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Fundamental Solutions of Stokes and Oseen Problem in Two Spatial Dimensions

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Abstract. Fundamental solutions for the linearizations of Stokes and Oseen of the Navier–Stokes time dependent equations in two spatial dimensions are determined. The derivation of these solutions is greatly simplified with the use of a trick known as centering in the probability literature. The relation of these time dependent solutions with their steady counterparts is also established.

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1. Introduction

The fundamental solution in two spatial dimensions of the steady Stokes and Oseen problems is known from the work of Lorentz [5] and Oseen [7]. Furthermore, a substantial knowledge of their properties is available. See e.g. [1], [2], [8], [3] and [4]. On the other hand, explicit formulae for the fundamental solutions of the corresponding time dependent problems do not seem to be known. The purpose of this brief note is two fold. One is to obtain explicit formulae for the fundamental solution of the time dependent problems. The other is to illustrate a method for the derivation of these fundamental solutions that might be of interest on its own. This method uses an adjustment of the standard heat kernel in two dimensions to render convergent, otherwise divergent integrals. This adjustment is sometimes referred as "centering" in the probability literature. Our knowledge of such a trick resulted from a visit by R. Bhattacharya to Oregon State University in the Fall of 2001.

Let's first illustrate the basic idea that makes this centering trick a useful

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tool for calculations. Let $\mathbf{\Gamma}(\mathbf{x};\mathbf{y},t)$ denote a fundamental solution of an evolution equation. That is, regarding $\mathbf{x}\in\mathsf{R}^d$ as a parameter, and with $\delta_{\mathbf{x}}(\mathbf{y})$ denoting the Dirac mass at $\mathbf{x}$, $\mathbf{\Gamma}$ satisfies

$$\frac{\partial\mathbf{\Gamma}}{\partial t}-\mathsf{L}_{\mathbf{y}}\mathbf{\Gamma}=0,\quad \mathbf{\Gamma}(\mathbf{x};\mathbf{y},0)=\delta_{\mathbf{x}}(\mathbf{y})$$

in the sense of distributions. Here L denotes a differential operator on R^d with coefficients that do not depend on t. Let $f(t)$ be such that

$$\mathsf{E}(\mathbf{x};\mathbf{y})=\int_0^\infty[\mathbf{\Gamma}(\mathbf{x};\mathbf{y},t)-f(t)]dt$$

is a convergent integral. If the convergence of this integral is such that the order of integration and differentiation can be interchanged one has

$$\mathsf{L}_{\mathbf{y}}\mathsf{E}=\int_0^\infty \mathsf{L}_{\mathbf{y}}\mathbf{\Gamma}dt=\int_0^\infty\frac{\partial\mathbf{\Gamma}}{\partial t}dt=-\delta_{\mathbf{x}}(\mathbf{y}).$$

Thus, E is a fundamental solution of the time independent problem. For example, if $\mathsf{L}_{\mathbf{y}}$ is the Laplace operator, $\mathbf{\Gamma}$ is the heat kernel and the steps described above can be justified if $d\geq 3$ with $f(t)=0$. However, while the heat kernel is not integrable in time for $d<3$, these steps can be justified using $f(t)=(4\pi t)^{-d/2}e^{-1/t}$. We illustrate this in the first section of this paper.

The same approach is applicable when one considers the time dependent linearizations of Stokes and Oseen of the Navier–Stokes equations. However, due to the incompressibility condition, one is naturally led to consider the projection, in the sense of distributions, on the divergence free vector fields of the Dirac delta function. Specifically, the time dependent Stokes and Oseen equations have the form

$$\frac{\partial\mathbf{u}}{\partial t}-\mathsf{L}\mathbf{u}+\nabla p=0,\quad \nabla\cdot\mathbf{u}=0$$

where

$$\mathsf{L}\mathbf{u}=\Delta\mathbf{u}-\mathsf{U}\cdot\nabla\mathbf{u}$$

and U denotes a constant vector in R^d corresponding to a far field velocity. We wish to determine $\mathbf{u}$ such that $\mathbf{u}(\mathbf{x},0)=\delta(\mathbf{x})$. Let P denote the projection onto divergent free vector fields. Then $\mathbf{w}=\mathsf{P}\mathbf{u}$, satisfies

$$\frac{\partial\mathbf{w}}{\partial t}-\mathsf{L}\mathbf{w}+\nabla p=0,\quad \nabla\cdot\mathbf{u}=0$$

with initial data

$$\mathbf{w}(\mathbf{x},0)=\mathsf{P}\delta(\mathbf{x}).$$

Throughout this paper, we denote by $\mathbf{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)$ the solution of this problem, suppressing the relation to the far field velocity in the case of the Stokes linearization corresponding to $\mathsf{U}=0$.

This note is composed of two sections. In the first section, devoted to the Stokes problem, a derivation of the steady state fundamental solution is presented as a time integral of the time dependent fundamental solution. As shown in that section, this result also requires the use of the centering trick since the fundamental solution of the time dependent Stokes problem is not integrable with respect to time. In the second section, the fundamental solution of the Oseen time dependent problem is obtained. Its time integral gives a representation of the steady state fundamental solution that appears to be different from those available in the literature (see e.g. [3]). So, as a point of comparison, the asymptotic behavior of the fundamental solution is also presented in this section. The Appendix contains the details of some of the calculations needed for the second section.

2. Stokes flow

The fundamental solution of the Stokes problem

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla \pi = 0 \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{for } t > 0, \quad \mathbf{y} \in \mathbf{R}^2 \tag{2.1}$$

can be written as

$$\mathbf{\Gamma}(\mathbf{x}; \mathbf{y}, t) = -\Delta_{\mathbf{y}} \Psi(\mathbf{x}; \mathbf{y}, t)\mathbf{I} + \mathrm{Hess}\Psi(\mathbf{x}; \mathbf{y}, t), \tag{2.2}$$

where for each $\mathbf{x} \in \mathbf{R}^2$, $t > 0$, Ψ satisfies

$$\Delta_{\mathbf{y}} \Psi(\mathbf{x}; \mathbf{y}, t) = -\mathbf{k}(\mathbf{x}; \mathbf{y}, t), \quad \mathbf{k}(\mathbf{x}; \mathbf{y}, t) = \frac{1}{4\pi t} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}}$$

is the fundamental solution of the heat equation in $\mathbf{R}^2$, $\mathrm{Hess}\Psi$ denotes the matrix of second order partial derivatives with respect to the $\mathbf{y}$ variable, and $\mathbf{I}$ denotes the 2×2 identity matrix. Oseen in [7] appears to be the first one to write the fundamental solution tensor for the steady problems in this form. More recently, Solonnikov [9] uses a similar expression in his analysis of the time dependent problem in $\mathbf{R}^3$, whereas the authors of this note used it in [10] to obtain an explicit formula for the fundamental solution of different linearizations of the Navier–Stokes equations also in $\mathbf{R}^3$.

To illustrate the centering idea mentioned above, one has

Lemma 2.1. *Let $a > 0$ be an arbitrary real number. Then*

$$\int_0^\infty \frac{1}{4\pi t} \left(e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}} - e^{-\frac{a}{4t}} \right) dt = \frac{1}{4\pi} \left[\ln a - \ln \|\mathbf{x} - \mathbf{y}\|^2 \right].$$

Proof.

$$\begin{aligned}\int_0^\infty \frac{1}{4\pi t}\left(e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}} - e^{-\frac{a}{4t}}\right)dt &= \int_0^\infty \frac{1}{4\pi t}\int_{\|\mathbf{x}-\mathbf{y}\|^2/4t}^{a/4t} e^{-s}ds dt \\ &= \frac{1}{4\pi}\int_0^\infty e^{-s}\int_{\|\mathbf{x}-\mathbf{y}\|^2/4s}^{a/4s}\frac{1}{t}dt ds \\ &= \frac{1}{4\pi}\int_0^\infty e^{-s}\left[\ln a - \ln\|\mathbf{x}-\mathbf{y}\|^2\right]ds \\ &= \frac{1}{4\pi}\left[\ln a - \ln\|\mathbf{x}-\mathbf{y}\|^2\right].\end{aligned}$$

□

In particular, with the choice of $a = 1$ one obtains the fundamental solution of the Laplace equation in two spatial dimensions, as the time integral of an adjusted heat kernel.

The representation of the fundamental solution of the Laplace equation in $\mathbf{R}^2$ obtained in the previous lemma, permits a simple calculation of Ψ. Indeed, from Lemma 2.1, one has

$$\begin{aligned}\Psi(\mathbf{x};\mathbf{y},t) &= \int_{\mathbf{R}^2}\frac{1}{4\pi t}e^{-\frac{\|\mathbf{x}-\mathbf{z}\|^2}{4t}}\frac{1}{4\pi}\ln(1/\|\mathbf{z}-\mathbf{y}\|^2)d\mathbf{z} \\ &= \int_{\mathbf{R}^2}\frac{1}{4\pi t}e^{-\frac{\|\mathbf{x}-\mathbf{z}\|^2}{4t}}\int_0^\infty\frac{1}{4\pi s}\left(e^{-\frac{\|\mathbf{z}-\mathbf{y}\|^2}{4s}} - e^{-\frac{1}{4s}}\right)dsd\mathbf{z}.\end{aligned} \tag{2.3}$$

Note that this last double integral is absolutely convergent since from Lemma 2.1 it follows that

$$\begin{aligned}&\int_{\mathbf{R}^2}\int_0^\infty\frac{1}{4\pi t}e^{-\frac{\|\mathbf{x}-\mathbf{z}\|^2}{4t}}\frac{1}{4\pi s}\left|e^{-\frac{\|\mathbf{z}-\mathbf{y}\|^2}{4s}} - e^{-\frac{1}{4s}}\right|dsd\mathbf{z} \\ &= \int_{\mathbf{R}^2}\frac{1}{4\pi t}e^{-\frac{\|\mathbf{x}-\mathbf{z}\|^2}{4t}}\frac{1}{4\pi}\left|\ln(\|\mathbf{z}-\mathbf{y}\|^2)\right|d\mathbf{z}\end{aligned}$$

which is clearly integrable. Exchanging the order of integration in (2.3) and using the semigroup property of the heat kernel, one has

$$\begin{aligned}\Psi(\mathbf{x};\mathbf{y},t) &= \int_0^\infty\left[\int_{\mathbf{R}^2}\frac{1}{4\pi t}\frac{1}{4\pi s}e^{-\frac{\|\mathbf{x}-\mathbf{z}\|^2}{4t}}e^{-\frac{\|\mathbf{z}-\mathbf{y}\|^2}{4s}}d\mathbf{z}\right. \\ &\quad\left. - \int_{\mathbf{R}^2}\frac{1}{4\pi t}e^{-\frac{\|\mathbf{x}-\mathbf{z}\|^2}{4t}}\frac{1}{4\pi s}e^{-\frac{1}{4s}}d\mathbf{z}\right]ds \\ &= \int_0^\infty\left[\frac{1}{4\pi(t+s)}e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(t+s)}} - \frac{1}{4\pi s}e^{-\frac{1}{4s}}\right]ds \\ &= \lim_{R\to\infty}\int_0^R\left[\frac{1}{4\pi(t+s)}e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(t+s)}} - \frac{1}{4\pi s}e^{-\frac{1}{4s}}\right]ds.\end{aligned}$$

To evaluate this limit, note that

$$\int_0^R \left[\frac{1}{4\pi(t+s)} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4(t+s)}} - \frac{1}{4\pi s} e^{-\frac{1}{4s}}\right] ds$$
$$= \int_t^{R+t} \frac{1}{4\pi s} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4s}} ds - \int_0^R \frac{1}{4\pi s} e^{-\frac{1}{4s}} ds$$
$$= \int_0^{R+t} \left[\frac{1}{4\pi s} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4s}} - \frac{1}{4\pi s} e^{-\frac{1}{4s}}\right] ds$$
$$- \int_0^t \frac{1}{4\pi s} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4s}} ds + \int_R^{R+t} \frac{1}{4\pi s} e^{-\frac{1}{4s}} ds.$$

Once again using Lemma 2.1 one has

$$\lim_{R\to\infty} \int_0^{R+t} \left[\frac{1}{4\pi s} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4s}} - \frac{1}{4\pi s} e^{-\frac{1}{4s}}\right] ds = -\frac{1}{4\pi} \ln \|\mathbf{x}-\mathbf{y}\|^2.$$

Since

$$\lim_{R\to\infty} \int_R^{R+t} \frac{1}{4\pi s} e^{-\frac{1}{4s}} ds \le \lim_{R\to\infty} \frac{t}{4\pi R} = 0,$$

one has

$$\Psi(\mathbf{x};\mathbf{y},t) = -\frac{1}{4\pi}\left[\ln \|\mathbf{x}-\mathbf{y}\|^2 + \int_0^t \frac{1}{s} e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4s}} ds\right].$$

In summary, with the substitution $u = \|\mathbf{x}-\mathbf{y}\|^2/4s$, one has established the following.

Proposition 2.2. *The function*

$$\Psi(\mathbf{x};\mathbf{y},t) = -\frac{1}{4\pi}\left[\ln \|\mathbf{x}-\mathbf{y}\|^2 + \int_{\|\mathbf{x}-\mathbf{y}\|^2/4t}^{\infty} \frac{e^{-u}}{u} du\right]$$

satisfies for fixed $\mathbf{x} \in \mathbf{R}^2$ *and* $t > 0$, $\Delta_{\mathbf{y}}\Psi(\mathbf{x};\mathbf{y},t) = \mathbf{k}(\mathbf{x};\mathbf{y},t)$.

A straightforward calculation gives

$$\begin{aligned}\text{Hess}\Psi \;=\; & -\frac{1}{2\pi}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^2}\left[1 - e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t}\right]\left[\mathbf{I} - 2\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}\right] \\ & -\frac{1}{4\pi t} e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t}\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}.\end{aligned}$$

Thus, from (2.2), the fundamental solution of the time dependent Stokes problem in $\mathbf{R}^2$ is given by

$$\begin{aligned}\mathbf{\Gamma}(\mathbf{x};\mathbf{y},t) &= \mathbf{k}(\mathbf{x};\mathbf{y},t)\left[\mathbf{I} - \frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}\right] \\ &\quad - \frac{1}{2\pi\|\mathbf{x}-\mathbf{y}\|^2}(1-e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t})\left[\mathbf{I} - 2\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}\right].\end{aligned} \tag{2.4}$$

The fundamental solution of the steady state Stokes problem,

$$\mathbf{E}(\mathbf{x};\mathbf{y}) = \frac{-1}{4\pi}\left[\ln\|\mathbf{x}-\mathbf{y}\|\ \mathbf{I} - \frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}\right]$$

is known from the work of Lorentz [5]. From (2.4), it easy to see that the asymptotic behavior of $\mathbf{\Gamma}$ for large values of t is given by

$$\mathbf{\Gamma}(\mathbf{x};\mathbf{y},t) \sim \frac{1}{8\pi t}\mathbf{I}.$$

It is then possible to recover $\mathbf{E}(\mathbf{x};\mathbf{y})$ as a time integral of a properly modified $\mathbf{\Gamma}(\mathbf{x};\mathbf{y},t)$.

Proposition 2.3. *Let* $\mathbf{\Gamma}$ *be defined by* (2.4). *Then*

$$\int_0^\infty \left[\mathbf{\Gamma}(\mathbf{x};\mathbf{y},t) - \frac{1}{2}\frac{1}{4\pi t}e^{-e/4t}\mathbf{I}\right]dt = \mathbf{E}(\mathbf{x};\mathbf{y}).$$

Proof. Rewrite the integral as the sum of the following terms

$$\frac{1}{2}\mathbf{I}\int_0^\infty \frac{1}{4\pi t}[e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t} - e^{-e/4t}]dt \tag{2.5}$$

and

$$\frac{1}{2\pi\|\mathbf{x}-\mathbf{y}\|^2}\left[\mathbf{I} - 2\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}\right]\int_0^\infty\left[\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t} - 1 + e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t}\right]dt. \tag{2.6}$$

From Lemma 2.1, (2.5) equals

$$\frac{1}{8\pi}\left[1 - \ln\|\mathbf{x}-\mathbf{y}\|^2\right]\mathbf{I}. \tag{2.7}$$

For the integral in (2.6), use the substitution $u = \|\mathbf{x}-\mathbf{y}\|^2/4t$ to get

$$\begin{aligned}&\frac{1}{2\pi}\frac{1}{\|\mathbf{x}-\mathbf{y}\|^2}\int_0^\infty\left[\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t} - 1 + e^{-\|\mathbf{x}-\mathbf{y}\|^2/4t}\right]dt \\ &= \frac{1}{8\pi}\int_0^\infty\left[ue^{-u} - 1 + e^{-u}\right]\frac{1}{u^2}du \\ &= \frac{-1}{8\pi}.\end{aligned} \tag{2.8}$$

The proposition follows from (2.7) and (2.8). □

Remark 2.4. It is simple to obtain the rate of convergence of the integral in this last proposition. Indeed,

$$\begin{aligned}\mathbf{E}(\mathbf{x};\mathbf{y})-\int_0^t \left[\boldsymbol{\Gamma}(\mathbf{x};\mathbf{y},s)-\frac{1}{8\pi t}e^{-e/4s}\mathbf{I}\right]ds &= \int_t^\infty \left[\boldsymbol{\Gamma}(\mathbf{x};\mathbf{y},s)-\frac{1}{8\pi t}e^{-e/4s}\mathbf{I}\right]ds \\ &= \frac{1}{4\pi}\int_0^{\|\mathbf{x}-\mathbf{y}\|^2/4t}\left(e^{-\lambda}-e^{-e\lambda/\|\mathbf{x}-\mathbf{y}\|^2}\right)\frac{d\lambda}{\lambda} \\ &= \left(e/\|\mathbf{x}-\mathbf{y}\|^2-1\right)\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}+O(t^{-2}).\end{aligned}$$

3. Oseen flow

The fundamental solution, $\boldsymbol{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)$ of the Oseen problem

$$\frac{\partial \mathbf{u}}{\partial t}-\mathsf{U}\cdot\nabla\mathbf{u}-\Delta\mathbf{u}+\nabla\pi=0 \qquad \nabla\cdot\mathbf{u}=0 \;\text{ for } t>0,\quad \mathbf{y}\in\mathbf{R}^2 \tag{3.1}$$

can also be written as

$$\boldsymbol{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)=\mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{y},t)\mathbf{I}+\mathrm{Hess}\Psi_{\mathsf{U}}(\mathbf{x};\mathbf{y},t),$$

where

$$\Delta_{\mathbf{y}}\Psi_{\mathsf{U}}=-\mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{y},t).$$

It is simple to see that in fact

$$\boldsymbol{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)=\boldsymbol{\Gamma}(\mathbf{x}+t\mathsf{U};\mathbf{y},t).$$

This follows by repeating the proof of Proposition 2.1 and noting that by uniqueness of solutions to the heat equation,

$$\int_{\mathbf{R}^2}\mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{z},t)\mathbf{k}(\mathbf{z};\mathbf{y},s)d\mathbf{z}=\mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{y},t+s).$$

Thus,

$$\begin{aligned}\boldsymbol{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)&=\mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{y},t)\left[\mathbf{I}-\frac{(\mathbf{x}+t\mathsf{U}-\mathbf{y})\otimes(\mathbf{x}+t\mathsf{U}-\mathbf{y})}{\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2}\right] \\ &-\frac{1}{2\pi\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2}(1-e^{-\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t})\left[\mathbf{I}-2\frac{(\mathbf{x}+t\mathsf{U}-\mathbf{y})\otimes(\mathbf{x}+t\mathsf{U}-\mathbf{y})}{\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2}\right].\end{aligned} \tag{3.2}$$

Unlike the case of the Stokes problem,

$$\mathbf{E}_{\mathsf{U}}(\mathbf{x};\mathbf{y})=\int_0^\infty \boldsymbol{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)dt \tag{3.3}$$

is well defined as the integral is absolutely convergent. The representation of $\mathbf{E}_\mathsf{U}$ obtained in this way provides a representation that clearly exhibits the symmetries of the Oseen problem in terms of K_ν, the modified Bessel functions of order ν. Recall that

$$K_\nu(s) = \frac{1}{2}\int_0^\infty t^{-(1+\nu)} \exp\left(-\frac{s}{2}\left(\frac{1}{t}+t\right)\right) dt \tag{3.4}$$

and let

$$q = \frac{(\mathbf{x}-\mathbf{y})\cdot \mathsf{U}}{\|\mathbf{x}-\mathbf{y}\|\|\mathsf{U}\|} \quad \text{and} \quad \sigma = \frac{\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|}{2}. \tag{3.5}$$

(See for example [6] for basic properties of the Bessel functions.)

Then, with detailed calculations provided in the Appendix,

$$\begin{aligned}\mathbf{E}_\mathsf{U}(\mathbf{x};\mathbf{y}) =& \frac{1}{2\pi}\left[e^{-q\sigma}K_0(\sigma) - \frac{1}{2\sigma}\int_0^\sigma e^{-qs}K_0(s)\ ds\right]\mathbf{I} \\ &+ \frac{1}{4\pi\sigma}\left[\frac{\mathsf{U}\otimes\mathsf{U}}{\|\mathsf{U}\|^2} + \frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}\right]\int_0^\sigma se^{-qs}K_1(s)\ ds \\ &+ \frac{1}{4\pi\sigma}\frac{(\mathbf{x}-\mathbf{y})\otimes\mathsf{U} + \mathsf{U}\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|\|\mathsf{U}\|}\int_0^\sigma se^{-qs}K_0(s)\ ds.\end{aligned} \tag{3.6}$$

Since this formula for $\mathbf{E}_\mathsf{U}$ is not clearly identical to the ones available in the literature (see e.g. [3]) the following proposition is needed.

Proposition 3.1. *Let* $\mathbf{E}_U(\mathbf{x};\mathbf{y})$ *be given by* (3.3) *and let* $\mathbf{Q} = \frac{1}{2\pi}\nabla \ln(1/\|\mathbf{x}-\mathbf{y}\|)$. *Then, in the sense of distributions*

$$U\cdot\nabla\mathbf{E}_U - \Delta\mathbf{E}_U + \nabla\mathbf{Q} = \delta_\mathbf{x}(\mathbf{y})\mathbf{I}, \quad \nabla\cdot\mathbf{E}_U = 0.$$

Proof. It is easier to show this result using elementary properties of the Fourier transform. Let $\mathsf{F}(g)(\xi) = \int_{\mathbf{R}^2} e^{i\mathbf{x}\cdot\xi}g(\mathbf{x})d\mathbf{x}$ denote the Fourier transform of g and recall that in the sense of distributions,

$$\mathsf{F}\left(\frac{1}{2\pi}\ln(1/\|\mathbf{x}\|)\right)(\xi) = \frac{1}{\|\xi\|^2}$$

and hence

$$\mathsf{F}\left(\nabla\nabla\left(\frac{1}{2\pi}\ln(1/\|\mathbf{x}\|)\right)\right)(\xi) = -\frac{\xi\otimes\xi}{\|\xi\|^2}.$$

In Fourier space, the projection operator onto divergence free vector fields is given by the matrix

$$\mathbf{I} - \frac{\xi\otimes\xi}{\|\xi\|^2}$$

and a calculation using (3.3) gives

$$\mathsf{F}(\mathbf{E}_{\mathsf{U}})(\xi) = \frac{1}{-i\mathsf{U}\cdot\xi + \|\xi\|^2}\left(\mathbf{I} - \frac{\xi\otimes\xi}{\|\xi\|^2}\right)$$

so that

$$\mathsf{F}(\mathsf{U}\cdot\mathbf{E}_{\mathsf{U}} - \Delta\mathbf{E}_{\mathsf{U}})(\xi) = \mathbf{I} - \frac{\xi\otimes\xi}{\|\xi\|^2}$$

and so

$$\mathsf{F}\left(\mathsf{U}\cdot\mathbf{E}_{\mathsf{U}} - \Delta\mathbf{E}_{\mathsf{U}} - \nabla\nabla\left(\frac{1}{2\pi}\ln(\|\mathbf{x}\|)\right)\right)(\xi) = \mathbf{I}$$

from which the result follows. □

Remark 3.2. Similar to the case of the Stokes problem, one has that for fixed $\mathbf{x}\neq\mathbf{y}$, $\mathbf{E}_{\mathsf{U}}(\mathbf{x};\mathbf{y}) - \int_0^t \mathbf{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},s)ds = O(1/t)$. Indeed, the second term in (3.2) determines the rate of convergence of the improper integral. This term can be estimated, with an appropriate constant C, by

$$\begin{aligned}\int_t^\infty \frac{C}{\|\mathbf{x}+s\mathsf{U}-\mathbf{y}\|^2}ds &= \int_t^\infty \frac{C}{\|\mathbf{x}-\mathbf{y}\|^2 + 2s(\mathbf{x}-\mathbf{y})\cdot\mathsf{U} + s^2\|\mathsf{U}\|^2}\,ds \\ &= \frac{C}{t}\int_1^\infty \frac{1}{\|\mathbf{x}-\mathbf{y}\|^2/t^2 + 2\lambda(\mathbf{x}-\mathbf{y})\cdot\mathsf{U}/t + \lambda^2\|\mathsf{U}\|^2}\,d\lambda \\ &\leq \frac{C}{\|\mathsf{U}\|^2 t}.\end{aligned}$$

An alternative formula for $\mathbf{E}_{\mathsf{U}}$, that is more amenable for establishing its asymptotic behavior, is given by

$$\begin{aligned}\mathbf{E}_{\mathsf{U}}(\mathbf{x};\mathbf{y}) &= \left[\mathbf{I} - \frac{1}{2}\left(\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2} + \frac{\mathsf{U}\otimes\mathsf{U}}{\|\mathsf{U}\|^2}\right)\right]\frac{1}{2\pi}e^{-q\sigma}K_0(\sigma) \\ &+ \left[\frac{(\mathbf{x}-\mathbf{y})\otimes\mathsf{U} + \mathsf{U}\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|\|\mathsf{U}\|} - q\left(\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2} + \frac{\mathsf{U}\otimes\mathsf{U}}{\|\mathsf{U}\|^2}\right)\right] \\ &\times \frac{1}{4\pi\sigma}\int_0^\sigma s e^{-qs}K_0(s)\,ds \\ &+ \left[-\mathbf{I} + \frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2} + \frac{\mathsf{U}\otimes\mathsf{U}}{\|\mathsf{U}\|^2}\right]\frac{1}{4\pi\sigma}\int_0^\sigma e^{-qs}K_0(s)\,ds.\end{aligned} \tag{3.7}$$

This follows from (3.6) recalling that

$$K_1(s) = K_{-1}(s) = -\frac{dK_0}{ds}$$

so, in particular

$$\int_0^\sigma \sigma e^{-q\sigma} K_1(\sigma)\ ds = -\sigma e^{-q\sigma} K_0(\sigma) + \int_0^\sigma e^{-qs} K_0(s)\ ds - q\int_0^\sigma s e^{-qs} K_0(s) ds.$$

The asymptotic behavior of the integrals involving the modified Bessel functions that appear in (3.7) is obtained in the following lemmata. Recall that as $s \to \infty$, $K_0(s) \sim \sqrt{\pi/2}(e^{-s}/\sqrt{s})$ so that for large σ,

$$e^{-q\sigma} K_0(\sigma) \sim \sqrt{\pi/2}\frac{e^{-(q+1)\sigma}}{\sqrt{\sigma}} \equiv \mathsf{m}(q;\sigma). \tag{3.8}$$

For $|q| < 1$, let

$$f_0(q) = \frac{\arccos q}{\sqrt{1-q^2}}, \qquad f_1(q) = -\frac{df_0}{dq} = \frac{1}{1-q^2}\left[1 - q\frac{\arccos q}{\sqrt{1-q^2}}\right] \tag{3.9}$$

whereas

$$f_0(1) = 1, \quad f_1(1) = 1/3.$$

Lemma 3.3. *Let $-1 < q \le 1$ and $\mathsf{m}(q;\sigma)$ be define by* (3.8)*. Then as $\sigma \to \infty$,*

$$\int_0^\sigma e^{-qs} K_0(s)\ ds \sim f_0(q) - \frac{\mathsf{m}(q;\sigma)}{(q+1)} + \frac{1}{q+1} O\left(\frac{1}{\sigma^{3/2}}\right)$$
$$\int_0^\sigma s e^{-qs} K_0(s)\ ds \sim f_1(q) - \frac{\sigma \mathsf{m}(q;\sigma)}{q+1} + \frac{1}{q+1} O\left(\frac{1}{\sqrt{\sigma}}\right).$$

Proof. First, we show that for $|q| < 1$

$$\int_0^\infty e^{-qs} K_0(s)\ ds = f_0(q), \qquad \int_0^\infty s e^{-qs} K_0(s)\ ds = f_1(q). \tag{3.10}$$

To see this, it is sufficient to establish the first equality. The second follow by differentiation with respect to q since the integrals are absolutely convergent and the exchange in the order of differentiation and integration can be easily justified. Note that from (3.4)

$$\begin{aligned}\int_0^\infty e^{-qs} K_0(s)\, ds &= \int_0^\infty \frac{1}{2t}\int_0^\infty \exp(-s(t^2+2qt+1)/2t)\, ds\, dt \\ &= \int_0^\infty \frac{1}{2t}\frac{2t}{t^2+2qt+1}\, dt\end{aligned}$$

from which the first equality in (3.10) follows.

A simple calculation shows that

$$\left[\int_\sigma^\infty e^{-qs} K_0(s)\, ds\right] \Big/ \left[\sqrt{\pi/2}\frac{e^{-(q+1)\sigma}}{(q+1)\sqrt{\sigma}}\right] = 1 + O\left(\frac{1}{\sigma}\right)$$

with the error term uniformly bounded with respect to q. Thus, the first asymptotic result follows using 3.10 since

$$\int_0^\sigma e^{-qs} K_0(s)\, ds = \int_0^\infty e^{-qs} K_0(s)\, ds - \int_\sigma^\infty e^{-qs} K_0(s)\, ds.$$

Similar considerations can be used to obtain the second asymptotic result. □

The arguments used in the proof of Lemma 3.3 can not be applied in the case $q = -1$, since the improper integral is divergent. Instead the following lemma is needed.

Lemma 3.4. *With the notation of* (3.4),

$$\begin{aligned}\frac{1}{2\pi}\left[e^\sigma K_0(\sigma) - \frac{1}{2\sigma}\int_0^\sigma e^s K_0(s)\ ds\right] &= \frac{1}{4\pi\sigma}\left[1 - \int_0^\infty \frac{e^{-\rho\sigma}}{\sqrt{\rho}(2+\rho)^{3/2}}\right] d\rho \\ &= \frac{1}{4\pi\sigma} + o\left(\frac{1}{\sigma}\right).\end{aligned}$$

Proof. Note that by (3.4) one has

$$\begin{aligned}\int_0^\sigma e^s K_0(s)\ ds &= \int_0^\sigma \int_0^\infty \exp(-s(1-t)^2/2t)\frac{dt}{2t}\ ds \\ &= 2\int_0^1 \int_0^\sigma \exp(-s(1-t)^2/2t)\ ds\frac{dt}{2t} \\ &= 2\int_0^1 \frac{1}{(1-t)^2}(1 - \exp(-\sigma(1-t)^2/2t))dt.\end{aligned}$$

A similar calculation gives

$$e^\sigma K_0(\sigma) = \int_0^\infty \exp(-\sigma(1-t)^2/2t)\ \frac{dt}{2t} = \int_0^1 \exp(-\sigma(1-t)^2/2t)\ \frac{dt}{t}.$$

With the substitution $\rho = (1-t)^2/2t$, one has

$$\frac{dt}{t} = \frac{d\rho}{\sqrt{\rho}\sqrt{2+\rho}}.$$

In particular

$$e^\sigma K_0(\sigma) = \int_0^\infty \frac{e^{-\sigma\rho}}{\sqrt{\rho}\sqrt{2+\rho}}\ d\rho, \tag{3.11}$$

whereas integration by parts gives

$$\begin{aligned}\int_0^\sigma e^s K_0(s)\ ds &= \int_0^\infty \frac{1-e^{-\sigma\rho}}{\rho^{3/2}\sqrt{2+\rho}}\ d\rho\\ &= 2\sigma\int_0^\infty \frac{e^{-\sigma\rho}}{\sqrt{\rho}\sqrt{2+\rho}}\ d\rho - \int_0^\infty \frac{1}{\sqrt{\rho}(2+\rho)^{3/2}}\ d\rho\\ &\quad + \int_0^\infty \frac{e^{-\sigma\rho}}{\sqrt{\rho}(2+\rho)^{3/2}}\ d\rho.\end{aligned}$$

Thus from (3.11),

$$\begin{aligned}e^\sigma K_0(\sigma) - \frac{1}{2\sigma}\int_0^\sigma e^s K_0(s)\ ds &= \frac{1}{2\sigma}\int_0^\infty \frac{1}{\sqrt{\rho}(2+\rho)^{3/2}}\ d\rho\\ &\quad - \frac{1}{2\sigma}\int_0^\infty \frac{e^{-\sigma\rho}}{\sqrt{\rho}(2+\rho)^{3/2}}\ d\rho.\end{aligned}$$

The lemma follows since

$$\int_0^\infty \frac{1}{\sqrt{\rho}(2+\rho)^{3/2}}\ d\rho = 1$$

and the other integral in the last equality is uniformly bounded by an integrable function and pointwise convergent to zero. □

A further consequence of the previous Lemma, is that

$$\frac{1}{4\pi\sigma}\int_0^\sigma e^s K_0(s)\, ds \sim \frac{1}{2\pi}\sqrt{\pi/2}\frac{1}{\sqrt{\sigma}} - \frac{1}{4\pi\sigma} + o\left(\frac{1}{\sigma}\right). \tag{3.12}$$

To obtain the asymptotic behavior of the fundamental solution tensor $\mathbf{E}_{\mathsf{U}}$, let

$$\mathbf{A}_{11} = \frac{\mathsf{U}\otimes\mathsf{U}}{\|\mathsf{U}\|^2},\quad \mathbf{A}_{12} = \mathbf{A}_{21} = \frac{\mathsf{U}\otimes\mathsf{U}^\perp + \mathsf{U}^\perp\otimes\mathsf{U}}{2\|\mathsf{U}\|^2},\quad \text{and}\quad \mathbf{A}_{22} = \frac{\mathsf{U}^\perp\otimes\mathsf{U}^\perp}{\|\mathsf{U}\|^2}.$$

The contraction

$$\mathsf{E}_{ij} = \mathbf{E}_{\mathsf{U}} : \mathbf{A}_{ij} = \sum_{jk} (\mathbf{E}_{\mathsf{U}})_{jk}(\mathbf{A}_{ij})_{jk} \tag{3.13}$$

is the ij component of $\mathbf{E}_{\mathsf{U}}$ in a system of coordinates such that $\mathbf{e}_1 = \mathsf{U}/\|\mathsf{U}\|$ and $\mathbf{e}_2 = \mathsf{U}^\perp/\|\mathsf{U}\|$. As a summary of the asymptotic behavior of the fundamental solution tensor one has;

Proposition 3.5. *For* $\mathbf{x}, \mathbf{y}, U \neq 0 \in \mathbf{R}^2$, *let* q, σ *and* $m(q;\sigma)$ *be given by* (3.5) *and* (3.8) *respectively. Then, the asymptotic behavior as* $\sigma \to \infty$ *of* E_{ij} *defined by* (3.13) *is given by*

$$E_{11}(q;\sigma) \sim \frac{q}{4\pi\sigma} + \frac{1-q}{4\pi} m(q;\sigma)$$

$$E_{12}(q;\sigma) = E_{21}(q;\sigma) \sim \frac{\sqrt{1-q^2}}{4\pi\sigma} - \frac{\sqrt{1-q^2}}{4\pi} m(q;\sigma)$$
$$E_{22}(q;\sigma) \sim -\frac{q}{4\pi\sigma} + \frac{(1+q)}{4\pi} m(q;\sigma).$$

Proof. A calculation using (3.7) gives

$$\begin{aligned}
\mathsf{E}_{11}(q;\sigma) &= \frac{1}{4\pi}(1-q^2)e^{-q\sigma}K_0(\sigma) \\
&\quad + \frac{1}{4\pi\sigma}q(1-q^2)\int_0^\sigma se^{-qs}K_0(s)\,ds + \frac{1}{4\pi\sigma}q^2\int_0^\sigma e^{-qs}K_0(s)\,ds \\
\mathsf{E}_{12}(q;\sigma) = \mathsf{E}_{21}(q;\sigma) &= -\frac{1}{4\pi}q\sqrt{1-q^2}e^{-q\sigma}K_0(\sigma) \\
&\quad + \frac{1}{4\pi\sigma}\sqrt{1-q^2}(1-q^2)\int_0^\sigma se^{-qs}K_0(s)\,ds \\
&\quad + \frac{1}{4\pi\sigma}q\sqrt{1-q^2}\int_0^\sigma e^{-qs}K_0(s)\,ds \\
\mathsf{E}_{22}(q;\sigma) &= \frac{1}{4\pi}(1+q^2)e^{-q\sigma}K_0(\sigma) \\
&\quad - \frac{1}{4\pi\sigma}q(1-q^2)\int_0^\sigma se^{-qs}K_0(s)\,ds - \frac{1}{4\pi\sigma}q^2\int_0^\sigma e^{-qs}K_0(s)\,ds.
\end{aligned}$$

The proposition follows using the estimates obtained in Lemma 3.3, Lemma 3.4 and (3.8). For example for $-1 < q \leq 1$, the behavior of E_{11} for large σ follows from Lemma 3.3, and (3.8) leading to

$$\begin{aligned}
\mathsf{E}_{11}(q,\sigma) &\sim \frac{1}{4\pi}(1-q^2)\mathsf{m}(q;\sigma) + \frac{1}{4\pi\sigma}q(1-q^2)\left[f_1(q) - \frac{\sigma\mathsf{m}(q;\sigma)}{(q+1)}\right] \\
&\quad + \frac{1}{4\pi\sigma}q^2\left[f_0(q) - \frac{\mathsf{m}(q;\sigma)}{(q+1)}\right] + O\left(\frac{1}{\sigma^{3/2}}\right) \\
&\sim \frac{1}{4\pi}\mathsf{m}(q;\sigma)(1-q) + \frac{1}{4\pi\sigma}\left[q(1-q^2)f_1(q) + q^2 f_0(q)\right] \\
&= \frac{(1-q)}{4\pi}\mathsf{m}(q;\sigma) + \frac{q}{4\pi\sigma}
\end{aligned}$$

where in the last equality we have used that $q(1-q^2)f_1(q) + q^2 f_0(q) = q$. Finally, using (3.12) one has that for $q = -1$,

$$\mathsf{E}_{11}(-1;\sigma) = \frac{1}{4\pi\sigma}\int_0^\sigma e^s K_0(s)\,ds \sim -\frac{1}{4\pi\sigma} + \frac{1}{2\pi}\mathsf{m}(-1;\sigma)$$

as claimed.

Similar considerations apply for the other components. Note that in particular, from Lemma 3.4 one has

$$\mathsf{E}_{22}(-1;\sigma) = \frac{1}{2\pi}\left[e^\sigma K_0(\sigma) - \frac{1}{2\sigma}\int_0^\sigma e^s K_0(s)\,ds\right] \sim \frac{1}{4\pi\sigma}. \qquad \square$$

The asymptotic behavior established in this proposition is in agreement with the known asymptotic behavior of $\mathbf{E}_{\mathsf{U}}$ (see e.g. [3], p. 376–378). In particular, using Proposition 3.5 the uniform bounds

$$\mathsf{E}_{i2}(q;\sigma) = O\left(\frac{1}{\sigma}\right), \quad i = 1, 2, \quad |q| \le 1,$$

and

$$\mathsf{E}_{11}(q;\sigma) = O\left(\frac{1}{\sqrt{\sigma}}\right), \quad |q| \le 1$$

follow immediately.

4. Appendix

The calculations leading to the expression (3.6) are presented here.

Note that $\mathbf{\Gamma}_{\mathsf{U}}(\mathbf{x};\mathbf{y},t)$ given in (3.2) can be written as

$$\left[\mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{y},t) - \mathbf{\Gamma}_{\mathsf{U}}^{(1)}(\mathbf{x};\mathbf{y},t)\right] \mathbf{I} + \mathbf{\Gamma}_{\mathsf{U}}^{(2)}(\mathbf{x};\mathbf{y},t)(\mathbf{x}+t\mathsf{U}-\mathbf{y}) \otimes (\mathbf{x}+t\mathsf{U}-\mathbf{y})$$

where

$$\mathbf{\Gamma}_{\mathsf{U}}^{(1)}(\mathbf{x};\mathbf{y},t) = \frac{1}{8\pi t}\frac{4t}{\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2}(1 - e^{-\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}) \tag{4.1}$$

and

$$\mathbf{\Gamma}_{\mathsf{U}}^{(2)}(\mathbf{x};\mathbf{y},t) = \frac{1}{4\pi t}\frac{1}{4t}\left[\frac{4t}{\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2}\left[\frac{4t}{\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2}(1 - e^{-\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}) - e^{-\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}\right]\right]. \tag{4.2}$$

The time integral of each of these terms is the content of the following lemmata.

From (3.4)

$$\begin{aligned}\int_0^\infty \mathbf{k}(\mathbf{x}+t\mathsf{U};\mathbf{y},t)\,dt &= \frac{1}{4\pi}e^{-(\mathbf{x}-\mathbf{y})\cdot\mathsf{U}/2}\int_0^\infty \frac{1}{t}\exp\left(-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t} - \frac{t\|\mathsf{U}\|^2}{4}\right)dt \\ &= \frac{1}{2\pi}e^{-(\mathbf{x}-\mathbf{y})\cdot\mathsf{U}/2}K_0(\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2) \\ &= \frac{1}{2\pi}e^{-q\sigma}K_0(\sigma).\end{aligned} \tag{4.3}$$

Lemma 4.1. *Let q, σ and $\mathbf{\Gamma}_U^{(1)}(\mathbf{x};\mathbf{y},t)$ be defined by (3.5) and (4.1) respectively. Then*

$$\int_0^\infty \mathbf{\Gamma}_U^{(1)}(\mathbf{x};\mathbf{y},t)\,dt = \frac{1}{4\pi\sigma}\int_0^\sigma e^{-qs}K_0(s)\;ds.$$

Proof. Since $\int_0^1 e^{-ps}\,ds = (1/p)(1-e^{-p})$ one has

$$\int_0^\infty \mathbf{\Gamma}_{\mathsf{U}}^{(1)}(\mathbf{x};\mathbf{y},t)\,dt = \frac{1}{8\pi}\int_0^\infty \frac{1}{t}\int_0^1 e^{-s\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}\,ds\;dt$$

$$\cdot\frac{1}{8\pi}\int_0^1 e^{-s(\mathbf{x}-\mathbf{y})\cdot\mathsf{U}/2}\int_0^\infty \frac{1}{t}\exp\left(-\frac{s\|\mathbf{x}-\mathbf{y}\|^2}{4t}-\frac{st\|\mathsf{U}\|^2}{4}\right)dt\,ds$$

$$= \frac{1}{4\pi}\int_0^1 e^{-s(\mathbf{x}-\mathbf{y})\cdot\mathsf{U}/2}K_0(s\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2)\,ds.$$

The lemma follows using the substitution $s' = s\sigma$. □

Lemma 4.2. *Let* $\mathbf{\Gamma}_U^{(2)}(\mathbf{x},\mathbf{y},t)$ *be defined by* (4.2). *Then*

$$\int_0^\infty \mathbf{\Gamma}_U^{(2)}(\mathbf{x};\mathbf{y},t)\;(\mathbf{x}+tU-\mathbf{y})\otimes(\mathbf{x}+tU-\mathbf{y})\;dt$$

$$= \left[\frac{(\mathbf{x}-\mathbf{y})\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|^2}+\frac{U\otimes U}{\|U\|^2}\right]\frac{1}{4\pi\sigma}\int_0^\sigma se^{-qs}K_1(s)\;ds$$

$$+\left[\frac{(\mathbf{x}-\mathbf{y})\otimes U+U\otimes(\mathbf{x}-\mathbf{y})}{\|\mathbf{x}-\mathbf{y}\|\|U\|}\right]\frac{1}{4\pi\sigma}\int_0^\sigma se^{-qs}K_0(s)\;ds.$$

Proof. Since

$$\int_0^1 se^{-sp}\;ds = \frac{1}{p}\left[\frac{1}{p}(1-e^{-p})-e^{-p}\right],$$

the integral involving $\mathbf{\Gamma}_{\mathsf{U}}^{(2)}$ is a combination of the following integrals

$$\frac{(\mathbf{x}_i-\mathbf{y}_i)(\mathbf{x}_j-\mathbf{y}_j)}{16\pi}\int_0^\infty \frac{1}{t^2}\int_0^1 se^{-s\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}\;ds\;dt$$

$$\frac{(\mathbf{x}_i-\mathbf{y}_i)\mathsf{U}_j}{16\pi}\int_0^\infty \frac{1}{t}\int_0^1 se^{-s\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}\;ds\;dt$$

$$\frac{\mathsf{U}_i\mathsf{U}_j}{16\pi}\int_0^\infty\int_0^1 se^{-s\|\mathbf{x}+t\mathsf{U}-\mathbf{y}\|^2/4t}\;ds\;dt.$$

Each of these integrals can be written in terms of modified Bessel functions after a change of order of integrations and using (3.4). Indeed,

$$\frac{(\mathbf{x}_i-\mathbf{y}_i)(\mathbf{x}_j-\mathbf{y}_j)}{8\pi}\frac{\|\mathsf{U}\|}{\|\mathbf{x}-\mathbf{y}\|}\int_0^1 se^{-s\mathsf{U}\cdot(\mathbf{x}-\mathbf{y})/2}K_1(s\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2)\ ds$$

$$=\frac{(\mathbf{x}_i-\mathbf{y}_i)(\mathbf{x}_j-\mathbf{y}_j)}{\|\mathbf{x}-\mathbf{y}\|^2}\frac{1}{2\pi\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|}\int_0^{\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2} se^{-qs}K_1(s)\ ds$$

$$\frac{(\mathbf{x}_i-\mathbf{y}_i)\mathsf{U}_j}{8\pi}\int_0^1 se^{-s\mathsf{U}\cdot(\mathbf{x}-\mathbf{y})/2}K_0(s\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2)\ ds$$

$$=\frac{(\mathbf{x}_i-\mathbf{y}_i)\mathsf{U}_j}{\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|}\frac{1}{2\pi\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|}\int_0^{\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2} se^{-qs}K_0(s)\ ds$$

and

$$\frac{\mathsf{U}_i\mathsf{U}_j}{8\pi}\frac{\|\mathbf{x}-\mathbf{y}\|}{\|\mathsf{U}\|}\int_0^1 se^{-s\mathsf{U}\cdot(\mathbf{x}-\mathbf{y})/2}K_{-1}(s\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2)\ ds$$

$$=\frac{\mathsf{U}_i\mathsf{U}_j}{\|\mathsf{U}\|^2}\frac{1}{2\pi\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|}\int_0^{\|\mathsf{U}\|\|\mathbf{x}-\mathbf{y}\|/2} se^{-qs}K_{-1}(s)\ ds.$$

□

Combining the results of (4.3) and Lemmas 4.1 and 4.2, one obtains the first expression of $\mathbf{E}_{\mathsf{U}}$ given in (3.6).

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