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On Leray's Self-Similar Solutions of the Navier-Stokes Equations Satisfying Local Energy Estimates

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Abstract

This paper proves that Leray's self-similar solutions of the three-dimensional Navier-Stokes equations must be trivial under very general assumptions, for example, if they satisfy local energy estimates.

1. Introduction

In 1934 LERAY [Le] raised the question of the existence of self-similar solutions of the Navier-Stokes equations. For a long time, self-similar solutions had appeared to be good candidates for constructing singular solutions of the Navier-Stokes equations. LERAY's question was unanswered until 1995, when NEČAS, RŮŽIČKA, & ŠVERÁK [NRS] showed, among other things, that the only self-similar solution satisfying the global energy estimates is zero. Although they answered LERAY's original problem, some important questions were left open. For example, can a self-similar solution satisfying *local* energy estimates exist? The goal of this paper is to show that the self-similar solutions must be zero under very general assumptions, for example, if they satisfy the local energy estimates.

For the Navier-Stokes equations

$$\left.\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{aligned}\right\} \quad \text{in } \mathbf{R}^3 \times (t_1, t_2) \tag{1.1}$$

with $\nu > 0$, LERAY's (backward) self-similar solutions are of the form

$$\begin{gathered} u(x,t) = \lambda(t)\; U(\lambda(t)x), \quad p(x,t) = \lambda^2(t) P(\lambda(t)x), \\ \text{with } \lambda(t) = \frac{1}{\sqrt{2a(T-t)}}, \end{gathered} \tag{1.2}$$

where $a > 0$, $U(y) = (U_1, U_2, U_3)(y)$ and $P(y)$ are defined in $\mathbf{R}^3$. We also require that certain natural energy norms of u be finite. (Otherwise there exist nontrivial solutions. See Remark 5.4.) The Navier-Stokes equations for u give the system

$$\left.\begin{array}{c} -\nu \Delta U + aU + a\,(y \cdot \nabla)\,U + (U \cdot \nabla)\,U + \nabla P = 0 \\ \operatorname{div} U = 0 \end{array}\right\} \quad \text{in } \mathbf{R}^3 \tag{1.3}$$

for U. As suggested by LERAY, a nonzero U would produce a solution u of (1.1) with a singularity at $(0, T)$. This would give a counterexample to the open question whether a solution of (1.1) satisfying natural energy estimates can develop a singularity.

In addition to yielding particular singular solutions, the study of self-similar solutions seems to be important also from a more general point of view. It is related to the scaling property of (1.1), the fact that if $u\,(x, t)$ satisfies (1.1), then so do the rescaled functions

$$u_r\,(x, t) := ru\left(rx, r^2\,(t - T) + T\right)$$

for each $r > 0$. If $(0, T)$ is a singular point, then the asymptotics of the singularity is encoded in the behavior of u_r as $r \to 0^+$. If u_r converges to a limit $\bar{u}$, the limit $\bar{u}$ must be self-similar, i.e. $(\bar{u})_r = \bar{u}$ for all $r > 0$, which implies that $\bar{u}$ is of the form (1.2). Of course, more complicated singularities may possibly exist. The study of self-similar solutions has proved to be very useful in the investigations of singularities of many equations with similar scaling properties, such as the harmonic map heat flow, semilinear heat equations, and nonlinear Schrödinger equations; see for example [Str1, GK, KL]. It is hoped that the study of (1.3) can shed some light on the regularity question for the Navier-Stokes equations.

The known regularity criteria for Navier-Stokes equations (such as [Se; FJR; vW, p 190; Gi; Str2; Ta; CF]) do not apply to self-similar singularities (unless certain quantities are small). The main result of [NRS] is that the only weak solution of (1.3) belonging to $L^3\left(\mathbf{R}^3\right)$ is $U \equiv 0$. Also see [MNPS] (who showed the same conclusion under a stronger assumption, but without using results from [CKN]). The L^3 integrability condition holds if the corresponding solution u of the Navier-Stokes equations satisfies the *global* energy estimates

$$\int_{\mathbf{R}^3} \tfrac{1}{2}|u(x,t)|^2 dx + \int_{t_1}^{t} \int_{\mathbf{R}^3} \nu |\nabla u(x,t)|^2 dx dt \leqq \int_{\mathbf{R}^3} \tfrac{1}{2}|u(x,t_1)|^2 dx$$

for all $t \in (t_1, t_2)$. On the other hand, if we only assume the *local* energy estimates

$$\operatorname*{ess\,sup}_{t_3 < t < T} \int_B \tfrac{1}{2}|u(x,t)|^2 dx + \int_{t_3}^{T} \int_B \nu |\nabla u(x,t)|^2\, dx\, dt < \infty \tag{1.4}$$

for some ball B and some $t_3 < T$, then we only get estimates of some weighted norms which do not imply that $U \in L^3$. (See Section 4 for more details.) Therefore, [NRS] left open the existence of self-similar singularities which satisfy the local energy estimates. For example, a solution with the following decay was not excluded:

$$U(y) = A\left(\frac{y}{|y|}\right)\frac{1}{|y|} + o\left(\frac{1}{|y|}\right) \quad \text{as } y \to \infty, \tag{1.5}$$

where $A : S^2 \to \mathbf{R}^3$ is smooth. At the same time, this seems to be a very natural candidate for a self-similar singularity: the function u given by $(1.2)_1$ satisfies the local energy estimates, and $u(x,t) \to u_T(x)$ as $t \to T^-$, where $u_T(x) = A(x/|x|)/|x|$ is homogeneous of degree -1. We might speculate that after the blowup time T, u would become a forward (or defocusing) self-similar solution (the existence of such solutions was studied in [GM, CP]), providing a rather nice interior singularity. Very recently a solution of this type was constructed for the harmonic map heat flow and other equations by Angenent, Ilmanen, & Velázquez [AIV]. (To make an analogy between the Navier-Stokes equations and the harmonic map heat flow, we should compare the velocity with the gradient of the solution of the harmonic map heat flow.) In addition, as suggested in [CKN], the blowup rate of a singularity of u at $(0, T)$ is (at least in "parabolic average")

$$|u(x,t)| \geqq \frac{C}{|x| + \sqrt{T-t}}, \tag{1.6}$$

which is satisfied by a solution u given by (1.2) if U has the decay (1.5).

In this paper we exclude the possibility of such self-similar singularities. In fact, we prove

Theorem 1. *If a weak solution U of* (1.3) *belongs to $L^q(\mathbf{R}^3)$, for some $q \in (3, \infty]$, then it must be constant* (*and hence identically zero if $q < \infty$*).

Theorem 2. *Suppose u is a weak solution of* (1.1) *satisfying the local energy estimates* (1.4) *in the cylinder $Q_1(0,T) = B_1(0) \times (T-1, T)$. If u is of the form* $(1.2)_1$, *then u is identically zero.*

We refer the reader to Section 2 for the definitions of weak solutions. A particular corollary of these results is that a weak solution U of (1.3) with the decay (1.5) must be zero.

Let us explain the main idea of the proof. We recall from [NRS] that the smooth function

$$\Pi(y) = \tfrac{1}{2}|U(y)|^2 + P(y) + ay \cdot U(y)$$

satisfies

$$-\nu\Delta\Pi(y) + (U(y) + ay)\cdot\nabla\Pi(y) = -\nu|\Omega(y)|^2 \leqq 0, \tag{1.7}$$

where $\Omega = \operatorname{curl} U$. (Hence Π satisfies the maximal principle.) One of the crucial steps in [NRS] was to show that

$$\Pi(y) = o(1) \quad \text{as } y \to \infty. \tag{1.8}$$

They achieved this by showing that

$$U(y) = O\big(|y|^{-3}\big), \quad P\big(y\big) = O\big(|y|^{-2}\big) \quad \text{as } y \to \infty$$

under the assumption that $U \in L^3(\mathbf{R}^3)$. Using their method in the case $U \in L^q(\mathbf{R}^3)$, $3 < q < 9$, we only get weaker estimates

$$U(y) = O\big(|y|^{-1}\big), \quad P\big(y\big) = O\big(|y|^{-2+\sigma}\big) \quad \text{as } y \to \infty \tag{1.9}$$

for arbitrary small $\sigma > 0$, from which (1.8) does not follow. This seems to be a serious obstacle for generalizing the method used in [NRS] to $q > 3$.

Our key observation is that (1.9) and, in fact, even a much weaker condition

$$U\big(y\big) = o\big(|y|\big), \quad P\big(y\big) = O\big(|y|^N\big) \quad \text{as } y \to \infty, \tag{1.10}$$

for some finite N (no matter how large), is sufficient to imply that Π is constant. Heuristically, for the differential equation

$$\Delta v(y) = y \cdot \nabla v(y), \tag{1.11}$$

the right-hand side is a "magnifying force" (cf. the 1-dimensional case: $v'' = xv'$). Therefore, a solution should either be constant, or grow unboundedly. In addition, in $\mathbf{R}^3$, a radial solution $v(y) = \phi(r)$, $r = |y|$, satisfies

$$\phi'(r) = c \cdot \frac{1}{r^2} \exp\left(\frac{r^2}{2}\right), \tag{1.12}$$

which suggests that a nonconstant solution Π blows up at the same rate as ϕ. This also suggests that we can find suitable comparison functions with very fast growth. Based on this observation, we will prove a Liouville-type lemma which implies that Π is constant under the assumption (1.10) in Section 5. In the other part of this paper, we establish the estimates (1.10).

It should be emphasized that Theorem 2 is purely *local* in the sense that we do not impose any boundary condition on u. This is related to one special aspect of our analysis. In the study of Navier-Stokes equations, the pressure is usually considered as a "global" term and can be difficult to deal with. See, for example, [CKN, SvW, Str2, LL, HW]. When there is no boundary assumption on u, in general one cannot obtain a "desired" estimate of p for every weak solution u. In our analysis, we overcome this difficulty by making use of the self-similarity: the *local* estimate of p corresponds to a *global* estimate of P, which we obtain by applying certain results on singular integrals on the space *BMO* and on some weighted spaces with the weights in the class A_p. In particular, we are able to show that every self-similar weak solution u in Theorem 2 is a "suitable weak solution" in the sense of [CKN], and then apply the partial regularity result in [CKN] to obtain (1.10).

Our plan for this paper is as follows. In Section 2 we recall the definitions and prove some results about the pressure. In Section 3 we prove the growth estimates (1.10) for Theorem 1, using the representation formula for the Stokes system and certain results from harmonic analysis. In Section 4 we prove the estimates (1.10) for Theorem 2, using A_p weights and a variant of Proposition 2 from [CKN]. In Section 5 we prove the Liouville-type lemma and conclude Theorems 1 and 2. To understand quickly the main idea, the reader may just assume (1.10) and go directly to Section 5.

2. Preliminaries

This section establishes notational conventions and some definitions. First we discuss the notation. We use Q_r to denote the parabolic cylinders

$$Q_r\,(x,T) = B_r\,(x) \times \left(T - r^2, T\right).$$

We also use the summation convention. We write $u_{i,j} = \partial_j u_i = \partial u_i / \partial x_j$, and use the letters C and c to denote generic constants which may change from line to line.

As far as the definitions of solutions of the Navier-Stokes equations (1.1) are concerned, we refer the reader to [CKN] for the concept of *Leray-Hopf weak solutions* [Le, Ho], and the concept of *suitable weak solutions* [Sch1, CKN]. Also see [CFo, Te, vW]. We recall that the main ingredients of the definition of suitable weak solutions in parabolic cylinders Q_r are

(i) the equation (1.1), interpreted in the sense of distributions,
(ii) the local energy estimates (1.4),
(iii) the assumption that the pressure p belong to $L^{5/4}(Q_r)$, and
(iv) the generalized energy inequality (see [CKN] p. 779).

When we consider Leray-Hopf weak solutions, only (i), (ii) and an energy inequality (instead of (iv)) are required. Among these conditions, (iv) is satisfied by any smooth solution of (1.1). Also, (iii) can be derived from (i), (ii), and (iv) if we impose a 0-boundary condition on u. See [SvW, LL]. However, in general (iii) is not a result of (i), (ii), and (iv), as can be seen from this example. We use SERRIN's idea [Se1, p. 187] and consider the vector field u defined by

$$u\,(x,t) = (T-t)^s\,(1,0,0)\,, \quad 0 < s < \tfrac{1}{5};$$

u is a Leray-Hopf weak solution in $Q_1(0,T)$ (with non-homogeneous boundary condition). (In fact, u is a Leray-Hopf weak solution in $B_1(0) \times (T-1, T+1)$ if we extend u to $B_1(0) \times (T, T+1)$ by zero.) However, since

$$p\,(x,t) = s\,(T-t)^{s-1}\,x_1 + \text{const.},$$

p does not belong to $L^{5/4}(Q_1)$ and hence (u, p) is not a suitable weak solution in Q_1. We remark that, in Theorem 2, we do not require the weak solution u to be a Leray-Hopf weak solution. Our only requirements (apart from self-similarity) are (i) and (ii): the Navier-Stokes equations and the local energy estimates.

A function U is called a *weak solution* of (1.3) if $U = (U_1, U_2, U_3) \in W^{1,2}_{\text{loc}}\left(\mathbf{R}^3\right)$, div $U = 0$, and

$$\int_{\mathbf{R}^3} \left(\nu \nabla U \cdot \nabla\varphi + [aU + a\,(y\cdot\nabla)\,U + (U\cdot\nabla)\,U]\cdot\varphi\right) dx = 0$$

for all $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C_c^\infty(\mathbf{R}^3)$, div $\varphi = 0$. By standard regularity theory of stationary Navier-Stokes equations, every weak solution U of (1.3) is actually smooth (see, for example, [Ga II, GiM, La, Te]).

Regarding the pressure, in the definition of a weak solution U, we do not require the specification of the pressure function. On the other hand, given a weak solution U, one can always locally define P such that (U, P) satisfies (1.3) in bounded regions. (For example, we can apply the results for Stokes system from [Ga I, p. 180], with the body force $f = aU + a\,(y \cdot \nabla)\,U + (U \cdot \nabla)\,U$.) In any given connected region, P is unique up to a constant. Therefore we can define a pressure P_R in each ball $B_R\,(0)$ and make them agree with each other by specifying $\int_{B_1} P_R = 0$ (or a fixed constant). In this way we can define P globally. Since U is smooth, P is also smooth.

Another way to define P is by considering its equation. We take the divergence of (1.3) and formally deduce that

$$-\Delta P = \sum \partial_i \partial_j (U_i U_j). \tag{2.1}$$

If $2 < q < \infty$, we can define

$$\widetilde{P} = \sum R_i R_j (U_i U_j) \tag{2.2}$$

(cf. [NRS]), where R_j, $j = 1, 2, 3$, are the classical Riesz transforms (see for example [St1]). We will show that $\widetilde{P}$ is smooth and differs from P by a constant. We can also define $\widetilde{P}$ by (2.2) for the case $q = \infty$. In that case, $\widetilde{P}$ is to be understood as a *BMO* function and is defined by duality; see [JN; FS; St2, p. 156]. We are interested in $\widetilde{P}$ since we can get a global control of $\widetilde{P}$ which will give us the desired local control of P. For clarification, we formulate

Lemma 2.1. *Let a weak solution U of* (1.3) *belong to $L^q\left(\mathbf{R}^3\right)$, $2 < q \leqq \infty$. Let $\widetilde{P}$ be defined by* (2.2)*. Then $\widetilde{P}$ satisfies* (2.1) *in the distributional sense, and hence is smooth. Moreover, we have*

$$\|\widetilde{P}\|_{q/2} \leqq C\|U\|_q^2 \quad \text{if } 2 < q < \infty, \tag{2.3}$$

$$\|\widetilde{P}\|_{BMO} \leqq C\|U\|_\infty^2 \quad \text{if } q = \infty. \tag{2.4}$$

Finally, if P is defined as earlier in this section, then $P - \widetilde{P}$ is constant if $2 < q < \infty$, and affine if $q = \infty$.

Proof. To show that $\widetilde{P}$ is a distributional solution of (2.1), we have to show that

$$-\int \widetilde{P}\Delta\varphi = -\int R_i R_j (U_i U_j)\Delta\varphi = \int U_i U_j \partial_i \partial_j \varphi \tag{2.5}$$

for all $\varphi \in C_c^\infty\left(\mathbf{R}^3\right)$. We recall that $R_i R_j$ are self-adjoint in $L^2(\mathbf{R}^3)$ (this can be checked easily by Fourier transforms; cf. [St1, p. 58]), and we have

$$R_i R_j \Delta\varphi = -\partial_i \partial_j \varphi$$

(see [St1, p. 59]). Therefore, for the case $2 < q < \infty$, (2.5) can be established for $U \in C_c^\infty \subset L^2$. We then extend this result to general $U \in L^q$ by approximation. For the case $q = \infty$, we first observe that all integrals in (2.5) converge absolutely since φ has compact support and $\widetilde{P} \in BMO \subset L^r_{\text{loc}}$ for all $r < \infty$. We next observe that $\Delta\varphi$ is in the Hardy space $\mathscr{H}^1(\mathbf{R}^3)$ (since $\Delta\varphi \in C_c^\infty$ and has mean value zero) and, since $\widetilde{P}$ is defined by duality (see [St2, p. 156]), we get

$$-\int \widetilde{P}\Delta\varphi = -\int U_i U_j \left(R_i R_j \Delta\varphi\right) = \int U_i U_j \partial_i \partial_j \varphi.$$

This shows that $\widetilde{P}$ is a distributional solution of (2.1).

We remark that, by definition and the fact that $\Delta\varphi \in C_c^\infty$ with mean value zero, $R_i R_j \Delta\varphi$ remains the same independently of whether $\Delta\varphi$ is considered as an L^2 function or as an $\mathscr{H}^1$ function.

Since U is smooth and $\widetilde{P}$ satisfies (2.1) in the distributional sense, $\widetilde{P}$ is also smooth by Weyl's lemma. Estimate (2.3) now follows from [CZ]. Equation (2.4) is due to SPANNE, PEETRE, & STEIN, (see [St2, p. 191] for references). See also [St1,2].

Finally, we show the last assertion by modifying the argument of [NRS, Lemma 3.1]. Let

$$F = -\nu\Delta U + aU + a\,(y\cdot\nabla)\,U + (U\cdot\nabla)\,U + \nabla\widetilde{P}. \tag{2.6}$$

We know that $F = \nabla\widetilde{P} - \nabla P$; hence $\Delta F = 0$ in $\mathbf{R}^3$ since both P and $\widetilde{P}$ satisfy (2.1). Therefore F is analytic. We now assert that $D^\alpha F(0) = 0$ for each α in case $2 < q < \infty$, and for each α with $|\alpha| \geqq 1$ in case $q = \infty$. (Clearly this is enough to conclude that F is respectively 0 or constant by its analyticity.) To prove this assertion, we note that, since $\Delta D^\alpha F = 0$, for every radial function $\varphi \in C_c^\infty$ with $\int \varphi = 1$, we have

$$D^\alpha F(0) = (-1)^{|\alpha|} \int_{\mathbf{R}^3} F(y)\,\varepsilon^{3+|\alpha|}\left(D^\alpha\varphi\right)(\varepsilon y)\,dy$$

([St1, p. 275]). We claim that, as $\varepsilon \to 0$, all terms obtained by substituting (2.6) into the above integral converge to zero. Since the proof is similar to that in [NRS], we only give an illustration and show how to deal with the terms involving $(y\cdot\nabla)\,U$ and $\nabla\widetilde{P}$ in the case $q = \infty$.

For the term involving $(y\cdot\nabla)\,U$,

$$\begin{aligned}
&\left|\int_{\mathbf{R}^3} (y\cdot\nabla)\,U\varepsilon^{3+|\alpha|}\left(D^\alpha\varphi\right)(\varepsilon y)\,dy\right| \\
&\quad= \left|-\varepsilon^{3+|\alpha|}\int_{\mathbf{R}^3} 3U\left(D^\alpha\varphi\right)(\varepsilon y)\,dy - \varepsilon^{3+|\alpha|}\int_{\mathbf{R}^3} U\varepsilon y_j\left(\partial_j D^\alpha\varphi\right)(\varepsilon y)\,dy\right| \\
&\quad\leqq 3\varepsilon^{3+|\alpha|}\|U\|_\infty\left(\int_{\mathbf{R}^3}\left|\left(D^\alpha\varphi\right)(\varepsilon y)\right|dy + \int_{\mathbf{R}^3}\left|\varepsilon y_j\left(\partial_j D^\alpha\varphi\right)(\varepsilon y)\right|dy\right) \\
&\quad= 3\varepsilon^{3+|\alpha|}\|U\|_\infty\left(\int_{\mathbf{R}^3}\left|\left(D^\alpha\varphi\right)(z)\right|\varepsilon^{-3}dz + \int_{\mathbf{R}^3}\left|z_j\left(\partial_j D^\alpha\varphi\right)(z)\right|\varepsilon^{-3}dz\right).
\end{aligned}$$

Therefore this term goes to zero if $|\alpha| \geqq 1$. (If $q < \infty$, we use the Hölder inequality in the third line and get the same conclusion for all α.)

Next we consider the term involving $\nabla \widetilde{P}$:

$$\int_{\mathbf{R}^3} \nabla \widetilde{P} \varepsilon^{3+|\alpha|} \left(D^\alpha \varphi \right) (\varepsilon y) \, dy = -\varepsilon^{3+|\alpha|+1} \int_{\mathbf{R}^3} \widetilde{P} \left(\operatorname{div} D^\alpha \varphi \right) (\varepsilon y) \, dy.$$

We notice that $(\operatorname{div} D^\alpha \varphi)(\varepsilon y)$ has mean value zero and compact support. Therefore it belongs to the Hardy space $\mathscr{H}^1(\mathbf{R}^3)$ and

$$\| \left(\operatorname{div} D^\alpha \varphi \right) (\varepsilon y) \|_{\mathscr{H}^1} = \varepsilon^{-3} \| \left(\operatorname{div} D^\alpha \varphi \right) (y) \|_{\mathscr{H}^1}.$$

By the *BMO*-$\mathscr{H}^1$ pairing we have

$$\left| \int_{\mathbf{R}^3} \nabla \widetilde{P} \varepsilon^{3+|\alpha|} \left(D^\alpha \varphi \right) (\varepsilon y) \, dy \right| \leqq \varepsilon^{3+|\alpha|+1} \| \widetilde{P} \|_{BMO} \cdot \varepsilon^{-3} \| \left(\operatorname{div} D^\alpha \varphi \right) (y) \|_{\mathscr{H}^1}.$$

Hence this term goes to zero for each α.

We have shown that F is some constant vector c ($c = 0$ if $q < \infty$). Hence $\nabla \left(\widetilde{P} - P \right) = c$. Therefore $\left(\widetilde{P} - P \right)(y) = \left(\widetilde{P} - P \right)(0) + c \cdot y$. The proof is complete. □

Since adding a constant to P does not have any effect, we can assume that $P = \widetilde{P}$ in the case $2 < q < \infty$. We remark that, in Section 4, we will again define $\widetilde{P}$ by (2.2) for the case when U is in certain weighted L^p space involving A_p-weights.

3. Growth Estimates: Theorem 1

In this section we establish the growth estimates for Theorem 1. We first derive a local gradient estimate for U. Then we use a bootstrap argument to obtain the polynomial growth of the pressure at infinity. Finally we use Green's representation formula for the Stokes system to improve the growth estimate of U. We remark that obtaining the local estimate of ∇U requires certain weak local control of U and P. We obtain this weak control of P by considering $\widetilde{P}$ given by (2.2). The global control of U gives us a (weak) *global* control of $\widetilde{P}$, which we then use to obtain a *local* control of $\widetilde{P}$ and P. It is also possible to obtain local estimates of P in terms of local norms of U. We will discuss related estimates of the Stokes system in a forthcoming paper [ST].

In this section, we work on balls $B = B_\rho(y_0)$ with center $y_0 \in \mathbf{R}^3$ and radius $\rho < 10$.

3.1. Gradient Estimate

In this subsection we prove

Lemma 3.1. *Let U be a weak solution of* (1.3)*. If $U \in L^q$, $3 \leqq q < \infty$, then*

$$\|\nabla U\|_{2,B_1(y_0)} + \|U\|_{6,B_1(y_0)} = o\left(|y_0|^{1/2} \right) \quad \textit{as} \quad |y_0| \to \infty.$$

Proof. We define P by (2.2). For a given center y_0, let ϕ be a cut-off function with compact support in $B_2(y_0)$, $\phi = 1$ in $B_1(y_0)$, $|\nabla\phi| + |\nabla^2\phi| < 20$. We take the dot product of (1.3) with ϕU and then integrate. Since U is smooth and $-U \cdot \Delta U = |\nabla U|^2 - \frac{1}{2}\Delta U^2$, we get

$$\int \nu\phi|\nabla U|^2 dy$$
$$= \int \left[\Delta \frac{\nu U^2}{2} - aU^2 - (ay + U) \cdot \nabla \frac{U^2}{2} - \nabla P \cdot U\right] \phi dy$$
$$= \int \tfrac{1}{2}\nu U^2 \Delta\phi - \int aU^2\phi + \int \tfrac{1}{2}U^2 (ay + U) \cdot \nabla\phi + \int \tfrac{3}{2}aU^2\phi + \int PU \cdot \nabla\phi.$$

Hence

$$\int_{B_1} \nu|\nabla U|^2 dy \leqq C \int_{B_2} \left[U^2 + |y_0|U^2 + |U|^3 + |PU|\right]. \tag{3.1}$$

Since $\|U\|_{3,B_2}$ and $\|P\|_{3/2,B_2}$ tend to zero as y_0 goes to infinity (we recall that P is defined by (2.2)), we conclude that

$$\|\nabla U\|_{2,B_1} = o\left(|y_0|^{1/2}\right).$$

By Sobolev imbedding, we get

$$\|U\|_{6,B_1} \leqq C\|\nabla U\|_{2,B_1} + C\|U\|_{q,B_1} = o\left(|y_0|^{1/2}\right). \quad \square$$

Remark 3.1. If $U \in L^\infty$, and $\|P\|_{1,B_1(y_0)} = O\left(|y_0|^N\right)$ at infinity for some $N \geqq 1$, then by (3.1) we have

$$\|\nabla U\|_{2,B_1(y_0)} = O\left(|y_0|^{N/2}\right) \quad \text{as} \quad |y_0| \to \infty.$$

3.2. Bootstrap

By iterating the Sobolev imbedding theorem and the interior L^p estimates for the Stokes system, we derive the polynomial growth of U and P in this subsection. We first recall the interior L^p estimates for Stokes system (see [Ga I p. 208]). If (v, π) is a solution of the Stokes system

$$\nu\Delta v - \nabla\pi = f, \quad \operatorname{div} v = 0$$

in B_{2R}, then

$$\|\nabla^2 v\|_{r,B_R} + \|\nabla\pi\|_{r,B_R} \leqq C\left(\|f\|_{r,B_{2R}} + \|v\|_{1,r,B_{2R}-B_R} + \|\pi\|_{r,B_{2R}-B_R}\right) \tag{3.2}$$

for $r \in (1, \infty)$, where $C = C(\nu, r, R)$. Our situation is

$$\nu\Delta U - \nabla P = F, \quad \operatorname{div} \quad U = 0,$$

$$F(y) = aU + a(y\cdot\nabla)U + (U\cdot\nabla)U.$$

For the case $3 \leqq q < \infty$, we have

$$\|F\|_{3/2,B_1} \leqq C\left(\|U\|_{2,B_1} + |y_0|\|\nabla U\|_{2,B_1} + \|U\|_{6,B_1}\cdot\|\nabla U\|_{2,B_1}\right) = o\left(|y_0|^{3/2}\right)$$

by Lemma 3.1. The interior estimate (3.2) then gives

$$\|\nabla^2 U\|_{3/2,B_{1/2}} + \|\nabla P\|_{3/2,B_{1/2}} = o\left(|y_0|^{3/2}\right)$$

by Lemma 3.1 and the fact that $\|P\|_{3/2,B_1} = o(1)$.

Now, using the Sobolev imbedding theorem, we get

$$\|\nabla U\|_{3,B_{1/2}} + \|P\|_{3,B_{1/2}} = o\left(|y_0|^{3/2}\right),$$

and hence

$$\|U\|_{r,B_{1/2}} = o\left(|y_0|^{3/2}\right) \qquad \text{for any } \; r < \infty. \tag{3.3}$$

We now go back to F and get

$$\begin{aligned}\|F\|_{2,B_{1/2}} &\leqq C\left(\|U\|_{2,B_{1/2}} + |y_0|\|\nabla U\|_{2,B_{1/2}} + \|U\|_{6,B_{1/2}}\cdot\|\nabla U\|_{3,B_{1/2}}\right)\\ &= o\left(|y_0|^2\right).\end{aligned}$$

By the interior estimate and another application of the imbedding, we get

$$\|\nabla^2 U\|_{2,B_{1/4}} + \|\nabla P\|_{2,B_{1/4}} = o\left(|y_0|^2\right),$$

$$\|\nabla U\|_{6,B_{1/4}} + \|P\|_{6,B_{1/4}} = o\left(|y_0|^2\right),$$

$$\operatorname{osc}\left(U, B_{1/4}\right) = o\left(|y_0|^2\right).$$

By bootstrapping again, we get

$$\operatorname{osc}\left(P, B_{1/8}\right) = o\left(|y_0|^4\right).$$

These give

$$|U(y_0)| = o\left(|y_0|^3\right), \quad |P(y_0)| = o\left(|y_0|^5\right). \tag{3.4}$$

Next we consider the case $q = \infty$, which requires more care. Lemma 2.1 tells us that $\widetilde{P}$ defined by (2.2) is in the *BMO* space. Hence by [JN, FS], we have (also see [St2, p. 141, 144])

$$\int_{\mathbf{R}^3} |\widetilde{P}(y) - \widetilde{P}_{B_1(0)}|\,(1+|y|)^{-3-1}\,dy \leqq C\|\widetilde{P}\|_{BMO},$$

$$\|\widetilde{P} - \widetilde{P}_{B_1(y_0)}\|_{2,B_1(y_0)} \leqq C\|\widetilde{P}\|_{BMO},$$

where $\widetilde{P}_{B_1(y_0)}$ denotes $\frac{1}{|B_1|}\int_{B_1(y_0)} \widetilde{P}dy$. In particular, we get

$$\|\widetilde{P}\|_{2,B_1(y_0)} = O\left(|y_0|^4\right).$$

Since $P - \widetilde{P}$ is affine, we conclude the same growth control for $\|P\|_{2,B_1(y_0)}$. Remark 3.1 then gives us

$$\|\nabla U\|_{2,B_1(y_0)} = O\left(|y_0|^2\right).$$

We now repeat the previous bootstrap argument and end up with

$$|P(y_0)| = O\left(|y_0|^N\right)$$

for some $N > 0$.

We summarize our discussion in

Lemma 3.2. *Let U be a weak solution of* (1.3) *and let P be defined as in Section 2. If $U \in L^q\left(\mathbf{R}^3\right)$, $3 \leqq q \leqq \infty$, then*

$$|P(y_0)| = O\left(|y_0|^N\right) \quad \text{as} \quad y_0 \to \infty \tag{3.5}$$

for some $N < \infty$.

3.3. Representation Formula

In this subsection we use Green's representation formula for the Stokes system to get a growth control of U. For simplicity we assume that $\nu = 1$. For the Stokes system

$$\Delta v - \nabla p = f, \quad \operatorname{div} v = 0$$

in a ball $B = B_\rho(y_0)$ in $\mathbf{R}^3$, we have the following representation formula (see [Ca; Va; Ga I, p. 234]):

$$v_j(y) = \int_B G_{ij}(y,z) f_i(z)\,dz - \int_{\partial B} v_i(z)\left[T_{il}\left(\mathbf{G}_j, g_j\right)(y,z)\right] n_l(z)\,d\sigma_z,$$

$$p(y) = -\int_B g_i(y,z) f_i(z)\,dz - 2\int_{\partial B} v_i(z)\,n_i(z)\,d\sigma_z + \text{const.},$$

where T_{il} denotes the stress tensor

$$T_{il}(w,\pi) = -\delta_{il}\pi + \left(w_{i,l} + w_{l,i}\right)$$

and $\mathbf{G}_j = \left(G_{1j}, G_{2j}, G_{3j}\right)$, G_{ij} and g_j are the Green's tensors ([Ga I, pp. 226–227]) which satisfy, for each fixed $z \in B$,

$$\Delta_y \mathbf{G}_j(y,z) + \nabla_y g_j(y,z) = \delta(y-z)\,\mathbf{e}_j,$$

$$\operatorname{div}_y \mathbf{G}_j(y,z) = 0,$$

$$\mathbf{G}_j(y,z)\big|_{y\in\partial B} = 0.$$

Moreover, we have $G_{ij}(y,z) = G_{ji}(z,y)$, and the estimates

$$|G_{ij}(y,z)| \leqq c_1 |y-z|^{-1}, \tag{3.6}$$

$$|\nabla_y G_{ij}(y,z)| + |\nabla_z G_{ij}(y,z)| + |g_j(y,z)| \leqq c_1 |y-z|^{-2} \tag{3.7}$$

for $y, z \in \bar{B}$ (see [Ca, pp. 335–336]). In addition, we can choose c_1 uniformly for $\rho \in \left[\frac{1}{10}, 10\right]$.

By the above estimates we have, for each fixed $y \in B$ (and we restrict $\rho \in \left[\frac{1}{10}, 10\right]$),

$$\|G_{ij}(y,\cdot)\|_{L^s(B)} \leqq c_2(s) \quad \text{for } s < 3, \tag{3.8}$$

$$\|\nabla_z G_{ij}(y,\cdot)\|_{L^s(B)} + \|g_j(y,\cdot)\|_{L^s(B)} \leqq c_3(s) \quad \text{for } s < \tfrac{3}{2}, \tag{3.9}$$

where $c_2(s)$ and $c_3(s)$ are independent of $y \in B$.

Now we can prove

Lemma 3.3. *Let U be a weak solution of* (1.3). *If $U \in L^q(\mathbf{R}^3)$, $3 < q < \infty$, then*

$$U(y) = o(|y|) \quad \textit{as} \quad y \to \infty.$$

Proof. We define P by (2.2). For every ball $B = B_\rho(y_0)$, $\rho \in \left[\frac{3}{4}, 1\right]$, by the representation formula,

$$U_j(y) = \int_B G_{ij}(y,z)\, aU_i\, dz + \int_B G_{ij}(y,z)\,(az+U)\cdot\nabla U_i(z)\, dz$$
$$- \int_{\partial B} U_i(z)\left[T_{il}\left(\mathbf{G}_j, g_j\right)(y,z)\right] n_l(z)\, d\sigma_z.$$

Let

$$I_1 = \int_B G_{ij}(y,z)\, aU_i dz,$$

$$I_2 = -\int_{\partial B} U_i(z)\left[T_{il}\left(\mathbf{G}_j, g_j\right)(y,z)\right] n_l(z)\, d\sigma_z;$$

then

$$\begin{aligned} U_j(y) &= \int_B G_{ij}(y,z)\,(az+U)\cdot\nabla U_i\, dz + I_1 + I_2 \\ &= \lim_{\varepsilon\to 0^+} \int_{B\setminus B_\varepsilon(y)} \frac{\partial}{\partial z_l}\left[G_{ij}(y,z)\,(az_l+U_l)\right] U_i\, dz \\ &\quad + \int_{\partial B} G_{ij}(y,z)\,(az_l+U_l)\, U_i n_l\, d\sigma_z \\ &\quad - \lim_{\varepsilon\to 0^+} \int_{\partial B_\varepsilon(y)} G_{ij}(y,z)\,(az_l+U_l)\, U_i n_l\, d\sigma_z + I_1 + I_2. \end{aligned}$$

The first limit equals $I_3 + I_4 + 3I_1$ where

$$I_3 = \int_B \frac{\partial}{\partial z_l} G_{ij}(y,z)\, a z_l U_i \, dz,$$

$$I_4 = \int_B \frac{\partial}{\partial z_l} G_{ij}(y,z)\, U_l U_i \, dz.$$

Let us call the middle term I_5,

$$I_5 = \int_{\partial B} G_{ij}(y,z)\,(a z_l + U_l)\, U_i n_l \, d\sigma_z.$$

The second limit is zero by (3.6). Hence

$$U_j(y) = 4I_1 + I_2 + I_3 + I_4 + I_5.$$

If we make the restriction that $|y - y_0| < \frac{1}{2}$ and $\frac{3}{4} \leqq \rho \leqq 1$, then

$$|I_2| \leqq C c_1 \int_{\partial B} |U|\, d\sigma_z,$$

$$|I_5| \leqq 4c_1 \int_{\partial B} a|y_0||U| + |U|^2 \, d\sigma_z.$$

Furthermore, since $q > 3$, $\left(\text{hence } q' < \frac{3}{2}\right)$, we have

$$|I_1| \leqq 4ac_2\left(q'\right) \|U\|_{q,B},$$

$$|I_3| \leqq ac_3\left(q'\right) \|U\|_{q,B} \cdot |y_0|$$

by the Hölder inequality and (3.8), (3.9).

Finally we deal with I_4, which requires more care. If $q > 6$, I_4 is $o(1)$ by the Hölder inequality. In the case $q \in (3, 6]$, if we use the Hölder inequality, we only get $I_4 = o\left(|y_0|^{1+\sigma}\right)$ for any small $\sigma > 0$ (the detail is left to the interested reader). Let us use the following weighted inequality in $\mathbf{R}^n$, which is due to LERAY and HARDY. For any $f \in C_c^\infty(\mathbf{R}^n)$, $1 \leqq r < n$, and any $y \in \mathbf{R}^n$, we have

$$\left\| \frac{f(z)}{|z-y|} \right\|_{r,\mathbf{R}^n} \leqq \frac{r}{n-r} \|\nabla f\|_{r,\mathbf{R}^n}. \tag{3.10}$$

(See [Ga I, p. 59] for a proof and the references; also see [La, p. 16] for the case $r = 2$.) Now we set $r = 2$ and choose a smooth cut-off function ϕ with compact support in $B_4(y)$, $\phi = 1$ in $B_2(y)$, $|\nabla\phi| < 1$. We substitute $f = \phi U$ into (3.10) and get

$$\left\| \frac{U(z)}{|z-y|} \right\|_{2,B_2(y)} \leqq C\|\nabla U\|_{2,B_4(y)} + C\|U\|_{2,B_4(y)} = o\left(|y_0|^{1/2}\right)$$

by Lemma 3.1. Hence

$$|I_4| \leqq C \int_{B_2(y)} \frac{|U(z)|^2}{|z-y|^2} dz = o\left(|y_0|^1\right).$$

We conclude that, for $|y - y_0| < \frac{1}{2}$ and $\rho \in \left[\frac{3}{4}, 1\right]$,

$$|U_j(y)| \leqq o\left(|y_0|^1\right) + C \int_{\partial B_\rho} a|y_0||U| + |U|^2 d\sigma_z.$$

Now we integrate this inequality with respect to ρ from $\frac{3}{4}$ to 1 and get

$$|U(y)| \leqq o\left(|y_0|^1\right) + C \int_{B_1 - B_{3/4}} a|y_0||U| + |U|^2 dz$$

$$= o\left(|y_0|^1\right). \quad \square$$

Remark 3.2. (i) We did not use the full power of the representation formula. It is enough to assume that $y = y_0$ in our estimates. (ii) When $q = 3$, we can only show that $|I_3| = o\left(|y_0|^{1+\sigma}\right)$ for any small $\sigma > 0$. This is the main difficulty we encounter if we try to use the same method to do the case $q = 3$. (iii) Our analysis begins with the estimate of $\|\nabla U\|_{2,B_1}$, which is based on the control of $\|U\|_{3,B_2}$. Hence to deal with the case $q < 3$ would seem to require a different idea. (iv) For $3 < q < \infty$, as we easily see, we can weaken the assumption $U \in L^q\left(\mathbf{R}^3\right)$ to

$$\|U\|_{q,B_2(y)} + \|P\|_{q/2,B_2(y)} \to 0 \quad \text{as} \quad |y| \to \infty.$$

3.4. Another Approach for $3 \leqq q < 9$

In the case $3 \leqq q < 9$, we can apply the same argument of [NRS] to show

$$|U(y)| = O\left(|y|^{-1}\right), \quad |P(y)| = O\left(|y|^{-2+\sigma}\right) \tag{3.11}$$

for arbitrary small $\sigma > 0$. The argument is as follows. We consider parabolic cylinders $Q_1(x_0, T)$ for $|x_0|$ large enough. We will show that the integral

$$\int_{Q_1(x_0,T)} |u|^3 + |p|^{3/2} dx\, dt$$

goes to zero as $|x_0|$ goes to infinity. This can be done by using the Hölder inequality (we recall that $\lambda = \lambda(t) = (2a(T-t))^{-1/2}$):

$$\begin{aligned}
\int_{Q_1(x_0,T)} |u|^3 dx\, dt &= \int_{T-1}^{T} dt \int_{B_1(x_0)} |U(\lambda x)|^3 \left(\lambda^3 dx\right) \\
&= \int_{T-1}^{T} dt \int_{B_\lambda(\lambda x_0)} |U(y)|^3 dy \\
&\leqq \int_{T-1}^{T} dt \left(\int_{B_\lambda(\lambda x_0)} |U(y)|^q dy\right)^{3/q} \cdot C\left(\lambda^3\right)^{1-3/q} \\
&= \int_{T-1}^{T} \lambda^{3-9/q}(t)\, dt \cdot o(1) \\
&= o(1) \quad \text{if } 3 \leqq q < 9.
\end{aligned}$$

A similar estimate holds for

$$\int_{Q_1(x_0,T)} |p|^{3/2} dx\, dt.$$

Hence by [NRS, Proposition 2.1], u is bounded in $Q_{1/2}(x_0, T)$ for $|x_0|$ bigger than some r_0, which is independent of the direction. Therefore, by [Se1, Oh], $\nabla^k u$ are uniformly bounded for $|x| \in \left[r_0 - \frac{1}{4}, r_0 + \frac{1}{4}\right]$, $t \in \left(T - \frac{1}{4}, T\right)$, for each $k = 0, 1, 2 \ldots$. In terms of U, we get the first part of (3.11). We also need to obtain decay estimates for P, which is slightly more difficult than the proof of a corresponding statement in [NRS]. Nevertheless, it is possible to prove the second part of (3.11) by using the integral form of $R_i R_j$ in $\mathbf{R}^3$:

$$R_i R_j f(y) = \lim_{\varepsilon \to 0^+} \int_{|z|>\varepsilon} \frac{3}{4\pi} \frac{K(z)}{|z|^5} f(y-z) dz - \frac{1}{3}\delta_{ij} f(y),$$

with $K(z) = z_i z_j - \frac{1}{3}\delta_{ij}|z|^2$ (cf. [St, p. 73, p. 58]) and $f = U_i U_j$, and by estimating this integral directly. Since we have proved Lemma 3.2 (which is enough for the proof of Theorem 1), we leave the details to the interested reader.

Another way to get (3.11) is to replace $Q_1(x_0, T)$ by $Q_1(0, T)$ in the previous computation and get

$$\int_{Q_1(0,T)} |u|^3 + |p|^{3/2} dx\, dt < \infty$$

if $3 \leqq q < 9$. This integral may not be small; hence we cannot use [NRS, Proposition 2.1] (which uses [CKN, Proposition 2]) to show that u is bounded in $Q_{1/2}(0, T)$. Nonetheless, the proof of the same proposition shows that (u, p) is a suitable weak solution in $Q_{3/4}(0, T)$, and therefore we can apply the results from Section 4.

We remark that G. TIAN & Z. XIN [TX] recently proved the boundedness of suitable weak solutions u in $Q_{R/2}(x_0, T)$ under the condition that

$$\sup_{r \leqq R} r^{-3} \iint_{Q_r(x,T)} |u(x,t)|^2 dxdt < \varepsilon_2, \tag{3.12}$$

where ε_2 is an absolute constant. If we could apply the previous computation, we would conclude the same estimates (3.11) for $2 < q < \infty$. However, [TX] assumes the local energy estimates (1.4) for u in Q_R which, in our case, is not a consequence of either (3.12) or the assumption $U \in L^q(\mathbf{R}^3)$.

4. Growth Estimates: Theorem 2

In this section we establish the growth estimates for Theorem 2, that is, under the assumption of the local energy estimates (1.4) for u. By [CKN, p. 781], (1.4) implies $\|u\|_{10/3,Q_1} < \infty$. We easily calculate that

$$\|u(\cdot,t)\|_{2,B_1} = \lambda(t)^{-1/2} \|U\|_{2,B_{\lambda(t)}(0)},$$

$$\int_{Q_1} |u|^{10/3}\, dx\, dt = \int_{\mathbf{R}^3} |U|^{10/3} \cdot A_1 \min\left(|y|^{-5/3}, \lambda_0^{-5/3}\right) dy, \tag{4.1}$$

$$\int_{Q_1} |\nabla u|^2\, dx\, dt = \int_{\mathbf{R}^3} |\nabla U|^2 \cdot A_2 \min\left(|y|^{-1}, \lambda_0^{-1}\right) dy,$$

where $\lambda_0 = (2a)^{-1/2}$, and A_1 and A_2 are some explicit constants. Hence all the right-hand sides are finite. An immediate consequence is that $U \in W^{1,2}_{\text{loc}}$. Since u is a weak solution of (1.1) by assumption, U is a weak solution of (1.3). Hence U and u are both smooth. In particular, u is a "Leray-Hopf weak solution" and it differs from a "suitable weak solution" only by lacking the estimate

$$p \in L^{5/4}(Q_1(0,T)). \tag{4.2}$$

From (4.1) we also have

$$\|\nabla U\|_{2,B_1(y_0)} + \|U\|_{10/3,B_1(y_0)} = o\left(|y_0|^{1/2}\right).$$

If we assume suitable control of $\|P\|_{3/2,B_1}$ (or of a certain weaker norm), we can follow the bootstrap argument in Subsection 3.2 to obtain certain polynomial growth control of U and P. Unfortunately, the growth control of U cannot be improved by using Green's representation formula because of the term I_3 in Subsection 3.3 (cf. Remark 3.2 (ii)).

Instead, we show that u is a suitable weak solution, and apply the partial regularity result from [CKN] to get the growth estimates of U. To show that u is a suitable weak solution we have to "find" a pressure p satisfying (4.2). We do this by using some weighted estimates involving A_p-weights. (See for example [St2, Chapter 5].) Since U is smooth, the condition $(4.1)_2 < \infty$ implies that

$$\int_{\mathbf{R}^3} |U|^{10/3} \cdot w(y)\, dy < \infty, \tag{4.3}$$

where

$$w(y) = |y|^{-5/3}.$$

It is well known (and easily verified) that $w(y)$ is an $A_{5/3}$ weight in $\mathbf{R}^3$, that is, the quantity

$$F(y_0, r) := \left(\frac{1}{|B|} \int_B w(y)\, dy \right) \cdot \left(\frac{1}{|B|} \int_B w(y)^{-3/2}\, dy \right)^{2/3}$$

is uniformly bounded for all balls $B = B_r(y_0)$. Using the results in [St2, pp. 204–211], we see that the operators $R_i R_j$ used in (2.2) to define $\widetilde{P}$ are continuous on $L_w^{5/3}(\mathbf{R}^3)$, the space of all functions f for which the norm

$$\|f\|_{w,5/3} = \left(\int_{\mathbf{R}^3} |f|^{5/3} \cdot w(y)\, dy \right)^{3/5}$$

is finite. Therefore, given U satisfying (4.3), we can set $\widetilde{P} = \sum R_i R_j \left(U_i U_j\right)$ and we have

$$\|\widetilde{P}\|_{w,5/3} \leqq C \sum \|U_i U_j\|_{w,5/3}.$$

Now, following the proof of Lemma 2.1, we want to show that $\widetilde{P}$ differs from P only by a constant. We first claim that $\widetilde{P}$ satisfies (2.1) in the distributional sense. This can be proved in the same way as in Lemma 2.1. The weight w does not cause any new difficulty since the test function φ used in the proof has a compact support. By Weyl's lemma, $\widetilde{P}$ is smooth. Next we have to verify that, for the function $F(y)$ defined in Lemma 2.1, $D^\alpha F(0) = 0$ for all α. We consider the term involving $y \cdot \nabla U$ as an example. The computations for other terms are essentially the same.

$$\begin{aligned}
&\left| \int_{\mathbf{R}^3} (y \cdot \nabla)\, U \varepsilon^{3+|\alpha|} \left(D^\alpha \varphi\right)(\varepsilon y)\, dy \right| \\
&= \left| -\varepsilon^{3+|\alpha|} \int_{\mathbf{R}^3} 3U \left(D^\alpha \varphi\right)(\varepsilon y)\, dy - \varepsilon^{3+|\alpha|} \int_{\mathbf{R}^3} U \varepsilon y_j \left(\partial_j D^\alpha \varphi\right)(\varepsilon y)\, dy \right| \\
&\leqq C\varepsilon^{3+|\alpha|} \left(\int_{\mathbf{R}^3} |U|^{10/3} w(y)\, dy \right)^{3/10} \cdot \\
&\quad \cdot \left(\int_{\mathbf{R}^3} \left| \left(D^\alpha \varphi\right)(\varepsilon y)\, |y|^{1/2} \right|^{10/7} + \left| \varepsilon y_j \left(\partial_j D^\alpha \varphi\right)(\varepsilon y)\, |y|^{1/2} \right|^{10/7} dy \right)^{7/10} \\
&\leqq C\varepsilon^{3+|\alpha|} \cdot \left(\int_{\mathbf{R}^3} \left[\left| \left(D^\alpha \varphi\right)(z) \right|^{10/7} + \left| z_j \left(\partial_j D^\alpha \varphi\right)(z) \right|^{10/7} \right] |z|^{5/7} \varepsilon^{-5/7-3}\, dz \right)^{7/10} \\
&= C\varepsilon^{3+|\alpha|} \cdot \varepsilon^{-13/5}.
\end{aligned}$$

Therefore, as ε goes to zero, this term goes to zero. This shows that $\widetilde{P}$ together with U satisfy Leray's equation (1.3).

Now we can define p by $(1.2)_2$. Clearly p together with u satisfy (1.1) and $p \in L^{5/3}(Q_1)$ since $\widetilde{P} \in L^{5/3}_w(\mathbf{R}^3)$. We summarize this discussion in

Lemma 4.1. *Let u be a weak solution of* (1.1) *in Q_1 satisfying the local energy estimates* (1.4)*. If u is of the form* $(1.2)_1$*, then u is smooth, and there is a smooth function $p \in L^{5/3}(Q_1)$ of the form* $(1.2)_2$ *such that (u, p) is a suitable weak solution of* (2.1) *in Q_1.*

We remark that, by construction, the $L^{5/3}$-norm of p is bounded by the local energy of u. More generally, by using similar arguments we can also bound $\|p\|_{r/2,Q_1}$ by $\|u\|^2_{r,Q_1}$ if we assume that $u \in L^r(Q_1)$ for $r \in (2, 6)$. (We use the $A_{r/2}$ weight $|y|^{-r/2}$.)

We will use the following lemma, which is a variant of Proposition 2 of [CKN].

Lemma 4.2. *Let (u, p) be a suitable weak solution of* (1.1)*. There is an absolute constant $\varepsilon_4 > 0$ such that, if*

$$\limsup_{r\to 0^+} r^{-1} \int_{Q_r(x,t)} |\nabla u|^2 \leqq \varepsilon_4,$$

then u is essentially bounded in $Q_{r_1}(x, t)$ for some $r_1 > 0$.

This lemma differs from Proposition 2 of [CKN] by replacing $Q^*_r(x, t)$ by $Q_r(x, t)$. (We recall that $Q^*_r(x, t) = Q_r\left(x, t + \frac{1}{8}r^2\right)$.) It assumes the information only at times previous to t, and gets control only at times previous to t. Since the original proof of Proposition 2 of [CKN] and the accompanying lemmas go through without change, we omit the details and refer the reader to [CKN]. As in the proof of Theorem B in [CKN, p. 807], this lemma implies that the singular set at the top of the parabolic cylinder also has one-dimensional Hausdorff measure zero.

With this lemma, we can prove

Corollary 4.3. *Let (u, p) be a suitable weak solution of the Navier-Stokes equations* (1.1) *in $Q_1(0, T)$, and let u be of the form* $(1.2)_1$*. Then $U(y) = O\left(|y|^{-1}\right)$.*

Remark. Heuristically, if the corollary were not true, there would be a direction along which $|y||U(y)|$ is not bounded. By the self-similarity, u would be singular at all points on that direction (at time T); hence we have a segment consisting of singular points. We know from [CKN] that this would be impossible if they were *interior* points. Using Lemma 4.2, we conclude that the top of the parabolic cylinder cannot contain a singular segment either.

Proof. We may assume that $T = 0$. If the corollary were not true, we could find $y_k \in \mathbf{R}^3$, $|y_k| \to \infty$, and

$$|U(y_k)| \cdot |y_k| > k.$$

Let us denote $y/|y|$ by $\widehat{y}$. The set $\{\widehat{y_k}\}$ has an accumulation point x_* on the unit sphere $\{|y| = 1\}$. We may assume that $\widehat{y_k} \to x_*$ by considering a subsequence. We assert that u is not bounded in any $Q_r(\sigma x_*, 0)$ for $\sigma \in (0, 1)$ and $r \in \left(0, \frac{1}{2}\right)$. To see this, let us fix σ and r, and let $\lambda_0 = \lambda\left(-r^2\right) = 1/\sqrt{2a}r$ (we recall that $\lambda(t) = (-2at)^{-1/2}$). For k large enough, we have $|y_k| > \sigma\lambda_0$ and $|\sigma\widehat{y_k} - \sigma x_*| < r$. Let t_k be the time such that $\lambda(t_k)\cdot\sigma = |y_k|$. We easily check that $\lambda(t_k) = \sigma^{-1}|y_k| > \lambda_0$ and hence $t_k \in \left(-r^2, 0\right)$. Therefore the point $(\sigma\widehat{y_k}, t_k)$ is contained in $Q_r(\sigma x_*, 0)$. On the other hand,

$$|u(\sigma\widehat{y_k}, t_k)| = \lambda(t_k)\cdot|U(\lambda(t_k)\sigma\widehat{y_k})| = \sigma^{-1}|y_k|\cdot|U(y_k)| > \sigma^{-1}k.$$

This shows the assertion that all points on the segment $\{(\sigma x_*, 0) : \sigma \in (0, 1)\}$ are singular. This is a contradiction to the fact that the singular set at the top of the parabolic cylinder has one-dimensional Hausdorff measure zero. This contradiction shows our corollary. □

To finish the proof of Theorem 2, we have several possibilities. The first way is to observe that, since U is smooth, $U \in L^q\left(\mathbf{R}^3\right)$ for $q > 3$ by Lemma 4.1 and Corollary 4.3. Therefore Theorem 2 follows from Theorem 1. The second way is to prove

$$P(y) = o\left(|y|^N\right) \quad \text{as} \quad y \to \infty \tag{4.4}$$

for some $N < \infty$. (Corollary 4.3 and (4.4), together with Lemma 5.1 in the next section, prove Theorem 2 without using Theorem 1.) To prove (4.4), we can use Lemma 4.1 and follow the proof of Subsections 3.1 and 3.2, as sketched at the beginning of this section. Alternatively, since Corollary 4.3 implies that u is uniformly bounded in $\{x : \frac{1}{4} < |x| < 1\} \times (-1, 0)$, the same argument of Subsection 3.4 gives the second part of (3.11).

5. The Liouville-Type Lemma and the Main Theorems

In this last section we prove the key Liouville-type lemma and the main theorems.

Lemma 5.1. *Let $\Pi : \mathbf{R}^3 \to \mathbf{R}$, and $U : \mathbf{R}^3 \to \mathbf{R}^3$ be smooth, and satisfy*

$$-\nu\Delta\Pi(y) + (U(y) + ay)\cdot\nabla\Pi(y) \leqq 0 \tag{5.1}$$

in $\mathbf{R}^3$. If $|U(y)| \leqq b|y|$ for some constant $b \in (0, a)$ and for $|y|$ sufficiently large, and

$$|\Pi(y)| = o\left(\int^{|y|} e^{cs^2/2}ds\right) \quad \text{as} \quad |y| \to \infty,$$

where $c = (a - b)/\nu$, then Π is constant.

Proof. Let $M(r) = \max_{|y|=r}\Pi(y)$. $M(r)$ is non-decreasing in r by the maximal principle. Let

$$\phi(r) = \frac{-1}{cr}e^{cr^2/2} + \int^r e^{cs^2/2}ds.$$

(It satisfies (1.12) up to a constant factor.) It is easy to see that the function $v(y) = \phi(|y|)$ is a supersolution for $|y|$ larger than some r_0, i.e.,

$$-\nu\Delta v(y) + (U(y) + ay) \cdot \nabla v(y) \geqq 0 \quad \text{for} \quad |y| > r_0.$$

Let

$$\psi_\varepsilon(y) = M(r_0) + \varepsilon \cdot [\phi(|y|) - \phi(r_0)] \quad \text{for} \quad \varepsilon > 0.$$

Then ψ_ε are all supersolutions for $|y| > r_0$. It is clear that $\psi_\varepsilon(y) \geqq \Pi(y)$ for $|y| = r_0$ and for $|y|$ near ∞ by the growth of Π. By the comparison principle we have

$$\Pi(y) \leqq M(r_0) + \varepsilon \cdot [\phi(|y|) - \phi(r_0)] \quad \text{for} \quad |y| \geqq r_0.$$

Now letting ε go to zero, we get $\Pi(y) \leqq M(r_0)$ for all $|y| \geqq r_0$, and hence $\max \Pi$ is attained at some y, $|y| = r_0$. By the strong maximal principle, Π must be constant. (Notice that our coefficients are bounded in bounded regions.) □

Remark 5.1. It is reasonable to expect that $(U + ay) \cdot \nabla\Pi$ acts as a "magnifying force", since we are looking for blow-up solutions of the Navier-Stokes equations, (cf. [GK, equation (3.2)] and [Gi2]). It is known that for Navier-Stokes equations there exist *forward* self-similar solutions which are defined on $\mathbf{R}^3 \times (0, \infty)$ and are singular at (0, 0) (see [GM, CP]). Its corresponding stationary problem behaves like $\Delta v(y) = -y \cdot \nabla v(y)$ at infinity, which does not blow up at infinity.

Remark 5.2. We can easily extend this lemma to $\mathbf{R}^n$, $n \geqq 2$. In that case, the comparison function ϕ takes the form

$$\phi(r) = \int^r s^{1-n} e^{cs^2/2} ds.$$

Now we prove Theorems 1 and 2.

Proofs of Theorems 1 and 2. Lemmas 3.2 and 3.3, Corollary 4.3 and the estimate (4.4), give us the growth estimates

$$U(y) = o(|y|), \; \Pi(y) = O\left(|y|^N\right) \quad \text{as} \quad |y| \to \infty$$

under either Theorem 1 or Theorem 2, for some $N < \infty$. Lemma 5.1 then implies that Π is constant. Therefore $\nabla\Pi$ is zero, i.e.,

$$U_j U_{j,i} + P_{,i} + aU_i + ay_j U_{j,i} = 0 \qquad \text{for each } i. \tag{5.2}$$

Besides (5.2), if we consider (1.7), we get $|\Omega(y)|^2 = 0$, that is,

$$\partial_i U_j = \partial_j U_i \quad \text{for all } i, j.$$

Comparing (5.2) with the equations (1.3) of U

$$-\nu\Delta U_i + U_j U_{i,j} + P_{,i} + aU_i + ay_j U_{i,j} = 0 \quad \text{for each} \quad i,$$

we get

$$-\nu\Delta U_i = 0 \quad \text{for each } i.$$

Since $U \in L^q(\mathbf{R}^3)$ in Theorem 1 and $U \to 0$ at infinity in Theorem 2, the usual Liouville theorem implies that the U_i are 0 (or constant if $U \in L^\infty$). The proof is complete. □

Remark 5.3. By Lemma 5.1, it is sufficient to have $|U(y)| \leqq b|y|$ for some $b < a$ and for $|y| > r_0$ (and $P(y) = O(|y|^N)$ for some $N > 0$), for the triviality of U. The corresponding condition for u (given by $(1.2)_1$) is

$$|u(x,t)| \leqq \frac{b|x|}{2a(T-t)} \quad \text{for} \quad |x| > (T-t)^{1/2} r_0.$$

It is interesting to compare it with (1.6).

Remark 5.4. To conclude that a solution of (1.3) is zero, certain assumptions on the growth of U are necessary, as can be seen from the following example. Let Φ be an arbitrary harmonic function on $\mathbf{R}^3$. Let $U = \nabla\Phi$ and $P = -\frac{1}{2}|U|^2 - ay \cdot U$, (i.e., $\Pi = 0$). Then (U, P) satisfies LERAY's equations (1.3). This gives us a certain heuristic reason for considering the quantity Π. That the quantity $\frac{1}{2}|u|^2 + p$ satisfies a maximal principle for the stationary Navier-Stokes equations is well known (see, e.g., [Se2, p. 261, GW]), and has played an important role in recent results ([FR1,2, Str3]) regarding the regularity of solutions of the stationary Navier-Stokes equations in higher dimensions.

Remark 5.5. SCHEFFER [Sch2] raised the question of the existence of nontrivial solutions of LERAY's equation with a "speed-reducing" force g:

$$-\nu\Delta U + aU + ay \cdot \nabla U + U \cdot \nabla U + \nabla P = g, \quad \text{div} \quad U = 0,$$

for some U, g with $U \cdot g \leqq 0$. By using the methods in this paper we can obtain some partial results on SCHEFFER's question, but the general case seems to remain open.

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