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Global well-posedness of the 3-dimensional Navier-Stokes initial value problem in $L^p \cap L^2$ with $3 < p < \infty$

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Abstract

By using the continuous induction method, we prove that the initial value problem of the three dimensional Navier-Stokes equations is globally well-posed in $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ for any $3 < p < \infty$. The proof is rather simple.

Keywords: Navier-Stokes equations, three dimension, initial value problem, global well-posedness.

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1 Introduction

In this paper we prove that the initial value problem of the three dimensional Navier-Stokes equations

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla P, \quad x \in \mathbb{R}^3, \ \ t > 0 \\ \operatorname{div}(\mathbf{u}) = 0, \quad x \in \mathbb{R}^3, \ \ t > 0 \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}^3 \end{cases}$$

is globally well-posed in $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ for any $3 < p < \infty$.

It is well-known that the pressure P can be expressed in terms of the velocity $\mathbf{u}$ by the formula

$$P = \sum_{j,k=1}^{3} R_j R_k (u_j u_k),$$

where (R_1, R_2, R_3) is the Riesz transform. Hence, letting $\mathbb{P}$ be the Helmholtz-Weyl projection operator, i.e., the 3×3 matrix pseudo-differential operator in $\mathbb{R}^3$ with the matrix symbol $\Big(\delta_{ij} - |\xi|^{-2}\xi_i\xi_j\Big)_{3\times 3}$, where δ_{ij} are the Kronecker symbols, the above problem can be rewritten into the following equivalent form:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \mathbb{P}\nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = 0, \quad x \in \mathbb{R}^3, \ \ t > 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\otimes$ denotes the tenser product between vectors, i.e., $\mathbf{u}\otimes\mathbf{u} = \mathbf{u}\mathbf{u}^T = (u_iu_j)_{i,j=1}^3$ for $\mathbf{u} = (u_1, u_2, u_3)^T$ (note that we always regard $\mathbf{u}$ as a column 3-vector, and identify all column 3-vectors with corresponding 3×1 matrices throughout this paper), and $\nabla\cdot(\mathbf{u}\otimes\mathbf{u})$ represents the (column) vector with each component being the divergence of the corresponding row vector of the matrix $\mathbf{u}\otimes\mathbf{u}$. Thus, throughout this paper we shall consider the problem (1.1). Before giving the precise statement of our main result, let us first make a short review on the study of this problem and state two known results.

The first important work on this problem was performed by Leray in 1934 in the reference [19], where he studied the general N-dimension case and proved that the problem (1.1) (with 3 replaced by N) has a global *weak solution* in the class

$$C_w([0,\infty), L^2(\mathbb{R}^N)) \cap L^\infty([0,\infty), L^2(\mathbb{R}^N)) \cap L^2([0,\infty), \dot{H}^1(\mathbb{R}^N))$$

for any initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^N)$ with $\mathrm{div}(\mathbf{u}_0) = 0$, where $C_w([0,\infty), L^2(\mathbb{R}^N))$ denotes the set of maps from $[0,\infty)$ to $L^2(\mathbb{R}^N)$ which are continuous with respect to the weak topology of $L^2(\mathbb{R}^N)$. Here and throughout this paper, for simplicity of notations we use the same notation to denote both a scaler function space and its corresponding N-vector counterpart; for instance, the notation $L^2(\mathbb{R}^N)$ denotes both the space of scaler L^2 functions and the space of N-vector L^2 functions. To obtain this result Leray used a smooth approximation approach based on weak compactness of bounded sets in separable Banach spaces and dual of Banach spaces. An important feature of the solution obtained by Leray [19] is that it possesses the following property: For $t_0 = 0$ and almost every $t_0 > 0$, there holds the *energy inequality*:

$$\|\mathbf{u}(t)\|_2^2 + \int_{t_0}^t \|\nabla\mathbf{u}(s)\|_2^2 ds \le \|\mathbf{u}(t_0)\|_2^2, \quad \forall t > t_0. \tag{1.2}$$

A different approach which uses the Picard iteration argument was introduced by Kato and Fujita in 1964 in [8], where they established local well-posedness of the problem (1.1) (again in the general N-dimension case) in the space $H^s(\mathbb{R}^N)$ for $s \ge \frac{N}{2} - 1$, and global well-posedness for small initial data in $H^{\frac{N}{2}-1}(\mathbb{R}^N)$. This approach was later extended to various other function spaces, such as the Lebesque space $L^p(\mathbb{R}^N)$ for $p \ge N$ by Kato in [12] (see also Fabes, Johns and Riviere [7] and Giga [10]), the critical and subcritical Sobolev spaces and Besov spaces of either positive or negative orders by Kato and Ponce [13], Planchon [22], Terraneo [26] and et al, the Lorentz spaces $L^{p,q}$ by Barraza [1], the Morrey-Campanato spaces $M^{p,q}$ by Giga and Miyakawa [11], the space BMO^{-1} of derivatives of BMO functions by Koch and Tataru [15], and general Sobolev and Besov spaces over shift-invariant Banach spaces of distributions that can be continuously embedded into the Besov space $B^{-1}_{\infty\infty}(\mathbb{R}^N)$ by Lemarié-Rieusset in his expository book [18]. The literatures listed here are far from being complete; we refer the reader to see [4] and [18] for expositions and references cited therein. A third approach which combines the above two approaches was introduced by Calderón in 1990 in [2]. He proved global existence of weak solutions in the class $C_w([0,\infty), L^2(\mathbb{R}^N)+L^p(\mathbb{R}^N))$ for initial data $\mathbf{u}_0 \in L^p(\mathbb{R}^N)$ for any $2 \le p < \infty$. We refer the reader to see Lemarié-Rieusset [17], [18] and the present author's recent work [5] for further results obtained from the third approach.

Solutions obtained from the second approach are usually called *mild solutions* or *strong solutions* to distinguish with those obtained by Leray and from the third approach. Since the famous work of Serrin [24], a large number of literatures have been devoted to the study of weak-strong uniqueness and weak-strong regularity, see e.g. [9], [16], [20], [25] and the references cited therein. From the results on this topic we know that if the Navier-Stokes initial value problem is (locally) well-posed in some function space X, then for any initial data $\mathbf{u}_0 \in X \cap L^2(\mathbb{R}^N)$, the weak solution coincides with the unique strong solution on the maximal existence interval of the strong solution.

For $p \ge 2$ we denote

$$L^p_\omega(\mathbb{R}^N) = \{\mathbf{u}_0 \in L^p(\mathbb{R}^N) : \operatorname{div}(u_0) = 0\}.$$

Related to this work, we particularly write down the following two theorems which follow from some of the above-mentioned literatures so that whose proofs we omit:

Theorem 1.1 *Let $p > N$ and $\mathbf{u}_0 \in L^p_\omega(\mathbb{R}^N)$. Then the following assertions hold:*

(1) *There exists a constant $T = T(\|\mathbf{u}_0\|_p) > 0$ such that the problem* (1.1) *has a unique mild solution $\mathbf{u} \in C([0,T], L^p_\omega(\mathbb{R}^N))$.*

(2) *Let T^* be the lifespan of the solution, i.e. the suprimum of all $T > 0$ such that the problem* (1.1) *has a mild solution $\mathbf{u} \in C([0,T], L^p_\omega(\mathbb{R}^N))$. If $T^* < \infty$ then*

$$\lim_{t\to T^*-0} \|\mathbf{u}(\cdot,t)\|_p = \infty.$$

(3) *For any $0 < T < T^*$ and $p < q \le \infty$ the solution belongs to $C((0,T], L^q(\mathbb{R}^N))$. Moreover,*

$$t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\mathbf{u} \in C([0,T], L^q(\mathbb{R}^N)) \quad \textit{and} \quad \lim_{t\to 0} t^{\frac{N}{2}(\frac{1}{p}-\frac{1}{q})}\|\mathbf{u}\|_q = 0.$$

(4) *For any $0 < T < T^*$ and $p \le q \le \infty$ and $1 < r \le \infty$ satisfying the relation*

$$\frac{2}{r} + \frac{N}{q} = \frac{N}{p},$$

we have

$$\mathbf{u} \in L^r([0,T], L^q(\mathbb{R}^N)).$$

(5) *The solution belongs to $C^\infty((0,T^*)\times\mathbb{R}^N)$. In fact, for any $0 < T < T^*$, $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}^N_+$ and $p \le q \le \infty$, we have $\partial_t^k\partial_x^\alpha \mathbf{u} \in C((0,T], L^q(\mathbb{R}^N))$, or more precisely,*

$$t^\sigma \partial_t^k\partial_x^\alpha \mathbf{u} \in C([0,T], L^q(\mathbb{R}^N)), \quad \textit{and} \quad \lim_{t\to 0} t^\sigma \|\partial_t^k\partial_x^\alpha \mathbf{u}\|_q = 0 \ \textit{in case} \ \sigma > 0,$$

where $\sigma = k + |\alpha|/2 + (N/2)(1/p - 1/q)$.

(6) *The solution map* $\Phi : \mathbf{u}_0 \mapsto \mathbf{u}$ *is Lipschitz continuous in the following sense: For any* $\mathbf{v}_0 \in L^p_\omega(\mathbb{R}^N)$ *and* $0 < T < T^*$, *where* T^* *is the lifespan of the solution* $\mathbf{v} = \Phi(t; \mathbf{v}_0)$, *there exists corresponding* $\varepsilon > 0$ *such that for any* $\mathbf{u}_0 \in L^p_\omega(\mathbb{R}^N)$ *such that* $\|\mathbf{u}_0 - \mathbf{v}_0\|_p < \varepsilon$, *the lifespan of the solution* $\mathbf{u} = \Phi(t; \mathbf{u}_0)$ *is larger than* T, *and there exists constant* $C > 0$ *such that*

$$\sup_{0 \le t \le T} \|\mathbf{u}(t) - \mathbf{v}(t)\|_p \le C \|\mathbf{u}_0 - \mathbf{v}_0\|_p.$$

Theorem 1.2 *Let* $\mathbf{u}_0 \in L^N_\omega(\mathbb{R}^N)$. *Then the following assertions hold:*

(1) *There exists* $T = T(\mathbf{u}_0) > 0$ *such that the problem* (1.1) *has a mild solution* $\mathbf{u} \in C([0,T], L^N_\omega(\mathbb{R}^N))$ *which satisfies the following properties:*

$$t^{\frac{1}{2}(1-\frac{N}{p})}\mathbf{u} \in C([0,T], L^p_\omega(\mathbb{R}^N)) \quad \text{for any } N < p \le \infty,$$

$$t^{1-\frac{N}{2p}}\nabla\mathbf{u} \in C([0,T], L^p_\omega(\mathbb{R}^N)) \quad \text{for any } N \le p \le \infty,$$

both with values zero at $t = 0$.

(2) *For any* $p > N$, *the solution is unique in the class*

$$\Big\{\mathbf{u} \in C((0,T], L^p_\omega(\mathbb{R}^N)) : \ t^{\frac{1}{2}(1-\frac{N}{p})}\mathbf{u} \in C([0,T], L^p(\mathbb{R}^N)) \text{ and } \lim_{t\to 0} t^{\frac{1}{2}(1-\frac{N}{p})}\|\mathbf{u}(t)\|_p = 0\Big\}.$$

(3) *Let* T^* *be the lifespan of the solution, i.e. the suprimum of all* $T > 0$ *such that the problem* (1.1) *has a mild solution* $\mathbf{u} \in C([0,T], L^3_\omega(\mathbb{R}^N))$. *In the case* $N = 3$, *if* $T^* < \infty$ *then*

$$\lim_{t \to T^*-0} \|\mathbf{u}(\cdot, t)\|_p = \infty.$$

(4) *For any* $0 < T < T^*$ *and* $N \le q \le \infty$ *and* $1 < r \le \infty$ *satisfying the relation*

$$\frac{2}{r} + \frac{N}{q} = 1,$$

we have

$$\mathbf{u} \in L^r([0,T], L^q_\omega(\mathbb{R}^N)).$$

(5) *The solution belongs to* $C^\infty((0,T^*) \times \mathbb{R}^N)$. *In fact, for any* $0 < T < T^*$, $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}^N_+$ *and* $N \le p \le \infty$, *we have* $\partial_t^k \partial_x^\alpha \mathbf{u} \in C((0,T], L^p_\omega(\mathbb{R}^N))$, *or more precisely,*

$$t^\sigma \partial_t^k \partial_x^\alpha \mathbf{u} \in C([0,T], L^p_\omega(\mathbb{R}^N)), \quad \text{and} \quad \lim_{t \to 0} t^\sigma \|\partial_t^k \partial_x^\alpha \mathbf{u}\|_p = 0 \text{ in case } \sigma > 0,$$

where $\sigma = k + |\alpha|/2 + (N/2)(1/N - 1/p)$.

(6) *The solution map* $\Phi : \mathbf{u}_0 \mapsto \mathbf{u}$ *is Lipschitz continuous in the following sense: For any* $\mathbf{v}_0 \in L^N_\omega(\mathbb{R}^N)$ *and* $0 < T < T^*$, *where* T^* *is the lifespan of the solution* $\mathbf{v} = \Phi(t; \mathbf{v}_0)$, *there exists*

corresponding $\varepsilon > 0$ *such that for any* $\mathbf{u}_0 \in L^N_\omega(\mathbb{R}^N)$ *such that* $\|\mathbf{u}_0 - \mathbf{v}_0\|_N < \varepsilon$*, the lifespan of the solution* $\mathbf{u} = \Phi(t;\mathbf{u}_0)$ *is larger than* T*, and there exists constant* $C > 0$ *such that*

$$\sup_{0\le t\le T} \|\mathbf{u}(t) - \mathbf{v}(t)\|_N \le C\|\mathbf{u}_0 - \mathbf{v}_0\|_N.$$

(7) *There exists* $\varepsilon > 0$ *such that if* $\|\mathbf{u}_0\|_N \le \varepsilon$*, then the solution is global, i.e.,* $T^* = \infty$*.*

Unlike the assertion (2) of Theorem 2.1 which is an immediate consequence of the assertion (1), the proof of the assertion (3) of Theorem 2.2 is very difficult, for which we refer the reader to see [6] and [23]; see also [14] for the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ version of a such result.

The question remaining unanswered till now is as follows: Is $T^* = \infty$ for all sufficiently smooth initial data or is it true that $T^* < \infty$ for certain initial data, no matter how smooth or how fast they decay at infinity? When comparing the Navier-Stokes equations with the other classical nonlinear partial differential equations of the evolutionary type, this question becomes quite remarkable, because for the nonlinear heat equations, the nonlinear wave equations, the nonlinear Schrödinger equations, the KdV equation and etc, we know that they are either globally well-posed in suitable function spaces or there exist finite-time blow-up solutions. For the Navier-Stokes equations, however, we know neither global well-posedness except for small initial data nor existence of a finite time blow-up solution.

The purpose of this paper is to prove that the problem (1.1) is globally well-posed in the space $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ for any $3 < p < \infty$. To give a precise statement of our main result, we introduce a notation. Given $p > 3$, for any $\mathbf{u}_0 \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ we denote

$$\kappa_p(\mathbf{u}_0) = \|\mathbf{u}_0\|_p^{\frac{p}{3(p-2)}} \|\mathbf{u}_0\|_2^{\frac{2(p-3)}{3(p-2)}},$$

and call it the κ_p*-value* of $\mathbf{u}_0$. Note that by the Hölder inequality we have

$$\|\mathbf{u}_0\|_3 \le \kappa_p(\mathbf{u}_0).$$

Note also that both $\|\cdot\|_3$ and κ_p are invariant under the NS-related scaling, namely,

$$\|\mathbf{u}_0^\lambda\|_3 = \|\mathbf{u}_0\|_3 \quad \text{and} \quad \kappa_p(\mathbf{u}_0^\lambda) = \kappa_p(\mathbf{u}_0),$$

where $\mathbf{u}_0^\lambda(x) = \lambda\mathbf{u}_0(\lambda x)$ $(\lambda > 0)$.

The main result of this paper is as follows:

Theorem 1.3 *Let* $3 < p < \infty$*. For any* $\mathbf{u}_0 \in L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3)$ *the problem* (1.1) *has a unique solution* $\mathbf{u} \in C([0,+\infty), L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3))$*. Moreover, there exists constant* $C[\kappa_p(\mathbf{u}_0)] > 0$ *depending only on* $\kappa_p(\mathbf{u}_0)$ *(not on specific* $\mathbf{u}_0$*), such that*

$$\sup_{t\ge 0} \|\mathbf{u}(t)\|_p \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p. \tag{1.3}$$

Corollary 1.4 *Under the assumptions of Theorem 1.3, we have* $\mathbf{u} \in C^\infty(\mathbb{R}^3 \times (0, +\infty))$.

Corollary 1.5 *Let* $\mathbf{u}_0 \in L^2_\omega(\mathbb{R}^3)$. *Let* $\mathbf{u}$ *be the Leray-Hopf weak solution of the problem* (1.1) *with initial data* $\mathbf{u}_0$. *Then* $\mathbf{u} \in C^\infty(\mathbb{R}^3 \times (0, +\infty))$.

Corollary 1.6 *Let* $\mathbf{u}_0 \in H^\infty(\mathbb{R}^3) = \bigcap_{m=0}^{\infty} H^m(\mathbb{R}^3)$ *and satisfy the condition* $\mathrm{div}(\mathbf{u}_0) = 0$. *Let* $\mathbf{u}$ *be the Leray-Hopf weak solution of the problem* (1.1) *with initial data* $\mathbf{u}_0$. *Then* $\mathbf{u} \in C^\infty(\mathbb{R}^3 \times [0, +\infty))$.

The idea of the proof of Theorem 1.3 is to show that global solutions of the problem (1.1) decay sufficiently fast as $t \to \infty$ so that there holds the estimate

$$\int_0^\infty \|\mathbf{u}(t)\|_p^{\frac{p(p-1)}{p-3}} dt \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p. \tag{1.4}$$

We shall use the continuous induction method to prove this estimate, namely, we first prove that this estimate holds for $\mathbf{u}_0$ in a small neighborhood of the origin of $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, and next prove that this estimate is stable under small perturbations. Since this estimate is clearly also stable under the limit procedure, we conclude that it holds for arbitrary initial data in $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Thus, the proof is very simple.

We point out that Theorem 1.3 can be immediately extended to the general N-dimension case. See Remark 4.1.

In the next section we deduce some basic estimates and prove that (1.4) holds for $\mathbf{u}_0$ in a small neighborhood of the origin of $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. In Section 3 we consider the perturbed problem. In the last section we complete the proof of Theorem 1.3.

2 Some basic estimates

In this section we deduce some basic estimates and prove that (1.4) holds for $\mathbf{u}_0$ in a small neighborhood of the origin of $L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$.

We shall frequently use the following inequality: For any $2 \le p < \infty$,

$$\|\mathbf{u}\|_{3p}^p \le C \int_{\mathbb{R}^3} |\mathbf{u}(x)|^{p-2} |\nabla \mathbf{u}(x)|^2 dx, \tag{2.1}$$

where C is a positive constant depending only on p. This follows from the well-known Sobolev embedding inequality

$$\|\varphi\|_{L^r(\mathbb{R}^n)} \le C(n,q)\|\nabla\varphi\|_{L^q(\mathbb{R}^n)} \qquad \Big(1 \le q < n,\ \frac{1}{r} = \frac{1}{q} - \frac{1}{n}\Big)$$

applied to $\varphi = |\mathbf{u}|^{\frac{p}{2}}$, $q = 2$, $r = 6$ and $n = 3$.

Lemma 2.1 *Let $3 \le p < \infty$. Let $\mathbf{u}$ be a $L^p \cap L^2$-solution of (1.1) on the time interval $[0, T)$. Then we have that for all $0 < t < T$,*

$$\frac{1}{p}\frac{d}{dt}\|\mathbf{u}(t)\|_p^p + \int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx \le C\|\mathbf{u}\|_p^{\frac{p-1}{2}}\Big(\int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx\Big)^{\frac{p+3}{2p}}, \tag{2.2}$$

where C is a positive constant depending only on p. Moreover, if $3 < p < \infty$ then we further have

$$\frac{1}{p}\frac{d}{dt}\|\mathbf{u}(t)\|_p^p + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx \le C\|\mathbf{u}(t)\|_p^{\frac{p(p-1)}{p-3}}. \tag{2.3}$$

Proof: Computing the L^2 inner product of both sides of the partial differential equations in (1.1) with $|\mathbf{u}|^{p-2}\mathbf{u}$, we get

$$\int_{\mathbb{R}^3} \partial_t\mathbf{u} \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx - \int_{\mathbb{R}^3} \Delta\mathbf{u} \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx = -\int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{u}\otimes\mathbf{u})] \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx. \tag{2.4}$$

We first note that

$$\int_{\mathbb{R}^3} \partial_t\mathbf{u} \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}\partial_t(|\mathbf{u}|^2) dx = \frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3} |\mathbf{u}|^p dx = \frac{1}{p}\frac{d}{dt}\|\mathbf{u}(t)\|_p^p, \tag{2.5}$$

$$\begin{aligned} -\int_{\mathbb{R}^3} \Delta\mathbf{u} \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx &= -\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2} u_k \partial_j^2 u_k dx \\ &= \int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla\mathbf{u}|^2 dx + (p-2)\sum_{j=1}^{3}\int_{\mathbb{R}^3} |\mathbf{u}|^{p-4}\Big(\sum_{k=1}^{3} u_k\partial_j u_k\Big)^2 dx \\ &\ge \int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla\mathbf{u}|^2 dx. \end{aligned} \tag{2.6}$$

Next we note that denoting by L the mapping from 3×3 matrix-valued functions to scaler functions defined by

$$LA(x) = \sum_{j,k=1}^{3} R_j R_k(f_{jk}(x)) \quad \text{for} \quad A(x) = (f_{jk}(x))_{3\times 3},$$

where (R_1, R_2, R_3) is the Riesz transform, we have

$$\mathbb{P}[\nabla \cdot (\mathbf{u}\otimes\mathbf{u})] = \nabla \cdot (\mathbf{u}\otimes\mathbf{u}) - \nabla L(\mathbf{u}\otimes\mathbf{u}),$$

so that by integration by parts we get

$$-\int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{u}\otimes\mathbf{u})] \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx$$

$$= \int_{\mathbb{R}^3} (\mathbf{u} \otimes \mathbf{u}) \cdot [\nabla \otimes (|\mathbf{u}|^{p-2}\mathbf{u})] dx - \int_{\mathbb{R}^3} L(\mathbf{u} \otimes \mathbf{u}) \cdot \mathrm{div}(|\mathbf{u}|^{p-2}\mathbf{u}) dx.$$

Here the first dot on the right-hand side represents the inner product between matrices, i.e., $(a_{jk})_{3\times 3} \cdot (b_{jk})_{3\times 3} = \sum_{j,k=1}^{3} a_{jk} b_{jk}$, and the second one represents the product between scaler functions. Hence, since R_j's are bounded in $L^p(\mathbb{R}^3)$ for any $1 < p < \infty$ and both $|\nabla \otimes (|\mathbf{u}|^{p-2}\mathbf{u})|$ and $|\mathrm{div}(|\mathbf{u}|^{p-2}\mathbf{u})|$ are bounded by $|\mathbf{u}|^{p-2}|\nabla \mathbf{u}|$, by using the Hölder inequality we have

$$\begin{aligned} & - \int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{u} \otimes \mathbf{u})] \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx \\ \le & C\Big(\int_{\mathbb{R}^3} |\mathbf{u} \otimes \mathbf{u}|^{\frac{p^2}{2(p-1)}} dx \Big)^{\frac{2(p-1)}{p^2}} \Big(\int_{\mathbb{R}^3} (|\mathbf{u}|^{p-2}|\nabla \mathbf{u}|)^{\frac{p^2}{p^2-2p+2}} dx \Big)^{\frac{p^2-2p+2}{p^2}} \\ \le & C\Big(\int_{\mathbb{R}^3} |\mathbf{u} \otimes \mathbf{u}|^{\frac{p^2}{2(p-1)}} dx \Big)^{\frac{2(p-1)}{p^2}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{\frac{p^2}{p-2}} dx \Big)^{\frac{(p-2)^2}{2p^2}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx \Big)^{\frac{1}{2}} \\ \le & C\Big(\int_{\mathbb{R}^3} |\mathbf{u}|^p dx \Big)^{\frac{1}{p}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{\frac{p^2}{p-2}} dx \Big)^{\frac{p-2}{2p}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx \Big)^{\frac{1}{2}}. \end{aligned}$$

By the Hölder inequality and the inequality (2.1) we have

$$\|\mathbf{u}\|_{\frac{p^2}{p-2}} \le \|\mathbf{u}\|_p^{\frac{p-3}{p}} \|\mathbf{u}\|_{3p}^{\frac{3}{p}} \le C\|\mathbf{u}\|_p^{\frac{p-3}{p}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx \Big)^{\frac{3}{p^2}}.$$

It follows that

$$\begin{aligned} - \int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{u} \otimes \mathbf{u})] \cdot |\mathbf{u}|^{p-2}\mathbf{u} dx \le & C\|\mathbf{u}\|_p \cdot \|\mathbf{u}\|_p^{\frac{p-3}{2}} \cdot \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx \Big)^{\frac{3}{2p}+\frac{1}{2}} \\ \le & C\|\mathbf{u}\|_p^{\frac{p-1}{2}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx \Big)^{\frac{p+3}{2p}}. \end{aligned} \tag{2.7}$$

Substituting (2.5)–(2.7) into (2.4), we see that (2.2) follows. Next, since

$$\|\mathbf{u}\|_p^{\frac{p-1}{2}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx \Big)^{\frac{p+3}{2p}} \le C\|\mathbf{u}\|_p^{\frac{p(p-1)}{p-3}} + \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla \mathbf{u}|^2 dx,$$

from (2.2) we obtain (2.3). □

Lemma 2.2 *Let $3 < p < \infty$. There exists $\varepsilon > 0$ such that for any $\mathbf{u}_0 \in L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3)$ with $\|\mathbf{u}_0\|_3 < \varepsilon$, the following assertions hold:*

(1) *The solution $\mathbf{u}(t) = \Phi(t, \mathbf{u}_0)$ is a global $L^p \cap L^2$-solution.*

(2) *$\|\mathbf{u}(t)\|_2$, $\|\mathbf{u}(t)\|_3$ and $\|\mathbf{u}(t)\|_p$ are monotone decreasing for all $t > 0$.*

(3) *There exists positive constant C depending only on p such that*

$$\int_0^\infty \|\mathbf{u}(t)\|_p^{\frac{p(p-1)}{p-3}} dt \le C\|\mathbf{u}_0\|_3^{\frac{2p}{p-3}} \|\mathbf{u}_0\|_p^p.$$

Proof: The assertion (1) is well-known (see also the argument below). The assertion that $\|\mathbf{u}(t)\|_2$ is monotone decreasing follows from the energy inequality. Next, letting $p = 3$ in (2.2), we see that

$$\frac{1}{3}\frac{d}{dt}\|\mathbf{u}(t)\|_3^3 + \int_{\mathbb{R}^3} |\mathbf{u}(t)||\nabla\mathbf{u}(t)|^2 dx \leq C\|\mathbf{u}(t)\|_3 \int_{\mathbb{R}^3} |\mathbf{u}(t)||\nabla\mathbf{u}(t)|^2 dx.$$

From this inequality we easily see that if $C\|\mathbf{u}(t)\|_3 \leq 1$, then $\|\mathbf{u}(t)\|_3$ is monotone decreasing. Thus, if $C\|\mathbf{u}_0\|_3 \leq 1$ then $\mathbf{u}$ is global and $\|\mathbf{u}(t)\|_3$ is monotone decreasing. Besides, since

$$\|\mathbf{u}\|_p \leq \|\mathbf{u}\|_3^{\frac{2}{p-1}} \|\mathbf{u}\|_{3p}^{\frac{p-3}{p-1}} \leq C\|\mathbf{u}\|_3^{\frac{2}{p-1}} \Big(\int_{\mathbb{R}^3} |\mathbf{u}|^{p-2}|\nabla\mathbf{u}|^2 dx\Big)^{\frac{p-3}{p(p-1)}},$$

from (2.2) we obtain

$$\frac{1}{p}\frac{d}{dt}\|\mathbf{u}(t)\|_p^p + \int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx \leq C\|\mathbf{u}(t)\|_3 \int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx.$$

It follows that if $C\|\mathbf{u}(t)\|_3 \leq \frac{1}{2}$ then

$$\frac{1}{p}\|\mathbf{u}(t)\|_p^p + \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx dt \leq \frac{1}{p}\|\mathbf{u}_0\|_p^p.$$

Hence, if $C\|\mathbf{u}_0\|_3 \leq \frac{1}{2}$, then $\|\mathbf{u}(t)\|_p$ is monotone decreasing. Finally, from (2.1) and the above inequality we have

$$\int_0^\infty \|\mathbf{u}(t)\|_{3p}^p dt \leq C\int_0^\infty \int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2 dx dt \leq C\|\mathbf{u}_0\|_p^p.$$

Hence, using the Hölder inequality and the fact that $\|\mathbf{u}(t)\|_3$ is monotone decreasing, we get

$$\begin{aligned}\int_0^\infty \|\mathbf{u}(t)\|_p^{\frac{p(p-1)}{p-3}} dt &\leq \int_0^\infty \|\mathbf{u}(t)\|_3^{\frac{2p}{p-3}} \|\mathbf{u}(t)\|_{3p}^p dt \\ &\leq \|\mathbf{u}_0\|_3^{\frac{2p}{p-3}} \int_0^\infty \|\mathbf{u}(t)\|_{3p}^p dt \\ &\leq C\|\mathbf{u}_0\|_3^{\frac{2p}{p-3}} \|\mathbf{u}_0\|_p^p.\end{aligned}$$

This proves the assertion (3). $\square$

Lemma 2.3 *Let* $3 < p < \infty$. *Assume that* $\mathbf{u}$ *is a global* $L^p \cap L^2$*-solution of the problem* (1.1) *and there exists a constant* $C[\kappa_p(\mathbf{u}_0)] > 0$ *depending only on* $\kappa_p(\mathbf{u}_0)$ (*not on the specific* $\mathbf{u}_0$) *such that*

$$\int_0^\infty \|\mathbf{u}(t)\|_p^{\frac{p(p-1)}{p-3}} dt \leq C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p. \tag{2.8}$$

Then for some similar but possibly larger constant $C[\kappa_p(\mathbf{u}_0)] > 0$ we have the following estimates:

$$\sup_{t\geq 0} \|\mathbf{u}(t)\|_p^p \leq C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \tag{2.9}$$

$$\int_0^\infty \int_{\mathbb{R}^3} |\mathbf{u}(t)|^{p-2}|\nabla \mathbf{u}(t)|^2 dxdt \leq C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \tag{2.10}$$

$$\int_0^\infty \|\mathbf{u}(t)\|_{3p}^p dt \leq C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \tag{2.11}$$

$$\int_0^\infty \|\mathbf{u}(t)\|_9^3 dt \leq C[\kappa_p(\mathbf{u}_0)], \tag{2.12}$$

$$\sup_{t\geq 0} \|\mathbf{u}(t)\|_3 \leq C[\kappa_p(\mathbf{u}_0)]. \tag{2.13}$$

Proof: (2.9) and (2.10) are immediate consequences of (2.3) and (2.8), and (2.11) follows from (2.1) and (2.10). Next, by using (2.1) for $p = 2$ and the energy inequality we get

$$\int_0^\infty \|\mathbf{u}(t)\|_6^2 dt \leq C \int_0^\infty \|\nabla \mathbf{u}(t)\|_2^2 dt \leq C\|\mathbf{u}_0\|_2^2.$$

Hence, using the Hölder inequality, the above inequality, and the inequality (2.11) we see that

$$\begin{aligned}\int_0^\infty \|\mathbf{u}(t)\|_9^3 dt \leq& \int_0^\infty \|\mathbf{u}(t)\|_6^{\frac{2(p-3)}{p-2}} \|\mathbf{u}(t)\|_{3p}^{\frac{p}{p-2}} dt \\ \leq& \Big(\int_0^\infty \|\mathbf{u}(t)\|_6^2 dt\Big)^{\frac{p-3}{p-2}} \Big(\int_0^\infty \|\mathbf{u}(t)\|_{3p}^p dt\Big)^{\frac{1}{p-2}} \\ \leq& C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_2^{\frac{2(p-3)}{p-2}} \|\mathbf{u}_0\|_p^{\frac{p}{p-2}}.\end{aligned}$$

Since $\|\mathbf{u}_0\|_2^{\frac{2(p-3)}{p-2}} \|\mathbf{u}_0\|_p^{\frac{p}{p-2}} = [\kappa_p(\mathbf{u}_0)]^3$, we obtain (2.12). Finally, (2.13) follows from (2.9) and the Hölder and energy inequalities. □

3 The perturbed problem

Let $3 < p < \infty$. Let $\mathbf{v} = \mathbf{v}(t)$ $(0 \leq t < +\infty)$ be a global $L^p \cap L^2$-solution of the problem (1.1) with initial data $\mathbf{v}_0 \in L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3)$, satisfying the following condition:

$$\int_0^\infty \|\mathbf{v}(t)\|_p^{\frac{p(p-1)}{p-3}} dt \leq C[\kappa_p(\mathbf{v}_0)]\|\mathbf{v}_0\|_p^p, \tag{3.1}$$

where $C[\kappa_p(\mathbf{v}_0)]$ is a positive constant depending only on the κ_p-value of $\mathbf{v}_0$ (not on the specific $\mathbf{v}_0$). We want to prove that there exists $\varepsilon > 0$ such that if $\mathbf{u}_0 \in L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3)$ satisfies

$$\|\mathbf{u}_0 - \mathbf{v}_0\|_p + \|\mathbf{u}_0 - \mathbf{v}_0\|_2 < \varepsilon,$$

then $\mathbf{u}(t) = \Phi(t, \mathbf{u}_0)$ is also a global $L^p \cap L^2$-solution satisfying a similar estimate.

Let $\mathbf{w}_0 = \mathbf{u}_0 - \mathbf{v}_0$ and consider the initial value problem:

$$\begin{cases} \partial_t \mathbf{w} - \Delta \mathbf{w} + \mathbb{P}[2\nabla \cdot (\mathbf{v} \otimes_s \mathbf{w}) + \nabla \cdot (\mathbf{w} \otimes \mathbf{w})] = 0, & x \in \mathbb{R}^3, \ \ t > 0, \\ \mathbf{w} = \mathbf{w}_0, \qquad x \in \mathbb{R}^3, \ \ t = 0, \end{cases} \tag{3.2}$$

where $\mathbf{v} \otimes_s \mathbf{w} = \dfrac{1}{2}(\mathbf{v} \otimes \mathbf{w} + \mathbf{w} \otimes \mathbf{v})$. Clearly, if $\mathbf{w}$ is a $L^p \cap L^2$-solution of this problem, then $\mathbf{v} + \mathbf{w}$ is a $L^p \cap L^2$-solution of the problem (1.1) with initial data $\mathbf{u}_0$, so that $\mathbf{u} = \mathbf{v} + \mathbf{w}$. We shall prove that there exists $\varepsilon > 0$, such that for any $\mathbf{w}_0 \in L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3)$ satisfying

$$\|\mathbf{w}_0\|_p < \varepsilon \quad \text{and} \quad \|\mathbf{w}_0\|_2 < \varepsilon,$$

the above problem has a unique global $L^p \cap L^2$-solution $\mathbf{w} \in C_b([0,\infty), L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ satisfying some nice estimates.

By virtue of Lemma 2.3, (3.1) implies that the following estimate holds:

$$\int_0^\infty \|\mathbf{v}(t)\|_9^3 dt \le C[\kappa_p(\mathbf{v}_0)]. \tag{3.3}$$

Lemma 3.1 *Let $3 < p < \infty$. Let $\mathbf{v} = \mathbf{v}(t)$ be a global $L^p \cap L^2$-solution of the problem* (1.1) *with initial data $\mathbf{v}_0 \in L^p_\omega(\mathbb{R}^3) \cap L^2_\omega(\mathbb{R}^3)$, satisfying the condition* (3.1)*. Then there exists $\varepsilon > 0$ such that for any $\mathbf{w}_0 \in L^3_\omega(\mathbb{R}^3)$ with $\|\mathbf{w}_0\|_3 < \varepsilon$, the problem* (3.2) *has a unique global L^3-solution $\mathbf{w} \in C([0,\infty), L^3(\mathbb{R}^3))$ satisfying the following estimate:*

$$\sup_{t \ge 0} \|\mathbf{w}(t)\|_3 \le C[\kappa_p(\mathbf{v}_0)] \|\mathbf{w}_0\|_3. \tag{3.4}$$

Proof: By a standard Banach fixed point argument, we can easily prove that the problem (3.2) is locally well-posed in $L^3(\mathbb{R}^3)$. Thus, we only need to prove the estimate (3.4), because if this is proved then it follows immediately that the solution is global.

Computing the L^2 inner product of both sides the equations in (3.2) with $|\mathbf{w}|\mathbf{w}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^3} \partial_t \mathbf{w} \cdot |\mathbf{w}|\mathbf{w} dx - \int_{\mathbb{R}^3} \Delta \mathbf{w} \cdot |\mathbf{w}|\mathbf{w} dx \\ = &- 2\int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{v} \otimes_s \mathbf{w})] \cdot |\mathbf{w}|\mathbf{w} dx - \int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{w} \otimes \mathbf{w})] \cdot |\mathbf{w}|\mathbf{w} dx. \end{aligned} \tag{3.5}$$

We have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^3} \mathbb{P}[\nabla \cdot (\mathbf{v} \otimes_s \mathbf{w})] \cdot |\mathbf{w}|\mathbf{w} dx \Big| \\ \le &C \int_{\mathbb{R}^3} [|\mathbf{v} \otimes_s \mathbf{w}| + |L(\mathbf{v} \otimes_s \mathbf{w})|] \cdot |\mathbf{w}||\nabla \mathbf{w}| dx \end{aligned}$$

$$\begin{aligned}
&\le C\Big(\int_{\mathbb{R}^3}|\mathbf{v}\otimes_s \mathbf{w}|^2|\mathbf{w}|dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx\Big)^{\frac12}\\
&\le C\|\mathbf{v}\otimes_s \mathbf{w}\|_{\frac94}\|\mathbf{w}\|_9^{\frac12}\Big(\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx\Big)^{\frac12}\\
&\le C\|\mathbf{v}\|_9\|\mathbf{w}\|_3\Big(\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx\Big)^{\frac23}\\
&\le C\|\mathbf{v}\|_9^3\|\mathbf{w}\|_3^3+\frac14\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx,
\end{aligned}$$

and, similarly,

$$\Big|\int_{\mathbb{R}^3}\mathbb{P}[\nabla\cdot(\mathbf{w}\otimes_s\mathbf{w})]\cdot|\mathbf{w}|\mathbf{w}dx\Big|\le C\|\mathbf{w}\|_3\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx.$$

Hence, similarly as in the proof of Lemma 2.1 we see that (3.5) implies that

$$\frac13\frac{d}{dt}\|\mathbf{w}\|_3^3+\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx\le C\|\mathbf{v}\|_9^3\|\mathbf{w}\|_3^3+\Big(\frac14+C\|\mathbf{w}\|_3\Big)\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx.$$

It follows that if $C\|\mathbf{w}\|_3\le\dfrac14$ then

$$\frac13\frac{d}{dt}\|\mathbf{w}\|_3^3+\frac12\int_{\mathbb{R}^3}|\mathbf{w}||\nabla\mathbf{w}|^2dx\le C\|\mathbf{v}\|_9^3\|\mathbf{w}\|_3^3,$$

so that by (3.3) we have

$$\|\mathbf{w}(t)\|_3^3+\frac32\int_0^t\int_{\mathbb{R}^3}|\mathbf{w}(t)||\nabla\mathbf{w}(t)|^2dxdt\le\|\mathbf{w}_0\|_3^3e^{3C\int_0^\infty\|\mathbf{v}(t)\|_9^3dt}\le\|\mathbf{w}_0\|_3^3e^{3C[\kappa_p(\mathbf{v}_0)]},$$

by which (3.4) follows. Note that in order for the condition $C\|\mathbf{w}(t)\|_3\le\dfrac14$ to be satisfied for all $t\ge0$, we only need to have $C\|\mathbf{w}_0\|_3e^{C[\kappa_p(\mathbf{v}_0)]}\le\dfrac14$. □

Lemma 3.2 *Let the conditions in Lemma 3.1 be satisfied. Then there exists $\varepsilon>0$ such that for any $\mathbf{w}_0\in L^2_\omega(\mathbb{R}^3)\cap L^p_\omega(\mathbb{R}^3)$ with $\|\mathbf{w}_0\|_2<\varepsilon$ and $\|\mathbf{w}_0\|_p<\varepsilon$, the problem (3.2) has a unique global $L^p\cap L^2$-solution $\mathbf{w}\in C_b([0,\infty),L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3))$, satisfying the following estimates:*

$$\sup_{t\ge0}\|\mathbf{w}(t)\|_2^2+\int_0^\infty\|\nabla\mathbf{w}(t)\|_2^2dt\le\|\mathbf{w}_0\|_2^2+C[\kappa_p(\mathbf{v}_0)]\|\mathbf{w}_0\|_3^2\|\mathbf{v}_0\|_2^2,\tag{3.6}$$

$$\sup_{t\ge0}\|\mathbf{w}(t)\|_p^p+\int_0^\infty\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dxdt\le\|\mathbf{w}_0\|_p^p+C[\kappa_p(\mathbf{v}_0)]\|\mathbf{w}_0\|_3^p\|\mathbf{v}_0\|_p^p,\tag{3.7}$$

$$\int_0^\infty\|\mathbf{w}(t)\|_p^{\frac{p(p-1)}{p-3}}dt\le C[\kappa_p(\mathbf{v}_0)](\|\mathbf{w}_0\|_p^p+\|\mathbf{w}_0\|_3^p\|\mathbf{v}_0\|_p^p).\tag{3.8}$$

Proof: Global existence of the solution $\mathbf{w}$ follows from Lemma 3.1 and a standard result on local existence of the solution. Thus we only need to prove the estimates (3.6)–(3.8). Since we shall not use (3.6) later on, we only give the proofs of (3.7) and (3.8), and leave the proof of (3.6) to the reader as an exercise.

By some similar argument as in the proof of Lemma 2.1, we have the following inequality:

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\mathbf{w}(t)\|_p^p+\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\\
\leq&C\|\mathbf{v}\|_p\|\mathbf{w}\|_p^{\frac{p-3}{2}}\Big(\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\Big)^{\frac{p+3}{2p}}\\
&+C\|\mathbf{w}\|_p^{\frac{p-1}{2}}\Big(\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\Big)^{\frac{p+3}{2p}}.
\end{aligned}$$

Since

$$\|\mathbf{v}\|_p\leq\|\mathbf{v}\|_3^{\frac{2}{p-1}}\|\mathbf{v}\|_{3p}^{\frac{p-3}{p-1}},\qquad \|\mathbf{w}\|_p\leq\|\mathbf{w}\|_3^{\frac{2}{p-1}}\|\mathbf{w}\|_{3p}^{\frac{p-3}{p-1}},\tag{3.9}$$

we get

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\mathbf{w}(t)\|_p^p+\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\\
\leq&C\|\mathbf{v}\|_3^{\frac{2}{p-1}}\|\mathbf{v}\|_{3p}^{\frac{p-3}{p-1}}\|\mathbf{w}\|_3^{\frac{p-3}{p-1}}\|\mathbf{w}\|_{3p}^{\frac{(p-3)^2}{2(p-1)}}\Big(\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\Big)^{\frac{p+3}{2p}}\\
&+C\|\mathbf{w}\|_3\|\mathbf{w}\|_{3p}^{\frac{p-3}{2}}\Big(\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\Big)^{\frac{p+3}{2p}}\\
\leq&C\|\mathbf{v}\|_3^{\frac{2}{p-1}}\|\mathbf{w}\|_3^{\frac{p-3}{p-1}}\Big(\int_{\mathbb{R}^3}|\mathbf{v}(t)|^{p-2}|\nabla\mathbf{v}(t)|^2dx\Big)^{\frac{p-3}{p(p-1)}}\Big(\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\Big)^{\frac{p^2-2p+3}{p(p-1)}}\\
&+C\|\mathbf{w}\|_3\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\\
\leq&C\|\mathbf{v}\|_3^{\frac{2p}{p-3}}\|\mathbf{w}\|_3^p\int_{\mathbb{R}^3}|\mathbf{v}(t)|^{p-2}|\nabla\mathbf{v}(t)|^2dx+\Big(\frac{1}{4}+C\|\mathbf{w}\|_3\Big)\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx.
\end{aligned}$$

Hence, if $C\|\mathbf{w}\|_3\leq\dfrac{1}{4}$ then

$$\frac{1}{p}\frac{d}{dt}\|\mathbf{w}(t)\|_p^p+\frac{1}{2}\int_{\mathbb{R}^3}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dx\leq C\|\mathbf{v}(t)\|_3^{\frac{2p}{p-3}}\|\mathbf{w}(t)\|_3^p\int_{\mathbb{R}^3}|\mathbf{v}(t)|^{p-2}|\nabla\mathbf{v}(t)|^2dx.$$

Integrating this inequality and using Lemmas 2.3 and 3.1, we see that (3.7) follows. The estimate (3.8) follows from the second inequality in (3.9), (3.4), (2.1) and (3.7). □

4 The proof of Theorem 1.3

Let $3<p<\infty$ be given. We denote by G_p the set of all $\mathbf{u}_0\in L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$ such that the problem (1.1) has a global $L^p\cap L^2$-solution $\mathbf{u}=\mathbf{u}(t)$ satisfying the following estimate:

$$\int_0^\infty\|\mathbf{u}(t)\|_p^{\frac{p(p-1)}{p-3}}dt\leq C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p.\tag{4.1}$$

In what follows we prove the following two assertions:

Assertion 1: G_p *is open* (as a subset of $L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$).
Assertion 2: G_p *is closed* (as a subset of $L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$).

To prove Assertion 1 we let $\mathbf{v}_0\in G_p$. If $\mathbf{v}_0=0$ then by Lemma 2.2 we see that there is a neighborhood of $\mathbf{v}_0$ which is contained in G_p. We now assume that $\mathbf{v}_0\neq 0$, so that $\|\mathbf{v}_0\|_p>0$, $\|\mathbf{v}_0\|_2>0$. It follows by Lemma 3.2 that there exists $\varepsilon>0$ sufficiently small, such that for any $\mathbf{w}_0\in L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$ satisfying the conditions $\|\mathbf{w}_0\|_p\leq\varepsilon\|\mathbf{v}_0\|_p$ and $\|\mathbf{w}_0\|_2\leq\varepsilon\|\mathbf{v}_0\|_2$, the problem (3.2) has a unique global solution $L^p\cap L^2$-solution $\mathbf{w}=\mathbf{w}(t)$ which satisfies the estimates (3.6)–(3.8). Hence, if $\mathbf{u}_0\in L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$ is so close to $\mathbf{v}_0$ that $\|\mathbf{u}_0-\mathbf{v}_0\|_p\leq\varepsilon\|\mathbf{v}_0\|_p$ and $\|\mathbf{u}_0-\mathbf{v}_0\|_2\leq\varepsilon\|\mathbf{v}_0\|_2$, then by letting $\mathbf{w}_0=\mathbf{u}_0-\mathbf{v}_0$ and then letting $\mathbf{u}(t)=\mathbf{v}(t)+\mathbf{w}(t)$ for all $t\geq 0$, we obtain a global $L^p\cap L^2$-solution $\mathbf{u}=\mathbf{u}(t)$ of the problem (1.1) with initial data $\mathbf{u}_0$. The estimate (4.1) follows from (3.1) and (3.8) by using the inequalities

$$(1-\varepsilon)\|\mathbf{v}_0\|_p\leq\|\mathbf{u}_0\|_p\leq(1+\varepsilon)\|\mathbf{v}_0\|_p$$

and choosing ε sufficiently small. To prove Assertion 2 we let $\{\mathbf{u}_{0n}\}_{n=1}^\infty$ be a sequence in G_p converging to $\mathbf{u}_0$ in $L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$. Let $\mathbf{u}_n=\mathbf{u}_n(t)$ be the global $L^p\cap L^2$-solution of (1.1) with initial data $\mathbf{u}_{0n}$, $n=1,2,\cdots$, and let $\mathbf{u}=\mathbf{u}(t)$ be the maximally extended $L^p\cap L^2$-solution of (1.1) with initial data $\mathbf{u}_0$. By continuous dependence of the solution of (1.1) on the initial data, we see that for any $0<T<T^*$, where T^* denotes the lifespan of $\mathbf{u}=\mathbf{u}(t)$, we have

$$\mathbf{u}_n\to\mathbf{u}\quad\text{strongly in }\ C([0,T],L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3))$$

as $n\to\infty$. Hence, by first letting $n\to\infty$ in the inequalities

$$\sup_{0\leq t\leq T}\|\mathbf{u}_n(t)\|_p\leq M,\quad n=1,2,\cdots,$$

where $M=\sup_n C[\kappa_p(\mathbf{u}_{0n})]\|\mathbf{u}_{0n}\|_p<\infty$, and then letting $T\to T^{*-}$, we see that $\mathbf{u}=\mathbf{u}(t)$ cannot blow-up at any finite time, so that it is a global $L^p\cap L^2$-solution. Moreover, by letting $n\to\infty$ in the inequalities

$$\int_0^\infty\|\mathbf{u}_n(t)\|_p^{\frac{p(p-1)}{p-3}}\,dt\leq C[\kappa_p(\mathbf{u}_{0n})]\|\mathbf{u}_{0n}\|_p^p,\quad n=1,2,\cdots,$$

we see that (4.1) holds. Hence $\mathbf{u}_0\in G_p$.

Having proved the above two assertions, we conclude that $G_p=L^p_\omega(\mathbb{R}^3)\cap L^2_\omega(\mathbb{R}^3)$, and Theorem 1.3 follows. □

Remark 4.1: Theorem 1.3 can be immediately extended to the general N-dimension case. To see this we only need to replace all the estimates established in previous two sections with their N-dimensional counterparts. We use the same equation number with $'$ to mark these estimates:

$$\frac{1}{p}\frac{d}{dt}\|\mathbf{u}(t)\|_p^p + \int_{\mathbb{R}^3}|\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2dx \le C\|\mathbf{u}\|_p^{1+\frac{p-N}{2}}\Big(\int_{\mathbb{R}^N}|\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2dx\Big)^{\frac{p+N}{2p}}, \tag{2.2$'$}$$

$$\frac{1}{p}\frac{d}{dt}\|\mathbf{u}(t)\|_p^p + \frac{1}{2}\int_{\mathbb{R}^N}|\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2dx \le C\|\mathbf{u}(t)\|_p^{\frac{p(p-N+2)}{p-N}}. \tag{2.3$'$}$$

$$\text{If}\quad \int_0^\infty \|\mathbf{u}(t)\|_p^{\frac{p(p-N+2)}{p-N}}dt \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \quad\text{then} \tag{2.8$'$}$$

$$\sup_{t\ge0}\|\mathbf{u}(t)\|_p^p \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \tag{2.9$'$}$$

$$\int_0^\infty\int_{\mathbb{R}^N}|\mathbf{u}(t)|^{p-2}|\nabla\mathbf{u}(t)|^2dxdt \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \tag{2.10$'$}$$

$$\int_0^\infty \|\mathbf{u}(t)\|_{\frac{Np}{N-2}}^p dt \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{u}_0\|_p^p, \tag{2.11$'$}$$

$$\int_0^\infty \|\mathbf{u}(t)\|_{\frac{N^2}{N-2}}^N dt \le \int_0^\infty \|\mathbf{u}(t)\|_{\frac{Np}{N-2}}^{\frac{p(N-2)}{p-2}}\|\mathbf{u}(t)\|_{\frac{2N}{N-2}}^{\frac{2(p-N)}{p-2}}dt \le C[\kappa_p(\mathbf{u}_0)], \tag{2.12$'$}$$

$$\sup_{t\ge0}\|\mathbf{u}(t)\|_N \le C[\kappa_p(\mathbf{u}_0)]. \tag{2.13$'$}$$

$$\text{If}\quad \int_0^\infty \|\mathbf{v}(t)\|_p^{\frac{p(p-N+2)}{p-N}}dt \le C[\kappa_p(\mathbf{u}_0)]\|\mathbf{v}_0\|_p^p, \quad\text{then} \tag{3.1$'$}$$

$$\sup_{t\ge0}\|\mathbf{w}(t)\|_N \le C[\kappa_p(\mathbf{v}_0)]\|\mathbf{w}_0\|_N, \tag{3.4$'$}$$

$$\sup_{t\ge0}\|\mathbf{w}(t)\|_2^2 + \int_0^\infty \|\nabla\mathbf{w}(t)\|_2^2dt \le \|\mathbf{w}_0\|_2^2 + C[\kappa_p(\mathbf{v}_0)]\|\mathbf{w}_0\|_N^2\|\mathbf{v}_0\|_2^2, \tag{3.6$'$}$$

$$\sup_{t\ge0}\|\mathbf{w}(t)\|_p^p + \int_0^\infty\int_{\mathbb{R}^N}|\mathbf{w}(t)|^{p-2}|\nabla\mathbf{w}(t)|^2dxdt \le \|\mathbf{w}_0\|_p^p + C[\kappa_p(\mathbf{v}_0)]\|\mathbf{w}_0\|_N^p\|\mathbf{v}_0\|_p^p, \tag{3.7$'$}$$

$$\int_0^\infty \|\mathbf{w}(t)\|_p^{\frac{p(p-N+2)}{p-N}}dt \le C[\kappa_p(\mathbf{v}_0)](\|\mathbf{w}_0\|_p^p + \|\mathbf{w}_0\|_N^p\|\mathbf{v}_0\|_p^p). \tag{3.8$'$}$$

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