

REGULARITY OF LERAY-HOPF SOLUTIONS TO NAVIER-STOKES EQUATIONS (II)–BLOW UP RATE WITH SMALL $L^2(\mathbb{R}^3)$ DATA

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ABSTRACT. An upper bound of blow up rate for incompressible Navier-Stokes equations with small data in $L^2(\mathbb{R}^3)$ is obtained.

1. INTRODUCTION

We consider the blow up rate of weak solutions to incompressible Navier-Stokes equations

$$(1.1) \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, & \text{in } \mathbb{R}^3 \times (0, T) \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^3 \times (0, T) \\ u(x, 0) = u_0(x), & \text{in } \mathbb{R}^3 \end{cases}$$

where u and p denote the unknown velocity and pressure of incompressible fluid respectively.

In this paper, we shall estimate the upper bound of blow up rate for the Navier-Stokes equations.

Theorem 1.1. *There is $\delta > 0$ such that if $\|u_0\|_{L^2(\mathbb{R}^3)} \leq \delta$, and if u is a Leray-Hopf solution to the problem (1.1) and blows up at $t = T$, then for any small $\epsilon > 0$, there is $t_0 \in (0, T)$, such that*

$$(1.2) \quad \|u(t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{\epsilon}{(T-t)^{1/2}}, \quad \text{for all } t \in (t_0, T).$$

Here $u : (x, t) \in \mathbb{R}^3 \times (0, T) \rightarrow \mathbb{R}^3$ is called a weak solution of (1.1) if it is a Leray-Hopf solution. Precisely, it satisfies

- (1) $u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))$,
- (2) $\operatorname{div} u = 0$ in $\mathbb{R}^3 \times (0, T)$,
- (3) $\int_0^T \int_{\mathbb{R}^3} \{-u \cdot \partial_t \phi + \nabla u \cdot \nabla \phi + (u \cdot \nabla u) \cdot \phi\} dx dt = 0$

for all $\phi \in C_0^\infty(\mathbb{R}^3 \times (0, T))$ with $\operatorname{div} \phi = 0$ in $\mathbb{R}^3 \times (0, T)$.

Combining Theorem 1.1 with my former result in [31], we have

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Corollary 1.2. *There is $\delta > 0$ such that if $\|u_0\|_{L^2(\mathbb{R}^3)} \leq \delta$, and if u is a Leray-Hopf solution of the Navier-Stokes equations (1.1), then u is regular in $\mathbb{R}^3 \times (0, \infty)$.*

Because (1.1) is invariant under a group action, if $(u(x, t), p(x, t))$ is a solution, the same is true for

$$u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) := \lambda^2 p(\lambda x, \lambda^2 t)$$

for all $\lambda \in (0, \infty)$. So for any $u_0 \in L^2(\mathbb{R}^3)$, there is $\lambda \in (0, \infty)$ such that the $L^2(\mathbb{R}^3)$ norm of $\lambda u_0(\lambda x)$ satisfies the condition of Corollary 1.2. Then we have

Corollary 1.3. *For all $u_0 \in L^2(\mathbb{R}^3)$, if u is a Leray-Hopf solution of the Navier-Stokes equations (1.1), then u is regular in $\mathbb{R}^3 \times (0, \infty)$. Moreover, u is the unique solution of the Navier-Stokes equations (1.1).*

Since Leray(1934)[19] and Hopf(1951)[15] obtained the global existence of weak solutions, it has been a fundamental open problem to prove the uniqueness and regularity of weak solutions to the Navier-Stokes equations.

2. ENERGY ESTIMATES

As in [7][8][9] where Giga and Kohn introduced similar transformations for the blow-up problem of semi-linear heat equations, we apply

$$(2.1) \quad y = \frac{1}{(T-t)^{1/2}}x, \quad \tau = -\ln(T-t), \quad w(y, \tau) = (T-t)^{1/2}u(x, t),$$

to (1.1) and consider the following new problem

$$(2.2) \quad \begin{cases} \partial_\tau w = \Delta_y w - \frac{y}{2} \cdot \nabla_y w - \frac{1}{2}w - w \cdot \nabla_y w - \nabla_y q, & \forall y \in \mathbb{R}^3, \quad \tau > -\ln T \\ \operatorname{div}_y w(y, \tau) = 0, & \text{in } \mathbb{R}^3 \times (-\ln T, \infty) \\ w(y, -\ln T) = T^{1/2}u_0(T^{1/2}y), & \text{in } \mathbb{R}^3 \end{cases}$$

where

$$q(y, \tau) = (T-t)p(x, t).$$

Without loss generality, in this section we take $T = 1$. Multiplying the first one of (2.2) by w and integrating it over \mathbb{R}^3 , by using the second equation of (2.2) we have

$$(2.3) \quad \begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} \partial_\tau |w(y, \tau)|^2 dy &= (-1) \int_{\mathbb{R}^3} |\nabla_y w(y, \tau)|^2 - \frac{1}{4} |w(y, \tau)|^2 dy \\ &\quad - \frac{1}{4} \int_{\mathbb{R}^3} \operatorname{div} (y |w(y, \tau)|^2) dy. \end{aligned}$$

Noting that

$$(2.4) \quad \int_{\mathbb{R}^3} \operatorname{div} (y |w(y, \tau)|^2) dy = \lim_{R \rightarrow \infty} \int_{\partial B_R} |y| |w(y, \tau)|^2 d\sigma(y) \geq 0$$

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we obtain

Lemma 2.1. *For any $\tau > 0$, we have*

$$(2.5) \quad \frac{1}{2} \frac{d}{d\tau} \|w(\tau)\|_{L^2(\mathbb{R}^3)}^2 \leq (-1) \{ \|\nabla_y w(\tau)\|_{L^2(\mathbb{R}^3)}^2 - \frac{1}{4} \|w(\tau)\|_{L^2(\mathbb{R}^3)}^2 \}.$$

Furthermore, we take differential in the equations of (2.2) and obtain

$$(2.6) \quad \begin{aligned} \partial_\tau \partial_j w &= \Delta \partial_j w - \frac{1}{2} y \cdot \nabla \partial_j w - \partial_j w \\ &\quad - (\partial_j w \cdot \nabla) w - (w \cdot \nabla) \partial_j w - \nabla_y \partial_j q. \end{aligned}$$

By the same strategy as in the proof of Lemma 2.1, from (2.6) as well as the equation

$$\partial_\tau \Delta w = \Delta^2 w - \frac{1}{2} (y \cdot \nabla) \Delta w - \frac{3}{2} \Delta w - \Delta((w \cdot \nabla) w) - \nabla \Delta q$$

by taking twice differential in (2.2), we have

Lemma 2.2. *For all $\tau > 0$*

$$(2.7) \quad \begin{aligned} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy &\leq -2 \int_{\mathbb{R}^3} |\nabla^2 w(y, \tau)|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy \\ &\quad - 2 \sum_{j,k,l=1}^3 \int_{\mathbb{R}^3} \partial_j w_k(y, \tau) \partial_j w_l(y, \tau) \partial_l w_k(y, \tau) dy \end{aligned}$$

and

$$(2.8) \quad \begin{aligned} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta w(y, \tau)|^2 dy &\leq -2 \int_{\mathbb{R}^3} |\nabla \Delta w(y, \tau)|^2 dy - \frac{3}{2} \int_{\mathbb{R}^3} |\Delta w(y, \tau)|^2 dy \\ &\quad - 2 \int_{\mathbb{R}^3} (\Delta w(y, \tau)) \cdot \Delta((w(y, \tau) \cdot \nabla) w(y, \tau)) dy. \end{aligned}$$

Remark 2.3. (1) For any $t_1 > 0$, there is $t_0 \in (0, t_1)$ such that $u(\cdot, t_0) \in H^1(\mathbb{R}^3)$. With the initial data $u(x, t_0)$, the Leray-Hopf solution $u(x, t)$ is regular at least in a short time interval after t_0 (see [19][24]). We are discussing the blow-up problem for these short time regular solutions.

(2) As a blow-up argument, we assume that $u(x, t)$ is bounded for $t < T$ and blows up at $t = T$. As a direct corollary, we can prove that $\|u(t)\|_{H^3(\mathbb{R}^3)}$ and $\|\partial_t u(t)\|_{H^m(\mathbb{R}^3)}$ ($m = 0, 1, 2$), as well as $\|\partial_t u(t)\|_{L^2(\mathbb{R}^3)}$, $\|\partial_t \nabla_x u(t)\|_{L^2(\mathbb{R}^3)}$ are bounded for $t < T$. So we have the same results for $\|w(\tau)\|_{H^3(\mathbb{R}^3)}$ and $\|\partial_\tau w(\tau)\|_{H^m(\mathbb{R}^3)}$ ($m = 0, 1, 2$) for $\tau < \infty$, as well as the similar results for q by the boundedness of Riesz transformation.

(3) Since $u(x, t)$, $\partial_t u(x, t) \in L^2(\mathbb{R}^3)$ for $t < T$,

$$\int_0^t \int_{\mathbb{R}^3} |\partial_h u(x, h)| |u(x, h)| dx dh < \infty,$$

we can use Fubini theorem to obtain

$$(2.9) \quad \begin{aligned} 2 \int_{\mathbb{R}^3} \partial_t u(x, t) \cdot u(x, t) dx &= \frac{d}{dt} \int_0^t \int_{\mathbb{R}^3} \partial_h |u(x, h)|^2 dx dh \\ &= \frac{d}{dt} \int_{\mathbb{R}^3} \int_0^t \partial_h |u(x, h)|^2 dh dx = \frac{d}{dt} \int_{\mathbb{R}^3} |u(x, t)|^2 dx. \end{aligned}$$

Noting that

$$\partial_t u(x, t) = (T-t)^{-\frac{3}{2}} \left\{ \partial_\tau w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) + \frac{x}{2(T-t)^{1/2}} \cdot \nabla_y w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) + \frac{1}{2} w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) \right\}$$

where $\tau = (-) \ln(T-t)$, from

$$(2.10) \quad \int_{\mathbb{R}^3} |\partial_t u(x, t)|^2 dx = (T-t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \left| \partial_\tau w(y, \tau) + \frac{y}{2} \cdot \nabla_y w(y, \tau) + \frac{1}{2} w(y, \tau) \right|^2 dy$$

and

$$(2.11) \quad \int_{\mathbb{R}^3} |u(x, t)|^2 dx = (T-t)^{1/2} \int_{\mathbb{R}^3} |w(y, \tau)|^2 dy$$

we have for $t < T$

$$(2.12) \quad \int_{\mathbb{R}^3} \left| \partial_\tau w(y, \tau) + \frac{y}{2} \cdot \nabla_y w(y, \tau) \right|^2 dy < \infty.$$

Moreover, from (2.9), we get

$$(2.13) \quad \begin{aligned} & (T-t)^{-\frac{1}{2}} \left\{ \partial_\tau \int_{\mathbb{R}^3} |w(y, \tau)|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |w(y, \tau)|^2 dy \right\} \\ &= \frac{d}{dt} \left\{ (T-t)^{\frac{1}{2}} \int_{\mathbb{R}^3} |w(y, \tau)|^2 dy \right\} = \frac{d}{dt} \int_{\mathbb{R}^3} |u(x, t)|^2 dx = 2 \int_{\mathbb{R}^3} \partial_t u(x, t) \cdot u(x, t) dx \\ &= 2 \int_{\mathbb{R}^3} (T-t)^{-\frac{3}{2}} \left\{ \partial_\tau w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) + \frac{x}{2(T-t)^{1/2}} \cdot \nabla_y w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) \right. \\ & \quad \left. + \frac{1}{2} w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) \right\} \cdot (T-t)^{-\frac{1}{2}} w\left(\frac{x}{(T-t)^{1/2}}, \tau\right) dx \\ &= (T-t)^{-\frac{1}{2}} \int_{\mathbb{R}^3} \left\{ 2 \partial_\tau w(y, \tau) \cdot w(y, \tau) + (y \cdot \nabla_y w(y, \tau)) \cdot w(y, \tau) + |w(y, \tau)|^2 \right\} dy. \end{aligned}$$

By using (2.13), from (2.3) we get (2.5) again.

(4) From Leray's theorem (see V. Scheffer, Pacific J. Math. Vol. 66, No.2, pp 535-552(1976)), there is a disjoint open interval sequence $\{J_q\}_q$ in $(0, \infty)$ such that the Lebesgue measure of $(0, \infty) \setminus \cup_q J_q$ is zero, and the Leray-Hopf solution u can be modified on a set of Lebesgue measure zero so that its restriction to each $\mathbb{R}^3 \times J_q$ becomes smooth. The first blow-up time T assumed in this paper may be considered as the right-side of an open interval J_q . From the arguments of this paper, we can see that the Leray-Hopf solution u can be extended smoothly over the right-side of J_q . So we get that u is smooth from the left-side of the open interval J_q . Since the Lebesgue measure of $(0, \infty) \setminus \cup_q J_q$ is zero, we get that for all $t > 0$, the Leray-Hopf solution $u(x, t)$ is smooth.

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3. (L^∞, L^2) -DECOMPOSITION OF w

In this section we shall prove that w can be decomposed as the sum of a $L^\infty(0, \infty; L^m(\mathbb{R}^3))$ ($m \in [4, \infty]$) part and a $L^\infty(0, \infty; L^2(\mathbb{R}^3)) \cap L^2(0, \infty; H^1(\mathbb{R}^3))$ part.

Let $\varphi \in C_0^\infty(\mathbb{R}^3, [0, 1])$ be a radial symmetrical function satisfying

$$(3.1) \quad \varphi(\xi) = 1 \quad \forall |\xi| \leq 1, \quad \varphi(\xi) = 0 \quad \forall |\xi| \geq 2, \quad \xi \cdot \nabla \varphi(\xi) \leq 0 \quad \forall \xi.$$

Like the Littlewood-Paley analysis, we define the operators

$$\Delta_{-1}f = \mathcal{F}^{-1}[\varphi(\xi)\mathcal{F}[f](\xi)], \quad \Delta_0f = \mathcal{F}^{-1}[(1 - \varphi(\xi))\mathcal{F}[f](\xi)].$$

Denote

$$(3.2) \quad \begin{aligned} \underline{w}(y, \tau) &= \Delta_{-1}w(y, \tau) = \mathcal{F}^{-1}[\varphi * w(y, \tau)], \\ \bar{w}(y, \tau) &= w(y, \tau) - \underline{w}(y, \tau) = \Delta_0w(y, \tau) = \mathcal{F}^{-1}[1 - \varphi] * w(y, \tau), \\ \tilde{w}(y, \tau) &= \mathcal{F}^{-1}[\sqrt{1 - \varphi^2}] * w(y, \tau). \end{aligned}$$

Notice that

$$\|w(\tau)\|_{L^2(\mathbb{R}^3)}^2 = \|\underline{w}(\tau)\|_{L^2(\mathbb{R}^3)}^2 + \|\tilde{w}(\tau)\|_{L^2(\mathbb{R}^3)}^2.$$

So (2.5) can be written as

$$(3.3) \quad \begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{w}(y, \tau)|^2 dy &\leq - \int_{\mathbb{R}^3} |\nabla \tilde{w}(y, \tau)|^2 - \frac{1}{4} |\tilde{w}(y, \tau)|^2 dy \\ &\quad - \int_{\mathbb{R}^3} |\nabla \underline{w}(y, \tau)|^2 - \frac{1}{4} |\underline{w}(y, \tau)|^2 dy \\ &\quad - \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\underline{w}(y, \tau)|^2 dy. \end{aligned}$$

Applying the operator Δ_{-1} to the first equation of (2.2), we have

$$(3.4) \quad \partial_\tau \Delta_{-1}w = \Delta \Delta_{-1}w - \frac{1}{2} \Delta_{-1}(y \cdot \nabla w) - \frac{1}{2} \Delta_{-1}w - \Delta_{-1}((w \cdot \nabla)w) - \nabla \Delta_{-1}q.$$

Multiplying (3.4) by $\Delta_{-1}w$ and integrating over \mathbb{R}^3 we get

$$(3.5) \quad \begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_{-1}w|^2 dy &= - \int_{\mathbb{R}^3} |\nabla \Delta_{-1}w|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\Delta_{-1}w|^2 dy \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^3} \Delta_{-1}(y \cdot \nabla w) \cdot \Delta_{-1}w dy - \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1}w dy, \end{aligned}$$

where $\operatorname{div} w = 0$ is used to cancel the term including q .

Because

$$\begin{aligned} \int_{\mathbb{R}^3} y \cdot \nabla |\Delta_{-1}w|^2 dy &= 2 \int_{\mathbb{R}^3} y_j \Delta_{-1}w \cdot \partial_j \Delta_{-1}w dy \\ &= 2 \int_{\mathbb{R}^3} \xi_j \varphi \mathcal{F}[w] \cdot \partial_j \overline{(\varphi \mathcal{F}[w])} d\xi \\ &= -3 \int_{\mathbb{R}^3} \varphi^2 |\mathcal{F}[w]|^2 dy, \end{aligned}$$

we have

$$\int_{\mathbb{R}^3} \partial_j \{y_j |\Delta_{-1} w|^2\} dy = 0.$$

So

$$\begin{aligned} & \int_{\mathbb{R}^3} \partial_j (\Delta_{-1}(y_j w) \cdot \Delta_{-1} w) dy \\ &= \int_{\mathbb{R}^3} \partial_j \{ \mathcal{F}^{-1}[\varphi] * (y_j w) \cdot \mathcal{F}^{-1}[\varphi] * w \} dy \\ &= (-1) \int_{\mathbb{R}^3} \partial_j \{ \tilde{\varphi}_j * w \cdot \mathcal{F}^{-1}[\varphi] * w \} dy + \int_{\mathbb{R}^3} \partial_j \{ y_j |\Delta_{-1} w|^2 \} dy \\ &= 0 \end{aligned}$$

where $\tilde{\varphi}_j(y) = y_j \mathcal{F}^{-1}[\varphi](y)$.

Noting that

$$\begin{aligned} & \int \Delta_{-1}(y \cdot \nabla w) \cdot \Delta_{-1} w dy \\ &= - \sum_{j=1}^3 \int \Delta_{-1}(y_j w) \cdot \Delta_{-1} \partial_j w dy - 3 \int |\Delta_{-1} w|^2 dy \\ &= - \sum_{j=1}^3 \int \varphi(\xi) \mathcal{F}[y_j w] \cdot \overline{\varphi(\xi) \mathcal{F}[\partial_j w]} d\xi - 3 \int |\Delta_{-1} w|^2 dy \end{aligned}$$

and

$$\mathcal{F}[y_j w] = i \frac{\partial}{\partial \xi_j} \mathcal{F}[w], \quad \mathcal{F}[\partial_j w] = i \xi_j \mathcal{F}[w],$$

we have

(3.6)

$$\begin{aligned} \int \Delta_{-1}(y \cdot \nabla w) \cdot \Delta_{-1} w dy &= - \sum_{j=1}^3 \int \varphi^2(\xi) \xi_j \frac{\partial}{\partial \xi_j} \mathcal{F}[w] \cdot \overline{\mathcal{F}[w]} d\xi - 3 \int |\Delta_{-1} w|^2 dy \\ &= - \sum_{j=1}^3 \frac{1}{2} \int \varphi^2(\xi) \xi_j \frac{\partial}{\partial \xi_j} |\mathcal{F}[w]|^2 d\xi - 3 \int |\Delta_{-1} w|^2 dy \\ &= \sum_{j=1}^3 \frac{1}{2} \int \xi_j \frac{\partial}{\partial \xi_j} \varphi^2(\xi) |\mathcal{F}[w]|^2 d\xi + \frac{3}{2} \int \varphi^2(\xi) |\mathcal{F}[w]|^2 d\xi - 3 \int |\Delta_{-1} w|^2 dy \\ &= \frac{1}{2} \int \xi \cdot \nabla \varphi^2(\xi) |\mathcal{F}[w]|^2 d\xi - \frac{3}{2} \int |\Delta_{-1} w|^2 dy. \end{aligned}$$

From (3.5)-(3.6), we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_{-1} w|^2 dy = - \int_{\mathbb{R}^3} |\nabla \Delta_{-1} w|^2 dy + \frac{1}{4} \int_{\mathbb{R}^3} |\Delta_{-1} w|^2 dy \\ (3.7) \quad & - \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \varphi^2(\xi) |\mathcal{F}[w](\xi, \tau)|^2 d\xi - \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1} w dy. \end{aligned}$$

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From (3.3) and (3.7), we have

$$\begin{aligned}
 (3.8) \quad & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{w}(y, \tau)|^2 dy \leq - \int_{\mathbb{R}^3} |\nabla \tilde{w}(y, \tau)|^2 - \frac{1}{4} |\tilde{w}(y, \tau)|^2 dy \\
 & + \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \varphi^2(\xi) |\mathcal{F}[w](\xi, \tau)|^2 d\xi + \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1}w dy \\
 & \leq -\frac{3}{4} \int_{\mathbb{R}^3} |\nabla \tilde{w}(y, \tau)|^2 dy \\
 & + \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \varphi^2(\xi) |\mathcal{F}[w](\xi, \tau)|^2 d\xi + \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1}w dy
 \end{aligned}$$

where $|\xi| |\mathcal{F}[\tilde{w}]|^2 \geq |\mathcal{F}[\tilde{w}]|^2$ is used in the last step.

Let $\alpha \in (0, \frac{1}{8})$ and define

$$\chi(\xi) = \begin{cases} |\xi|^{\frac{1}{2}+2\alpha} \varphi(\xi), & \forall |\xi| \leq \frac{1}{2} + \alpha \\ (\frac{1}{2} + \alpha)^{\frac{1}{2}+2\alpha} \varphi(\xi), & \forall |\xi| \geq \frac{1}{2} + \alpha. \end{cases}$$

Instead of φ by χ , we define the operator

$$\tilde{\Delta}_{-1}f = \mathcal{F}^{-1}[\chi(\xi)\mathcal{F}[f](\xi)].$$

Applying $\tilde{\Delta}_{-1}$ to (2.2), as (3.7) we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 dy \\
 & = - \int_{\mathbb{R}^3} |\nabla \tilde{\Delta}_{-1}w|^2 dy + \frac{1}{4} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 dy \\
 & - \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla \chi^2(\xi) |\mathcal{F}[w](\xi, \tau)|^2 d\xi - \int_{\mathbb{R}^3} \tilde{\Delta}_{-1}((w \cdot \nabla)w) \cdot \tilde{\Delta}_{-1}w dy.
 \end{aligned}$$

Combining it with (3.8), we have

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 + |\tilde{w}|^2 dy \\
 & \leq -\frac{3}{4} \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 dy - \alpha \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 dy \\
 & + \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1}w - \tilde{\Delta}_{-1}((w \cdot \nabla)w) \cdot \tilde{\Delta}_{-1}w dy \\
 & + \frac{1}{4} \int_{\mathbb{R}^3} \xi \cdot \nabla (\varphi^2 - \chi^2) |\mathcal{F}[w]|^2 d\xi - \int_{\mathbb{R}^3} |\nabla \tilde{\Delta}_{-1}w|^2 dy + (\frac{1}{4} + \alpha) \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1}w|^2 dy.
 \end{aligned}$$

For $|\xi| \leq 1$, the last term is written as

$$A = \int \left\{ \frac{1}{4} \xi \cdot \nabla (-\chi^2) - |\xi|^2 \chi^2 + (\frac{1}{4} + \alpha) \chi^2 \right\} |\mathcal{F}[w]|^2 d\xi$$

and noting that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ as well as the definition of χ , $A \leq 0$. For $|\xi| \in [1, 2]$, the last term is written as

$$B = \int \left\{ \frac{1}{4} \left(1 - \left(\frac{1}{2} + \alpha \right)^{1+4\alpha} \right) \xi \cdot \nabla \varphi^2 - \left(|\xi|^2 - \left(\frac{1}{4} + \alpha \right) \right) \left(\frac{1}{2} + \alpha \right)^{1+4\alpha} \varphi^2 \right\} |\mathcal{F}[w]|^2 d\xi,$$

and $B \leq 0$. So we get

$$(3.9) \quad \begin{aligned} \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 + |\tilde{w}|^2 dy &\leq -\frac{3}{4} \int_{\mathbb{R}^3} |\nabla \tilde{w}|^2 dy - \alpha \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w|^2 dy \\ &+ \int_{\mathbb{R}^3} \Delta_{-1}((w \cdot \nabla)w) \cdot \Delta_{-1} w - \tilde{\Delta}_{-1}((w \cdot \nabla)w) \cdot \tilde{\Delta}_{-1} w dy. \end{aligned}$$

Lemma 3.1. (1) For any $m \in [4, \infty]$,

$$\|\Delta_{-1} f\|_{L^m(\mathbb{R}^3)} \leq C(\alpha) \|\tilde{\Delta}_{-1} f\|_{L^2(\mathbb{R}^3)}, \quad \forall f \in L^2(\mathbb{R}^3)$$

where the constant $C(\alpha) < \infty$ depends only on α .

(2) For all $\beta = (\beta_1, \beta_2, \beta_3)$ ($\beta_j \in \mathbb{N}$, $j = 1, 2, 3$)

$$\|D^\beta \Delta_0 w(\cdot, \tau)\|_{L^2(\mathbb{R}^3)} \leq \|D^\beta \tilde{w}(\cdot, \tau)\|_{L^2(\mathbb{R}^3)}$$

where $D^\beta = \partial_1^{\beta_1} \partial_2^{\beta_2} \partial_3^{\beta_3}$.

Proof. From Hausdorff-Young inequality

$$\begin{aligned} &\|\Delta_{-1} f\|_{L^m(\mathbb{R}^3)} \\ &\leq (2\pi)^{3/m'} \left(\int_{\mathbb{R}^3} |\varphi(\xi) \mathcal{F}[f](\xi)|^{m'} d\xi \right)^{1/m'} \\ &\leq (2\pi)^{3/m'} \left(\int_{\mathbb{R}^3} \|\xi\|^{\frac{1}{2}+2\alpha} \varphi(\xi) |\mathcal{F}[f](\xi)|^2 \right)^{1/2} \left(\int_{|\xi| \leq 2} |\xi|^{-\left(\frac{1}{2}+2\alpha\right) \frac{2m'}{2-m'}} d\xi \right)^{\frac{2-m'}{2m'}} \\ &\leq C(\alpha) \left(\int_{\mathbb{R}^3} |\chi(\xi) \mathcal{F}[f](\xi)|^2 \right)^{1/2} \end{aligned}$$

for $\alpha \in (0, \frac{1}{8})$. So we have (1).

To prove (2), we only need to consider the case $|\beta| = \sum_{1 \leq j \leq 3} \beta_j = 0$. Since $0 \leq \varphi \leq 1$ and $1 - \varphi^2 = (1 - \varphi)(1 + \varphi) \geq (1 - \varphi)^2$, in this case we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\Delta_0 w(y, \tau)|^2 dy &= \int_{\mathbb{R}^3} (1 - \varphi(\xi))^2 |\mathcal{F}[w](\xi, \tau)|^2 d\xi \\ &\leq \int_{\mathbb{R}^3} (1 - \varphi^2(\xi)) |\mathcal{F}[w](\xi, \tau)|^2 d\xi = \int_{\mathbb{R}^3} |\tilde{w}(y, \tau)|^2 dy. \quad \square \end{aligned}$$

Now we estimate the last term in the right of (3.9). We only need to consider the integration for the first function in the last term, because for another function

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the proof is same. Notice that

$$\begin{aligned} & \int \Delta_{-1}(w_j \partial_j w) \cdot \Delta_{-1} w dy = - \int \Delta_{-1}(w_j w) \cdot \Delta_{-1}(\partial_j w) dy \\ & = - \int \Delta_{-1}(\Delta_{-1} w_j \Delta_{-1} w) \cdot \Delta_{-1}(\partial_j w) dy - \int \Delta_{-1}(\Delta_{-1} w_j \Delta_0 w) \cdot \Delta_{-1} \partial_j w dy \\ & \quad - \int \Delta_{-1}(\Delta_0 w_j \Delta_{-1} w) \cdot \Delta_{-1}(\partial_j w) dy - \int \Delta_{-1}(\Delta_0 w_j \Delta_0 w) \cdot \Delta_{-1}(\partial_j w) dy. \end{aligned}$$

Because

$$\begin{aligned} & \left| \int \Delta_{-1}(\Delta_{-1} w_j \Delta_{-1} w) \cdot \Delta_{-1}(\partial_j w) dy \right| \\ & \leq \left(\int |\Delta_{-1} w|^4 dy \right)^{1/2} \left(\int \varphi^2(\xi) |\xi_j \mathcal{F}[w](\xi, \tau)|^2 dx \right)^{1/2} \\ & \leq C \|\tilde{\Delta}_{-1} w\|_{L^2(\mathbb{R}^3)}^3, \quad (\text{by Lemma 3.1 (1) and the definition of } \tilde{\Delta}_{-1}) \end{aligned}$$

and

$$\begin{aligned} & \left| \int \Delta_{-1}(\Delta_{-1} w_j \Delta_0 w) \cdot \Delta_{-1} \partial_j w dy \right| \\ & \leq \left(\int |\Delta_{-1} w|^4 dy \right)^{1/2} \left(\int |\Delta_0 w|^2 dy \right)^{1/2} \\ & \leq C \|\tilde{\Delta}_{-1} w\|_{L^2(\mathbb{R}^3)}^2 \|\tilde{w}\|_{L^2(\mathbb{R}^3)} \quad (\text{by Lemma 3.1 (1)-(2)}) \end{aligned}$$

as well as

$$\begin{aligned} & \left| \int \Delta_{-1}(\Delta_0 w_j \Delta_0 w) \cdot \Delta_{-1}(\partial_j w) dy \right| \\ & \leq \|\Delta_{-1}(\partial_j w)\|_{L^\infty(\mathbb{R}^3)} \|\Delta_0 w\|_{L^2(\mathbb{R}^3)}^2 \\ & \leq C \|\tilde{\Delta}_{-1} w\|_{L^2(\mathbb{R}^3)} \|\tilde{w}\|_{L^2(\mathbb{R}^3)}^2 \quad (\text{by Lemma 3.1 (1)-(2)}) \end{aligned}$$

the last term in the right of (3.9) can be estimated by

$$C \{ \|\tilde{\Delta}_{-1} w\|_{L^2(\mathbb{R}^3)}^3 + \|\tilde{\Delta}_{-1} w\|_{L^2(\mathbb{R}^3)} \|\tilde{w}\|_{L^2(\mathbb{R}^3)} + \|\tilde{\Delta}_{-1} w\|_{L^2(\mathbb{R}^3)} \|\tilde{w}\|_{L^2(\mathbb{R}^3)}^2 \}.$$

So we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, \tau)|^2 + |\tilde{w}(y, \tau)|^2 dy + \left(\frac{3}{4} - \alpha\right) \int_{\mathbb{R}^3} |\nabla \tilde{w}(y, \tau)|^2 dy \\ (3.10) \quad & \leq -\alpha \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, \tau)|^2 + |\tilde{w}(y, \tau)|^2 dy \\ & \quad + C \left(\int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, \tau)|^2 + |\tilde{w}(y, \tau)|^2 dy \right)^{3/2} \end{aligned}$$

Proposition 3.2. *There is $\delta > 0$ such that if*

$$(3.11) \quad \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, 0)|^2 + |\tilde{w}(y, 0)|^2 dy \leq \delta$$

then for all $\tau > 0$

$$(3.12) \quad \frac{d}{d\tau} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, \tau)|^2 + |\tilde{w}(y, \tau)|^2 dy \leq -\alpha \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, \tau)|^2 + |\tilde{w}(y, \tau)|^2 dy.$$

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Moreover $w(y, \tau) = \underline{w}(y, \tau) + \overline{w}(y, \tau)$, and for all $m \in [4, \infty]$,

$$(3.13) \quad \begin{aligned} \|D^\beta \underline{w}(\tau)\|_{L^m(\mathbb{R}^3)} &\leq C(\beta)\delta, \quad \forall \tau > 0, \quad \forall \beta, \\ \lim_{\tau \rightarrow \infty} \|\underline{w}(\tau)\|_{L^m(\mathbb{R}^3)} &= 0, \end{aligned}$$

$$(3.14) \quad \begin{aligned} \sup_{\tau \geq 0} \int_{\mathbb{R}^3} |\overline{w}(y, \tau)|^2 dy + \int_0^\infty d\tau \int_{\mathbb{R}^3} |\nabla \overline{w}(y, \tau)|^2 dy &\leq C\delta, \\ \lim_{\tau \rightarrow \infty} \int_{\mathbb{R}^3} |\overline{w}(y, \tau)|^2 dy &= 0. \end{aligned}$$

For example, we may take $\delta \leq (\frac{\alpha}{2C})^2$. Proposition 3.2 follows from (3.10) and Lemma 3.1. Note that

$$(3.15) \quad \begin{aligned} \int_{\mathbb{R}^3} |\tilde{\Delta}_{-1} w(y, 0)|^2 + |\tilde{w}(y, 0)|^2 dy &= \int_{\mathbb{R}^3} (\chi^2(\xi) + 1 - \varphi^2(\xi)) |\mathcal{F}[w](\xi, 0)|^2 d\xi \\ &\leq \int_{\mathbb{R}^3} |\mathcal{F}[w](\xi, 0)|^2 d\xi = \int_{\mathbb{R}^3} |w(y, 0)|^2 dy = \int_{\mathbb{R}^3} |u_0(x)|^2 dx. \end{aligned}$$

So we have

Corollary 3.3. *There is $\delta > 0$ such that if $\|u_0\|_{L^2(\mathbb{R}^3)} \leq \delta^{1/2}$, then we have (3.12)-(3.14).*

Remark 3.4. Suppose ψ is a function satisfying

$$(3.16) \quad \psi \in C(\mathbb{R}^3, [0, 1]), \quad \xi \cdot \nabla_\xi \psi(\xi) \in L^\infty(\mathbb{R}^3).$$

Since $\psi(\xi)\mathcal{F}[w](\xi, \tau) \in L^2(\mathbb{R}^3)$, we have $\mathcal{F}^{-1}[\psi] * w = \mathcal{F}^{-1}[\psi\mathcal{F}[w]] \in L^2(\mathbb{R}^3)$ and

$$(3.17) \quad \begin{aligned} &\int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy (T-t)^{\frac{3}{2}} \\ &= \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(\frac{\mu}{(T-t)^{1/2}}, \tau)|^2 d\mu \\ &= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi]\left(\frac{\mu}{(T-t)^{1/2}} - z\right) w(z, \tau) dz \right|^2 d\mu \\ &= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \mathcal{F}[\psi]\left(\frac{\mu}{(T-t)^{1/2}} - z\right) (T-t)^{1/2} u((T-t)^{1/2}z, t) dz \right|^2 d\mu \\ &= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi]\left(\frac{\mu-x}{(T-t)^{1/2}}\right) u(x, t) dx \right|^2 d\mu (T-t)^{-2} \end{aligned}$$

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where $\tau = (-) \ln(T - t)$. Note that

$$\begin{aligned}
 (3.18) \quad & \partial_t \{ (T - t)^{3/2} \psi((T - t)^{1/2} \xi) \mathcal{F}[u](\xi, t) \} \\
 & = \partial_t \mathcal{F} \left[\int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu - x}{(T - t)^{1/2}} \right) u(x, t) dx \right] \\
 & = \mathcal{F} \left[\int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu - x}{(T - t)^{1/2}} \right) \partial_t u(x, t) + \left\{ \frac{\mu - x}{2(T - t)^{3/2}} \cdot \mathcal{F}^{-1}[\psi]' \left(\frac{\mu - x}{(T - t)^{1/2}} \right) \right\} u(x, t) dx \right] \\
 & = \mathcal{F} \left[\int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu - x}{(T - t)^{1/2}} \right) \left\{ \partial_\tau w \left(\frac{x}{(T - t)^{1/2}}, \tau \right) + \frac{x}{2(T - t)^{1/2}} \cdot \nabla_y w \left(\frac{x}{(T - t)^{1/2}}, \tau \right) \right. \right. \\
 & \quad \left. \left. + \frac{1}{2} w \left(\frac{x}{(T - t)^{1/2}}, \tau \right) \right\} (T - t)^{-\frac{3}{2}} \right. \\
 & \quad \left. + \left\{ \frac{\mu - x}{2(T - t)^{3/2}} \cdot \mathcal{F}^{-1}[\psi]' \left(\frac{\mu - x}{(T - t)^{1/2}} \right) \right\} w \left(\frac{x}{(T - t)^{1/2}}, \tau \right) (T - t)^{-\frac{1}{2}} dx \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (3.19) \quad & \frac{d}{dt} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu - x}{(T - t)^{1/2}} \right) u(x, t) dx \right|^2 d\mu \\
 & = \frac{d}{dt} \int_{\mathbb{R}^3} |(T - t)^{3/2} \psi((T - t)^{1/2} \xi) \mathcal{F}[u](\xi, t)|^2 d\xi \\
 & = \frac{d}{dt} \int_{\mathbb{R}^3} \int_0^t \partial_h |(T - h)^{3/2} \psi((T - h)^{1/2} \xi) \mathcal{F}[u](\xi, h)|^2 dh d\xi \\
 & = \frac{d}{dt} \int_0^t \int_{\mathbb{R}^3} \partial_h |(T - h)^{3/2} \psi((T - h)^{1/2} \xi) \mathcal{F}[u](\xi, h)|^2 d\xi dh \\
 & = \int_{\mathbb{R}^3} \partial_t |(T - t)^{3/2} \psi((T - t)^{1/2} \xi) \mathcal{F}[u](\xi, t)|^2 d\xi \\
 & = 2 \int_{\mathbb{R}^3} \left\{ (-) \frac{3}{2} (T - t)^{1/2} \psi((T - t)^{1/2} \xi) \mathcal{F}[u](\xi, t) - (T - t)^{\frac{\xi}{2}} \cdot \psi'((T - t)^{1/2} \xi) \mathcal{F}[u](\xi, t) \right. \\
 & \quad \left. + (T - t)^{3/2} \psi((T - t)^{1/2} \xi) \partial_t \mathcal{F}[u](\xi, t) \right\} \cdot (T - t)^{3/2} \psi((T - t)^{1/2} \xi) \overline{\mathcal{F}[u](\xi, t)} d\xi
 \end{aligned}$$

where noting that $\mathcal{F}[u](\xi, t), \partial_t \mathcal{F}[u](\xi, t) \in L^2(\mathbb{R}^3)$ for $t < T$ and ψ satisfies (3.16), we have

$$\int_{\mathbb{R}^3} |\partial_t |(T - t)^{3/2} \psi((T - t)^{1/2} \xi) \mathcal{F}[u](\xi, t)|^2| d\xi < \infty$$

and Fubini theorem can be used.

From (3.17)-(3.19), we get

$$\begin{aligned}
 (3.20) \quad & (T-t)^{5/2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau) - \frac{7}{2}(T-t)^{5/2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy \\
 & = \frac{d}{dt} \left\{ (T-t)^{7/2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy \right\} \\
 & = \frac{d}{dt} \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu-x}{(T-t)^{1/2}} \right) u(x, t) dx \right|^2 d\mu \\
 & = 2(T-t)^{5/2} \int_{\mathbb{R}^3} \left\{ (-) \frac{3}{2} \psi(\xi) \mathcal{F}[w](\xi, \tau) - \frac{\xi}{2} \cdot \psi'(\xi) \mathcal{F}[w](\xi, \tau) \right\} \cdot \psi(\xi) \overline{\mathcal{F}[w]}(\xi, \tau) d\xi \\
 & + 2 \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu-x}{(T-t)^{1/2}} \right) \partial_t u(x, t) dx \right\} \cdot \left\{ \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] \left(\frac{\mu-x}{(T-t)^{1/2}} \right) u(x, t) dx \right\} d\mu \\
 & = (-3)(T-t)^{5/2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy \\
 & \quad - \frac{(T-t)^{5/2}}{2} \int_{\mathbb{R}^3} (\xi \cdot \nabla_\xi \psi^2(\xi)) |\mathcal{F}[w](\xi, \tau)|^2 d\xi \\
 & + 2(T-t)^{5/2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi](y-z) \left\{ \partial_\tau w(z, \tau) + \frac{z}{2} \cdot \nabla_z w(z, \tau) + \frac{1}{2} w(z, \tau) \right\} dz \\
 & \quad \cdot \{ \mathcal{F}^{-1}[\psi] * w(y, \tau) \} dy
 \end{aligned}$$

So we have

$$\begin{aligned}
 (3.21) \quad & 2 \int_{\mathbb{R}^3} \mathcal{F}^{-1}[\psi] * \left\{ \partial_\tau w + \frac{y}{2} \cdot \nabla_y w \right\} (y, \tau) \cdot \mathcal{F}^{-1}[\psi] * w(y, \tau) dy \\
 & = \frac{d}{d\tau} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy - \frac{3}{2} \int_{\mathbb{R}^3} |\mathcal{F}^{-1}[\psi] * w(y, \tau)|^2 dy \\
 & \quad + \frac{1}{2} \int_{\mathbb{R}^3} (\xi \cdot \nabla_\xi \psi^2(\xi)) |\mathcal{F}[w](\xi, \tau)|^2 d\xi.
 \end{aligned}$$

Note that φ and χ satisfy (3.16), and we can use (3.21) to obtain (3.7) for φ and χ again. Furthermore, notice that $1 - \varphi$ satisfies (3.16) and $\|\partial_t \nabla_x u(t)\|_{L^2(\mathbb{R}^3)}$ is bounded for $t < T$, we can prove the same equation as (3.21) for $(1 - \varphi)$ and $\nabla_y w$ instead of ψ and w , which can be used to obtain (4.4) of section 4 from (4.1) too.

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4. L^∞ -ESTIMATE OF \bar{w}

Applying the operator Δ_0 (see (3.2)) to (2.6), and integrating over \mathbb{R}^3 we have

$$\begin{aligned}
 (4.1) \quad & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy = - \int_{\mathbb{R}^3} |\nabla^2 \Delta_0 w|^2 dy - \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \\
 & - \sum_{j=1}^3 \frac{1}{2} \int_{\mathbb{R}^3} \Delta_0 (y \cdot \nabla \partial_j w) \cdot \Delta_0 \partial_j w dy \\
 & - \sum_{j=1}^3 \int_{\mathbb{R}^3} \Delta_0 ((\partial_j w \cdot \nabla) w) \cdot \Delta_0 \partial_j w + \Delta_0 ((w \cdot \nabla) \partial_j w) \cdot \Delta_0 \partial_j w dy.
 \end{aligned}$$

Since the support set of $1 - \varphi$ is not compact, we can not do the same thing as in (3.6) for the 3rd term in the right side of (4.1). But with more patient, by using $\Delta_0 f = f - \Delta_{-1} f$, we have

$$\begin{aligned}
 (4.2) \quad & \int \Delta_0 ((y \cdot \nabla) \partial_j w) \cdot \Delta_0 \partial_j w dy \\
 & = \int ((y \cdot \nabla) \partial_j w) \cdot \partial_j w dy - \int ((y \cdot \nabla) \partial_j w) \cdot \Delta_{-1} \partial_j w dy \\
 & - \int \Delta_{-1} ((y \cdot \nabla) \partial_j w) \cdot \partial_j w dy + \int \Delta_{-1} ((y \cdot \nabla) \partial_j w) \cdot \Delta_{-1} (\partial_j w) dy.
 \end{aligned}$$

As in (2.4), we have

$$\int ((y \cdot \nabla) \partial_j w) \cdot \partial_j w dy \geq -\frac{3}{2} \int |\partial_j w|^2 dy.$$

On the other hand, as in (3.6) we have

$$\int \Delta_{-1} ((y \cdot \nabla) \partial_j w) \cdot \Delta_{-1} (\partial_j w) dy = \frac{1}{2} \int \xi \cdot \nabla \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi - \frac{3}{2} \int \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi.$$

The remainder in the right of (4.2) is

$$\begin{aligned}
 & 2 \int \varphi \partial_k (\xi_k \mathcal{F}[\partial_j w]) \cdot \overline{\mathcal{F}[\partial_j w]} d\xi \\
 & = - \int 2(\xi \cdot \nabla \varphi) |\mathcal{F}[\partial_j w]|^2 + \varphi \xi \cdot \nabla |\mathcal{F}[\partial_j w]|^2 d\xi \\
 & = - \int (\xi \cdot \nabla \varphi) |\mathcal{F}[\partial_j w]|^2 d\xi + 3 \int \varphi |\mathcal{F}[\partial_j w]|^2 d\xi.
 \end{aligned}$$

Then the right of (4.2) is larger than

$$\begin{aligned}
 (4.3) \quad & -\frac{3}{2} \int |\partial_j w|^2 dy - \frac{3}{2} \int \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi + \frac{1}{2} \int \xi \cdot \nabla \varphi^2 |\mathcal{F}[\partial_j w]|^2 d\xi \\
 & - \int (\xi \cdot \nabla \varphi) |\mathcal{F}[\partial_j w]|^2 d\xi + 3 \int \varphi |\mathcal{F}[\partial_j w]|^2 d\xi \\
 & = \frac{1}{2} \int \xi \cdot \nabla (1 - \varphi(\xi))^2 |\mathcal{F}[\partial_j w]|^2 d\xi - \frac{3}{2} \int |\Delta_0 \partial_j w|^2 dy.
 \end{aligned}$$

Since from (3.1)

$$\xi \cdot \nabla(1 - \varphi(\xi))^2 = |\xi| \frac{d}{d|\xi|} (1 - \varphi(\xi))^2 \geq 0$$

Instead of the 3rd term in the right side of (4.1) by (4.2)-(4.3), we get

$$(4.4) \quad \begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \leq - \int_{\mathbb{R}^3} |\nabla^2 \Delta_0 w|^2 dy - \frac{1}{4} \int_{\mathbb{R}^3} |\Delta_0 \nabla w|^2 dy \\ & - \sum_{j=1}^3 \int_{\mathbb{R}^3} \Delta_0 ((\partial_j w \cdot \nabla) w) \cdot \Delta_0 \partial_j w + \Delta_0 ((w \cdot \nabla) \partial_j w) \cdot \Delta_0 \partial_j w dy. \end{aligned}$$

Decompose the last integration of the right side of (4.4) by $w = \underline{w} + \bar{w}$ and note that

$$\begin{aligned} | \int ((\bar{w} \cdot \nabla) \partial_j \bar{w}) \cdot \partial_j \bar{w} dy | & \leq \| \nabla^2 \bar{w} \|_{L^2(\mathbb{R}^3)} \left(\int |\bar{w}|^2 |\nabla \bar{w}|^2 dy \right)^{1/2} \\ & \leq C \| \nabla^2 \bar{w} \|_{L^2(\mathbb{R}^3)}^{3/2} \| \nabla \bar{w} \|_{L^2(\mathbb{R}^3)}^{3/2}, \\ | \int ((\underline{w} \cdot \nabla) \partial_j \bar{w}) \cdot \partial_j \bar{w} dy | & \leq \| \underline{w} \|_{L^\infty(\mathbb{R}^3)} \| \nabla^2 \bar{w} \|_{L^2(\mathbb{R}^3)} \| \nabla \bar{w} \|_{L^2(\mathbb{R}^3)}, \\ | \int ((\bar{w} \cdot \nabla) \partial_j \underline{w}) \cdot \partial_j \bar{w} dy | & \leq C \| \underline{w} \|_{L^\infty(\mathbb{R}^3)} \| \bar{w} \|_{L^2(\mathbb{R}^3)} \| \nabla \bar{w} \|_{L^2(\mathbb{R}^3)}, \end{aligned}$$

and

$$| \int ((\underline{w} \cdot \nabla) \partial_j \underline{w}) \cdot \partial_j \bar{w} dy | \leq C \left(\int |\underline{w}|^4 dy \right)^{1/2} \| \nabla \bar{w} \|_{L^2(\mathbb{R}^3)}$$

as well as the same estimates for another one. Then by Proposition 3.2, we have

$$(4.5) \quad \begin{aligned} & \frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla \bar{w}(y, \tau)|^2 dy \leq - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla^2 \bar{w}(y, \tau)|^2 dy - \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \bar{w}(y, \tau)|^2 dy \\ & + C \| \nabla \bar{w}(\tau) \|_{L^2(\mathbb{R}^3)} \{ C\delta - \| \nabla \bar{w}(\tau) \|_{L^2(\mathbb{R}^3)} + C \| \nabla \bar{w}(\tau) \|_{L^2(\mathbb{R}^3)}^5 \} \end{aligned}$$

Note that (see Remark 4.4) there is $\delta_1 > 0$ such that if for some $\tau_0 \geq 0$

$$(4.6) \quad \| \nabla \bar{w}(\tau_0) \|_{L^2(\mathbb{R}^3)} \leq \delta_1$$

then

$$\| \nabla \bar{w}(\tau) \|_{L^2(\mathbb{R}^3)} \leq \delta_1, \quad \forall \tau \geq \tau_0.$$

From (3.14), (4.6) can be satisfied provided that (3.11) is satisfied. So we have

Lemma 4.1. *Suppose (3.11) is satisfied. Then there is $\delta_1 > 0$ ($\delta_1 \downarrow 0$ as $\delta \downarrow 0$) and $\tau_0 > 0$ such that*

$$\| \nabla \bar{w}(\tau) \|_{L^2(\mathbb{R}^3)} \leq \delta_1, \quad \forall \tau \geq \tau_0.$$

Estimate the last term in the right side of (2.7) by using $w = \underline{w} + \bar{w}$, and note that

$$\begin{aligned} | \int \partial_j \underline{w}_k \partial_j \underline{w}_l \partial_l \underline{w}_k dy | & \leq \| \nabla \underline{w} \|_{L^\infty(\mathbb{R}^3)} \int |\nabla \underline{w}|^2 dy \\ & \leq C\delta \int |\nabla \underline{w}|^2 dy, \end{aligned}$$

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$$\begin{aligned} \left| \int \partial_j \underline{w}_k \partial_j \underline{w}_l \partial_l \bar{w}_k dy \right| &\leq \left(\int |\nabla \underline{w}|^4 dy \right)^{1/2} \left(\int |\nabla \bar{w}|^2 dy \right)^{1/2} \\ &\leq C\delta \left(\int |\nabla \bar{w}|^2 dy \right)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \left| \int \partial_j \underline{w}_k \partial_j \bar{w}_l \partial_l \bar{w}_k dy \right| &\leq \|\nabla \underline{w}\|_{L^\infty(\mathbb{R}^3)} \int |\nabla \bar{w}|^2 dy \\ &\leq C\delta \int |\nabla \bar{w}|^2 dy, \end{aligned}$$

as well as

$$\begin{aligned} \left| \int \partial_j \bar{w}_k \partial_j \bar{w}_l \partial_l \bar{w}_k dy \right| &\leq C \|\nabla \bar{w}\|_{L^2(\mathbb{R}^3)}^{3/2} \|\nabla^2 \bar{w}\|_{L^2(\mathbb{R}^3)}^{3/2} \\ &\leq \|\nabla^2 \bar{w}\|_{L^2(\mathbb{R}^3)}^2 + C\delta_1. \end{aligned}$$

So we have

$$(4.7) \quad \frac{d}{d\tau} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy \leq - \int_{\mathbb{R}^3} |\nabla^2 w(y, \tau)|^2 dy - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy + C\delta_1.$$

Lemma 4.2. *Suppose (3.11) is satisfied. Then there is $\delta_1 > 0$ ($\delta_1 \downarrow 0$ as $\delta \downarrow 0$) and $\tau_0 > 0$ such that for all $\tau \geq \tau_0$,*

$$(4.8) \quad \int_{\mathbb{R}^3} |\nabla w(y, \tau)|^2 dy \leq e^{-\frac{1}{2}(\tau-\tau_0)} \int_{\mathbb{R}^3} |\nabla w(y, \tau_0)|^2 dy + 2C\delta_1(1 - e^{-\frac{1}{2}(\tau-\tau_0)}).$$

Considering (2.8), and noting that $\operatorname{div} \Delta w = 0$ implies

$$\int (\Delta w) \cdot ((w \cdot \nabla) \Delta w) dy = 0,$$

the last term in the right side of (2.8) can be written as the sum of the following terms

$$\int |\nabla^2 w|^2 |\nabla w| dy.$$

Since it can be estimated by

$$\begin{aligned} &\left(\int |\nabla w|^2 dy \right)^{1/2} \left(\int |\nabla^2 w|^4 dy \right)^{1/2} \\ &\leq C \left(\int |\nabla w|^2 dy \right)^{1/2} \left(\int |\Delta w|^2 dy \right)^{1/4} \left(\int |\nabla \Delta w|^2 dy \right)^{3/4} \end{aligned}$$

by (2.8) and Lemma 4.2 we have

Lemma 4.3. *Suppose (3.11) is satisfied. Then there is $\delta_1 > 0$ ($\delta_1 \downarrow 0$ as $\delta \downarrow 0$) and $\tau_0 > 0$ such that for all $\tau \geq \tau_0$,*

$$\int_{\mathbb{R}^3} |\Delta w(y, \tau)|^2 dy \leq e^{-(\frac{3}{2}-C\delta_1)(\tau-\tau_0)} \int_{\mathbb{R}^3} |\Delta w(y, \tau_0)|^2 dy$$

From Lemma 4.1-4.3 and Corollary 3.3, we proved the Theorem 1.1.

Remark 4.4. Suppose a nonnegative continuous function $h(\tau)$ satisfies

$$\frac{d}{d\tau}h(\tau) \leq F(h(\tau)) := C\delta - Bh(\tau) + h^5(\tau), \quad \forall \tau > 0,$$

where C , B and δ are positive constants. If δ is small enough so that

$$h_- := \frac{1}{2}(B - \sqrt{B^2 - 4C\delta}) \in (0, 1),$$

and if $h(0) < h_-$, then for all $\tau > 0$, $F(h(\tau)) \leq C\delta - Bh(\tau) + h^2(\tau)$ and

$$h(\tau) \in [0, h_-].$$

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