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NAVIER-STOKES EQUATIONS IN EXTERIOR DOMAINS

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SUNTO. — Si dimostra l'esistenza e regolarità delle soluzioni di Navier-Stokes in dominio aperto.

The Navier-Stokes equations

$$(1) \qquad \begin{aligned} \mathbf{v}_t - \nu \Delta \mathbf{v} + \nabla p + (\mathbf{v} \cdot \nabla) \mathbf{v} &= \mathbf{f} \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned}$$

are assumed to govern the motion of a viscous, incompressible fluid which is described by the vector field $\mathbf{v}(x, t)$ denoting the velocity of a particle at position x and time t and by the scalar function p, the pressure. As usual, $\nu > 0$ is a constant which is called kinematic viscosity, and $\mathbf{f}$ is the density of the exterior forces. To consider the equations of motion in an exterior domain $\mathcal{E} \subset \mathbb{R}^3$ (which is the complement of a bounded domain Ω with boundary Σ) has proved useful in the study of flows past rigid bodies that move in a big reservoir of fluid. For if one wants to derive physical or geometrical properties of the flow the assumption of an infinite container makes the flow independent of the particular shape of the container's walls; they are always far away from the body and hence are assumed to have nearly no influence on the behavior of the flow in its neighborhood.

As an example of this type we consider a rigid body, represented by Ω, that moves with prescribed velocity $\mathbf{U}$; $\mathbf{U}$ generally depends on time. Let $\mathbf{v}_0(x)$ be a solution to the stationary Navier-Stokes equations

$$(2) \qquad \begin{aligned} -\nu\Delta \mathbf{v} + \nabla p + (\mathbf{v}\cdot\nabla)\mathbf{v} &= \mathbf{f} \\ \nabla\cdot\mathbf{v} &= 0 \end{aligned}$$

in the complement of Ω when the body moves with constant velocity $\mathbf{U}_0$. If we assume that the reference frame is attached to Ω the velocity of the fluid tends to a constant limit $-\mathbf{U}_0$ at infinity. As the Navier-Stokes equations are invariant under Galilean transformations this leads to (2) with the boundary conditions

$$(3) \qquad \mathbf{v}(x) = 0 \qquad \forall x \in \Sigma$$

$$(4) \qquad \mathbf{v}(x) \to -\mathbf{U}_0 \quad \text{as} \quad |x| \to \infty .$$

We now ask whether a solution $\mathbf{v}_0(x)$ of (2)-(4) is stable with respect to small disturbances $\mathbf{v}'$. The means, does there exist a solution $\mathbf{v}(x, t)$ to (1) with initial values $\mathbf{v}_0(x) + \mathbf{v}'(x)$ and does $\mathbf{v}(x, t)$ tend to $\mathbf{v}_0(x)$ for $t \to \infty$? Another example is the starting problem: Let at time $t = 0$ both the fluid and the body be at rest; then accelerate the body within some finite interval of time $(0, t_0)$ such that its velocity at t_0 equals $\mathbf{U}_0$, and maintain this velocity for all $t > t_0$. Does the solution $\mathbf{v}(x, t)$ of this acceleration process tend to $\mathbf{v}_0$, the solution of the stationary problem?

These two questions involve problems of existence, uniqueness, and asumptotic behavior for large t of solutions to the non-stationary system (1); but before we present the results for the initial value problems we briefly recall some basic facts about solutions to the stationary equations because their properties require special methods for the time-dependent case.

The first contribution to (2)-(4) is due to J. Leray [7] who proved the existence of at least one solution for arbitrary but smooth data provided the condition at infinity is weakened in the following sense

$$(5) \qquad \int_{\mathcal{E}} \frac{|\mathbf{v}_0(x) - \mathbf{U}_0|^2}{|x-y|^2}\, dx < C \quad \text{uniformly in } y \in \mathcal{E} .$$

The solution is characterized by finite Dirichlet's integral $\int_{\mathcal{E}} |\nabla \mathbf{v}_0|^2 \, dx$. Later R. Finn proved that $\mathbf{v}_0$ is continuous at infinity. He also introduced another function class by

$$(6) \qquad |\mathbf{v}_0(x) - \mathbf{U}_0| < C\,|x|^{-1/2-\varepsilon} \quad \text{as} \quad |x| \to \infty$$

with positive constants C and ε, and Finn showed (¹) that there exists a solution to (2), (3), (6) for small data. These solutions have significant physical and geometrical properties; first there is an expansion at infinity

$$(7) \qquad \mathbf{v}_0(x) = \mathbf{U}_0 + \mathbf{F}_\Sigma \cdot \mathbf{E}\,(x, 0) + 0(|x|^{-2}) \qquad \text{for } |x| \text{ large,}$$

where

$$(8) \qquad \mathbf{F}_\Sigma = \oint_\Sigma \mathbf{T} \cdot \mathbf{n}\; d\sigma, \; T_{ij} = -p_0\,\delta_{ij} + \nu \left(\frac{\partial v^i_0}{\partial x^j} + \frac{\partial v^j_0}{\partial x^i}\right),$$

is the force exerted on the body, and E_{ij} is the fundamental solution tensor to the Oseen equations

$$(9) \qquad \begin{aligned} -\nu \triangle \mathbf{v} + \nabla p + (\mathbf{U}_0 \cdot \nabla)\,\mathbf{v} &= 0 \\ \nabla \cdot \mathbf{v} &= 0\,. \end{aligned}$$

From (7) and the estimates for $\mathbf{E}$ at infinity one deduces the existence of a wake behind Ω: There is a region of the form of a paraboloid with axis in the direction of $\mathbf{U}_0$ in which $\mathbf{v}_0$ differs much more from the limiting velocity $\mathbf{U}_0$ than outside of it. Another consequence of (7) concerns the kinetic energy associated with the flows; as $\mathbf{E}$ is not square integrable we get

$$(10) \qquad E_{kin} = \frac{\varrho}{2} \int_{\mathcal{E}} |\mathbf{v}_0(x) - \mathbf{U}_0|^2 \, dx = +\infty$$

unless $\mathbf{F}_\Sigma = 0$; this can happen only in some cases with boundary values different from (3), for instance if Ω propels itself by rotating about an axis; such situations will not be considered here. Solutions satisfying (6) at infinity were termed « physically reasonable » (*PR*-solutions) by Finn.

(¹) For the proofs we refer to [4] and earlier papers cited there; see also Finn's survey articles [2], [3] where our problems are formulated for the first time.

These properties of steady state solutions $\mathbf{v}_0$ are clearly important for the investigation of our problems mentioned above. As we want the time-dependent solution $\mathbf{v}(x, t)$ to converge to $\mathbf{v}_0(x)$ we have to make sure that the method we may apply in the existence proof does not exclude the solution a priori from converging to $\mathbf{v}_0(x)$. This would be the case, for instance, if the existence theorem for $\mathbf{v}(x, t)$ were based on energy estimates, probably the best known method for proving existence at all. For a solution $\mathbf{v}(x, t)$ whose kinetic energy (10) is bounded for all t by some fixed constant cannot tend to a stationary solution of class PR.

We now formulate the main result of [1]. Let A denote Oseen's operator, defined for $C^{2+\alpha}$-functions by the boundary value problem

$$(1)\qquad \begin{cases} -\nu\Delta\mathbf{v} + \nabla p + (\mathbf{U}_0\cdot\nabla)\mathbf{v} = f \\ \qquad\qquad\qquad \nabla\cdot\mathbf{v} = 0 \end{cases} \text{in } \mathcal{E}$$
$$\mathbf{v}(x) = 0 \text{ on } \Sigma,\quad \mathbf{v}(x)\to 0 \text{ as } |x|\to\infty,$$

and A^γ a fractional power of A.

THEOREM 1. - Consider the initial value problem

$$(12)\qquad \begin{cases} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla p + (\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{f} \\ \qquad\qquad\qquad \nabla\cdot\mathbf{v} = 0 \qquad \text{in } \mathcal{E}\times(0,T) \\ \mathbf{v}(x,t) = 0 \quad \forall x\in\Sigma \quad \forall t\in[0,T] \\ |\mathbf{v}(x,t) - \mathbf{U}_0| \leqq C|x|^{-1/2-\varepsilon}, \text{ as } |x|\to\infty \\ \mathbf{v}(x,0) = \mathbf{v}^*(x) \quad \forall x\in\mathcal{E}. \end{cases}$$

There exists a unique classical solution for arbitrary values of T if the data are small in the following sense

$$\mathbf{v}^*\in D(A^\delta) \quad\text{and}\quad \|\mathbf{v}^*\|_{C^{1+\beta}} \ll 1,$$

where $D(A^\gamma)$ denotes the domain of definition of A^γ. The solution is an element of $C^0((0,T), C^{2+\alpha}(\mathcal{E}))\cap C^1((0,T), C^{0+\alpha}(\mathcal{E}))$ and attains the initial data continuously in the $C^{0+\alpha}(\mathcal{E})$-norm.

The proof uses an approach by semigroups in Hölder-spaces, that is the generator of the semigroup — in our case Osen's operator A — acts on suitable subspaces of $C^{0+\alpha}(\mathcal{E})$. In this way we avoid solutions with finite integral norms of $\mathbf{v} - \mathbf{U}_0$, especially with finite kinetic energy, cf. (10). For parabolic equations and systems this approach was used first by W. von Wahl [8].

The main differences to the usual semigroup concept when the generators are defined on subspaces of L_p lies in the resolvent estimates. If L is an elliptic operator of second order with zero Dirichlet data there holds the resolvent estimate of S. Agmon

$$(13) \qquad \|(L-\lambda)^{-1}\|_{L_p} \leqq \frac{C}{|\lambda|}$$

if λ is not in the spectrum of L. If we regard L as an operator on $C^{0+\alpha}$ we get instead of (13)

$$(14) \qquad \|(L-\lambda)^{-1}\|_{C^{0+\alpha}} \leqq \frac{C}{|\lambda|^{1-\alpha/2}},$$

and von Wahl proved that this estimate is sharp with respect to the decay in $|\lambda|$. For the linearization (11) we prove inequality (14), that is

THEOREM 2. - Let $P \subset \mathbb{C}$ be the domain which is bounded by the parabola $|\alpha| = c(U, \nu)\beta^2$, $-\infty < \alpha < 0$, where $\alpha + i\beta$ denotes a point in $\mathbb{C}$. Then

$$(15) \qquad \begin{cases} -\nu\Delta\mathbf{v} + \nabla p + (\mathbf{U}_0 \cdot \nabla)\mathbf{v} + \lambda\,\mathbf{v} = f \\ \qquad\qquad\qquad\qquad \nabla \cdot \mathbf{v} = 0 \end{cases} \text{ in } \mathcal{E}$$
$$\mathbf{v}(x) = 0 \text{ on } \Sigma, \ |\mathbf{v}| \to 0 \text{ as } |x| \to \infty$$

possesses a unique solution for $\lambda \notin P$, and for large $|x|$ we have the decay $|\mathbf{v}| \leqq C|x|^{-1}$ if $\lambda = 0$ and $|\mathbf{v}| \leqq C|x|^{-3}$ if $\lambda \neq 0$.

$$(16) \qquad |\lambda|\,\|\mathbf{v}\|_0 + |\lambda|^{1-\alpha/2}\|\mathbf{v}\|_{0+\alpha} + |\lambda|^{1/2}\|\mathbf{v}\|_1 + |\lambda|^{1/2-\alpha/2}\|\mathbf{v}\|_{1+\alpha} + \\ + \|\mathbf{v}\|_2 + |\lambda|^{-\alpha/2}\|\mathbf{v}\|_{2+\alpha} \leqq C\|\mathbf{f}\|_{0+\alpha},$$

where $\|\ \|_k$ denotes the $C^k(\mathcal{E})$-norm.

The proof uses methods of potential theory; one starts with the construction of a fundamental solution tensor to (15) and introduces potentials of single and double layer. As in classical potential theory one is led to integral equations of Fredholm type; for $\lambda \notin P$ they are uniquely solvable. From this procedure there follows a representation of the solution by integrals over Σ and $\mathcal{E}$ and from this the estimate (16) can be derived.

From the proof we can also get some insight into the special structure of (16) and therefore also of (14). The fundamental solution tensor E_{ij} of the Oseen system (15) is for small $|x-y|$ of the form

$$E_{ij}(x, y; \lambda) = \varphi(x-y)\, e^{-\sqrt{\lambda}|x-y|} + \text{higher order terms,} \tag{17}$$

where φ grows like $|x-y|^{-1}$ (independently of λ), and the terms of higher order have no influence on the decay in $|\lambda|$. So for an arbitrary component of $\nabla^2 \mathbf{E}$ we have to study integrals of the form $\int [\nabla^2 \varphi + \psi]\, f(y)\, dy$ when we separate the strongly singular part. By the Hölder-Korn-Lichtenstein inequality we get for the singularity

$$\left[\int \frac{\omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^3} f(y)\, dy \right]_\alpha \leqq C[f]_\alpha ,$$

and there is left a weakly singular kernel

$$|\lambda| e^{-\sqrt{|\lambda|}|x-y|}\, |x-y|^{-1} + |\lambda|^{-1/2} e^{-\sqrt{|\lambda|}\,|x-y|}\, |x-y|^{-2}.$$

So consider for example

$$\left| \int |\lambda|^{1/2} \frac{e^{-\sqrt{|\lambda|}\,|x-y|}}{|x-y|^2} f(y)\, dy - \int |\lambda|^{1/2} \frac{e^{-\sqrt{|\lambda|}\,|z-y|}}{|z-y|^2} f(y)\, dy \right| \leqq$$

$$\leqq \left| \int |\lambda|^{3/2} \left\{ \frac{e^{-\sqrt{|\lambda|}\,|x-y|}}{||\lambda|^{1/2}(x-y)|^2} - \frac{e^{-\sqrt{|\lambda|}\,|z-y|}}{||\lambda|^{1/2}(z-y)|^2} f(y)\, dy \right\} \right| \leqq$$

$$\leqq C|\lambda|^{\alpha/2}\, |z-x|^\alpha\, ||f||_{C^0},$$

because $|\lambda|^{3/2}$ gives exactly the volume element of the transformation $y \to |\lambda|^{1/2} y \equiv \tilde{y}$, and $|\lambda|^{\alpha/2}$ is the $C^{0+\alpha}$-seminorm of $e^{-|\lambda^{1/2}|x|}$. The last estimate explains the difference between L_p- and $C^{0+\alpha}$-bounds: The $C^{0+\alpha}$-norm of $e^{-|\lambda|^{1/2}|x|}$ gives the factor $|\lambda|^{\alpha/2}$ whereas the L_p-norm is independent of λ, $|\lambda|$ large. That the various bounds in $C^{0+\alpha}$, $C^{1+\alpha}$, and $C^{2+\alpha}$ differ by $|\lambda|^{1/2}$, follows from (17), too, because each differentiation gives $|\lambda|^{1/2}$.

Once the existence of the semigroup e^{-tA} is established we get a local solution of (12) by investigating the integral equation

$$\mathbf{v}(t) = e^{-tA}\, \mathbf{v}_0 + \int_0^t e^{-\tau A} N(\mathbf{v}(\tau))\, d\tau \qquad \text{in } \mathcal{X}, \tag{18}$$

where $N(\mathbf{v})$ is the nonlinearity $(\mathbf{v} \cdot \nabla)\mathbf{v}$ and $\mathcal{X} = C^{0+\alpha}(\mathcal{E}) \cap \cap \{|\mathbf{v}| < C|x|^{-1}$ as $|x| \to \infty\}$. Exploiting the interpolation property

$$(19) \qquad \|\mathbf{v}\|_{C^{1+\beta}} \leqq C \, \|A^{\gamma} \mathbf{v}\|_{C^{0+\alpha}}$$

with suitable α, β, and γ for the fractional power $A^{\gamma} = [A^{-\gamma}]^{-1}$, where

$$A^{-\gamma}\mathbf{v} = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} e^{-sA} \, \mathbf{v} \, s^{\gamma-1} \, ds ,$$

we are led to

$$(20) \qquad \mathbf{w}(t) = e^{-tA} \, \mathbf{w}_0 + \int_0^t e^{-\tau A} N(A^{-\gamma} \mathbf{v}) \, d\tau .$$

This linear equation is solvable for $\mathbf{v} \in C^0([0, T], \mathcal{X})$ because $N(A^{-\gamma}\mathbf{v})$ is an element of $\mathcal{X}$, due to (19). The nonlinearity $N(A^{-\gamma}\mathbf{v})$ is compact in $\mathbf{v}$ and Lipschitz in $\nabla\mathbf{v}$, so we can use a fixed point theorem of H. Kielhöfer [6] to show that the following mapping F has a fixed point: F: $\mathbf{v} \to T\mathbf{v} \equiv \mathbf{w}$, and $\mathbf{w} = A^{\gamma} \tilde{\mathbf{w}}$, where $\tilde{\mathbf{w}}$ is a solution of (20).

The existence of a global solution (in time) follows from a bound of the form

$$(21) \qquad |e^{-tA}\mathbf{v}| \leqq C \, |x|^{-1} \quad \text{as} \quad |x| \to \infty$$

if $\mathbf{v}$ decays at infinity in the same way. The proof follows again from the fact that in the expression for the semigroup

$$e^{-tA} = \int_{\Gamma} e^{\lambda t} R(A ; \lambda) \, d\lambda$$

we have a representation for the resolvent $R(A ; \lambda)$ of Oseen's operator, which again can be estimated for $|x| \to \infty$. This concludes the proof of Theorem 1.

The questions we raised in the introduction concern the behavior of the solutions of (12) for $t \to \infty$; Do the disturbances of a steady state solutions decay or does the solution of the acceleration process tend to a PR-solution. Convergence for $t \to \infty$ holds in these cases, and the method of proof follows closely the analogous results for parabolic problems. We do not go into details here because our theorem is weaker as known results as they include also explicit estimates on the decay with respect to t.

Instead of this we want to investigate the conditions we posed on the initial value of $\mathbf{v}_0$ in Theorem 1. That the data have to be small in an appropriate norm was assumed in every existence theorem where regular global solutions were proved. In addition to this we have to assume that $\mathbf{v}_0$ lies in the domain of definition of a suitable fractional power of A. These domains $D(A^\gamma)$ cannot be characterized analytically so far, and therefore we have to show that this condition is satisfied, at least in the starting problem in which we are interested primarily.

During the acceleration process in the interval $(0, t_0)$ the space-time domain in which we have to solve the Navier-Stokes equations in noncylindrical whereas this property holds (after a Galilean transformation) if the immersed body moves with constant velocity; so Theorem 1 can be applied for all $t \geqq t_0$. On the other hand, the kinetic energy of the flow for $t \leqq t_0$ is certainly finite, and existence theorems based on energy estimates may be applied, cf. Heywood [5]. The question then arises: Is this solution $\mathbf{u}(x, t)$ for $t = t_0$ a suitable initial value for Theorem 1 such that $\mathbf{u}$ can be continued for all t? Smallness of $\mathbf{u}(x, t_0)$ can always be achieved by assuming $\mathbf{U}$ to be small, but to show $\mathbf{u}(t_0) \in D(A\ \ )$ we need the following result.

LEMMA. - Consider the initial value problem

$$(22)\qquad \begin{cases} \mathbf{u}_t - \nu\Delta\mathbf{u} + \nabla p + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{g} \\ \qquad\qquad\qquad\qquad \nabla\cdot\mathbf{u} = 0 \end{cases} \text{in } \mathcal{C}\times[0, t_0]$$
$$\mathbf{u}(x, t) = \mathbf{u}_0 \quad \forall x \in \Sigma \ \forall t \in [0, t_0]$$
$$\mathbf{u}_t(x, t) \to \mathbf{g} \text{ as } |x| \to \infty, \quad \forall t \in [0, t_0]$$
$$\mathbf{u}(x, 0) = 0 \quad \forall x \in \mathcal{C}.$$

Here $\mathbf{g}$ is a fictitious exterior force which corresponds to the fact that we describe the acceleration process in a reference frame that is attached to the body (and hence undergoes acceleration), such that (22) is an initial value problem in the space-time cylinder. Then (22) has a unique solution $\mathbf{u}(x, t)$ which fulfills the same regularity properties as the solution of Theorem 1. $\mathbf{u}(t_0)$ is an element of $D(S^\eta)$ with a suitable $\eta > 0$, where S is the Stokes-operator, and $D(S^\eta) \subset D(A^{\eta-\varepsilon})$ for any $\varepsilon > 0$ (2).

(2) This inclusion was suggested to me by Professor W. von Wahl.

S is defined on subspaces of $C^{0+\alpha}(\mathcal{E})$ by the boundary value problem for the Stokes equations

$$\left\{\begin{array}{ll} -\nu\Delta\, \mathbf{u} + \nabla\, p = \mathbf{f} & \\ \qquad \nabla \cdot u = 0 & \text{in } \mathcal{E} \\ \mathbf{u}(x) = 0 \text{ on } \Sigma;\ |\mathbf{u}(x)| \to 0 \text{ as } |x| \to \infty\,. \end{array}\right. \tag{23}$$

The existence of a solution of (22) can be shown in the same way as for (12); it is much simpler because solutions decay much more rapidly at spatial infinity. Especially we prove the existence of a Green's function $\mathcal{S}$ to the Stokes system (23); if we call the corresponding Green's function for the Oseen equations $\mathcal{A}$, we get for the resolvents representations by Green's functions

$$R(A\,;\lambda)\,\mathbf{f} = \int_{\mathcal{E}} \mathcal{S}(x, y\,; \lambda)\,\mathbf{f}\,(y)\,dy$$

$$R(S;\lambda)\ \mathbf{f} = \int_{\mathcal{E}} \mathcal{A}(x, y\,; \lambda)\,\mathbf{f}(y)\,dy\,.$$

From this explicit formulas we deduce the last proposition of the lemma.

Combining (22) and (12) we have solved the starting problem for all t but with different leariations for $t \leqq t_0$ and for $t \geqq t_0$. As $\mathbf{v}\,(x, t)$ assumes the initial values $\mathbf{u}\,(x, t_0)$ continuously it remains to show that the solution is differentiable with respect to time in t_0. To prove this we consider solutions $\tilde{\mathbf{u}}$ of (22) in $[0, t_0 + \delta]$ and $\tilde{\mathbf{v}}$ of (12) in $[t_0 + \delta, \infty)$ and we assume that δ is so small that $\tilde{\mathbf{u}}$ is a permissible initial value to (12). Let

$$\mathbf{U}\,(x, t) = \left\{\begin{array}{l}\mathbf{u}(x, t)\,,\ t \in [0, t_0] \\ \mathbf{v}(x, t)\,,\ t \in [t_0, \infty)\end{array}\right.;\ \tilde{\mathbf{U}}(x, t) = \left\{\begin{array}{l}\tilde{\mathbf{u}}(x, t)\,,\ t \in [0, t_0 + \delta] \\ \mathbf{v}(x, t)\,,\ t \in [t_0 + \delta, \infty)\,.\end{array}\right.$$

One can show now that $\mathbf{U}$ and $\tilde{\mathbf{U}}$ coincide, hence t_0 is a point interior to the interval $[0, t_0 + \delta]$, and $\tilde{\mathbf{u}}$ is differentiable in t; this completes the proof.

SUMMARY. — We prove existence and regularity of solutions to the non-stationary Navier-Stokes equations in exterior domains.

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