

A note on the uniqueness of weak solutions for the Navier-Stokes equations

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ABSTRACT. Consider the Navier-Stokes equations with the initial data $a \in L^2_\sigma(\mathbb{R}^d)$. Let u and v be two weak solutions with the same initial value a . If $\nabla u \in L^{\frac{2}{2-r}}((0, T); \dot{X}_r(\mathbb{R}^d)^d)$ where $\dot{X}_r(\mathbb{R}^d)$ is the multiplier space (see the definition in the text), then we have $u = v$.

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1. Introduction

Consider the Navier-Stokes equations in $(0, T) \times \mathbb{R}^d$ with $0 < T < \infty$ and $d \geq 3$

$$(1.1) \quad \begin{aligned} \partial_t u + (u \cdot \nabla) u - \Delta u + \nabla p &= 0, & (x, t) \in \mathbb{R}^d \times (0, \infty), \\ \nabla \cdot u &= 0, & (x, t) \in \mathbb{R}^d \times (0, \infty), \\ u(x, 0) &= a(x), & x \in \mathbb{R}^d, \end{aligned}$$

where $u = u(x, t)$ is the velocity field, $p = p(x, t)$ is the scalar pressure and $a(x)$ with $\operatorname{div} a = 0$ in the sense of distribution is the initial velocity field. For simplicity, we assume that the external force has a scalar potential and is included into the pressure gradient.

In their famous paper, Leray [8] and Hopf [3] constructed a weak solution u of (1.1) for arbitrary $a \in L^2_\sigma$. The solution is called the Leray-Hopf weak solution. In the general case the problem on uniqueness of Leray-Hopf's weak solutions is still open question. Masuda [9] extended Serrin's class for uniqueness of weak solutions

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and made it clear that the class $L^\infty((0, T); L^d(\mathbb{R}^d))$ plays an important role for uniqueness of weak solutions. Kozono-Sohr [5] showed that the uniqueness holds in $L^\infty((0, T); L^d)$.

Foias [1] and Serrin [10] introduced the class $L^\alpha((0, \infty); L^q)$ and showed that under the additional assumption

$$u \in L^\alpha((0, \infty); L^q) \quad \text{for} \quad \frac{2}{\alpha} + \frac{d}{q} = 1 \quad \text{with} \quad q > d,$$

u is the only weak solution.

The purpose of this note is to improve the criterion on uniqueness of weak solutions to in the class $L^{\frac{2}{2-r}}((0, T); \dot{X}_r(\mathbb{R}^d))$. We know that for every $a \in L^2_\sigma(\mathbb{R}^d)$, there is at least one weak solution u of (1.1) satisfying the energy inequality. Here we mean by the weak solution a function u in $u \in L^\infty((0, T); L^2_\sigma) \cap L^2((0, T); \dot{H}^1_\sigma)$ satisfying (1.1) in the sense of distributions (Definition 2). For more facts concerning uniqueness of weak solutions, we refer to a celebrated paper of Kozono and Sohr [5] (see also [2]).

Now, we give a description of the multiplier space \dot{X}_r introduced recently by P.G. Lemarié-Rieusset in his work [6] (see also [7]). The space \dot{X}_r of pointwise multipliers which map L^2 into \dot{H}^{-r} is defined in the following way

DEFINITION 1. For $0 \leq r < \frac{d}{2}$, we define the homogeneous space \dot{X}_r by

$$\dot{X}_r = \left\{ f \in L^2_{loc} : \forall g \in \dot{H}^r \quad fg \in L^2 \right\}$$

where we denote by $\dot{H}^r(\mathbb{R}^d)$ the completion of the space $\mathcal{D}(\mathbb{R}^d)$ with respect to the norm $\|u\|_{\dot{H}^r} = \left\| (-\Delta)^{\frac{r}{2}} u \right\|_{L^2}$.

The norm of \dot{X}_r is given by the operator norm of pointwise multiplication

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leq 1} \|fg\|_{L^2}$$

We have the homogeneity properties : $\forall x_0 \in \mathbb{R}^d$

$$\begin{aligned} \|f(\cdot + x_0)\|_{\dot{X}_r} &= \|f\|_{\dot{X}_r} \\ \|f(\lambda \cdot)\|_{\dot{X}_r} &= \frac{1}{\lambda^r} \|f\|_{\dot{X}_r}, \quad \lambda > 0. \end{aligned}$$

Additionally, for $0 \leq r < \frac{d}{2}$, we have the following inclusion relations :

$$L^{\frac{d}{r}}(\mathbb{R}^d) \subset L^{\frac{d}{r}, \infty}(\mathbb{R}^d) \subset \dot{X}_r(\mathbb{R}^d).$$

where $L^{p, \infty}$ denotes the usual Lorentz (weak L^p) space. For the definition and basic properties of Lorentz spaces $L^{p, q}$ we refer to [11].

2. Uniqueness theorem

Before turning our attention to uniqueness issues, we start with some prerequisites for our main result. Let

$$C_{0, \sigma}^\infty(\mathbb{R}^d) = \left\{ \varphi \in (C_0^\infty(\mathbb{R}^d))^d : \operatorname{div} \varphi = 0 \right\} \subseteq (C_0^\infty(\mathbb{R}^d))^d.$$

The subspace

$$L^2_\sigma(\mathbb{R}^d) = \overline{C^\infty_{0,\sigma}(\mathbb{R}^d)}^{\|\cdot\|_{L^2}} = \left\{ u \in L^2(\mathbb{R}^d)^d : \operatorname{div} u = 0 \right\}$$

obtained as the closure of $C^\infty_{0,\sigma}$ with respect to L^2 -norm $\|\cdot\|_{L^2}$. H^r_σ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|u\|_{H^r} = \|u\|_{L^2} + \left\| (1 - \Delta)^{\frac{r}{2}} u \right\|_{L^2}, \text{ for } r \geq 0.$$

Our definition of Leray-Hopf weak solutions (see e.g. [5]) now reads :

DEFINITION 2 (weak solutions). *Let $a \in L^2_\sigma$ and $T > 0$. A measurable function u is called a weak solution of (1.1) on $(0, T)$ if u satisfies the following properties*

- (1): $u \in L^\infty((0, T); L^2_\sigma) \cap L^2\left((0, T); \dot{H}^1_\sigma\right)$ for all $T > 0$;
- (2): $u(t)$ is continuous in time in the weak topology of L^2_σ with

$$\langle u(t), \phi \rangle \rightarrow \langle a, \phi \rangle \text{ as } t \rightarrow 0^+$$

for all $\phi \in L^2_\sigma$;

- (3): for any $0 \leq s \leq t \leq T$, u satisfies the identity

$$(2.1) \quad \int_s^t \left\{ -\langle u, \partial_\tau \phi \rangle + \langle u, \nabla u, \phi \rangle + \langle \nabla u, \nabla \phi \rangle \right\} d\tau = -\langle u(t), \phi(t) \rangle + \langle u(s), \phi(s) \rangle,$$

for all $\phi \in H^1((s, t); H^1_\sigma)$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product and $\|\cdot\|_{L^2}$ denotes the norm in $L^2(\mathbb{R}^d)^d$.

REMARK 1. For u and ϕ as above, the integral

$$\int_0^T \langle u, \nabla u, \phi \rangle d\tau$$

is well defined since we have by the Sobolev inequality

$$\|u\|_{L^{\frac{2d}{d-2}}} \leq C \|\nabla u\|_{L^2}$$

that

$$\begin{aligned} \left| \int_0^T \langle u, \nabla u, \phi \rangle d\tau \right| &\leq \int_0^T \|u\|_{L^{\frac{2d}{d-2}}} \|\nabla u\|_{L^2} \|\phi\|_{L^d} d\tau \\ &\leq C \sup_{0 < t < T} \|\phi\|_{L^d} \int_0^T \|\nabla u\|_{L^2}^2 d\tau \end{aligned}$$

Existence of weak solutions has been established by Leray in [8] for initial velocity in $L^2_\sigma(\mathbb{R}^d)$. The result is the following

THEOREM 1 (Leray - Hopf). *Let $T > 0$. Then, for any given $a \in L^2_\sigma(\mathbb{R}^d)$, there exists at least one weak solution u to (1.1) on $(0, T)$ such that*

$$(2.2) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds \leq \|a\|_{L^2}^2, \quad 0 \leq t < T.$$

and

$$\|u(t) - a\|_{L^2} \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Let us introduced the class $L^s((0, T); L^\gamma)$ with the norm $\|\cdot\|_{L^s((0, T); L^\gamma)}$

$$\|u\|_{L^s((0, T); L^\gamma)} = \left(\int_0^T \|u(t)\|_{L^\gamma}^s dt \right)^{\frac{1}{s}}.$$

The classical result on uniqueness of weak solutions in the class $L^s((0, T); L^\gamma)$ was given by Foias, Serrin and Masuda [1], [10], [9].

THEOREM 2 (Foias-Serrin-Masuda). *Let $a \in L^2_\sigma(\mathbb{R}^d)$. Let u and v are two weak solutions of (1.1) on $(0, T)$. Suppose that u satisfies*

$$(2.3) \quad u \in L^s((0, T); L^\gamma) \quad \text{for} \quad \frac{2}{s} + \frac{d}{\gamma} = 1 \quad \text{with} \quad d < \gamma < \infty.$$

Assume that v fulfills the energy inequality (2.2) for $0 \leq t < T$. Then we have $u = v$ on $[0, T)$.

REMARK 2. *In Theorem 2, v not need belong to the class (2.3). On the other hand, every weak solution u with (2.3) fulfills the energy identity*

$$(2.4) \quad \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds = \|a\|_{L^2}^2, \quad 0 \leq t \leq T.$$

It seems to be an interesting question whether every weak solution satisfies the energy inequality (2.2).

REMARK 3. *The class (2.3) is important from the view point of scaling invariance for the Navier-Stokes equations. It can be easily seen that if is a pair of the solution to (1.1) on $\mathbb{R}^d \times (0, T)$, then so is the family $\{u_\lambda, p_\lambda\}_{\lambda > 0}$ where*

$$u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t), \quad p_\lambda(x, t) = \lambda^2 u(\lambda x, \lambda^2 t).$$

Scaling invariance means that there holds

$$\|u_\lambda\|_{L^s((0, \infty); L^\gamma)} = \left(\lambda^{1 - (\frac{2}{s} + \frac{d}{\gamma})} \|u\|_{L^s((0, \infty); L^\gamma)} \right) = \|u\|_{L^s((0, \infty); L^\gamma)} \quad \text{for all } \lambda > 0$$

if and only if

$$\frac{2}{s} + \frac{d}{\gamma} = 1.$$

We shall next deal with the critical case with $s = \infty$ and $\gamma = d$ in (2.3).

THEOREM 3 (Masuda [9], Kozono-Sohr [5]). *Let $a \in L^2_\sigma(\mathbb{R}^d)$. Let u and v be two weak solutions of (1.1) on $(0, T)$. Suppose that*

$$(2.5) \quad u \in L^\infty((0, T); L^d)$$

and that v fulfills the energy inequality (2.2) for all $0 \leq t < T$. Then we have $u = v$ on $[0, T)$.

REMARK 4. *Masuda [9] proved that if $u \in L^\infty((0, T); L^d)$ is continuous from the right on $[0, T)$ in the norm of L^d , then there holds $u = v$ on $[0, T)$. Later on, Kozono-Sohr [5] showed that every weak solution in $L^\infty((0, T); L^d)$ of (1.1) on $(0, T)$ becomes necessarily continuous from the right in the norm of L^d .*

The same result holds when, for $\gamma = +\infty$, we replace the assumption

$$u \in L^2((0, T); L^\infty)$$

by the weaker assumption

$$\nabla u \in L^2((0, T); \dot{X}_1(\mathbb{R}^d)^d).$$

The replacement of hypothesis $u \in L^2((0, T); L^\infty)$ by $\nabla u \in L^2((0, T); \dot{X}_1(\mathbb{R}^d)^d)$ was recently discussed in a similar context by Gala [2]. Moreover, we have

THEOREM 4 (Gala). *Let $a \in L^2_\sigma(\mathbb{R}^d)$ and let u, v be two weak solutions of (1.1) on $(0, T)$. Suppose that*

$$(2.6) \quad \nabla u \in L^2((0, T); \dot{X}_1(\mathbb{R}^d)^d)$$

and that v fulfills the energy inequality (2.2) for $0 \leq t < T$. Then we have $u = v$ on $[0, T]$.

REMARK 5. *By Theorem 2, every weak solution in $L^2((0, T); L^\infty)$ is unique.*

Our result on uniqueness of the weak solution now reads :

THEOREM 5. *Let $a \in L^2(\mathbb{R}^d)^d$ with $\nabla \cdot a = 0$. Assume that there exists a solution u for the Navier-Stokes equations on $(0, T) \times \mathbb{R}^d$ (for some $T \in (0, +\infty]$) with some initial data a so that*

$$u \in L^\infty((0, T); L^2_\sigma(\mathbb{R}^d)^d) \cap L^2((0, T); \dot{H}^1_\sigma(\mathbb{R}^d)^d),$$

and

$$\nabla u \in L^{\frac{2}{2-r}}((0, T); \dot{X}_r(\mathbb{R}^d)^d) \quad \text{for all } 0 \leq r \leq 1.$$

Then, u is the unique Leray-Hopf solution associated with a on $[0, T]$.

The following corollary, which is an immediate consequence of Theorem 5 gives a simpler sufficient condition in term of Lorentz spaces.

COROLLARY 1. *Let $a \in L^2(\mathbb{R}^d)^d$ with $\nabla \cdot a = 0$. Assume that there exists a solution u for the Navier-Stokes equations on $(0, T) \times \mathbb{R}^d$ (for some $T \in (0, +\infty]$) with some initial data a so that*

$$u \in L^\infty((0, T); L^2_\sigma(\mathbb{R}^d)^d) \cap L^2((0, T); \dot{H}^1_\sigma(\mathbb{R}^d)^d),$$

and

$$\nabla u \in L^{\frac{2}{2-r}}((0, T); L^{\frac{d}{r}, \infty}(\mathbb{R}^d)^d),$$

where $L^{p, \infty}$ denotes the usual Lorentz (weak L^p) space. Then, u is the unique Leray-Hopf solution associated with a on $[0, T]$.

The same result again holds when the assumption

$$\nabla u \in L^{\frac{2}{2-r}}((0, T); L^{\frac{d}{r}, \infty}(\mathbb{R}^d)^d)$$

is replaced by

$$u \in L^{\frac{2}{2-r}}((0, T); L^{\frac{d}{r}}(\mathbb{R}^d)^d).$$

We are now in a position to proof the main result.

PROOF. Let v be another weak solution of (1.1) associated to a on $(0, T)$ (with associated pressure p) such that

$$v \in L^\infty \left((0, T); L^2_\sigma(\mathbb{R}^d)^d \right) \cap L^2 \left((0, T); \dot{H}^1_\sigma(\mathbb{R}^d)^d \right)$$

and

$$\nabla u \in L^{\frac{2}{2-r}} \left((0, T); \dot{X}_r(\mathbb{R}^d)^d \right).$$

We consider the difference $w = u - v$ and we obtain

$$\begin{aligned} \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds &\leq \|a\|_{L^2}^2, \\ \|v(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla v(s)\|_{L^2}^2 ds &\leq \|a\|_{L^2}^2. \end{aligned}$$

On the other hand, we have

$$\langle u(t), v(t) \rangle + 2 \int_0^t \langle \nabla u(s), \nabla v(s) \rangle ds = \|a\|_{L^2}^2 + \int_0^t \langle w \cdot \nabla u, w \rangle (s) ds$$

for all $0 \leq t < T$. Combining the above inequalities, we obtain

$$\begin{aligned} &\|w(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla w(s)\|_{L^2}^2 ds \\ &= \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s)\|_{L^2}^2 ds + \|v(t)\|_{L^2}^2 \\ &\quad + 2 \int_0^t \|\nabla v(s)\|_{L^2}^2 ds - 2 \langle u(t), v(t) \rangle - 4 \int_0^t \langle \nabla u(s), \nabla v(s) \rangle ds \\ (2.7) \quad &\leq -2 \int_0^t \langle w \cdot \nabla u, w \rangle ds. \end{aligned}$$

We thus observe that by Young inequality

$$(a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq a + b \text{ with } a, b \geq 0 \text{ and } 0 \leq \alpha \leq 1),$$

it follows that

$$\begin{aligned} \left| \int_0^t \langle w \cdot \nabla u, w \rangle ds \right| &\leq \int_0^t \|w \cdot \nabla u(s)\|_{L^2} \|w(s)\|_{L^2} ds \\ &\leq \int_0^t \|w(s)\|_{\dot{H}^r} \|\nabla u(s)\|_{\dot{X}_r} \|w(s)\|_{L^2} ds \\ &\leq \int_0^t \|w(s)\|_{L^2}^{1-r} \|\nabla w(s)\|_{L^2}^r \|\nabla u\|_{\dot{X}_r} \|w(s)\|_{L^2} ds \\ &\leq \int_0^t \left(\|w(s)\|_{L^2}^2 \|\nabla u\|_{\dot{X}_r}^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \left(\|\nabla w(s)\|_{L^2}^2 \right)^{\frac{r}{2}} ds \\ &\leq \frac{1}{2} \int_0^t \|\nabla w\|_{L^2(\mathbb{R}^d)}^2 ds + \frac{C}{2} \int_0^t \|w\|_{L^2(\mathbb{R}^d)^d}^2 \|\nabla u\|_{\dot{X}_r(\mathbb{R}^d)}^{\frac{2}{2-r}} ds. \end{aligned}$$

where we used the following ones ($0 \leq r \leq 1$)

$$\|\omega\|_{\dot{H}^r} = \frac{1}{(2\pi)^{\frac{d}{2}}} \| |\xi|^r \widehat{\omega} \|_{L^2} \leq \|\omega\|_{L^2}^{1-r} \|\nabla \omega\|_{L^2}^r.$$

Hence by (2.7) there holds

$$\|w(t)\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 d\tau \leq C \int_0^t \|w\|_{L^2(\mathbb{R}^d)}^2 \|\nabla v\|_{\dot{X}_r(\mathbb{R}^d)}^{\frac{2}{2-r}} d\tau$$

for all $t > 0$. Since $\nabla u \in L^{\frac{2}{2-r}}((0, T); \dot{X}_r(\mathbb{R}^d))$ and since $w(0) = 0$, it follows from the Gronwall inequality that

$$\|w(t)\|_{L^2}^2 \leq \|w(0)\|_{L^2}^2 \exp\left(C \int_0^t \|\nabla u\|_{\dot{X}_r(\mathbb{R}^d)}^{\frac{2}{2-r}} ds\right),$$

and thus

$$\|w(t)\|_{L^2}^2 = 0, \quad 0 \leq t < T$$

and implies uniqueness of weak solutions. \square

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