V$ is a solution of (5.1). □

## Example 8: Ginsburg-Landau Evolution Without Diffusion

The nondiffusive complex Ginzburg-Landau equation in $\mathbb{R}^N$,

$$\begin{cases} \dot{u}(t) - i\alpha\triangle u + \gamma|u|^{q-1}u + i\beta u - wu = 0 & \text{on } \mathbb{R}^N, \\ u(x, 0) = u_0, \end{cases} \tag{5.12}$$

is a direct consequence of the following self-dual principle for evolutions driven by essentially linear operators, established in [6]. It does not require the linear skew-adjoint operator to have a large domain in $X$, while the linear term $-wu$ can be handled by using an exponential shift.

THEOREM 5.3 *Let $X \subset H \subset X^*$ be an evolution triple, and let $A : \mathrm{dom}(A) \subseteq H \to X^*$ be a skew-adjoint operator. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a uniformly convex, lower-semicontinuous, and proper function on $X$ that is bounded on the* *bounded sets of $X$ and that is also coercive on $X$. Assume that $x_0 \in D(A)$ and that $\partial\Phi(x_0) \cap H$ is not empty. Then for all $w \in \mathbb{R}$ and for all $T > 0$, the functional*

$$I(x) = \int_0^T e^{2wt}\{\Phi(e^{-wt}x(t)) + \Phi^*(-Ae^{-wt}x(t) - e^{-wt}\dot{x}(t))\}dt \\ + \frac{1}{2}(|x(0)|^2 + |x(T)|^2) - 2\langle x(0), v_0\rangle + |v_0|^2,$$

*is self-dual on $A_H^2$ and attains its minimum at $\tilde{v} \in A_H^2$ such that $\tilde{v}(t) \in \mathrm{dom}(A)$ for almost all $t \in [0,T]$, and $I(\tilde{v}) = \inf_{x\in A_H^2} I(x) = 0$. Moreover, the path $v(t) = e^{-wt}\tilde{v}(t)$ solves the equation*

$$-\dot{v}(t) - Av(t) - wv(t) \in \partial\Phi(v(t)) \text{ for a.e. } t \in [0,T], \quad v(0) = v_0.$$

In order to find a solution for equation (5.12), it suffices to let $H := L^2(\mathbb{R}^N)$, $X = H \cap L^q(\mathbb{R}^N)$, and to consider the linear operator $Au := -i\alpha\triangle u + 2i\beta u$ and the convex function $\Phi(u) = \gamma/(q+1)\int_{\mathbb{R}^N} |u|^{q+1}\,dx$. We then obtain the following:

COROLLARY 5.4 *For every $u_0 \in D(A)$ with $\partial\Phi(u_0) \cap H \neq \varnothing$ there exists a solution $u \in A_H^2$ for equation* (5.12).

The following consequence is immediate.

COROLLARY 5.5 *For $q > 1$, $u_0 \in L^2(\mathbb{R}^N)$ with $\triangle u_0 \in L^2(\mathbb{R}^N)$ and $|u_0|^{q-1}u_0 \in L^2(\mathbb{R}^N)$, the equation*

$$\begin{cases} \dot{u}(t) - i\triangle u + |u|^{q-1}u = 0 & \text{on } \mathbb{R}^N, \\ u(x,0) = u_0, \end{cases} \tag{5.13}$$

*has a solution $u \in A_H^2$ such that $\triangle u(t) \in L^2(\mathbb{R}^N)$ for all $t \in [0,T]$. It can be obtained by minimizing the following self-dual functional on $A_H^2$:*

$$I(u) = \int_0^T \left\{\frac{1}{p+1}\int_{\mathbb{R}^N} |u(t,x)|^{p+1}\,dx \right. \\ \left. + \frac{p+1}{p}\int_{\mathbb{R}^N}\left|-i\Delta u(t,x) - \frac{\partial u}{\partial t}(t,x)\right|^{\frac{p}{p+1}} dx\right\}dt - \\ - 2\int_{\mathbb{R}^N} u(0,x)u_0(x)dx + \int_{\mathbb{R}^N} |u_0(x)|^2\,dx \\ + \frac{1}{2}\int_{\mathbb{R}^N}(|u(0,x)|^2 + |u(T,x)|^2)dx.$$

*Remark* 5.6. The global existence of unique strong solutions to (5.13) was first proved by Pecher and von Wahl [13] under the conditions $1 \le q \le \infty$ if $N = 1, 2$ and $1 \le q \le (N+2)/(N-2)$ for dimensions $3 \le N \le 8$. They conjectured that $(N+2)/(N-2)$ is the largest possible exponent (if $N > 2$) for global existence of strong solutions (see [13], remark 1.3). Shigeta managed in [14] to remove the restriction $N \leq 8$ on the dimension, but since the arguments in both [13] and [14] are based on the Gagliardo-Nirenberg inequality, they could not handle the case when $q > (N+2)/(N-2)$. This was done recently by Okazawa and Yokota [12], who proved the existence of strong solutions for all exponents $q \geq 1$. However, unlike the global argument above, their proof seems to work only for convex functions of power type.

## Example 9: Navier-Stokes Evolutions

Consider the evolution equation associated to a fluid driven by its boundary:

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u + f = \nu \Delta u - \nabla p & \text{on } [0,T] \times \Omega, \\ \operatorname{div} u = 0 & \text{on } [0,T] \times \Omega, \\ u(t,x) = u^0(x) & \text{on } [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{on } \Omega, \end{cases} \tag{5.14}$$

where $\int_{\partial\Omega} u^0 \cdot \mathbf{n}\, d\sigma = 0$, $\nu > 0$, and $f \in L^p(\Omega; \mathbb{R}^n)$. Assuming that $u^0 \in H^{3/2}(\partial\Omega)$ and that $\partial\Omega$ is connected, Hopf's extension theorem again yields the existence of $v^0 \in H^2(\Omega)$ such that

$$\begin{gathered} v^0 = u^0 \quad \text{on } \partial\Omega, \qquad \operatorname{div} v^0 = 0, \\ \int_\Omega \sum_{j,k=1}^n u_k \frac{\partial v_j^0}{\partial x_k} u_j \, dx \leq \epsilon \|u\|_X^2 \quad \text{for all } u \in X, \end{gathered} \tag{5.15}$$

where $V = \{u \in H^1(\Omega; \mathbb{R}^n) : \operatorname{div} v = 0\}$. Setting $v = u + v^0$ and then solving (5.14) reduces to finding a solution in the path space $\mathcal{X}_2$ corresponding to the Banach space $X = \{u \in H_0^1(\Omega; \mathbb{R}^n) : \operatorname{div} v = 0\}$ and the Hilbert space $H = L^2(\Omega)$ for

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + (v^0 \cdot \nabla)u + (u \cdot \nabla)v^0 + g \in -\partial\Phi(u), \quad u(0) = u_0 - v^0, \tag{5.16}$$

where $\Phi$ is again as in (3.2) and $g := f - \nu\Delta v^0 + (v^0 \cdot \nabla)v^0 \in X^*$.

In other words, this is an equation of the form

$$\frac{\partial u}{\partial t} + \Lambda u + Bu + g \in -\partial\Phi(u) \tag{5.17}$$

where $\Lambda u := (u \cdot \nabla)u$ is a nonlinear map, and where $Bu = (v^0 \cdot \nabla)u + (u \cdot \nabla)v^0$ lifts to a bounded linear operator from $L_X^2$ to $L_{X^*}^2$.

If we consider the linear version of (5.16), i.e., without the operator $\Lambda$,

$$\frac{\partial u}{\partial t} + (v^0 \cdot \nabla)u + (u \cdot \nabla)v^0 + g \in -\partial\Phi(u), \quad u(0) = u_0 - v^0, \tag{5.18}$$

and recalling again that the component $B^1 u := (v^0 \cdot \nabla)u$ of $B$ is skew-symmetric, which means, in view of Hopf's estimate,

$$C\|u\|_V^2 \geq \Psi(u) := \Phi(u) + \frac{1}{2}\langle Bu, u\rangle \geq \frac{1}{2}(\nu - \epsilon)\|u\|^2 \quad \text{for all } u \in X.$$

Letting $A^a$ be the antisymmetric part of the operator $Au = (u \cdot \nabla)v^0$, we can now apply Corollary 4.6 (with $\Lambda = 0$) to obtain a unique solution for (5.18) as the minimum of the functional

$$\begin{aligned} I(u) = \int_0^T \Big\{ \Psi(u) + \Psi^*(-B^a u + f - \dot{u}) - \int_\Omega \langle f, u\rangle dx \Big\} dt \\ + \int_\Omega \Big\{ \frac{1}{2}(|u(0,x)|^2 + |u(x,T)|^2) - 2\langle u(0,x), u_0(x) - v^0(x)\rangle \\ + |u_0(x) - v^0(x)|^2 \Big\} dx \end{aligned}$$

on $\mathcal{X}_2$ whose infimum is equal to 0.

If we now consider the full Navier-Stokes evolution, we see that—at least in dimension $n = 2$—the operator $\Lambda$ satisfies the following two properties [15]:

(1) If $u \in L^2_X \cap L^\infty_H$, then $\Lambda u \in L^2_{X^*}$; hence $\mathcal{X}_2 \subset D(\Lambda)$.
(2) If $u^k \to u$ weakly in $\mathcal{X}_2([0,T])$, then $u^k \to u$ strongly in $L^2_H$ and $\Lambda u^k \to \Lambda u$ weakly in $L^2_{X^*}$. In other words, $\Lambda$ is a regular operator on $\mathcal{X}_2$.

However, $\Lambda$ is not regular on $L^2_X$ and Corollary 4.6 does not readily apply. Still the functional

$$\begin{aligned} I(u) = \int_0^T \Big\{ \Psi(u) + \Psi^*(-(u \cdot \nabla)u - B^a u + f - \dot{u}) - \int_\Omega \langle f, u\rangle dx \Big\} dt \\ + \int_\Omega \Big\{ \frac{1}{2}(|u(0,x)|^2 + |u(x,T)|^2) - 2\langle u(0,x), u_0(x) - v^0(x)\rangle \\ + |u_0(x) - v^0(x)|^2 \Big\} dx \end{aligned}$$

is coercive and weakly lower-semicontinuous on $\mathcal{X}_2$ and therefore attains its infimum. However, in order to obtain a solution of the equation (5.16), we need to show that the infimum is actually 0. The argument requires a further refinement of Theorem 2.8 and is postponed to a forthcoming paper [9].

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