Functions spaces usefull for 3D periodic Navier-Stokes equations

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This work will be the first chapter of a book in progress. This project aims to study the 3D periodic Navier-Stokes equations, existence and regularity up-date results. We also shall study some LES related models like Leray-alpha, Bardina, ADM and deconvolution models, including the most recent results on these models.

We have take care in this work to make rigorous the mathematical foundations of functional analysis that we shall use for studying the 3D periodic Navier-Stokes equations.

1 Periodic function spaces

1.1 Lebesgue Spaces

Let $L \in \mathbb{R}_+^\star$, $\Omega = [0, L]^3 \subset \mathbb{R}^3$. We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis of $\mathbb{R}^3$, $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ the standard point in $\mathbb{R}^3$. Let us first start with some basic definitions.

[1.i] The space $L^p_{loc}(\mathbb{R}^3)$ denotes the space of all functions $u : \mathbb{R}^3 \to \mathbb{C}$ such that for all bounded set B, the restriction of u to B is in the classical Lebesgue space $L^p(B)$.

[1.ii] Let $s \in \mathbb{R}$. The space $H^s_{loc}(\mathbb{R}^3)$ denotes the space of all functions $u : \mathbb{R}^3 \to \mathbb{C}$ such that for all bounded set B, the restriction of u to B is in the classical Sobolev space $H^s(B)$.

[1.iii] A function $u : \mathbb{R}^3 \to \mathbb{C}$ is said to be Ω-periodic if and only if for all $\mathbf{x} \in \mathbb{R}^3$, for all $(p, q, r) \in \mathbb{Z}^3$ one has $u(\mathbf{x} + L(p\,\mathbf{e}_1 + q\,\mathbf{e}_2 + r\,\mathbf{e}_3)) = u(\mathbf{x})$.

[1.iv] $\mathcal{D}_{per}$ denotes all functions Ω-periodic of class C^∞.

When $p \in [1, \infty[$, we denote by $\mathbb{L}_p$ the space function defined by

$$\mathbb{L}_p = \{u : \mathbb{R}^3 \to \mathbb{C},\ u \in L^p_{loc}(\mathbb{R}^3),\ u \text{ is } \Omega-\text{periodic}\}, \tag{1.1}$$

equipped with the norm

$$||u||_{\mathbb{L}_p} = \left(\frac{1}{L^3}\int_\Omega |u(\mathbf{x})|^p d\mathbf{x}\right)^{\frac{1}{p}}. \tag{1.2}$$

When $p = 2$, $\mathbb{L}_2$ is an Hermitian space with the hermitian product

$$(u, v) = \frac{1}{L^3}\int_\Omega u(x)\overline{v}(x)dx. \tag{1.3}$$

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[1.v] Let $p \geq 1$; p' denotes its conjugate exponent given by the formula

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

[1.vi] We put $\mathcal{T}_3 = 2\pi\mathbb{Z}^3/L$. Let $\mathbb{T}_3$ be the torus defined by $\mathbb{T}_3 = \big(\mathbb{R}^3/\mathcal{T}_3\big)$. Let $C(\mathbb{T}_3)$ be the set of all continous fonction on $\mathbb{T}_3$, a set which can also be viewed as the set of all continous functions on $\mathbb{R}^3$ which are Ω-periodic.

[1.vii] Let λ be the Lesbegue measure on $\mathbb{T}_3$. Then $\mathbb{L}_p = L^p(\mathbb{T}_3, \lambda)$. The torus $\mathbb{T}_3$ is compact. Therefore thanks to Lusin's Lemma [Bib[5.i]], $C(\mathbb{T}_3)$ is everywhere dense in $\mathbb{L}_p$.

We also define $\mathbb{L}_\infty$ to be the space defined by

$$\mathbb{L}_\infty = \{u : \mathbb{R}^3 \to \mathbb{C},\, u \in L^\infty_{loc}(\mathbb{R}^3),\, u \text{ is } \Omega - \text{periodic}\}, \tag{1.4}$$

equipped with the norm

$$||u||_{\mathbb{L}_\infty} = ess \sup_{\mathbf{x}\in\Omega} |u(\mathbf{x})|. \tag{1.5}$$

Remark 1.1 *The factor $1/L^3$ involved in definition (1.2) is a normalization's factor which guaranties that for each $p \in [1,\infty]$ and each $\mathbf{k} \in \mathcal{T}_3$, one has $||e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}||_{\mathbb{L}_p} = 1$. It also makes $||u||_{\mathbb{L}_p}$ to be of the same physical dimension than u itself.*

1.2 Sobolev Spaces : version 1

[1.viii] A differentiable (weak or strong) function $u : \mathbb{R}^3 \to \mathbb{C}$ being given, we shall put

$$\partial_i u = \frac{\partial u}{\partial x_i},$$

and for a differentiable vector field $\mathbf{u} = (u^1, u^2, u^3)$, the divergence operator is defined by

$$\nabla \cdot \mathbf{u} = \sum_{i=1}^{3} \partial_i u^i = \partial_i u^i,$$

using the convention of reapeated index.

[1.ix] For u having m derivative (weak or strong), we define $\partial_{i_m,\ldots,i_1} u = \partial_{i_m}(\partial_{i_{m-1},\ldots,i_1} u)$, and the Laplace operator is defined by $\Delta u = \partial_{ii} u$.

[1.x] Let $m \in \mathbb{N}^\star$, and define the set

$$\mathcal{J}_m = \{\alpha = (i_1, ..., i_m),\, \forall\, k = 1, ..., m,\, i_k \in \{1, 2, 3\}\}.$$

We shall denote by $\nabla^m u$ the tensor $(\partial_{i_1,\ldots,i_m} u)_{(i_1,\ldots,i_m)\in\mathcal{J}_m}$. We set

$$|\nabla^m u|^2 = \sum_{(i_1,\ldots,i_m)\in\mathcal{J}_m} |\partial_{i_1,\ldots,i_m} u|^2. \tag{1.6}$$

We denote by $H^m_{per,0}(\mathbb{R}^3)$, $m \in \mathbb{N}$, the space

$$H^m_{per,0}(\mathbb{R}^3) = \{u : \mathbb{R}^3 \to \mathbb{C},\, u \in H^m_{loc}(\mathbb{R}^3),\, u \text{ is } \Omega - \text{periodic}, \int_\Omega u(\mathbf{x})d\mathbf{x} = 0\}. \tag{1.7}$$

In particular if $u \in H^m_{per,0}(\mathbb{R}^3)$, for every $q \leq m$ and every $(i_1, ..., i_q) \in \mathcal{J}_q$, $\partial_{i_1,...,i_q} u \in L^2_{loc}(\mathbb{R}^3)$ and is also periodic.

Lemma 1.1 *Let $m \in \mathbb{N}$. The application*

$$u \in H^m_{per,0}(\mathbb{R}^3) \to ||u||_{H_m} = \left(\frac{1}{L^3} \int_\Omega |\nabla^m u(\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}} \tag{1.8}$$

defines a norm on $H^m_{per,0}(\mathbb{R}^3)$.

Proof. Without loss of generality, we assume that $L = 1$. The case $m = 0$ is obvious since $H^0_{per,0}(\mathbb{R}^3)$ is a close subset of $\mathbb{L}_2$. Notice that $||u||_{H_0} = ||u||_{\mathbb{L}_2}$. We now assume that $m \geq 1$ and let $u \in H^m_{per,0}(\mathbb{R}^3)$. Then one has $|\nabla^m u| \in \mathbb{L}_2$. Therefore, the application (1.8) defines a semi norm since

$$v \to \left(\int_\Omega |v(\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}}$$

is a norm on the space $\mathbb{L}_2$. One has to check that $||u||_{H_m} = 0$ yields $u = 0$.
To do this, we start by showing that for $w = \partial_i v$, where v is a Ω-periodic smooth function, $i \in 1, 2, 3$,

$$\int_\Omega w = 0. \tag{1.9}$$

Indeed, let $\mathbf{n} = (n_1, n_2, n_3)$ be the normal outward vector to Ω, defined except on the corners and the edges of Ω. We encode the sides of Ω by Γ_i where

$$\begin{array}{lll} \Gamma_1 = \Omega \cap \{y = 0\}, & \Gamma_2 = \Omega \cap \{y = L\}, & \Gamma_3 = \Omega \cap \{z = 0\}, \\ \Gamma_4 = \Omega \cap \{z = L\}, & \Gamma_5 = \Omega \cap \{z = 0\}, & \Gamma_6 = \Omega \cap \{z = L\}. \end{array}$$

Then one has

$$\begin{array}{lll} \mathbf{n}_{\Gamma_1} = (0, -1, 0), & \mathbf{n}_{\Gamma_2} = (0, 1, 0), & \mathbf{n}_{\Gamma_3} = (0, 0, -1), \\ \mathbf{n}_{\Gamma_4} = (0, 0, 1), & \mathbf{n}_{\Gamma_5} = (-1, 0, 0), & \mathbf{n}_{\Gamma_6} = (1, 0, 0). \end{array}$$

Notice that one has $\mathbf{n}_{\Gamma_j} = -\mathbf{n}_{\Gamma_{j+1}}$ for $j = 1, 3, 5$. We now apply the Stokes formula in writing

$$\int_\Omega w = \int_{\partial\Omega} v\, n_i = \int_{\Gamma_1 \cup \Gamma_2} v\, n_i + \int_{\Gamma_3 \cup \Gamma_4} v\, n_i + \int_{\Gamma_5 \cup \Gamma_6} v\, n_i.$$

Thanks to the Ω-periodicity of v, one has $v_{\Gamma_j} = v_{\Gamma_{j+1}}$ for $j = 1, 3, 5$. Therefore, the considerations above on $\mathbf{n}$ makes sure that

$$\int_{\Gamma_j \cup \Gamma_{j+1}} v\, n_i = 0, \quad j = 1, 3, 5.$$

which yields (1.9) as claimed.

We now are in order to finish the proof. Let us assume that for $u \in H^m_{per,0}(\mathbb{R}^3)$, $||u||_{H_m} = 0$, meaning $\nabla^m u = 0$ *a.e.* in Ω, that can be read as $\partial_{i_m, i_{m-1} \ldots, i_1} u = 0$ for each $(i_m, i_{m-1} \ldots, i_1) \in J_m$. Therefore $\partial_{i_{m-1} \ldots, i_1} u$ is almost everywhere equal to a constant in the sense of the distributions Bib[5.ii], and this holds for every $(i_{m-1} \ldots, i_1) \in J_{m-1}$. Since its mean value equal to zero thanks to the result above, is it equal to zero almost everywhere. A finite induction yields $u = 0$ *a.e.* □

[1.xi] The associated Hermitian product in $H^m_{per,0}(\mathbb{R}^3)$ is defined by

$$(1.10) \qquad (u, v)_{H^m_{per,0}(\mathbb{R}^3)} = \frac{1}{L^3} \int_\Omega \nabla^m u \cdot \nabla^m \overline{v} = (\nabla^m u, \nabla^m v),$$

where

$$(1.11) \qquad \nabla^m u \cdot \nabla^m \overline{v} = \sum_{(i_1, \ldots, i_m) \in J_m} \partial_{i_1, \ldots, i_m} u \, . \, \partial_{i_1, \ldots, i_m} \overline{v}.$$

2 Definitions via Fourier Series

2.1 The ℓ_p spaces

[2.i] For $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{T}_3$, we put

$$(2.1) \qquad |\mathbf{k}|^2 = k_1^2 + k_2^2 + k_3^2, \quad |k|_\infty = \sup_i |k_i|,$$

$I_n = \{\mathbf{k} \in \mathcal{T}_3; \ |\mathbf{k}|_\infty \le n\}$.

[2.ii] We say that a Ω-periodic function P is a trigonometric polynomial if there exists $n \in \mathbb{N}$ and coefficients $a_\mathbf{k}$, $\mathbf{k} \in I_n$, and such that $P = \sum_{\mathbf{k} \in I_n} a_\mathbf{k} e^{\mathrm{i}\, \mathbf{k} \cdot \mathbf{x}}$. The degree of P is the greatest q such that there is a $\mathbf{k}$ with $|\mathbf{k}|_\infty = q$ and $a_\mathbf{k} \neq 0$.

[2.iii] We note V_n the finite dimensional space of all trigonometric polynomial of degree less than n with mean value equal to zero,

$$V_n = \{u = \sum_{\mathbf{k} \in I_n} u_\mathbf{k} e^{\mathrm{i}\, \mathbf{k} \cdot \mathbf{x}}, \, u_\mathbf{0} = 0\},$$

and $\mathbb{P}_n$ the orthogonal projection from $\mathbb{L}_2$ onto his closed subspace V_n.

[2.iv] Finally, let us put $\mathcal{I}_3 = \mathcal{T}_3^\star = (2\pi \mathbb{Z}^3 / L) \setminus \{0\}$.

Let $p \in [1, \infty]$ and

$$(2.2) \qquad \ell_p = \left\{ u : \mathbb{R}^3 \to \mathbb{C}, \quad u = \sum_{\mathbf{k} \in \mathcal{T}_3} u_\mathbf{k} e^{\mathrm{i}\, \mathbf{k} \cdot \mathbf{x}}, \quad \sum_{\mathbf{k} \in \mathcal{T}_3} |u_\mathbf{k}|^p < \infty \right\},$$

where above $\mathrm{i} \in \mathbb{C}$ with $\mathrm{i}^2 = -1$. We set

$$(2.3) \qquad ||u||_{\ell_p} = \left(\sum_{\mathbf{k} \in \mathcal{T}_3} |u_\mathbf{k}|^p \right)^{\frac{1}{p}}.$$

Lemma 2.1 *The formula (2.3) defines a norm and with this norm ℓ_p is a Banach space.*

Proof. Let ν be the "counting" measure on the discrete space T, equipped with the standard discrete σ-algebra. The counting measure is defined by $\nu(\mathbf{k}) = 1$ for each $\mathbf{k} \in \mathcal{T}_3$. Therefore, the space ℓ_p is the Lebesgue Space $L^p(\mathcal{T}_3, \nu)$. This concludes the proof thanks to standard results on Lebesgue Spaces. □

Remark 2.1 *Let $u \in \ell_1$. The function's serie which general term is $u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$ is a normal convergent serie. Therefore its sum is a continuous Ω-periodic function.*

We now establish the link between the space ℓ_p and the space $\mathbb{L}_p$. For it, let $f \in \mathbb{L}_p$. It makes sense to consider its Fourier's coefficient for $\mathbf{k} \in \mathcal{T}_3$,

$$\hat{f}_{\mathbf{k}} = (f, e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}) = \frac{1}{L^3} \int_\Omega f(\mathbf{x}) e^{-\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} d\mathbf{x},$$

and to introduce the formal serie defined by

$$Tf = \sum_{\mathbf{k} \in \mathcal{T}_3} \hat{f}_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{2.4}$$

Lemma 2.2 *Assume that $f \in \mathbb{L}_1$. Then $Tf \in \ell_\infty$ and one has*

$$||Tf||_{\ell_\infty} \leq ||f||_{\mathbb{L}_1}. \tag{2.5}$$

Proof. From the definition (2.4) and $|e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}| = 1$ one directly get for every $\mathbf{k} \in \mathcal{T}_3$,

$$|\hat{f}_{\mathbf{k}}| \leq \left| \frac{1}{L^3} \int_\Omega |f(\mathbf{x})| d\mathbf{x} \right| = ||f||_{\mathbb{L}_1},$$

and the result easely follows. □

Lemma 2.3 *The operator T is an isometry between $\mathbb{L}_2$ and ℓ_2 and one has for each $f \in \mathbb{L}_2$,*

$$||Tf||_{\ell_2} = ||f||_{\mathbb{L}_2}. \tag{2.6}$$

Proof. The key of the present proof is the density of $C(\mathbb{T}_3)$ in $\mathbb{L}_2$ (see 1.vi and 1.vii above). Therefore, we only need to prove (2.6) for any Ω-periodic continous function, and the rest of the claim in Lemma 2.3 will become straightforward.

We know from the Stone-Weirstrass Theorem [Bib[5.iii]] that each Ω-periodic continous function f can be uniformly approached by a sequence of trigonometric polynomials, $(f_j)_{j \in \mathbb{N}}$. An integer j being given, let n_j be the degree of f_j. We shall assume that f is not a trigometrical polynomial, else (2.6) is obvious. Therefore we are in the case where n_j goes to infinity when j goes to infinity. Since $\mathbb{P}_{n_j}$ is an orthogonal projection (see 2.iii), one has

$$||f - \mathbb{P}_{n_j} f||_{\mathbb{L}_2} \leq ||f - f_j||_{\mathbb{L}_2} \leq ||f - f_j||_\infty.$$

We deduce from the inequalities above

$$\lim_{j\to\infty} ||f - \mathbb{P}_{n_j} f||_{\mathbb{L}_2} = 0. \tag{2.7}$$

Moreover, combining the following decomposition

$$f = (f - \mathbb{P}_{n_j} f) + \mathbb{P}_{n_j} f,$$

and (2.7), we see that $\mathbb{P}_{n_j} f$ converges to f in $\mathbb{L}_2$ when j goes to infinity. Therefore, one has

$$\lim_{j\to\infty} ||\mathbb{P}_{n_j} f||_{\mathbb{L}_2} = ||f||_{\mathbb{L}_2}.$$

We finish the proof in observing that

$$\mathbb{P}_{n_j} f = \sum_{|\mathbf{k}|_\infty \le n_j} (f, e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}) e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} = \sum_{|\mathbf{k}|_\infty \le n_j} \hat{f}_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},$$

which yields

$$||\mathbb{P}_{n_j} f||^2_{\mathbb{L}_2} = \sum_{|\mathbf{k}|_\infty \le n_j} |\hat{f}_{\mathbf{k}}|^2 \longrightarrow ||Tf||^2_{\ell_2} = ||f||^2_{\mathbb{L}_2}.$$

when $j \to \infty$. □

Theorem 2.1 *Suppose that $1 \le p \le 2$.*
(1) If $f \in \mathbb{L}_p$ then $Tf \in \ell_{p'}$ and one has

$$||Tf||_{\ell_{p'}} \le ||f||_{\mathbb{L}_p} \tag{2.8}$$

(2) If $u \in \ell_p$, then there exists $f \in \mathbb{L}_{p'}$ be such that $u = Tf$ and one has

$$||f||_{\mathbb{L}_{p'}} \le ||u||_{\ell_p}. \tag{2.9}$$

Proof. The inequality (2.8) follows from (2.5), (2.6) combined with the interpolation Theorem due to Riesz-Thorin [Bib[5.vi]]. Indeed, T is of type $[1,\infty]$ (see (2.5) and also Lemma 2.1 together with the definition [5.vi] below). Moreover, T is also of type $[2,2]$ (from (2.6). Applying the Riesz-Thorin Theorem, T is of type $[p,q]$ where $1/p = (1-t)/2 + t/1$, $1/q = (1-t)/2 + t/\infty$, $t \in [0,1]$. It easy checked that $1/p + 1/q = 1$ and the constant is less than the infimum of both constants involved in the case $[2,2]$ and $[1,\infty]$, than means 1 in the present case.

For proving the second point, we start with the case $1 < p \le 2$, and let $u \in \ell_p$ and consider $U_n = \sum_{|\mathbf{k}|_\infty \le n} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$ (note that in the case $p = 2$, $\ell_2 = \mathbb{L}_2$ and $U_n = \mathbb{P}_n u$). Let $h \in \mathbb{L}_p$; one has

$$\frac{1}{L^3} \int_\Omega h(\mathbf{x}) \overline{U}_n(\mathbf{x}) d\mathbf{x} = \sum_{|\mathbf{k}|_\infty \le n} \overline{u}_{\mathbf{k}} \hat{h}_{\mathbf{k}}.$$

Using Hölder inequality, we deduce

$$\left| \frac{1}{L^3} \int_\Omega h(\mathbf{x}) \overline{U}_n(\mathbf{x}) d\mathbf{x} \right| \le \left(\sum_{|\mathbf{k}|_\infty \le n} |u_{\mathbf{k}}|^p \right)^{\frac{1}{p}} \left(\sum_{|\mathbf{k}|_\infty \le n} |\hat{h}_{\mathbf{k}}|^{p'} \right)^{\frac{1}{p'}}.$$

Using (2.8), one gets

$$\left| \frac{1}{L^3} \int_\Omega h(\mathbf{x}) \overline{U}_n(\mathbf{x}) d\mathbf{x} \right| \le ||u||_{\ell_p} ||h||_{\mathbb{L}_p}.$$

Therefore $U_n \in \mathbb{L}_{p'}$ and one has

$$||U_n||_{\mathbb{L}_{p'}} \le ||u||_{\ell_p}. \tag{2.10}$$

The same argument also yields for $m > n$

$$||U_m - U_n||_{\mathbb{L}_{p'}} \le \left(\sum_{n \le |\mathbf{k}|_\infty \le m} |u_\mathbf{k}|^p \right)^{\frac{1}{p}}.$$

Since $u \in \ell_p$, the real serie with general term $|u_\mathbf{k}|^p$ is a convergent serie. Therefore $(U_n)_{n\in\mathbb{N}}$ is a Cauchy's sequence in the space $\mathbb{L}_{p'}$ and consequently it is a convergent sequence in $\mathbb{L}_{p'}$. We denote by $f \in \mathbb{L}_{p'}$ its limit. It remains to prove that $Tf = u$. Let $\mathbf{k} \in \mathcal{T}_3$ and observe that one has for $n \ge |\mathbf{k}|_\infty$,

$$|\hat{f}_\mathbf{k} - u_\mathbf{k}| = \left| \frac{1}{L^3} \int_\Omega [f(\mathbf{x}) - U_n(\mathbf{x})] e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} d\mathbf{x} \right| \le ||f - U_n||_{\mathbb{L}_{p'}}. \tag{2.11}$$

We deduce that $\hat{f}_\mathbf{k} = u_\mathbf{k}$ by letting n go to infinity. Therefore one get the claimed equality, $Tf = u$. Moreover, (2.9) is a consequence of (2.10) when n goes to infinity.

It remains to check the case $p = 1$. As we already have said in Remark 2.1, the sum of the uniformly convergent serie $u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}$ is a Ω-periodic continuous function, that we denote by f. Since one has for all $\mathbf{x} \in \mathbb{R}$

$$|f(\mathbf{x})| \le \sum_{\mathbf{k}\in\mathcal{I}_3} |u_\mathbf{k}| = ||u||_{\ell_1},$$

we obtain$||f||_\infty \le ||u||_{\ell_1}$. The fact that $u = Tf$ proceeds in the same way than in (2.11) when replacing p' by ∞. □

2.2 Sobolev spaces : version 2

A real number s being given, we consider the space function $\mathbb{H}_s$ defined by

$$\mathbb{H}_s = \left\{ u : \mathbb{R}^3 \to \mathbb{C}, \quad u = \sum_{\mathbf{k}\in\mathcal{T}_3} u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, \quad u_0 = 0, \quad \sum_{\mathbf{k}\in\mathcal{T}_3} |\mathbf{k}|^{2s} |u_\mathbf{k}|^2 < \infty \right\}, \tag{2.12}$$

We put

$$||u||_s = \left(\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} |u_\mathbf{k}|^2 \right)^{\frac{1}{2}}, \quad (u,v)_s = \sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} u_\mathbf{k} \overline{v}_\mathbf{k}. \tag{2.13}$$

Lemme 2.1 *Let $s \in \mathbb{R}$. Then $\mathbb{H}_s$ endowed with the structure (2.13) is an Hermitian space.*

Proof. As in the proof of Lemma (2.1), we consider $\mathcal{I}_3 = \mathcal{T}_3^\star$ endowed with its discrete σ-algebra but now with the measure ν_s defined by $\nu_s(\{\mathbf{k}\}) = |\mathbf{k}|^{2s}$ (recall that $\mathbf{0} \notin \mathcal{I}_3$). Therefore $\mathbb{H}_s = L^2(\mathcal{I}_3, \nu_s)$ and the result is straightforward. □

Lemme 2.2 *Let $s \in \mathbb{R}$. There exists an isometry between $\mathbb{H}_{-s}$ and $\mathbb{H}'_s$.*

Proof. Let $u \in \mathbb{H}_{-s}$. We associate to u the form f defined by

$$\forall\, v \in \mathbb{H}_s, \quad (f, v) = \sum_{\mathbf{k}\in\mathcal{I}_3} u_{\mathbf{k}} \overline{v}_{\mathbf{k}}.$$

Notice that $(f, e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}) = u_{\mathbf{k}}$. Using the Cauchy-Schwarz inequality, we get when writing $1 = |\mathbf{k}|^s |\mathbf{k}|^{-s}$,

$$|(f,v)| \le \left(\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{-2s} |u_{\mathbf{k}}|^2 \right)^{1/2} \left(\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{2s} |v_{\mathbf{k}}|^2 \right)^{1/2} = ||u||_{-s} ||v||_s.$$

Therefore $f \in \mathbb{H}'_s$ and one has $||f||_{\mathbb{H}'_s} \le ||u||_{-s}$. This leads to introduce the map

$$\Psi : \begin{cases} \mathbb{H}_{-s} \longrightarrow \mathbb{H}'_s, \\ \quad u \longrightarrow f. \end{cases}$$

This map is linear and continuous with $||\Psi|| \le 1$. We have to show that Ψ is invertible and that its norm is equal to one.
We firstly note that one naturally has $(\Psi(u), e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}) = u_{\mathbf{k}}$. Let $f \in \mathbb{H}'_s$ and for $\mathbf{k} \in \mathcal{I}_3$, let us put $u_{\mathbf{k}} = (f, e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}})$. For a given integer n, consider $v \in \mathbb{H}_s$ defined by $v_{\mathbf{k}} = |\mathbf{k}|^{-2s} \overline{u}_{\mathbf{k}}$ when $|\mathbf{k}| \le n$, $v_{\mathbf{k}} = 0$ else. This guy lives in $\mathbb{H}_s$ and one has

$$(f,v) = \sum_{\mathbf{k}\in I_n} |\mathbf{k}|^{-2s} |u_{\mathbf{k}}|^2 \le ||f||_{\mathbb{H}'_s} ||v||_{\mathbb{H}_s} = ||f||_{\mathbb{H}'_s} \left(\sum_{\mathbf{k}\in I_n} |\mathbf{k}|^{-2s} |u_{\mathbf{k}}|^2 \right)^{1/2}. \tag{2.14}$$

If $f = 0$, there is nothing to prove. Assume that $f \neq 0$. Then for n_0 large enough there is some $\mathbf{k} \in \mathcal{I}_3$ with $|\mathbf{k}|_\infty \le n_0$ and such that $u_{\mathbf{k}} \neq 0$. Then, for every $n \ge n_0$, we deduce from (2.14)

$$\left(\sum_{\mathbf{k}\in I_n} |\mathbf{k}|^{-2s} |u_{\mathbf{k}}|^2 \right)^{1/2} \le ||f||_{\mathbb{H}'_s}. \tag{2.15}$$

This inequality remains true when n goes to infinity. Then

$$u = \sum_{\mathbf{k}\in\mathcal{I}_3} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} \in \mathbb{H}_s, \text{ and one has } \Psi(u) = f.$$

Therefore Ψ is invertible and (2.15) shows that $||\psi^{-1}|| \le 1$ when n goes to infinity, which together with $||\Psi|| \le 1$ yields $||\Psi|| = 1$, and the proof is complete. □

Theorem 2.2 *Let $m \in \mathbb{N}$. The operator T is a continous isomorphism between $H^m_{per,0}(\mathbb{R}^3)$ and $\mathbb{H}_m$.*

Proof. Let us consider the operator $(-\Delta)^m = (-\Delta) \circ (-\Delta)^{m-1}$. We notice that

$$(-\Delta)^m e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}} = |\mathbf{k}|^{2m} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \tag{2.16}$$

Using this remark and the result stated in the proof of Lemma 2.3, we shall first prove that $\mathcal{F} = (e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}})_{\mathbf{k}\in\mathcal{I}_3}$ is a total familly in $H^m_{per,0}(\mathbb{R}^3)$. Let us indeed consider $f \in H^m_{per,0}(\mathbb{R}^3)$ be such that $(e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, f)_{H^m_{per,0}(\mathbb{R}^3)} = 0$ (see Remark 1.xi above). We then use f as test function in (2.16) and we remember that $H^m_{per,0}(\mathbb{R}^3)$ is by definition a subspace of $\mathbb{L}_2$. This yields

$$\int_\Omega \nabla^m(e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}) \cdot \nabla^m f = |\mathbf{k}|^{2m}(e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}, f)_{\mathbb{L}_2} = 0.$$

Since we already know that $\mathcal{F}$ is a total familly in $\mathbb{L}_2$, we deduce that $f = 0$. We now have to show that $T(H^m_{per,0}(\mathbb{R}^3)) = \mathbb{H}_m$.
Recall that the set $\mathcal{J}_m$ is defined in [1.x]. Let $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{I}_3$, $m \in \mathbb{N}^\star$. We denote by $K_m(\mathbf{k})$ the set

$$K_m(\mathbf{k}) = \{(k_{i_1}, ..., k_{i_m}),\ (i_1, ..., i_m) \in \mathcal{J}_m,\ k_{i_j} \in \{k_1, k_2, k_3\}\}. \tag{2.17}$$

Notice that the previous result combined with the definition 1.8 of the norm that we first have considered on the space $H^m_{per,0}(\mathbb{R}^3)$ yields

$$T(H^m_{per,0}(\mathbb{R}^3)) = \Bigg\{ u : \mathbb{R}^3 \to \mathbb{C},\ u = \sum_{\mathbf{k}\in I} u_{\mathbf{k}} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}},\ u_0 = 0, \\ \sum_{\mathbf{k}\in I} \Big(\sum_{K_m(\mathbf{k})} |k_{i_1}|^2 ....|k_{i_m}|^2 \Big) |u_{\mathbf{k}}|^2 < \infty \Bigg\},$$

The end of the proof is technical. It remains to prove the equivalence of norms on formal series to be sure that the norm

$$\left(\sum_{\mathbf{k}\in I} \Big(\sum_{K_m(\mathbf{k})} |k_{i_1}|^2 ....|k_{i_m}|^2 \Big) |u_{\mathbf{k}}|^2 \right)^2$$

defines the same toplogy than the norm

$$\left(\sum_{\mathbf{k}\in I} |\mathbf{k}|^{2m} |u_{\mathbf{k}}|^2 \right)^{1/2}.$$

This is done in in section iii.i where the inequality (3.2) is carrefully proved. □

In the remainder we shall identify $H^m_{per,0}(\mathbb{R}^3)$ and $\mathbb{H}_m$ for an integer[Bib 5.vii].

Remark 2.2 *So far functions in $\mathbb{H}_m$ are limit of trigonometric polynomials which are also in $\mathcal{D}_{per}$ (see definition [1.iv] above), therefore $\mathcal{D}_{per,0}$ is everywhere dense in $\mathbb{H}_m$, by noting $\mathcal{D}_{per,0}$ the subspace of $\mathcal{D}_{per}$ of all function having a mean value equal to zero.*

2.3 Compactness results

Theorem 2.3 *Assume* $6/5 < p < 2$*. Then there exists a compact injection from* $\mathbb{H}_1$ *onto* ℓ_p *and there exists a constant* C_p *such that one has*

$$\forall\, u \in \mathbb{H}_1, \quad ||u||_{\ell_p} \le C_p ||u||_1. \tag{2.18}$$

Proof. Let p be such that $6/5 < p < 2$. We use the Hölder inequality (with the usual notations) and get

$$\begin{aligned} &||u||_{\ell_p}^p = \sum_{\mathbf{k}\in\mathcal{I}_3} |u_\mathbf{k}|^p = \\ &\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^{-p}|\mathbf{k}|^p|u_\mathbf{k}|^p \le \left(\sum_{\mathbf{k}\in\mathcal{I}_3} \frac{1}{|\mathbf{k}|^{\frac{2p}{2-p}}}\right)^{\frac{2-p}{2}} \left(\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^2|u_\mathbf{k}|^2\right)^{\frac{p}{2}}. \end{aligned} \tag{2.19}$$

It is well konwn that the serie

$$\sum_{\mathbf{k}\in\mathcal{I}_3} \frac{1}{|\mathbf{k}|^{\frac{2p}{2-p}}}$$

converges if and only if $(2p/2-p) > 3$ (we are working in the special 3D case), that means if and only if $p > 6/5$. We denote by ξ_p its limi. tHence for each $p \in]6/5, 2[$ one has

$$||u||_{\ell_p} \le \xi_p^{\frac{2-p}{2p}} ||u||_1,$$

and there is indeed a continous injection I_p mapping $\mathbb{H}_1$ into ℓ_p. Let us show that I_p is compact. To this end, let us consider

$$I_{p,n} : \left\{ \begin{array}{l} \mathbb{H}_1 \longrightarrow \ell_p, \\ u \longrightarrow \displaystyle\sum_{\mathbf{k}\in I_n} u_\mathbf{k} e^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}. \end{array} \right. \tag{2.20}$$

The map $I_{p,n}$ has a finite rank in the sense that $\dim(I_{p,n}(\mathbb{H}_1)) = \dim V_n$, where C is a constant which do not depend on n. Moreover, using the same trick than in (2.19), one has

$$||(I_p - I_{p,n})u||_{\ell_p}^p = \sum_{|\mathbf{k}|_\infty > n} |u_\mathbf{k}|^p \le \left(\sum_{|\mathbf{k}|_\infty > n} \frac{1}{|\mathbf{k}|^{\frac{2p}{2-p}}}\right)^{\frac{2-p}{2}} \left(\sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^2|u_\mathbf{k}|^2\right)^{\frac{p}{2}}, \tag{2.21}$$

which yields

$$||I_p - I_{p,n}|| \le \left(\sum_{|\mathbf{k}|_\infty > n} \frac{1}{|\mathbf{k}|^{\frac{2p}{2-p}}}\right)^{\frac{2-p}{2}} \longrightarrow 0 \quad \text{when } n \to \infty.$$

We have proved that I_n is a limit of finite rank operators, therefore it is a compact operator[Bib [5.iv]]. □

Corollary 2.1 *Let* $2 < p < 6$*. Then there exists a compact injection from* $\mathbb{H}_1$ *onto* $\mathbb{L}_p$*, and there exists a constant* $C_p > 0$ *such that*

$$\forall\, u \in \mathbb{H}_1, \quad ||u||_{\mathbb{L}_p} \le C_p ||u||_1. \tag{2.22}$$

Proof. This a direct consequence of Theorem 2.3 and inequality (2.18) combined with Theorem 2.1. □

It remains to treat the case $p = 2$.

Lemme 2.3 *The following Poincaré's inequality holds*

$$||u||_{\ell_2} \leq \frac{L}{2\pi}||u||_1, \tag{2.23}$$

and the injection from $\mathbb{H}_1$ *in* $\mathbb{L}_2 = \ell_2$ *is compact.*

Proof. Since for every $\mathbf{k} \in \mathcal{I}_3$, $|\mathbf{k}| \geq 2\pi/L$, then

$$(2\pi/L)^2||u||_{\ell_2}^2 = (2\pi/L)^2 \sum_{\mathbf{k}\in\mathcal{I}_3} |u_{\mathbf{k}}|^2 \leq \sum_{\mathbf{k}\in\mathcal{I}_3} |\mathbf{k}|^2|u_{\mathbf{k}}|^2 = ||u||_1^2.$$

Moreover the compactness of the injection of $\mathbb{H}_1$ onto $\mathbb{L}_4$ combined with the continuity of the injection of $\mathbb{L}_4$ into $\mathbb{L}_2$ guaranties that the injection from $\mathbb{H}_1$ onto $\mathbb{L}_2$ is compact. □

Remark 2.3 *We can be more occurate in what preceeds. Indeed, let* $u \in \mathbb{H}_1$*; using the Cauchy-Schwarz inequality combined with* (2.9)*, one has*

$$||u - \mathbb{P}_n u||_{\mathbb{L}_2} \leq L^{3/4}||u - \mathbb{P}_n u||_{\mathbb{L}_4} \leq L^{3/4}||u - \mathbb{P}_n u||_{\ell_{4/3}} \leq C_n||u - \mathbb{P}_n u||_1, \tag{2.24}$$

where C_n *goes to zero when* n *goes to infinity, and where we have used* (2.21) *for* $p = 4$*, identifying* u *with* Tu*.*

Remark 2.4 *It is easy checked that the constant* $L/2\pi$ *in (2.23) is the best constant.*

Lemme 2.4 *There exists a constant* S *which do not depend on* L *be such that the following Sobolev inequality holds:*

$$\forall\, u \in \mathbb{H}_1 \quad ||u||_{\mathbb{L}_6} \leq S||u||_1. \tag{2.25}$$

Proof. We start by noting that thanks to the periodicity, we can work with the cube $\tilde{\Omega} = [-L/2, L/2]^3$ instead of $\Omega = [0, L]^3$, without changing the values of the integrals that we study. We also denote by Γ_i the face of the cube $\tilde{\Omega}$ defined by $\Gamma_i = \tilde{\Omega} \cap \{x_i = -L/2\}$. Let $u \in \mathcal{D}_{per,0}$, u_i its trace on Γ_i. We write for $i = 1, 2, 3$, $\mathbf{x} = (x_1, x_2, x_3) \in \tilde{\Omega}$,

$$u(x_1, x_2, x_3) = u_i(\check{x}_i) + \int_{-L/2}^{x_i} \frac{\partial u}{\partial x_i}(x_1, x_2, x_3)dx_i,$$

where $\check{x}_1 = (-L/2, x_2, x_3)$, $\check{x}_2 = (x_1, -L/2, x_3)$, $\check{x}_3 = (x_1, x_2, -L/2)$. Therefore

$$|u|(x_1, x_2, x_3) \leq |u_i(\check{x}_i)| + \int_{-L/2}^{L/2} \left|\frac{\partial u}{\partial x_i}(x_1, x_2, x_3)dx_i\right| = f_i(\tilde{\mathbf{x}}_i), \tag{2.26}$$

where $\tilde{x}_1 = (x_2, x_3)$, $\tilde{x}_2 = (x_1, x_3)$, $\tilde{x}_3 = (x_1, x_2)$. We deduce the following inequality

$$|u|^{\frac{3}{2}}(x_1, x_2, x_3) \leq f_1(\tilde{x}_1)^{1/2} f_2(\tilde{x}_2)^{1/2} f_3(\tilde{x}_3)^{1/2}.$$

Integrating this inequality with respect to x_3 yields

$$\int_{-L/2}^{L/2} |u|^{3/2}(x_1,x_2,x_3)dx_3 \le f_3(\tilde{x}_3)^{1/2} \int_{-L/2}^{L/2} f_2(\tilde{\mathbf{x}}_2)^{1/2} f_1(\tilde{\mathbf{x}}_1)^{1/2} dx_3.$$

We then apply the Cauchy-Schwarz inequality to get

$$\int_{-L/2}^{L/2} |u|^{3/2}(x_1,x_2,x_3)dx_3 \le f_3(\tilde{x}_3)^{1/2} \left(\int_{-L/2}^{L/2} f_2(\tilde{\mathbf{x}}_2)dx_3\right)^{1/2} \left(\int_{-L/2}^{L/2} f_1(\tilde{\mathbf{x}}_1)dx_3\right)^{1/2},$$

We now integrate this with respect to x_2 and get

$$\int_{-L/2}^{L/2} |u|^{3/2}(x_1,x_2,x_3)dx_2dx_3 \le \left(\int_{-L/2}^{L/2} f_2(\tilde{\mathbf{x}}_2)dx_3\right)^{1/2} \int_{-L/2}^{L/2} f_3(\tilde{x}_3)^{1/2} \left(\int_{-L/2}^{L/2} f_1(\tilde{\mathbf{x}}_1)dx_3\right)^{1/2} dx_2,$$

which yields by using the Cauchy-Schwarz inequality

$$\int_{-L/2}^{L/2} |u|^{3/2}(x_1,x_2,x_3)dx_2dx_3 \le \left(\int_{-L/2}^{L/2} f_2(\tilde{\mathbf{x}}_2)dx_3\right)^{1/2} \left(\int_{-L/2}^{L/2} f_3(\tilde{x}_3)dx_2\right)^{1/2} \left(\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} f_1(\tilde{\mathbf{x}}_1)dx_2dx_3\right)^{1/2}.$$

We now integrate this last inequality with respect to x_1, we use again the Cauchy-Schwarz inequality and we get

$$||u||_{\mathbb{L}_{3/2}}^{3/2} \le \prod_{i=1}^{3} ||f_i||_{L^1([-L/2,L/2]^2)}^{1/2} \le \prod_{i=1}^{3} \left(\left|\left|\frac{\partial u}{\partial x_i}\right|\right|_{\mathbb{L}_1}^{1/2} + ||u_i||_{L^1(\Gamma_i)}^{1/2} \right). \tag{2.27}$$

We now put $u = v^4$ and insert this change of variables in inequality (2.27) to obtain

$$||v||_{\mathbb{L}_6}^{6} \le \prod_{i=1}^{3} \left(4\left|\left|v^3\frac{\partial v}{\partial x_i}\right|\right|_{\mathbb{L}_1}^{1/2} + ||v_i^4||_{L^1(\Gamma_i)}^{1/2} \right). \tag{2.28}$$

We firstly notice that the Cauchy-Schwarz inequality yields

$$\left|\left|v^3\frac{\partial v}{\partial x_i}\right|\right|_{\mathbb{L}_1}^{1/2} \le ||v||_{\mathbb{L}_6}^{3/2} \left|\left|\frac{\partial v}{\partial x_i}\right|\right|_{\mathbb{L}_2}^{1/2} \le ||v||_{\mathbb{L}_6}^{3/2} ||v||_1^{1/2}. \tag{2.29}$$

We now have to deal with the boundary term $||v_i^4||_{L^1(\Gamma_i)}^{1/2}$ appearing in (2.28). An integration by parts yields

$$3\int_{\tilde{\Omega}} |v|^4 = \int_{\partial\tilde{\Omega}} |v|^4(\mathbf{x}\cdot\mathbf{n}) - 2\int_{\tilde{\Omega}} (v^2\overline{v}\nabla\overline{v} + \overline{v}^2 v\nabla v)\cdot\mathbf{x},$$

where $\mathbf{n}$ denotes the outwards normal to $\tilde{\Omega}$, and where we have used the identity $\nabla\cdot\mathbf{x} = 3$. We notice that $\tilde{\Omega}$ is strictly star shaped with respect to the origin, and we one can check that for all $\mathbf{x} \in \partial\tilde{\Omega}$, $\mathbf{x}\cdot\mathbf{n} = L/2$ as well as $|\mathbf{x}| \le (3/2)L$. We get from the previous inequality

$$\int_{\partial\tilde{\Omega}} |v|^4 \le \frac{2}{L}\left(3\int_{\tilde{\Omega}} |v|^4 + 6L\int_{\Omega} |v|^3|\nabla v|\right).$$

Thanks to the Cauchy-Schwarz inequality together with the Poincaré inequality, one has

$$\int_{\tilde{\Omega}} |v|^4 \leq ||v||^3_{\mathbb{L}_6} ||v||_{\mathbb{L}_2} \leq \frac{L}{2\pi} ||v||^3_{\mathbb{L}_6} ||v||_1.$$

In particular, when using the same estimate than in (2.29) the following estimate holds

$$||v_i^4||^{1/2}_{L^1(\Gamma_i)} \leq (12 + 3/\pi)^{1/2} ||v||^{3/2}_{\mathbb{L}_6} ||v||^{1/2}_1. \tag{2.30}$$

We now combine together (2.28), (2.29) with (2.30) to get

$$||v||^6_{\mathbb{L}_6} \leq [4 + (12 + 3/\pi)^{1/2}] ||v||^{9/2}_{\mathbb{L}_6} ||v||^{3/2}_1,$$

which yields the required Sobolev inequality (2.25) after an elementary simplification. Notice that

$$S \leq [4 + (12 + 3/\pi)]^{1/3},$$

and the bound do not depend on L. □

Theorem 2.4 *Let $u \in \mathbb{H}_2$. Then u is continuous and there exists a constant C be such that for all $u \in \mathbb{H}_2$ one has*

$$||u||_{\mathbb{L}_\infty} \leq C ||u||_2. \tag{2.31}$$

Moreover the injection from $\mathbb{H}_2$ onto $C(\mathbb{T}_3)$ is compact.

Proof. We start from the result of Theorem 2.1 which states at most formally that one has

$$||u||_{\mathbb{L}_\infty} \leq \sum_{\mathbf{k} \in \mathcal{I}_3} |u_\mathbf{k}|.$$

Writing $|\mathbf{k}|^2 |\mathbf{k}|^{-2}$ for $\mathbf{k} \in \mathcal{I}_3$ and using again the Cauchy-Schwarz inequality, we have

$$\sum_{\mathbf{k} \in \mathcal{I}_3} |u_\mathbf{k}| \leq \left(\sum_{\mathbf{k} \in \mathcal{I}_3} \frac{1}{|\mathbf{k}|^4} \right)^{1/2} (\sum_{\mathbf{k} \in \mathcal{I}_3} |\mathbf{k}|^4 |u_\mathbf{k}|)^{1/2} = \left(\sum_{\mathbf{k} \in \mathcal{I}_3} \frac{1}{|\mathbf{k}|^4} \right)^{1/2} ||u||_2.$$

This inequality is valid since the serie with general term $|\mathbf{k}|^{-4}$ is convergent in the 3D case. Therefore, there is well an injection from $\mathbb{H}_2$ onto ℓ_1, and we alraedy know that ℓ_1 is naturally embeded in $C(\mathbb{T}_3)$. The compactness of the injection will be proved in the same way than the proof of Theorem 2.3. □

3 Technical results

3.1. We reacall that the set $\mathcal{J}_m$ is defined in [1.x]. Let $\mathbf{k} = (k_1, k_2, k_3) \in I$, $m \in \mathbb{N}^\star$. We denote by $K_m(\mathbf{k})$ the set

$$K_m(\mathbf{k}) = \{(k_{i_1}, ..., k_{i_m}),\ (i_1, ..., i_m) \in \mathcal{J}_m,\ k_{i_j} \in \{k_1, k_2, k_3\}\}. \tag{3.1}$$

We prove in what follows that there exists two constants $C_{1,m}$ and $C_{2,m}$ such that for every $\mathbf{k} \in I$ one has

$$C_{1,m} |\mathbf{k}|^{2m} \leq \sum_{K_m(\mathbf{k})} |k_{i_1}|^2 .... |k_{i_m}|^2 \leq C_{2,m} |\mathbf{k}|^{2m} \tag{3.2}$$

For convenience, we put $\tilde{K}_m(\mathbf{k}) = \{(k_{i_1}, ..., k_{i_m}) \in K_m(\mathbf{k}),\ \exists 1 \leq r, s \leq m \text{ s.t } k_{i_r} \neq k_{i_s}\}$. Therefore one can write

$$S_m(\mathbf{k}) = \sum_{K_m(\mathbf{k})} |k_{i_1}|^2....|k_{i_m}|^2 = |k_1|^{2m} + |k_2|^{2m} + |k_3|^{2m} + \sum_{\tilde{K}_m(\mathbf{k})} |k_{i_1}|^2....|k_{i_m}|^2,$$

which yields

$$S_m(\mathbf{k}) \geq |k_1|^{2m} + |k_2|^{2m} + |k_3|^{2m} = |\mathbf{k}|_{2m}^{2m},$$

where $|\mathbf{k}|_p = (|k_1|^p + |k_2|^p + |k_3|^p)^{1/p}$, $|\mathbf{k}|_2 = |\mathbf{k}|$. Using the equivalence of the norms in $\mathbb{R}^3$, we easely get the existence of the constant $C_{1,m}$. To prove the second part of (3.2), we prove in the following that there exists a constant β_m such that the general inequality is satisfied:

$$|k_{i_1}|^2....|k_{i_m}|^2 \leq \beta_m \sum_{s=1}^{m} |k_{i_s}|^{2m}. \tag{3.3}$$

Inequality 3.3 is satisfied when $m = 1$. We argue by induction and we assume that we have at the rank $m - 1$ the inequality

$$|k_{i_1}|^2....|k_{i_{m-1}}|^2 \leq \beta_{m-1} \sum_{s=1}^{m-1} |k_{i_s}|^{2(m-1)}. \tag{3.4}$$

Therefore, 3.4 yields

$$|k_{i_1}|^2....|k_{i_m}|^2 = (|k_{i_1}|^2....|k_{i_{m-1}}|^2)|k_{i_m}|^2 \leq \beta_{m-1} \sum_{s=1}^{m-1} |k_{i_s}|^{2(m-1)}|k_{i_m}|^2.$$

In order to conclude, it remains to prove the following inequality,

$$a^{2(m-1)}b^2 \leq \gamma_m(a^{2m} + b^{2m}), \tag{3.5}$$

where γ_m is a constant, and the inequality (3.5) must be satisfied for all $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+$. Let us consider de function f defined on $\mathbb{R}_+ \times \mathbb{R}_+$,

$$f(a, b) = \frac{a^{2(m-1)}b^2}{a^{2m} + b^{2m}}.$$

The change of variables $a = r\ cos\theta$, $b = r \sin\theta$ yields

$$f(a, b) = \frac{(\cos\theta)^{2(m-1)}(\sin\theta)^2}{(\cos\theta)^{2m} + (\sin\theta)^{2m}}.$$

Therefore f is clearly a bounded function. Hence, (3.5) follows as well as the second part of (3.2) thanks to the principle of the equivalence of norms in finite dimension. □

3.2. The following inequality holds

$$\forall \varepsilon > 0, \forall u \in \mathbb{H}_1, \quad ||u||_0 \leq \frac{\varepsilon}{\sqrt{2}}||u||_1 + \frac{1}{\sqrt{2}\,\varepsilon}||u||_{-1}. \tag{3.6}$$

Indeed, when $\mathbf{k} \in \mathcal{I}_3$, one writes $|u_\mathbf{k}|^2 = |\mathbf{k}||u_\mathbf{k}|.|\mathbf{k}|^{-1}|u_\mathbf{k}|$ and next we use the Young inequality. We get the following inequality:

$$|u_\mathbf{k}|^2 \leq \frac{\varepsilon^2}{2}|\mathbf{k}|^2|u_\mathbf{k}|^2 + \frac{2}{\varepsilon^2}|\mathbf{k}|^{-2}|u_\mathbf{k}|^2.$$

We obtain 3.6 by summing up this inequality on $\mathcal{I}_3$.

4 Exercices

[4.i] Let $\mathcal{S}_m$ be the set of the permutations on $\{1, ..., m\}$. Consider $(\alpha, \beta) \in \mathcal{J}_m^2$, $\alpha = (i_1, ..., i_m)$, $\beta = (j_1, ..., j_m)$, $\sigma \in \mathcal{S}_n$, we put $\sigma\beta = (j_{\sigma(1)}, ..., j_{\sigma(m)})$. We say that $\alpha \mathcal{R} \beta$ if and only if there exists $\sigma \in \mathcal{S}_n$ be such that $\alpha = \sigma\beta$, and we denote by $J_m = \mathcal{J}_m / \mathcal{R}$ the quotient set. Compute cardJ_m and deduce the order G_n of the tensor $\nabla^n u$ defined in [1.x].

[4.ii] For any Banach Space E, we denote by E' its topological dual space.
(1) Assume $1 < p < \infty$. Show that $(\mathbb{L}_p)' = \mathbb{L}_{p'}$ and $\ell_{p'} = (\ell_p)'$.
(2) Show that $(\mathbb{L}_1)' = \mathbb{L}_\infty$, $(\ell_1)' = \ell_\infty$. What's about $\mathbb{L}'_\infty$ and ℓ'_∞ ?

[4.iii] Compute the dimension of the space V_n of trigonometric polynomial of degree less than n introduced in 2.iii.

[4.iv] assume for the simplicity that $L = 2\pi$, and for $n \in \mathbb{N}$, consider

$$K_n(x) = \frac{1}{n+1} \left(\frac{\sin\left((2n+1)(x/2)\right)}{\sin(x/2)} \right)^2,$$

$\mathbb{K}_n(x_1, x_2, x_3) = K_n(x_1)K_n(x_2)K_n(x_3)$. Prove first that $\mathbb{K}_n$ is a trigonometric polynomial. What is its degree ? Let $u \in \mathbb{L}_2$,

$$u_n(\mathbf{x}) = \frac{1}{(2\pi)^3} \int_\Omega \mathbb{K}_n(\mathbf{x} - \mathbf{y}) u(\mathbf{y}) d\mathbf{y}.$$

Prove that u_n is a trigonometric polynomial and that it converges to u in $\mathbb{L}_2$ when n goes to infty.

[4.v] Prove that for any $s \geq 0$, not necessarily an integer, T is an isometry between $H^s_{per,0}(\mathbb{R}^3)$ and $\mathbb{H}_s$.

[4.vi] Prove that the constant involved in inequality (2.25) do not depend on L. More generally, prove that if an inequality of the type $||u||_{\mathbb{L}_p} \leq C||u||_1$ holds, the constant C do not depend on L if and only if $p = 6$.

[4.vii] Generalise the results of this chapter to every dimensions.

5 Bibliographical complements

[5.i] The Lusin's Theorem is proved for instance in the book of W. Rudin [4] as well as the general integration theory.

[5.ii] We use in the proof of Lemma 1.1 that when the gradient of a distribution is equal to zero almost everywhere, then it is a constant *a.e.* This is proved in the book of L. Schwarz [6].

[5.iii] On can find a version of the Stone-Weierstrass Theorem in an other book of Walter Rudin [5]. The general version of this very deep result can be stated as follows. Let us consider a topological compact space X which has the property that for each distinct points $a \in X$ and $b \in X$ there exists two open subset in X V_a and V_b with $V_a \cap V_b = \emptyset$ and $a \in V_a$, $b \in V_b$. Let A be an algebra in $C(X)$ wich contain at most one constant and such that for every each distinct points $a \in X$ and $b \in X$ there exists $p \in A$ be such that $p(a) \neq p(b)$. Then A is everywhere dense in $C(X)$ for the topology of the uniform norm. In the present case $X = \mathbb{T}_3$, A is the set of all trigonometric polynomials functions.

[5.iv] The theory of the compact operators can be founded in the book of H. Brézis [1] chapter VI. Let E ad F be two Banach spaces. We say that an operator $A : E \to F$ is compact operator if and only if for each $B \subset E$ is bounded, then $A(B)$ is compact in F. If there exists a sequence of operators $A_n : E \to F$ with a finite rank, $\dim A_n(E) < \infty$, and such that $||A - A_n|| \to 0$ when n goes to infinity, then A is a compact operator. In the special case E and F are Hilbert spaces (or Hermitian spaces), there is an easy way to check that A is compact. Indeed, it suffises to show that for any sequence $(u_n)_{n\in\mathbb{N}}$ which converges weakly to u in E, then $(Au_n)_{n\in\mathbb{N}}$ converges to Au strongly in F.

[5.v] The argument that we use to prove Bessel-Parseval formula takes inspiration in Chapter IX of [1]. Roughly speaking, on one side the operator $-\Delta$ is the inverse of a compact operator, and on the other side, the Laplace equation $-\Delta u = f$, increases two times the regularity in the sens u has two derivative more than f. Therefore, when solving $-\Delta u = \lambda u$, for $u \in$ "H^1", then one gets $u \in$ "H^3" and by induction $u \in C^\infty$.

[5.vi] The Riesz-Thorin Theorem is a well known interpolation Theorem that can be for instance be founded in [2]. We give in the following its statement. Let (X, μ) and (Y, ν) be measure spaces, $L^p(\mu)$ and $L^q(\nu)$ the correponsding Lebesgue Spaces. Let us consider $T : X \to Y$. We shall say that T is of type (p, q) if and only if there exists a constant C be such that for all $f \in L^p(\mu)$, one has $Tf \in L^q(\nu)$ and $||Tf||_{L^q(\nu)} \leq C||f||_{L^p(\mu)}$.

Theorem 5.1 *Assume T is simultaneously of type (p_j, q_j) for $j = 1, 2$ and $1 \leq p_j$, $q_j \leq \infty$. Then for every pair (p, q) of the form*

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}, \quad t \in [0, 1],$$

T is of type (p, q).

[5.vii] We have limited our investigations about Sobolev spaces defined thanks to Fourier series to the case of integer exponents. The spaces of H^s type when s is not an integer are studied in the book of J.-L. Lions and E. Magenes [3] as to be interpolation spaces bewteen $H^{[s]}$ and $H^{[s]+1}$, where $[s]$ is the integer part of s. They where be intensively studied by L. Tartar in the 70's. There are also various definitions, and the most up-date reference in this topics is the book of L. Tartar [7].

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