ON THE FINITE ELEMENT APPROXIMATION OF THE NONSTATIONARY NAVIER-STOKES PROBLEM

Rolf Rannacher
Institut für Angewandte Mathematik
der Universität Bonn, Beringstr. 4-6
5300 Bonn 1, GERMANY

In this note we report some basic convergence results for the semi-discrete finite element Galerkin approximation of the nonstationary Navier-Stokes problem. Asymptotic error estimates are established for a wide class of so-called conforming and nonconforming elements as described in the literature for modelling incompressible flows. Since the proofs are lengthy and very technical the present contribution concentrates on a precise statement of the results and only gives some of the key ideas of the argument for proving them. Complete proofs for the case of conforming finite elements may be found in a joint paper of J. Heywood, R. Rautmann and the author [5], whereas the nonconforming case will be treated in detail elsewhere.

1. The Navier-Stokes problem

We consider the nonstationary Navier-Stokes problem

$$(1)\qquad \left.\begin{aligned} u_t - \nu\Delta u + u\cdot\nabla u - \nabla p &= f \\ \nabla\cdot u &= o \end{aligned}\right\} \quad \text{in} \quad \Omega\times(o,\infty)$$

$$u_{|\partial\Omega} = o\ , \quad u_{|t=o} = a\ ,$$

where $\Omega \subset R^n$, $n = 2,3$, is a bounded domain, $u = u(x,t)$ is the velocity field in R^n and $p = p(x,t)$ the corresponding pressure function. For simplicity we assume homogeneous boundary data and the domain Ω to be convex polyhedral.

Throughout the paper $L^p(\Omega)^n$ denotes the Lebesgue space of n-vector functions with components being to the p-th power integrabel over Ω. $H^m(\Omega)^n$ is the m-th order Sobolev space of L^2-functions having generalized derivatives up to order m in $L^2(\Omega)$. The corresponding norms are

$$\|u\|_{L^p} = \left(\int_\Omega |u|^p\,dx\right)^{1/p}\ ,\quad \|u\|_{H^m} = \left(\sum_{k=o}^{m} \|\nabla^k u\|^2_{L^2}\right)^{1/2}\ ,$$

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|\ ,$$

where $\nabla^k u$ is the tensor of all k-th order derivatives of u. In the case p = 2 we set for convenience

$$(u,v) = \int_\Omega u\cdot v \ dx \ , \ \|u\| = \|u\|_{L^2} = (u,u)^{1/2} \ .$$

$H_o^1(\Omega)^n$ denotes the closure of the space $C_o^\infty(\Omega)^n$ of C^∞-vector functions having compact support in Ω and J_o is the subspace of all solenoidal functions in $H_o^1(\Omega)^n$:

$$J_o = \{v \in H_o^1(\Omega)^n : \quad \nabla\cdot v = o \quad \text{a.e. in} \quad \Omega\} \quad .$$

For time dependent functions into some Banach space X we use the notation

$$L^p(o,T;X) = \{u = u(t) : (o,T) \to X \ \text{measurable} : \int_o^T \|u(\tau)\|_X^p \, d\tau < \infty\}$$

with the usual modification for $p = \infty$.

Finally we introduce the bilinear and trilinear forms, respectively,

$$a(u,v) = \nu \int_\Omega \nabla u\cdot \nabla v \ dx \ , \quad b(u,v,w) = \int_\Omega u\cdot\nabla v \cdot w \ dx \ ,$$

where the dot "·" denotes the usual tensor multiplication.

Using these notation the weak formulation of problem (1) is as follows

(2) <u>Find some</u> $u = u(t) \in J_o$ <u>such that</u> $u(o) = a$ <u>and</u>

$$(u_t,\phi) + a(u,\phi) + b(u,u,\phi) = (f,\phi) \ , \quad \forall\phi \in J_o \ .$$

It is well known (see Ladyzhenskaya [5] and Heywood [3]) that under the assumptions

$$a \in J_o \ , \quad f \in L^2(o,\infty;L^2(\Omega)^n)$$

there is an unique solution $u \in L^\infty(o,T;J_o)$ of (2) on some time interval $[o,T)$ where $T > o$. Furthermore,

$$u \in L^2(o,T;H^2(\Omega)^n) \quad , \ u_t \in L^2(o,T;L^2(\Omega)^n)$$

and the corresponding pressure satisfying equation (1) is

$$p \in L^2(o,T;H^1(\Omega)).$$

Under the additional stronger conditions

$$a \in H^2(\Omega)^n \quad , \quad f_t \in L^2(o,\infty;L^2(\Omega)^n)$$

one even has

$$u \in L^\infty(o,T;H^2(\Omega)^n)\ ,\ u_t \in L^\infty(o,T;L^2(\Omega)^n) \cap L^2(o,T;H^1(\Omega)^n)$$
$$p \in L^\infty(o,T;H^1(\Omega)).$$

In the following we shall study the convergence behavior of the finite element Galerkin method for approximating the weak solution u of problem (1) under the above minimal assumptions.

2. Finite element Galerkin method

Let $\Pi_h = \{K\}$ be finite "triangulations" of the polyhedral domain Ω which satisfy the usual regularity conditions (see [1] and [2]) for mesh size h tending to zero, namely that each $K \in \Pi_h$ contains a n-ball of radius κh and is contained in a n-ball of radius $\kappa^{-1}h$.

We consider finite element spaces $J_{o,h}$ consisting of piecewise polynomial functions which are proper approximations of the basic space J_o in the following sense:

(3) Each function $v_h \in J_{o,h}$ satisfies

(i) $\int_{\partial K \cap \partial K'} \{v_{h|K} - v_{h|K'}\}\, ds = o, \quad \forall\, K,K' \in \Pi_h\ ,$

(ii) $\int_{\partial K \cap \partial\Omega} v_{h|K}\, ds = o, \quad \forall\, K \in \Pi_h\ ,$

(iii) $\int_K \nabla\cdot v_h\, dx = o, \quad \forall\, K \in \Pi_h\ .$

Furthermore, there are operators $r_h : \{J_o \cap H^2(\Omega)^n\} \oplus J_{o,h} \to J_{o,h}$ such that for $1 \le p \le \infty$

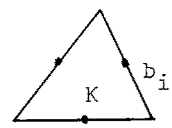
(iv) $r_h v_h = v_h\ , \quad \forall v_h \in J_{o,h}\ ,$

(v) $\| v - r_h v \|_{L^p} \le c\, h^{2+n/p-n/2} \| v \|_{H^2}\ , \quad \forall v \in J_o \cap H^2(\Omega)^n\ .$

By these conditions the spaces $J_{o,h}$ are approximations of J_o of order m = 1. They include a wide class of conforming and even nonconforming finite elements for modelling incompressible flows as studied for instance by Crouxeiz and Raviart [2] and Fortin [3]. By (3i, ii) it is guaranteed that the functions in $J_{o,h}$ are at least approximately H_o^1-functions, i.e. their jumps along the element boundaries ∂K and their boundary values on $\partial\Omega$ are in some sense small. Condition (3iii) means that even the divergence of functions in $J_{o,h}$ is in some sense small. Hence the spaces $J_{o,h}$ are approximately admissible with respect to the space J_o. The conditions (3iv,v) ensure that each function in J_o can be approximated arbitrarily close by a sequence of functions in $J_{o,h}$ for h tending to zero.

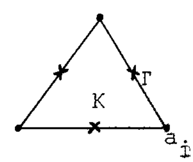
As examples of elements satisfying all the conditions (3i-v) we mention the nonconforming linear element in two or even three dimensions with the corresponding nodal values:

values $v_h(b_i)$ at the centers b_i of (n-1)-faces of all $K \in \Pi_h$



and the conforming quadratic element in two dimensions with the corresponding nodal values:

values $v_h(a_i)$ at the vertices a_i and line integrals $\int_\Gamma v_h \, ds$ over the sides of all triangles $K \in \Pi_h$.



Using a so-called "bulb-function" the order of the quadratic element can be raised to m = 2 without increasing the dimension of the space $J_{o,h}$. In all these cases the spaces $J_{o,h}$ are of the type

$$J_{o,h} = \{ \phi \in L^2(\Omega)^n : \text{i) } \phi|_K \in P_m \text{ for some } m \geq 1,\ K \in \Pi_h,$$

ii) ϕ is continuous with respect to the prescribed nodal values;

iii) ϕ has vanishing nodal values along the boundary $\partial\Omega$;

iv) $\int_K \nabla \cdot \phi \, dx = o \ , \ \forall K \in \Pi_h \}$.

As operator r_h one can choose the usual interpolation operator with respect to the prescribed nodal values which maps the space $J_o \cap H^2(\Omega)^n$ into $J_{o,h}$ and leaves the spaces $J_{o,h}$ invariant by its special construction. For a detailed description of these finite element spaces and for further examples of even higher order m > 1 we refer to the literature [2] , [3] and [8].

For functions $\phi \in J_o \oplus J_{o,h}$ the discrete gradient $\nabla_h \phi$ is defined piecewise with respect to the patchs $K \in \Pi_h$. In this sense we can define the following bilinear and trilinear forms, respectively, on the direct sum $J_o \oplus J_{o,h}$ of the spaces J_o and $J_{o,h}$:

$$a_h(v,w) = \nu(\nabla_h v, \nabla_h w) = \sum_{K \in \Pi_h} \int_K \nabla v \cdot \nabla w \ dx$$

$$b_h(u,v,w) = \frac{1}{2} \sum_{K \in \Pi_h} \int_K \{u \cdot \nabla v \cdot w - u \cdot \nabla w \cdot v\} \ dx .$$

Obviously the forms b_h are compatible with b in the sence

$$b_h(u,v,w) = b(u,v,w) \quad , \quad u,v,w \in J_o ,$$

and they even satisfy

$$b_h(u,v,v) = o \ , \quad u,v \in J_o \oplus J_{o,h} .$$

Using these notations the semi-discrete analogues of problem (2) are

(4) <u>Find some</u> $u_h = u_h(t) \in J_{o,h}$ <u>such that</u> $u_h(o) = a_h$ <u>and</u>

$$(u_{ht}, \phi_h) + a_h(u_h, \phi_h) + b_h(u_h, u_h, \phi_h) = (f, \phi_h) \ , \quad \forall \phi_h \in J_{o,h} \quad ,$$

where $a_h \in J_{o,h}$ are appropriate approximations to the initial data $a \in J_o$ satisfying uniformly for $h \to o$

$$\| a - a_h \| \le c \ h^k \ \| a \|_{H^k} \quad , \ k = 1,2 \ ,$$

provided that $a \in H^k(\Omega)^m$.

If $\{\phi_h^{(i)}, \ i = 1,\dots,N_h = \dim(J_{o,h})\}$ is a basis of $J_{o,h}$, then problem (4) is equivalent to a system of first-order ordinary differential equations for the coefficient functions $\xi_i(t)$ in the representation

$$u_h(t) = \sum_{i=1}^{N_h} \xi_i(t) \phi_h^{(i)} \qquad .$$

For some of the spaces satisfying the conditions (3i-v) quasi-local bases are known at least in two dimensions (see [3]). This problem which is crucial for the numerical realization of the scheme (4) will be discussed in a subsequent paper.

Setting $\phi_h = u_h(t)$ in (4), we obtain the discrete energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u_h(t)\|^2 + \|\nabla_h u_h(t)\|^2 \le \|f(t)\|\,\|u_h(t)\| \quad , \quad t \ge o,$$

and from that via Gronwall's inequality the bound

$$\|u_h(t)\|_{L^\infty} \le c(h)\|u_h(t)\| \le c(t) \quad , t \ge o .$$

This guarantees the existence of unique discrete solutions u_h of problem (4) which are in $L^\infty(o,T';J_{o,h})$ for all times $T' > o$.

For these approximate solutions u_h we have the following basic convergence results:

Theorem. Assume that the finite element spaces $J_{o,h}$ *are first-order approximations of the space* J_o *in the sense described above. Further assume that the data of problem* (1) *satisfy*

$$a \in J_o \quad , \quad f \in L^2(o,\infty;L^2(\Omega)^n),$$

and let $u \in L^\infty(o,T;J_o)$ *and* $u_h \in L^\infty(o,T;J_{o,h})$ *for some* $T > o$ *be the corresponding unique solutions of problem* (2) *and* (4), *respectively. Then the error function* $e = u - u_h$ *satisfies the estimate*

$$(5) \qquad \|e(t)\| + \left(\int_o^t \|\nabla_h e\|^2\, d\tau\right)^{1/2} \le c(t)\, h \;, \quad t \in [o,T),$$

and, if additionally

$$a \in H^2(\Omega)^n \quad , \quad f_t \in L^2(o,\infty;L^2(\Omega)^n) \; ,$$

even the pointwise estimate

$$(6) \qquad \|e(t)\|_{L^\infty} \le c(t) \begin{cases} h^{1/4} & \text{for } n = 3 \\ h^{1/2}|\ln h|^{1/2} & \text{for } n = 2 \end{cases} , \quad t \in [o,T) .$$

Moreover, if the solution u *also satisfies*

$$u_t \in L^2(o,T;H^2(\Omega)^n) \quad , \quad p_t \in L^2(o,T;H^1(\Omega)) \; ,$$

then we have

$$(7) \qquad \|\nabla_h e(t)\| + \left(\int_o^t \|e_t\|^2\, d\tau\right)^{1/2} \le c(t)\, h \;, \quad t \in [o,T),$$

and finally the improved°*pointwise estimate*

$$(8) \qquad \|e(t)\|_{L^\infty} \le c(t) \begin{cases} h^{1/2} & \text{for } n = 3 \\ h|\ln h|^{1/2} & \text{for } n = 2 \end{cases} , \quad t \in [o,T).$$

All the constants $c(t)$ *depend continuously on the specified data but are independent on the mesh size* h.

In order to illustrate the statements of the theorem we add the following remarks.

1. Note that the estimates (5) - (8) hold for any time $T > o$ such that the solution of problem (2) is known to be $u \in L^{\infty}(o,T;J_o)$, regardless where this information comes from. In the case $n = 2$ or, if the data of problem (1) are in a certain sense small enough, even in the case $n = 3$ one has $T = \infty$ and the constants $c(t)$ remain bounded for $t \to \infty$ (see [7] for the behaviour of the solution u of problem (1) and [5] for the behaviour of the discrete solutions u_h for time t tending to infinity).

2. The estimates (5) and (6) obviously hold under the above mentioned minimal assumptions on the data a and f. The sharper results (7) and (8) are a little problematic since the corresponding assumptions on u_t and p_t seem to be realistic only if certain global compatibility conditions for the initial data a are satisfied (see [3]). These unnatural strong assumptions on u_t, p_t are essentially forced by the allowed nonconformity of the spaces $J_{o,h}$. If the spaces $J_{o,h}$ are fully conforming, i.e. $J_{o,h} \subset J_o$, then the estimates (7), (8) even hold under the same natural assumptions as made for the estimate (6).(This corresponds with the estimates given by Rautmann [8] for the approximation of problem (1) by means of eigenfunctions of the Stokes operator.) It would be desirable to remove this weak point in our results.

3. For finite element spaces $J_{o,h}$ of higher order satisfying the conditions (3i-v) in a stronger sense corresponding higher order error estimates hold provided the solution u is regular enough (see[4]). But again this a priori assumption might be problematic unless certain compatibility conditions are satisfied by the initial data (see [3]).

4. Having already computed the approximation u_h to the velocity vector u one can generate approximations p_h to the corresponding pressure p by a suitable Galerkin Ansatz. Since this procedure is just the same as for the simplest steady state Stokes problem we only refer to the literature [2], [9] for a further discussion of pressure computation.

5. All the above results remain valid if the usual techniques for

approximating curved boundary and nonhomogeneous boundary data are used. The additionally required technical argument is again just the same as for the well known steady state Stokes problem or even as for any of the usual elliptic model problems (see [1]).

6. For time discretization several of the methods known for parabolic problems may also be used for the Navier-Stokes problem (see [9]). For instance the (nonlinear) Crank-Nicolson scheme is unconditionally stable and of second order convergent. It has the form

$$\frac{1}{\Delta t}(U_h^k - U_h^{k-1}, \phi_h) + a_h(\tilde{U}_h^k, \phi_h) + b_h(\tilde{U}_h^k, \tilde{U}_h^k, \phi_h) = (f^k, \phi_h), \quad \forall \phi_h \in J_{o,h},$$

where U_h^k is the discrete solution for the time level $k\cdot\Delta t$ and

$$\tilde{U}_h^k = \tfrac{1}{2}(U_h^k + U_h^{k-1}), \quad U_h^o = a_h, \quad f^k = \frac{1}{\Delta t}\int_{(k-1)\cdot\Delta t}^{k\cdot\Delta t} f(\tau)\,d\tau .$$

The error estimate

$$\|u_h(k\cdot\Delta t) - U_h^k\| \le c(t)\,\Delta t^2, \quad t = k\cdot\Delta t \in [o,T),$$

holds provided that $u_{htt} \in L^2(o,T;L^2(\Omega)^n)$. This may be shown by standard techniques for time discretization of parabolic problems. The linearized C.-N. scheme using the term $b_h(U_h^{k-1}, \tilde{U}_h^k, \phi_h)$ instead of $b_h(\tilde{U}_h^k, \tilde{U}_h^k, \phi_h)$ is also stable (see [9]) but only of first order convergent.

3. Proof of the theorem

In the following we present the key ideas of the argument for proving the error estimates (5) - (8). Most of the technical complications arise from the allowed nonconformity of the spaces $J_{o,h}$ with respect to the divergence condition "$\nabla\cdot v = o$" as well as with respect to the continuity requirement "$v \in H_o^1(\Omega)^n$". Since the techniques for overcoming these problems are essentially the same as for the steady state case (see [2] and [6]) we mainly concentrate on the argument concerning the nonstationarity of the problem.

A) Technical preliminaries

At first we provide some more technical tools which will be used below. Let the L^2-projections $L_h : L^2(\Omega)^n \to J_{o,h}$ be defined by

$$(v-L_h v,\phi_h) = o\ ,\quad \forall\phi_h \in J_{o,h}\ ;\ v \in L^2(\Omega)^n\ ,$$

and the (generalized) Stokes projections $S_h : J_o \oplus J_{o,h} \to J_{o,h}$ by

$$a_h(v-S_h v,\phi_h) = o\ ,\quad \forall\phi_h \in J_{o,h}\ ;\quad v \in J_o \oplus J_{o,h}\ ,$$

where again $J_o \oplus J_{o,h}$ denotes the direct sum of the vector spaces J_o and $J_{o,h}$. Both of the projection operators commute with time differentiation:

$$L_h v_t = (L_h v)_t\ ,\quad S_h v_t = (S_h v)_t\ .$$

Furthermore, under our assumptions on the spaces $J_{o,h}$, one may prove the following estimates (see for instance [2] and [5])(+)

$$(9)\qquad \begin{aligned} &\| L_h v\| + \|S_h v\| \le c\|v\| + ch\|\nabla_h v\| \\ &\|\nabla_h L_h v\| + \|\nabla_h S_h v\| \le c\|\nabla_h v\| \end{aligned}\ ,\qquad v \in J_o \oplus J_{o,h}\ ,$$

and

$$(1o)\qquad \begin{aligned} &\|v - L_h v\| + h\|\nabla_h(v-L_h v)\| \le c\, h^2\|v\|_{H^2} \\ &\|v - S_h v\| + h\|\nabla_h(v-S_h v)\| \le c\, h^2\|v\|_{H^2} \end{aligned}\ ,\ v \in J_o \cap H^2(\Omega)^n.$$

Below we shall frequently use the Sobolev inequality for $n \le 3$

$$\|v\|_{L^6} \le c\,\|v\|_{H^1}\ ,\quad v \in H^1(\Omega)^n\ ,$$

the Poincare inequality

$$\|v\|_{H^1} \le c\|\nabla v\|\ ,\quad v \in H^1_o(\Omega)^n\ ,$$

and the a priori estimate

$$\|v\|_{H^2} \le c\ \|\Delta v\|\ ,\ v \in H^1_o(\Omega)^n \cap H^2(\Omega)^n\ ,$$

where the latter holds on any bounded convex domain. This leads together with the Hölder inequality

$$\|v\|_{L^3} \le \|v\|^{1/2}\|v\|^{1/2}_{L^6}$$

to the estimates

$$\|v\|_{L^3} \le c\,\|\nabla v\|^{1/2}\ \|\Delta v\|^{1/2}\ ,\ v \in H^1_o(\Omega)^n \cap H^2(\Omega)^n\ ,$$

$$\|v\|_{L^3} \le c\|v\|^{1/2}\ \|\nabla v\|^{1/2}\ ,\ v \in H^1_o(\Omega)^n\ ,$$

which are useful for handling the nonlinear term $u\cdot\nabla u$ in equation (1).

(+) "c" always denotes a generic constant which may change with the context.

Similar inequalities as stated above even hold for the functions in the discrete spaces $J_{o,h}$, namely for $n \le 3$:

(11) $$\|v_h\|_{L^6} \le c\|\nabla_h v_h\| \quad , \quad v_h \in J_{o,h} ,$$

and

(12) $$\|\nabla_h v\|_{L^6} \le c\|\Delta_h v_h\| \quad , \quad v_h \in J_{o,h} .$$

Here Δ_h denotes a discrete analogue of the Laplacian Δ which is defined by means of the eigenvalues $\{\lambda_h^{(i)}\}$ and the corresponding eigenvector systems $\{w_h^{(i)}\} \subset J_{o,h}$, $i = 1,\dots,N_h = \dim(J_{o,h})$, of the discrete Stokes operator:

$$a_h(w_h^{(i)}, \phi_h) = \lambda_h^{(i)}(w_h^{(i)},\phi_h) \quad , \quad \forall\phi_h \in J_{o,h} .$$

Using this notation we set for $v_h \in J_{o,h}$

(13) $$\Delta_h v_h := \sum_{i=1}^{N_h} \lambda_h^{(i)}(v_h,w_h^{(i)})w_h^{(i)} .$$

By definition we have the discrete Green's formula

(14) $$(\nabla_h v_h,\nabla_h w_h) = -(\Delta_h v_h,w_h) \quad , \quad v_h,w_h \in J_{o,h} .$$

The estimates (11) and (12) will be proved for the conforming case, $J_{o,h} \subset H_o^1(\Omega)^n$, in [4] and for the more complicated nonconforming case in a forthcoming paper of the author (see also some similar estimates derived in [6] and [9]). A sketch of the proof of the presumably most surprising estimate (12) will be given below in step (C).

B) Proof of the estimates (5) and (7)

Because of their nonconformity the functions $\phi_h \in J_{o,h}$ may not be used directly as test functions in the weak formulation (2) of problem (1). To overcome this complication we note that under our assumptions equation (1) even holds in the strong sense

$$u_t - \nu\Delta u + u\cdot\nabla u + \nabla p = f \quad \text{a.e. in } \Omega \times (o,T).$$

Then multiplying this identity with $\phi_h \in J_{o,h}$ and integrating then by parts leads to

$$(u_t,\phi_h) + a_h(u,\phi_h) + b_h(u,u,\phi_h) = (f,\phi_h) + (p,\nabla_h\cdot\phi_n) + r_h(u,u,\phi_h) ,$$

where

$$\Gamma_h(u,u,\phi_h) = \sum_{K\in\Pi_h} \int_{\partial K} \{\nu u_n + \tfrac{1}{2}(u\cdot n)u - pn\}\cdot\phi_h \, ds$$

(n = outer normal unit vector to ∂K, $u_n = \frac{\partial u}{\partial n}$ normal derivative).

Combining this with the corresponding relation (4) for the discrete solutions u_h, we get for the error $e = u-u_h$ the identity

$$(15) \qquad (e_t,\phi_h) + a_h(e,\phi_h) = b_h(u_h,u_h,\phi_h) - b_h(u,u,\phi_h) + (p,\nabla_h\cdot\phi_h) + \\ + \Gamma_h(u,u,\phi_h) \;,\; \forall\, \phi_h \in J_{o,h} \;.$$

The two terms on the right hand side coming from the nonconformity of $J_{o,h}$ will be estimated as follows:

By assumption (3iii) we have

$$(p,\nabla_h\cdot\phi_h) = (p-q_h,\nabla_h\cdot\phi_h) \le \|p-q_h\| \; \|\nabla_h\cdot\phi_h\| \quad ,$$

where q_h is any piecewise constant approximation to the pressure p, and hence

$$(16) \qquad (p,\nabla_h\cdot\phi_h) \le c\, h\, \|\nabla_h\phi_h\| \; \|\nabla p\| \quad .$$

The boundary term may be rewritten as

$$\Gamma_h(u,u,\phi_h) = \sum_{\Gamma}\int_{\Gamma} \{\nu u_n + \tfrac{1}{2}(u\cdot n)u - pn\}\cdot[\phi_h]ds \quad ,$$

where the summation is taken over all (n-1)-faces of the $K \in \Pi_h$ and $[\phi_h]$ is the jump of ϕ_h along such a face Γ . By assumption (3i,ii) these jumps $[\phi_h]$ have vanishing mean value on Γ and hence allow us to insert appropriate mean values ω_Γ of the sum set in brackets:

$$\Gamma_h(u,u,\phi_h) = \sum_{\Gamma}\int_{\Gamma} \{\nu u_n + \tfrac{1}{2}(u\cdot n)u - pn - \omega_\Gamma\}\cdot[\phi_h]ds \quad .$$

Applying a Poincare type inequality to the term in brackets as well as to the jumps $[\phi_h]$ we conclude

$$\Gamma_h(u,u,\phi_h) \le c \sum_{K\in\Pi_h} h^{1/2}\, \|\nabla_h\phi_h\|\, h^{1/2}(\|\nabla^2 u\| + \|\nabla u^2\| + \|\nabla p\|).$$

Finally the estimates provided in step (A) lead us to

$$(17) \qquad \Gamma_h(u,u,\phi_h) \le c\, h\, \|\nabla_h\phi_h\|\, (\|\Delta u\| + \|\nabla u\|^3 + \|\nabla p\|).$$

Now in order to prove the estimate (5) we insert $\phi_h = L_h e \in J_{o,h}$ into the identity (12) and get by a simple rearrangement of terms

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \nu\|\nabla_h e\|^2 = (e_t, u-L_h u) + a_h(e, u-L_h u) + b_h(u_h, u_h, L_h e) -$$

$$- b_h(u,u,L_h e) + (p, \nabla_h \cdot L_h e) + \Gamma_h(u,u,L_h e)$$

and furthermore

$$b_h(u_h,u_h,L_h e) - b_h(u,u,L_h e) = b_h(e,e,L_h e) - b_h(e,u,L_h e) - b_h(u,e,L_h e).$$

Now applying the estimates preserved in part (A), one concludes by a somewhat lengthy but straightforward calculation that for any $\varepsilon \in (o,1]$.

$$(2o) \qquad b_h(u_h,u_h,L_h e) - b_h(u,u,L_h e) \le \varepsilon \|\nabla_h e\|^2 + c\|\nabla u\|^4 \|e\|^2 .$$

From the estimates (16) and (17) we have

$$(p,\nabla_h \cdot L_h e) + \Gamma_h(u,u,L_h e) \le \varepsilon\|\nabla_h e\|^2 + c\, h^2(\|\Delta u\|^2 + \|\nabla u\|^6 + \|\nabla p\|^2).$$

Choosing ε sufficiently small we arrive at

$$\frac{d}{dt}\|e\|^2 + \|\nabla_h e\|^2 \le c\, h^2 \frac{d}{dt}\|\nabla u\|^2 + c\, h^2(\|\Delta u\|^2 + \|\nabla u\|^6 + \|\nabla p\|^2).$$

From that the estimate (5) immediately follows by applying Gronwall's inequality.

To prove the estimate (7) we insert $\phi_h = S_h e_t$ into the identity (15) and get again by a simple rearranging of terms

$$\|e_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla_h e\|^2 = (e_t, u_t - S_h u_t) + a_h(e, u_t - S_h u_t) + b_h(u_h, u_h, S_h e_t) -$$

$$- b_h(u,u,S_h e_t) + (p, \nabla_h \cdot S_h e_t) + \Gamma_h(u,u,S_h e_t) .$$

By the definition of the Stokes projection we have

$$(21) \qquad a_h(e, u_t - S_h u_t) = a_h(u - S_h u, u_t - S_h u_t) = \frac{1}{2}\frac{d}{dt}\|\nabla_h(u - S_h u)\|^2$$

$$= h^2 \frac{d}{dt}[c_h \|\Delta u\|^2] , \quad c_h := \frac{\|\nabla_h(u-S_h u)\|^2}{h^2\ \|\Delta u\|^2} \le c .$$

Furthermore, with any $\varepsilon \in (o,1]$,

$$(22) \qquad (e_t, u_t - S_h u_t) \le \varepsilon\|e_t\|^2 + c\, h^2\|\nabla u_t\|^2 .$$

The terms representing the nonlinearity may be rearranged to

$$b_h(u_h,u_h,S_he_t)-b_h(u,u,S_he_t) = b_h(e,e,S_he_t)-b_h(e,u,S_he_t)-b_h(u,e,S_he_t)$$

$$= \frac{d}{dt}[b_h(e,e,S_he_t)-b_h(e,u,S_he_t)-b_h(u,e,S_he_t)] -b_h(e_t,e,S_he_t) -$$

$$- b_h(e,e_t,S_he_t)+b_h(e_t,u,S_he_t)+b_h(e,u_t,S_he_t)+b_h(u_t,e,S_he_t) +$$

$$+ b_h(u,e_t,S_he_t) .$$

Now again a rather lenghty calculation using the estimates provided in part (A) and in addition the already proved basic error estimate (5) leads us to ($\varepsilon \in (o,1]$)

$$(23) \quad b_h(u_h,u_h,S_he_t)-b_h(u,u,S_he_t) \le \frac{d}{dt}[c_o(t)\|\nabla_he\|^2] + \varepsilon\|e_t\|^2 + $$

$$+ c_\varepsilon(t)\|\nabla_he\|^2 + c_\varepsilon(t)\, h^2 ,$$

where the constants $c_o(t)$ and $c_\varepsilon(t)$ depend on $\|\Delta u(t)\|$ and $\|\Delta u_t\|$, respectively, but are bounded with respect to h.

The two terms coming from the nonconformity of the spaces $J_{o,h}$ are estimated in a similar way using the estimates (16) and (17) as follows

$$(24) \quad (p,\nabla_h\cdot S_he_t) = \frac{d}{dt}(p,\nabla_h\cdot S_he_t) - (p_t,\nabla_h\cdot S_he)$$

$$\le c\,\frac{d}{dt}(h^2\|\nabla p\|^2 + \|\nabla_he\|^2) + c\,h^2\,\|\nabla p_t\|^2 + c\,\|\nabla_he\|^2$$

and

$$(25) \quad \Gamma_h(u,u,S_he_t) = \frac{d}{dt}\Gamma_h(u,u,S_he) - \Gamma_h(u_t,u,S_he) - \Gamma_h(u,u_t,S_he)$$

$$\le \frac{d}{dt}[c_o(t)h^2 + \|\nabla_he\|^2] + c(t)h^2 + c\,\|\nabla_he\|^2 ,$$

where again the constants $c_o(t)$ and $c(t)$ depend on the norms $\|\Delta u_t\|$ and $\|\nabla p_t\|$, respectively, but are bounded with respect to h.

Collecting all the estimates (21) - (25) we obtain for ε, sufficiently small,

$$\|e_t\|^2 + \frac{d}{dt}\|\nabla_he\|^2 \le \frac{d}{dt}[c_o(t)\|\nabla_he\|^2 + c_1(t)h^2] + c_2(t)\|\nabla_he\|^2 .$$

From that one gets the desired estimate (7) again by applying the Gronwall inequality.

C) Proof of the pointwise estimates (6) and (8)

At first we give a short proof of the estimate (12) for the discrete Laplacian Δ_h.

To any given $v_h \in J_{o,h}$, which clearly has a representation of the form

$$v_h = \sum_{i=1}^{N_h} \alpha_i w_h^{(i)} \quad , \quad \alpha_i \in R ,$$

we attach a function $v \in J_o$ and a corresponding pressure q by solving the Stokes problem

$$(26) \qquad \left. \begin{aligned} -\nu \Delta v + \nabla q &= -\Delta_h v_h \\ v &= o \end{aligned} \right\} \quad \text{in } \Omega , \quad v = o \text{ on } \partial\Omega.$$

Then v_h turns out to be just the finite element approximation to v which is defined by the discretized Stokes problem

$$(27) \qquad a_h(v_h,\phi_h) = -(\Delta_h v_h,\phi_h) \quad , \quad \forall\ \phi_h \in J_{o,h} \quad .$$

For that there are the following error estimates available (see [2])

$$(28) \quad \|v-v_h\| + h\|\nabla_h(v-v_h)\| \leq ch(\|\Delta v\| + \|\nabla q\|) .$$

Furthermore, we have

$$\|\nabla_h v_h\|_{L^6} \leq \|\nabla_h(v_h-r_h v)\|_{L^6} + \|\nabla_h(r_h v-v)\|_{L^6} + \|\nabla v\|_{L^6} \quad ,$$

and so-called "inverse" property of finite elements (see [1]) gives us

$$\begin{aligned} \|\nabla_h(v_h-r_h v)\|_{L^6} &\leq c\ h^{-1}\|v_h-r_h v\|_{L^6} \\ &\leq c\ h^{-1}\|v_h-v\|_{L^6} + c\ h^{-1}\|v-r_h v\|_{L^6} \quad . \end{aligned}$$

Applying Sobolev's inequality

$$\|\nabla v\|_{L^6} \leq c\|v\|_{H^2}$$

and the estimate in assumption (3v)

$$\|v-r_h v\|_{L^6} + h\|\nabla_h(v-r_h v)\|_{L^6} \leq c\ h\ \|v\|_{H^2} \quad ,$$

we find

$$\|\nabla_h v_h\|_{L^6} \leq c\ \|v\|_{H^2} \quad .$$

Now the usual a priori estimate for the Stokes problem (see [7]) yields

$$\| v \|_{H^2} + \| \nabla q \| \le c \| \Delta_h v_h \|$$

and hence the desired estimate

$$\| \nabla_h v_h \|_{L^6} \le c \| \Delta_h v_h \| \quad , \qquad v_h \in J_{o,h} \; .$$

For proving the pointwise error estimates we shall use the following set of inequalities for functions $v_h \in J_{o,h}$

$$(29) \qquad \| v_h \|_{L^\infty} \le \begin{cases} h^{1/4} \; \| \nabla_h v_h \|_{L^6} + c h^{-3/4} \| v_h \| & (n=3) \\ h^{1/2} \; |\ln h|^{1/2} \| \nabla_h v_h \|_{L^6} + c \; h^{-1/2} \| v_h \| & (n=2) \end{cases}$$

and

$$(3o) \qquad \| v_h \|_{L^\infty} \le c \begin{cases} h^{-1/2} \| \nabla_h v_h \| & (n=3) \\ |\ln h|^{1/2} \| \nabla_h v_h \| & (n=2) \end{cases} \; .$$

Proofs of these estimates will be given for the conforming case in [5] and for the again more troublesome nonconforming case in a subsequent paper of the author already announced above.

Now we use the discrete Sobolev inequalities (11) and (12) from part (A) and conclude for the error $e = u-u_h$ in the case $n = 3$

$$\begin{aligned} \| e \|_{L^\infty} &\le \| u-r_h u \|_{L^\infty} + \| r_h e \|_{L^\infty} + h^{1/4} \| \nabla_h r_h e \|_{L^6} + c \; h^{-3/4} \| r_h e \| \\ &\le \| u-r_h u \|_{L^\infty} + h^{1/4} (\| \Delta u \| + \| \Delta_h u_h \|) + \\ &\quad + \; c \; h^{-3/4} (\| u-r_h u \| + \| e \|) \end{aligned}$$

and even

$$\| e \|_{L^\infty} \le \| u-r_h u \|_{L^\infty} + c \; h^{-1/2} (\| \nabla_h (u-r_h u) \| + \| \nabla_h e \|) .$$

Then assumption (3v), combined with the already proved error estimates (5) and (7), lead to

$$\| e \|_{L^\infty} \le c \; h^{1/4} (\| \Delta u \| + \| \Delta_h u_h \|) + c(t) h^{1/4}$$

and to the desired estimate (8)

$$\| e \|_{L^\infty} \le c(t) h^{1/2} \; .$$

In two dimensions, n = 2, one proceeds analoguously. So, in order to prove also the estimate (6), we have to bound the norm $\|\Delta_h u_h(t)\|$ for all times $t \in [o,T)$. This may be done in a similar way as for the norm $\|\Delta u(t)\|$ by inserting $\phi_h = -\Delta_h u_{ht}$ as test function into equation (4) (see [4] for the conforming case). This gives

$$\|\nabla_h u_{ht}\|^2 + \frac{1}{2}\frac{d}{dt}\|\Delta_h u_h\|^2 = \frac{d}{dt} b_h(u_h,u_h,\Delta_h u_h) + \frac{d}{dt}(f,\Delta_h u_h) -$$

$$- (f_t,\Delta_h u_h) - b_h(u_{ht},u_h,\Delta_h u_h) - b_h(u_h,u_{ht},\Delta_h u_h) .$$

By a somewhat complicated calculation the terms on the right hand side may be estimated in terms of $\|\nabla_h u_{ht}\|$ and $\|\Delta_h u_h\|$ such that again Gronwall's inequality yields the desired bound

$$\|\Delta_h u_h(t)\| \le c(t) \quad , t \in [o,T) .$$

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Address: The University of Michigan
Department of Mathematics
Ann Arbor, Michigan 481o9 (USA)

Institut für Angewandte Mathematik
der Universität Bonn
Beringstraße 4-6
D-53oo Bonn 1 (West-Germany)