FREE BOUNDARY PROBLEMS FOR THE NAVIER-STOKES EQUATIONS

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Abstract: A free boundary problem for the Navier-Stokes equations describes the flow of a viscous, incompressible fluid in a domain that is unknown or partially unknown. In this paper several results for flows in drops or in vessels are presented. The free boundary is governed by self-attraction or surface tension, and dynamic contact angles may occur.

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§ 1. The Equations of Motion

To determine the shape of a fluid body is a classical problem in mathematical physics. If the liquid rotates about a fixed axis and is moreover subject to self-attraction the problem was already investigated by I.Newton as a model for the figure of the earth. Since it was treated for the first time in the Philosophiae Naturalis Principia Mathematica 300 years ago it has stimulated research in various branches of mathematical analysis as for example potential theory, bifurcation theory for nonlinear integral equations, and more recently it was taken up again in connection with variational methods for free boundary problems, see e.g. Friedman [F2] Chap.4.

According to Newton's law the force of self-attraction equals

$$DU(x) = D \int_\Omega \frac{\rho g}{|x-y|}\, dy ,$$

where ρ = const in the density, $\Omega \subset \mathbb{R}^3$ the domain occupied by the fluid, and g the gravitational constant. If the body rotates about the x^3-axis the centrifugal force is

$$DR(x) = D \frac{\omega^2}{2} r^2(x) ,$$

where ω denotes the angular velocity and $r(x) = \left[(x^1)^2 + (x^2)^2\right]^{1/2}$ the distance of a point x from the axis of rotation. With no other forces present the boundary Σ of Ω must be an equipotential surface:

$$(1.1) \qquad \int_{\Omega} \frac{\rho g}{|x-y|}\, dy + \frac{\omega^2}{2} r^2(x) = \text{const} \quad \forall x \in \Sigma .$$

As the total mass is prescribed, too, we have the side condition

$$(1.2) \qquad \text{meas}\ \Omega = V_o .$$

Relation (1.1) can easily be modified to cover other physical situations, too, like compressible fluids, figures with prescribed angular momentum or variable angular velocity. In this sense (1.1) can be regarded as the basis for all investigations on equilibrium figures if treated as problems in hydrostatics.

A related problem concerns rotating drops that are held together by surface tension rather than self-attraction. The boundary is now determined by

$$(1.3) \qquad 2\kappa H(x) + \rho\omega^2 r^2(x) = \text{const} \quad \forall x \in \Sigma ,$$

where $H(x)$ denotes the mean curvature of Ω at x, and κ is a material constant. If Σ is assumed to be of a specific topological type, (1.3) can be transformed into a differential equation for a scalar function whose graph is Σ. Solutions of the topological type of the sphere were investigated by E.Hölder [H]; toroidal figures are treated by R.Gulliver [G].

In §§3,4 we present some of the author's work on free boundary problems for the Navier-Stokes equations that can be regarded as dynamical versions of (1.1) or (1.3) because now we allow relative motions inside the fluid body. For a viscous and incompressible fluid a stationary flow inside the unknown domain Ω is governed by the following equations

$$(1.4) \qquad -\nu\Delta v + Dp + (v\cdot D)v = f \quad \text{in } \Omega$$
$$\text{div}\ v = 0$$

$$(1.5) \qquad v\cdot n = 0 , \quad t_k\cdot T(v,p)\cdot n = 0 \quad \text{on } \Sigma$$

together with one of the following conditions on the free boundary

$$(1.6) \qquad n\cdot T(v,p)\cdot n = 2\kappa H \quad \text{on } \Sigma ,$$

or

$$(1.7) \qquad n\cdot T(v,p)\cdot n = 0 \quad \text{on } \Sigma ,$$

depending whether surface tension is present or not. Here v and p denote the velocity and the pressure at x, ν is the kinematic viscosity, and $T(v,p)$ the stress tensor

$$(1.8) \qquad T_{ij}(v,p) = -p\delta_{ij} + \nu\{D_i v^j + D_j v^i\} .$$

The exterior normal to Σ is denoted by n, and t_1, t_2 span the tangent plane. The exterior force density f that generates the motion will be specified later.

It is easily checked that for $v = 0$ (with respect to a suitable reference frame) the free boundary problem (1.4) - (1.6) reduces to

(1.1) if we only set $f = DU$, and similarly for (1.4), (1.5) and (1.7). The physical assumption then is that the hydrostatic pressure is replaced by the normal stress if we pass from a static to a dynamic problem.

In the analogues to (1.1) and (1.3) the fluid occupies a bounded domain Ω , and its boundary Σ is a closed surface. The methods can be extended to treat also a layer of fluid where the capillary surface is a graph over all of $\mathbb{R}^2$. In these problems there is no contact between the free boundary and a rigid wall, which means that contact angle phenomena are excluded. For such a situation, namely the steady flow in a capillary tube that is partly filled with liquid the free boundary value problem was first solved by D.H.Sattinger [S]; he assumed the contact angle under which the free surface meets the wall of the cylindrical tube to be $\pi/2$. More general angles were studied by V.A.Solonnikov [SO]. In §5 we present some of the recent results obtained by D.Kröner [K] who studies the following two-dimensional problem:

$$(1.9) \qquad \left\{\begin{aligned} -\nu\Delta v + Dp + (v\cdot D)v &= 0 \\ \operatorname{div} v &= 0 \end{aligned}\right. \qquad \text{in } G$$

$$(1.10) \qquad v\cdot n = 0 \qquad \text{on } \partial G$$

$$(1.11) \qquad \left\{\begin{aligned} t\cdot T(v,p)\cdot n &= 0 \\ n\cdot T(v,p)\cdot n &= -\kappa H \end{aligned}\right. \qquad \text{on } \Sigma$$

$$(1.12) \qquad \nu D_1 v^2 + \gamma_o v^2 = 0 \qquad \text{on } \Gamma_o$$

$$(1.13) \qquad \nu D_2 v^1 + \gamma v^1 = -\gamma S \qquad \text{on } \Gamma$$

$$(1.14) \qquad g(1) = 0 \ , \ g(0) = 0 \ .$$

The domain G which is occupied by the fluid is given as

$$(1.15) \qquad G = \{(x^1,x^2) \in \mathbb{R}^2 : 0 < x^2 < 1 \ ; \ g(x^2) < x^1 < x_o^1\} \ ,$$

and its boundary consists of

$$(1.16) \qquad \left\{\begin{aligned} \Gamma_o &= \{(x^1,x^2) \in \mathbb{R}^2 : 0 < x^2 < 1 \ , \ x^1 = x_o^1\} \\ \Gamma &= \{(x^1,x^2) \in \mathbb{R}^2 : 0 < x^1 < x_o^1 \ , \ y \in \{0,1\}\} \\ \Sigma &= \{(x^1,x^2) \in \mathbb{R}^2 : 0 < x^2 < 1 \ , \ x^1 = g(x^2)\} \ . \end{aligned}\right.$$

The domain G is bounded by a capillary surface Σ that is given as the graph of a function g , and by rigid walls Γ and Γ_o . It is assumed that Γ_o moves with constant velocity S through the infinite cylinder $\{(x^1,x^2) \in \mathbb{R}^2 : -\infty < x^1 < \infty \ , \ 0 < x^2 < 1\}$ and pushes the fluid into the negative x^1-direction; the boundary-value problem (1.9) - (1.14) then describes this flow in a coordinate system that moves with fluid. Of particular interest is the contact angle φ at (0,0) and (0,1) , especially how it depends on S and on the boundary conditions on Γ . In (1.9) γ is a friction coefficient, and therefore (1.13) is derived under the assumption that there is a force on Γ which is proportional to the tangential velocity. If one imposes Dirichlet data on Γ , i.e. $v^1 = -S$, then only for $\varphi = 0$ and $\varphi = \pi$ physically reasonable solutions $v \in H^1_2(G)$ may exist, cf. V.V.Puchnachev - V.A.Solonnikov [PS] .

§ 2. Approximation schemes

Free boundary problems to the Navier-Stokes equations have been solved so far only under the assumption of small data. This contrasts the situation in fixed domains where according to Leray's existence theorem at least one solution exists to arbitrary data; the proof is based on an a priori bound for Dirichlet's integral $\int_\Omega |Dv|^2 dx \leq C(\nu,\Omega,f,v^*)$ which holds for any solution to (1.4) that satisfies $v = v^*$ on $\partial\Omega$. If instead of Dirichlet data v^* a condition of Neumann type is imposed, Dirichlet's integral may not be finite any longer; a counter-example has been given by T.A.McCready [M] who showed the existence of solutions (v_n,p_n) to the Navier-Stokes equations $-\Delta v_n + D_{p_n} + \lambda_n(v_n \cdot D)v_n = 0$, $\operatorname{div} v_n = 0$ in $\Omega = \{(x^1,x^2) \in \mathbb{R}^2 : r^2 < (x^1)^2 + (x^2)^2 < \mathbb{R}^2\}$ such that $\int_\Omega |Dv_n|^2 dx \longrightarrow \infty$ as $\lambda_n \longrightarrow \infty$.

Although this example depends strongly on the fact that the underlying domain is an annulus and therefore does not apply to the situations considered in this paper it suggests that a global estimate for $\int_\Omega |Dv|^2 dx$ does not hold in general. Also physically it seems quite plausible because Dirichlet's integral measures the deformation energy, and by imposing $v = v^*$ on $\partial\Omega$ one assumes that the rigid boundary can resist arbitrary large stresses. For a drop as considered before, however, large stresses might result in large deformations of the shape and eventually the drop might break, a phenomenon that was already investigated by J.Plateau [P]. Clearly this implies that certain norms become unbounded.

Therefore we investigate solutions that are perturbations of a known static configuration. As examples we may take a spherical drop held together by surface tension or self-attraction but without any interior relative motion. We then construct a sequence of successive approximations $\{(v_n,p_n,\Sigma_n)\}$, starting with the static figure $v_o \equiv 0$, $p_o \equiv \text{const}$, $\Sigma_o = S = \{x \in \mathbb{R}^3 : |x| < 1\}$. In the first step we solve

(1.4) - (1.5) in Ω_o , the domain that is bounded by Σ_o ; this solution (v_1,p_1) is then inserted into $n\cdot T(v,p)\cdot n$ in (1.6) or (1.7), and from this equation we determine Σ_1 . Then we solve (1.4) - (1.5) in Ω_1 and obtain in this way the sequence $\{(v_n,p_n,\Sigma_n)\}$.
As a solution is sought in a neighborhood of (v_o,p_o,Σ_o) we can restrict Σ to be a graph over Σ_o . For $\Sigma_o = S$ the free boundary will then be of the form

$$\Sigma = \{(\xi,\rho) \in \mathbb{R}^3 : \xi \in S \ , \ \rho = 1 + \zeta(\xi) \ , \ \zeta : S \longrightarrow \mathbb{R}\} \ . \tag{2.1}$$

$\overline{\Omega} = \{(\xi,\rho):\xi \in S \ , \ 0 \le \rho \le \zeta(\xi)\}$ can then be mapped onto $\overline{B} = \{x \in \mathbb{R}^3 : |x| \le 1\}$ by the transformation

$$y = \sigma(x) = \sigma(\xi,\rho) = \left[\xi, \frac{\rho}{1+\zeta(\xi)}\right] \ . \tag{2.2}$$

On $\overline{B}$ we introduce the new independent variables

$$\begin{cases} u^i(y) = \left[\det\dfrac{\partial\sigma^i}{\partial x^j}\right]^{-1} \dfrac{\partial\sigma^i}{\partial x^j}(x) \ v^j(x) \\ \\ q(y) = p(x) \end{cases} \tag{2.3}$$,

where $x \in \Omega$ and $y \in B$ are related by $y = \sigma(x)$. For two boundaries Σ_n and Σ_{n-1} that are graphs of functions ζ_n and ζ_{n-1} their difference $\Sigma_n - \Sigma_{n-1}$ is defined to be $\{(\xi,\rho): \xi \in S, \ \rho = 1 + \zeta_n(\xi) - \zeta_{n-1}(\xi)\}$; and furthermore we choose $u_n(y) - u_{n-1}(y)$ as difference between $v_n\left[\sigma_n^{-1}(y)\right]$ and $v_{n-1}\left[\sigma_{n-1}^{-1}(y)\right]$.
A straightforward calculation gives for the transformed Navier-Stokes equations

$$Lu^i + \bar{a}_{ij}D_jq + N_i(u,Du) = \tilde{a}_{ij}f^j \qquad \text{in } B \tag{2.4}$$

$$D_ju^j = 0 \qquad \text{in } B \tag{2.5}$$

with

$$Lu^i = -\nu D_k(a_{kl}D_lu^i) + b_{ikl}D_ku^l + c_{ij}u^j \tag{2.6}$$

$$N_i(u,Du) = a^{-1}u^jD_ju^i + \tilde{b}_{ikl}u^ku^l \ , \tag{2.7}$$

where D^j means now partial differentiation with respect to the new variables y^j . The coefficients depend on σ and its derivatives, namely

$$(2.8)\quad \begin{cases} a_{ij} = \frac{\partial\sigma^i}{\partial x^h}\frac{\partial\sigma^j}{\partial x^h}\ ,\ a = \left[\det \frac{\partial\sigma^i}{\partial x^j}\right]^{-1},\ \bar{a}_{ij} = a a_{ij} \\ b_{ikl} = \delta_{li}\frac{\partial}{\partial y^n}a_{nk} - \delta_{li}\Lambda_x\sigma^k - 2\frac{\partial\sigma^i}{\partial x^r}\frac{\partial\sigma^k}{\partial x^s}a\frac{\partial}{\partial x^s}\left[a^{-1}\frac{\partial(\sigma^{-1})^r}{\partial y^l}\right] \\ c_{ij} = -\frac{\partial\sigma^i}{\partial x^n}a\Lambda_x\left[a^{-1}\frac{\partial(\sigma^{-1})^n}{\partial y^j}\right] \\ \tilde{a}_{ij} = \frac{\partial\sigma^i}{\partial x^j}a\ ,\ \tilde{b}_{ikl} = \frac{\partial\sigma^i}{\partial x^n}\frac{\partial(\sigma^{-1})^m}{\partial y^k}\frac{\partial}{\partial x^m}\left[a^{-1}\frac{\partial(\sigma^{-1})^n}{\partial y^l}\right] \end{cases}$$

To indicate that L and its coefficients depend on σ (and therefore on ζ) we sometimes write $L(\zeta)$, $a_{ij}(\zeta)$ etc.

It is understood that the coefficients a_{ij} , a , and α_{ijk} etc. (see below) depend on σ and its first derivatives, b_{ikl} , β_{ij} etc. on σ , $D\sigma$, $D^2\sigma$, and finally c_{ij} on σ and its derivatives up to order three. As new boundary conditions we obtain

$$(2.9)\qquad \alpha_i u^i = 0\ ,\ \alpha_{ijk}D_i u^j + \beta_{jk}u^j = 0 \quad \text{on } S, k = 1,2\ .$$

In principle, this reasoning applies also to (1.9) - (1.14), but due to the corners of the domain some modifications have to be made. If G is given and $\hat{G}$ is a domain of the same type, i.e.
$\hat{G} = \{(x^1,x^2) \in \mathbb{R}^2 : 0 < x^2 < 1,\quad \hat{g}(x^2) < x^1 < x_o^1\}$, again with $\hat{g}(0) = \hat{g}(1) = 0$, then one can use the transformation $\sigma:\hat{G} \longrightarrow G$, defined by

$$(2.10)\qquad (y^1,y^2) = \sigma(x^1,x^2) = (x^1 + \chi(x^1)[g(x^2)-\hat{g}(x^2)],x^2)$$

$\forall (x^1,x^2) \in \hat{G}$, with a cut-off function $\chi \in C(-\infty,x_o^1)$ that satisfies $\chi(x^1) \equiv 1$ on $(-\infty,\frac{1}{4}x_o^1)$ and $\chi(x^1) \equiv 0$ for $x^1 > \frac{3}{4}x_o^1$. (It is assumed without loss of generality that $\inf g$, $\inf \hat{g} < \frac{1}{4}x_o^1$.) As the equations of motion in (1.9) - (1.14) are two-dimensional, one can reduce the problem to a fourth-order equation for the stream function ψ ; therefore we can define $\varphi = \psi\circ\sigma$ as transformation of the dependent variable. In the corner $(0,0)$ for instance one uses a local transformation such that the free boundary becomes a straight line segment. Then ψ can be controlled up to the boundary in suitably weighted Hölder spaces, which are defined as

$$C_s^k(G,M) := \{u:\Omega \longrightarrow \mathbb{R} : \|u\|_{C_s^k(G,M)} := \sum_{|\beta|\le k}\sup_{x\in\Omega}|\rho(x)|^{-s+|\beta|}\cdot|D^\beta u(x)| < \infty\}$$

$$C_s^{k+\mu}(G,M) := \{u \in C_s^k(G,M) : \|u\|_{C_s^{k+\mu}(G,M)} = \|u\|_{C_s^{k+\mu}(G,M)} + \sum_{|\beta|=k}\sup_{x\in G}|\rho(x)|^{-s+k+\mu}\sup_{|x-x'|\le\frac{\rho(x)}{2}}\frac{|D^\beta u(x)-D^\beta u(x')|}{|x-x'|^\mu} < \infty\}.$$

As usual we have $k \in \mathbb{N}$, $\mu \in (0,1)$, β is a multi-index, and $\rho(x) = \operatorname{dist}(x,M)$, $s \in \mathbb{R}$.

We now state in what spaces the successive approximations will converge to a solution. If the free boundary is governed by surface tension we can solve (1.4) - (1.6) in $C^{2+\mu} \times C^{1+\mu} \times C^{3+\mu}$ because in this case we have the estimates

$$(2.11)\qquad \begin{cases} \|u_{n+1}-u_n\|_{C^{2+\mu}}+\|q_{n+1}-q_n\|_{C^{1+\mu}} \le C\|\zeta_n-\zeta_{n+1}\|_{C^{3+\mu}} \\ \|\zeta_n-\zeta_{n-1}\|_{C^{3+\mu}} \le C^*\{\|u_n-u_{n-1}\|_{C^{2+\mu}}+\|q_n-q_{n-1}\|_{C^{1+\mu}}\} \end{cases} .$$

For small data there holds $C\cdot C^* < 1$, and therefore we have convergence for $\{(u_n,q_n,\zeta_n)\}$. A solution (u,q) to (2.4) - (2.9) can be estimated as in (2.11) because on the right hand side of the Schauder estimates the $C^{0+\mu}$-norm of the coefficients of L occurs, and this clearly contains third derivatives of ζ ; similarly the $C^{2+\mu}$-norm of the coefficients in the Dirichlet boundary condition (2.9) enters into it and this again leads to $\|\zeta\|_{C^{3+\mu}}$. On the other hand, the equation for the free boundary (1.6) is of the form

$$(2.12)\qquad \frac{1}{\sqrt{g}}\left\{ D_i \frac{g^{ij}D_j\zeta}{\sqrt{1+|\mathfrak{D}\zeta|^2}} - \frac{\partial}{\partial\zeta}\sqrt{g}\sqrt{1+|\mathfrak{D}\zeta|^2}\right\} = n\cdot T\cdot n ,$$

$\forall\xi \in S$, where $g_{ij}(\xi,\rho)$ is the metric on $\partial B_\rho(0)$, $g = \det g_{ij}$, and g^{ij} is the matrix of the adjoints; $|\mathfrak{D}\zeta|^2 := g^{ij}D_i\zeta D_j\zeta$. Equation (2.12) is a non-uniformly elliptic equation of second order, and $u \in C^{2+\mu}$, $q \in C^{1+\mu}$ implies $T \in C^{1+\mu}$, and consequently $\|\zeta\|_{C^{3+\mu}}$ can be estimated by $\|u\|_{C^{2+\mu}} + \|q\|_{C^{1+\mu}}$ as stated in (2.11). In this way the successive approximations all lie in the same function space, hence for small data $\{(v_n,p_n,\zeta_n)\}$ forms a Cauchy sequence.

In the problem (1.9) - (1.14) we proceed basically in the same way; it becomes necessary, however, to give additional estimates of the behaviour in the corners. The stream function ψ satisfies

(2.13)	$\nu\Delta^2\psi = D_2\psi D_1\Delta\psi - D_1\psi D_2\Delta\psi$	in G
(2.14)	$\psi = 0$	on ∂G
(2.15)	$\nu D_2^2\psi + \gamma D_2\psi = -\gamma S$	on Γ
(2.16)	$\nu D_1^2\psi + \gamma_o D_1\psi = 0$	on Γ_o
(2.17)	$D_2^2\psi - HD_2\psi = 0$	on Σ

$$(2.18)\qquad \frac{1}{\sqrt{1+|g'|^2}}(-\kappa H + \beta g') = D_2\psi \frac{\partial}{\partial n} D_1\psi - D_1\psi \frac{\partial}{\partial n} D_2\psi$$

$$+ \nu \frac{\partial}{\partial n} \Delta\psi + 2\nu \frac{\partial^2}{\partial t^2} \frac{\partial}{\partial n}\psi \quad \text{on } \Sigma$$

$$(2.19)\qquad g(1) = g(0) = 0$$

The estimate which is analogues to (2.11) is

$$(2.20)\qquad \|\psi_{n+1}-\psi_n\|_{\bar{C}^{4+\mu}_{1+\delta}(G,M)} \le C\|g_{n+1}-g_n\|_{\bar{C}^{4+\mu}_{1+\delta}((0,1),M)} ,$$

where M consist of the corners of ∂G . If $\omega(x^2) := g(x^2) - g_o(x^2)$ denotes the deviation of Σ from the static configuration Σ_o (that is the graph of a function g_o) then (2.18) implies for ω

$$-(\omega' F'(g_o'))'' + \beta\omega' = Q_o' + Q \qquad \text{in } (0,1)$$

(2.21)

$$\omega(x^2) = 0 \qquad \text{for } x^2 = 0,1$$

where $F(t) = \frac{t}{\sqrt{1+t^2}}$ and Q_o depends on ω and g_o and their derivatives up to second order; Q is a nonlinear function in ψ and its derivatives up to third order, too. The solution to (2.21) with E as its right-hand side satisfies

$$(2.22)\qquad \|\omega\|_{\bar{C}^{4+\mu}_{1+\delta}} \le C\|E\|_{C^{1+\mu}_{-2+\delta}} ,$$

and apart from the weights which we will discuss later the estimates (2.20) and (2.22) are to be expected from Schauder's theory for elliptic equations and the fact that (2.22) is a third order equation for ω . If we insert $Q_o + Q(\psi,\ldots,D^3\psi)$ into (2.22) then the estimates show that also for the free boundary problem (2.13) - (2.19) the successive approximations converge to a solution.

In case there is no surface tension force the scheme from before will not yield approximations that lie all in the same space. If (1.7) can be solved at all for given $v \in C^{2+\mu}$, $p \in C^{1+\mu}$, its solution ζ will not be more regular than $T(v,p)$, i.e. $\zeta \in C^{1+\mu}$, and consequently we encounter the "loss of derivatives", a phenomenon to which hard implicit function theorems are especially suited. But (1.7) will generally not admit any solution, as it describes Σ only as a level set. We can turn (1.7) into an integral equation, however, for which existence can be shown, if we assume f to be of the form $f = f_o + h$ with $f_o(x) = DU(x)$. As the force of self-attraction f_o can be absorbed into the pressure, (1.7) becomes

$$(2.23)\qquad \int \frac{g}{|x-y|}\, dy = -p(x) + \nu\left[D_i v^j + D_j v^i\right](x) .$$

Now the unknown ζ appears in the domain of integration Ω . According to Lichtenstein [L] the integral can be written in the form

$$c_o\zeta(\xi) + \oint_S \frac{\zeta(\eta)}{d(\xi,\eta)}\, d\sigma(\eta) + N(\zeta)(\xi) \tag{2.24}$$

where c_o = const and $d(\xi,\eta)$ denotes the Euclidean distance between two points $\xi,\eta \in S$.
$N(\zeta)$ is a nonlinear operator which we will discuss in §3. If f_o is the dominating force, i.e. $\|h\| << \|f_o\|$, then $n\cdot T\cdot n$ will be small, too, and the solvability of (1.8) follows from the fact that $c_o\zeta + \oint \frac{\zeta}{d}\, d\sigma$ is invertible. In this way the introduction of f_o as dominating force leads to an equation for the free boundary that can be handled. But also for physical reasons f must be regarded as necessary. Self-attraction tends to hold the drop togethter and therefore balances other forces that possibly act in the opposite way.

<u>§ 3. Equilibrium figures with self-attraction</u>

In [B4] we proved the following result.

<u>Theorem 1:</u> Let $f_o(x) = DU(x)$ be the force of self attraction. For $f = f_o + h$, $h \in C^{\lambda+\mu}$ with $\lambda > 6$ and (in cylindrical coordinates r,θ,x^3)

$$\begin{aligned} h^\theta &= h^\theta(r,x^3) = -h^\theta(r,-x^3)\ ,\quad r^2 = (x^1)^2 + (x^2)^2 \\ h^3 &= h^r = 0\ , \end{aligned} \tag{3.1}$$

$\|h\|_{C^{\lambda+\mu}}$ small enough, there exists a unique solution $v \in C^{5+\mu}(\bar\Omega)$, $p \in C^{4+\mu}(\bar\Omega)$, and $\Sigma \in C^{6+\mu}$ to the free boundary problem (1.4), (1.5), (1.7); v and p are small in the sense that

$$\|v\|_{C^{5+\mu}} + \|p-U\|_{C^{4+\mu}} \leq C\|h\|_{C^{\lambda+\mu}}\ , \tag{3.2}$$

and Σ lies in a $C^{6+\mu}$ neighborhood of the unit sphere S; the $C^{6+\mu}$-norm of the distance of Σ from S can again be estimated by $C\|h\|_{C^{\lambda+\mu}}$.

The proof is based on a suitable version of the hard implicit function theorem; we regard (1.4), (1.5), (1.7) as a nonlinear mapping $F:\mathcal{Y}_o \times \mathcal{Z}_o \longrightarrow \mathfrak{X}_o$ which is defined by associating to $z = (u,q,\zeta) \in \mathcal{Z}_o := C^{2+\mu}(\bar B;\mathbb{R}^3) \times C^{3+\mu}(S;\mathbb{R})$ the right-hand side of these equations which then is an element of $\mathfrak{X}_o := C^{0+\mu}(\bar B;\mathbb{R}^3) \times C^{0+\mu}(\bar B;\mathbb{R}) \times C^{0+\mu}(S;\mathbb{R})$. Here we have identified functions (v,p) and (u,q) that are related by the transformation σ as in (2.3). In this way F is defined on a set which admits an affine structure, and consequently one can compute the linearisation $DF(f,z)$ of F with respect to the second argument, cf. [B4] (40) - (42).

$$
\begin{aligned}
DF^i(f,z)\tilde{z} &= L(\zeta)\tilde{u}^i + \bar{a}_{ij}(\zeta)D_j\tilde{q} + l_{ij}(u,\zeta)\tilde{u}^j \\
&\quad + l_j(u,\zeta)D_j\tilde{u}^i + \sum_{|\gamma|\le 3} l_\gamma(u,q.\zeta)D^\gamma\tilde{\sigma} \qquad (3.3)\\
&\quad + \sum_{|\gamma|\le 1} m_\gamma(f,\zeta)D^\gamma\tilde{\sigma}\ , \qquad i = 1,2,3
\end{aligned}
$$

$$DF^4(f,z)\tilde{z} = D_j\tilde{u}^j \qquad (3.4)$$

$$
\begin{aligned}
DF^5(f,z)\tilde{z} &= M\tilde{\zeta} + m_o(\zeta)\tilde{\zeta} + \sum_{|\gamma|\le 2} r_\gamma(u,q,\zeta)D^\gamma\tilde{\zeta} \qquad (3.5)\\
&\quad + m_{ij}(\zeta)D_i\tilde{u}^j + m(\zeta)\tilde{q}\ .
\end{aligned}
$$

The boundary conditions for $(\tilde{u},\tilde{q})$ are of the form (2.9).
We note that $DF^i(f,z)$, $i = 1,\ldots,4$ is not just the Stokes linearization of (2.4), (2.5) - which would consist only of $D_jF_i(f,z)$, $i,j = 1,\ldots,4$ - but contains the derivative of F^i with respect to $z^5 \equiv \zeta$, too; this results then in the third order operators in $\tilde{\sigma}$, which is the transformation belonging to $\tilde{\zeta}$. Similarly, in $DF^5(f,z)$ also operators in $\tilde{u}$ and $\tilde{q}$ occur. So in contrast to the approximation scheme from before the equations (3.3) - (3.5) no longer split into a boundary value problem for the velocity and the pressure and in a separate equation for the free boundary. On the other hand, it is not known how to invert $DF(f,z)$, and therefore we will use a variant of Moser's implicit function theorem which is due to E.Zehnder [Z], and which allows to work with (2.4), (2.5), (2.9), (2.24) instead of (3.3) - (3.5). It requires only the existence of an approximate inverse $H(f,z)$ to $DF(f,z)$ in the sense that

$$DF(f,z_n)\circ H(f,z_n) \longrightarrow \mathbb{1}\ ,$$

as z_n tends to the solution z of the nonlinear equation. More precisely, the hypotheses for this implicit function theorem are as follows. Let $\{\mathcal{Z}_t\}_{t\ge 0}$ be a one parameter family of Banach spaces with norms $|\cdot|_t$ such that for all t,t' with $0 \le t' \le t < \infty$ there holds

$$\mathcal{Z}_o \supset \mathcal{Z}_{t'} \supset \mathcal{Z}_t \supset \mathcal{Z}_\infty \equiv \bigcap_{t>0}$$

and

$$|z|_{t'} \le |z|_t \qquad \forall z \in \mathcal{Z}_{t'}\ ,\ t' \le t\ .$$

The same properties are assumed to hold for $\{\mathcal{Y}_t\}_{t\ge 0}$ and $\{\mathcal{X}_t\}_{t\ge 0}$. $z_o = (0,U_o(x),o)$, where $U_o(x)$ is the gravity potential of $\Omega_o = B(0)$, satisfies $F(f_o,z_o) = 0$. We then postulate

(H.1) F is continuous in (f,z) and two times differentiable in z ; in $\mathcal{B}_o = \{(f,z) : |f-f_o|_0 + |z-z_o|_0 < 1\}$ these derivatives are bounded.

(H.2) F is Lipschitz continuous in the first argument.

(H.3) F is of order s , where s is related to the loss of derivatives in (H.4); this means that if (f,z) becomes more regular its image $F(f,z)$ is more regular, too: $F(\mathcal{B}_o \cap (\mathcal{Y}_t \times \mathcal{Z}_t)) \subset \mathcal{X}_t$ $\forall\ t \in [1,s]$.

Hypotheses (H.1) - (H.3) can easily be verified because they are consequences of the regularity of the coefficients in (2.4), (2.5), (2.9), (2.24) .

(H.4) For every $(f,z) \in \mathcal{B}_\gamma$ there exists a linear map $H(f,z) : \mathcal{X}_\gamma \longrightarrow \mathcal{Z}_o$ such that $|H(f,z)(\varphi)|_0 \leq M_o |\varphi_\gamma| \ \forall\ \varphi \in \mathcal{X}_\gamma$; $H(f,z)$ is furthermore continuous from $\mathcal{X}_t$ into $\mathcal{Z}_{t-\nu}$. $H(f,z)$ is an approximate inverse in the sense that

(3.6) $$|[D_2F(f,z) \circ H(f,z) - \mathbb{1}](\varphi)|_0 \leq M_o |F(f,z)|_\gamma |\varphi|_\gamma$$

for all $\varphi \in \mathcal{X}_\gamma$.

<u>Theorem 2:</u> (E.Zehnder [Z]) Let F satisfy (H.1) - (H.4). Then there exists an open neighborhood $\mathcal{D}_\lambda = \{f \in \mathcal{Y}_\lambda : |f-f_o|_\lambda < C\}$ and a mapping $\psi : \mathcal{D}_\lambda \longrightarrow \mathcal{Z}_\rho$ such that for all $f \in \mathcal{D}_\lambda$

(3.7) $$F(f,z) = 0 \qquad \text{with} \quad z = \psi(f)$$

and

(3.8) $$|z-z_o|_\rho \leq C^{-1} |f-f_o|_\lambda \ .$$

The numbers λ and ρ can be chosen to be $\rho = 3$, $\lambda > 6$.

To verify (H.1) - (H.4) we choose first the underlying function spaces to be

$$\mathcal{X}_t = C^{t+\mu}(\bar{B};\mathbb{R}^3) \times C^{t+\mu}(\bar{B};\mathbb{R}^3) \times C^{t+\mu}(S;\mathbb{R})$$
$$\mathcal{Y}_t = C^{t+\mu}(\mathbb{R}^3;\mathbb{R}^3)$$
$$\mathcal{Z}_t = C^{t+2+\mu}(\bar{B};\mathbb{R}^3) \times C^{t+1+\mu}(\bar{B};\mathbb{R}^3) \times C^{t+3+\mu}(S;\mathbb{R}) \ .$$

To define the approximate inverse $H(f,z)$ we first consider the operator $D^*F(f,z)$ which consists of the linearized equations (2.4), (2.5), (2.9) in its first four components and of the linearization of (2.24) in $\tilde{\zeta}$. It is of the form (3.3) - (3.5), but with $l_\gamma D^\gamma \tilde{\sigma}$, $m_\gamma D^\gamma \tilde{\sigma}$, $r_\gamma D^\gamma \tilde{\zeta}$, $m_{ij} D_i \tilde{u}^j$ and $m\tilde{q}$ left out. $D^*F(f,z)$ is invertible, and we call its inverse $H(f,z)$. To prove the estimate (3.6) we showed in [B4] that the terms $l_\gamma D^\gamma \tilde{\sigma}$ etc. tend to zero if $\{z_n\}$ approaches its limit; the main idea in doing so consists in choosing suitable representations: when z_{n+1} is constructed we choose Ω_{n-1} as reference domain, such that ζ_{n+1} measures the distance between Σ_{n+1} and Σ_{n-1} along the normals to Σ_{n-1}.

That $D^*F(f,z)$ is invertible or equivalently that the approximations (u_n, q_n) and ζ_n can be constructed as claimed in §2 follows from Lemma 3 and Lemma 4.

<u>Lemma 3:</u> Let $u \in C^{2+\mu}(\bar{\Omega})$, $\zeta \in C^{3+\mu}(\partial\Omega)$ be given, when Ω is a domain with boundary of class $C^{3+\mu}$. Then the boundary value problem

$$(3.9)\quad \begin{cases} L(\zeta)\tilde{u}^i + \bar{a}_{ij}(\zeta)D_j\tilde{q} + l_{ij}(u,\zeta)\tilde{u}^j + l_j(u,\zeta)D_j\tilde{u}^i = \varphi^i \\ D_j\tilde{u}^j = 0 \quad \text{in } \Omega, \\ i=1,2,3 \end{cases}$$

$$(3.10)\quad \alpha_i(\zeta)\tilde{u}^i = 0\ ,\ \alpha_{ijk}(\zeta)D_i\tilde{u}^j + \beta_{kj}(\zeta)u^j = 0 \quad \text{on } \partial\Omega\ ,\ k = 1,2$$

with operators as defined in (2.6), (2.8), admits to $\varphi \in C^{0+\mu}$ a classical solution $\tilde{u} \in C^{2+\mu}(\bar{\Omega})$, $\tilde{q} \in C^{1+\mu}(\bar{\Omega})$ as long as $\|u\|_{C^{2+\mu}}$ and $\|\zeta\|_{C^{3+\mu}}$ are small enough. The solution can be estimated by

$$(3.11)\quad \|\tilde{u}\|_{C^{k+2+\mu}} + \|\tilde{q}\|_{C^{k+1+\mu}} \le C\left[\nu, k, \|\partial\Omega\|_{C^{k+3+\mu}}, \|\zeta\|_{C^{k+3+\mu}}, \|u\|_{C^{k+2+\mu}}\right] \|\varphi\|_{C^{k+\mu}}$$

for all $k \ge 0$.

The lemma states essentially that the Stokes equations are solvable if mixed boundary conditions as in (1.5) are prescribed rather than Dirichlet data; for a proof see [SS], [B1].

<u>Lemma 4:</u> Let Σ be a closed surface in a $C^{2+\mu}$-neighborhood of S. Then

$$(3.12)\quad \psi_\Sigma(\xi)\tilde{\zeta}(\xi) + \oint_\Sigma \frac{\tilde{\zeta}(\eta)}{d(\xi,\zeta)}\, d\sigma(\eta) = 0$$

for at most 6 eigensolutions $\tilde{\zeta}_1,\dots,\tilde{\zeta}_6$. Here ψ_Σ is the normal derivative of the Newtonian potential of the body Ω that is bounded by Σ.

$\tilde{\zeta}_1, \tilde{\zeta}_2, \tilde{\zeta}_3$ are the infinitesimal translations in the directions of the coordinate axes, and $\tilde{\zeta}_4, \tilde{\zeta}_5, \tilde{\zeta}_6$ are the rotations about these axes. If Σ is rotationally symmetric with respect to the x^3-direction, then $\tilde{\zeta}_{3+i}$ is not an eigensolution.

The proof is classically known for equilibrium figures, and requires therefore only some perturbation arguments.

Remark: (i) Lemma 4 is not restricted to surfaces near S . Actually the proof only uses that the sphere is an equilibrium figure to a value ω for which no bifurcation occurs. So if Σ lies near an equilibrium figure that is locally unique, Lemma 4 holds, too.
(ii) The restriction (3.1) on h in Theorem 1 guarantees that there holds $\int_\Omega h dx = 0$. Physically this means that the resultant of the forces vanishes which is obviously a necessary condition for the existence of stationary configurations. We will investigate this question again in § 4.

§ 4. Closed surfaces of prescribed mean curvature and free boundaries governed by surface tension

The solution to the free boundary problem (1.4) - (1.6) where the free boundary is now determined by surface tension can be obtained by the approximation scheme that we outlined in §2. Therefore it remains to prove existence for solutions (v_n, p_n) to (1.4), (1.5) in a fixed domain Ω_{n-1} and Σ_n to (1.6) or equivalently ζ_n to (2.12) where T is evaluated at (v_n, p_n). To show existence, uniqueness and regularity of solutions to the Navier-Stokes equations with mixed boundary conditions one can proceed in a way that is very close to the case of Dirichlet data.
To indicate the main difficulty in the problem of closed surfaces with prescribed mean curvature we start with the following formula for integration by parts on the surface:

$$(4.1) \qquad -2 \oint_\Sigma Hgn d\sigma = \oint_\Sigma \delta g d\sigma \qquad \forall\ g \in C^1_c(U(\Sigma)) \ .$$

Here $U(\Sigma)$ is a three-dimensional neighborhood of Σ , and $\delta g = Dg - (Dg \cdot n)n$ denotes the tangential part of the gradient of g . If Σ is a closed surface we can choose g to be one on Σ , and hence

$$(4.2) \qquad \oint_\Sigma Hn d\sigma = 0 \ .$$

It turns out that (4.2) poses a restriction to the data H , for which a surface with this H is its mean curvature exists.[1] For $H = -1 + \epsilon x^3$, which is only a perturbation to the mean curvature of the unit sphere, equation (4.2) obviously cannot hold.

If H, however, can be interpreted in physical terms as in (1.6), condition (4.2) is always satisfied, cf. [B2]. As in the remark to Lemma 4 we require the force f in (1.4) that generates a motion inside of Ω to be balanced: $\int_\Omega f(x)dx = 0$. Therefore

$$\begin{aligned} 0 &= \int_\Omega -\nu\Delta v + Dpdx + \int_\Omega (v\cdot D)vdx \\ &= \oint_\Sigma T(v\cdot p)\cdot n\, d\sigma . \end{aligned}$$

As the tangential part of $T\cdot n$ vanishes pointwise on Σ, cf. (1.5), this means $0 = \oint_\Sigma (n\cdot T\cdot n)nd\sigma \equiv 2\kappa \oint_\Sigma Hnd\sigma$. The restriction (4.2) to the purely geometric problem of constructing a closed surface of prescribed mean curvature is eventually quite natural if interpreted in physical terms. As (4.2) involves the data H and the solution Σ it still remains to find conditions on H alone such that (2.12) is solvable.

As there is a volume constraint (1.2) to be satisfied by the solution to (2.12) we will apply variational methods. (2.12) is the Euler-Lagrange equation to

$$I(\zeta) = \oint_S \sqrt{1+|\mathcal{D}\zeta|^2}\,\sqrt{g}\,d\xi + \oint_S \mathcal{H}(\xi,\zeta)\,\sqrt{g^*}\,d\xi \tag{4.3}$$

where

$$\mathcal{H}(\xi,\zeta) = \int_0^{\zeta(\xi)} - 2h(\xi,t)t^2dt \; ; \tag{4.4}$$

for h we have to insert the prescribed mean curvature $n\cdot t(v,p)\cdot n$ which after it is calculated for a specific approximation $(v_k,p_k)|\Sigma_k$ we may extend to be constant along rays.

The area integral $A(\zeta) = \oint_S \sqrt{1+|\mathcal{D}\zeta|^2}\,\sqrt{g}\,d\xi$ in (4.3) grows linearly in $|D\zeta|$, but not uniformly with respect to ζ:

$$c_o\zeta^2 + c_1\zeta^2|D\zeta| \le \sqrt{g}\,\sqrt{1+|\mathcal{D}\zeta|^2} \le c_o\zeta^2 + c_1\zeta^2 + c_1\zeta^2|D\zeta| \; .$$

Therfore the space of functions of bounded variation does not seem to

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[1] The rôle of (4.2) and an example for H that is even constant on rays such that there is no graph over S whose mean curvature is H was communicated to me by Henry C. Wente.

be appropriate as in the case of the Euclidean area functional $\int_{\Omega} \sqrt{1+|Du|^2}\, dx$. Hence we introduce another function space which is a variant of $BV(\Omega)$ by exploiting the following (formal) relation

$$\sqrt{g(\xi,\zeta)}\ \sqrt{1+g^{ij}(\xi,\zeta)D_i\zeta D_j\zeta} = \sqrt{\zeta^4+\zeta^2 g^{*ij}D_i\zeta D_j\zeta}\ \sqrt{g^*}$$
$$= \sqrt{(\zeta^2)^2+\tfrac{1}{4}g^{*ij}D_i(\zeta^2)D_j(\zeta^2)}\ .$$

If we now regard ζ^2 as the new dependent variable we can extend $A(\zeta)$ in terms of ζ^2 onto the function space

$BV_R(S) = \{\zeta\in L_2(S) : \oint_S |\mathcal{D}^*(\zeta^2)|\ \sqrt{g^*} < \infty\}$ where

$$\oint_S |\mathcal{D}^*(\zeta^2)|\ \sqrt{g^*} = \sup\left\{\oint_S \zeta^2 D_i(\sqrt{g^*}\ g^{*ij}\varphi^j)\,d\xi :\ \varphi^1,\varphi^2\in C^1(S)\ ,\ \sqrt{g^{*ij}\varphi^i\varphi^j} \le 1\right\} . \tag{4.5}$$

On $BV_R(S)$ we now define the area integral to be

$$\oint_S \sqrt{\zeta^4+\tfrac{1}{4}g^{*ij}D_i\zeta^2 D_j\zeta^2}\ \sqrt{g^*} = \sup\left\{\oint_S \zeta^2\varphi^o_*\ \sqrt{g^*} + \tfrac{1}{2}\zeta^2 D_i(g^{*ij}\varphi^j)\,d\xi :\ \varphi^o,\varphi^1,\varphi^2 \in C^1\ (S)\ ,\ (\varphi^o)^2 + g^{*ij}\varphi^i\varphi^j \le 1\right\} . \tag{4.6}$$

The approximations to the free boundary can now be obtained by the following variational problem: minimize $I(\zeta)$ in the class of functions

$$C = BV_R(S) \cap \left\{\zeta :\ \tfrac{1}{3}\oint_S \zeta\ \sqrt{g^*}\ d\xi = V_o\right\} \cap \left\{\zeta :\ \oint_S (\zeta-1)\zeta_i\ \sqrt{g^*}\ d\xi = 0\right\}$$

where ζ_i , $i = 1,2,3$ are the eigenfunctions to the Laplace-Beltrami operator Δ^* on S to the eigenvalue 2.

<u>Remarks:</u> (i) Because we use $BV_R(S)$ instead of BV , the volume constraint $\frac{1}{3}\oint \zeta^3\ \sqrt{g^*}\ d\xi = V_o$ becomes a compact side condition; for according to the Sobolev embedding theorem $BV_R(S)$ is continuously embedded in $L_4(S)$ and hence compactly in $L_p(S)$, $p < 4$.

(ii) The side condition $\oint (\zeta-1)\zeta_i\ \sqrt{g^*}\ d\xi = 0$ guarantees that the center of mass of the fluid body stays in the origin, for ζ_i are the infinitesimal translations in the coordinate axes. This side condition leads to Langrange multipliers but as we will restrict the exterior forces to be balanced, the corresponding Lagrange multipliers will vanish if (v_n,p_n,Σ_n) tends to the solution.

(iii) The introduction of spaces of BV-type where instead of the function u itself an expression $\varphi(u)$ has bounded variation turns out to be useful in other variational problems, too. See e.g. [BD], where the degenerate variational integral $\int_\Omega u\sqrt{1+|Du|^2}\,dx$, $u \geq 0$ a.e. in $\Omega \subset \mathbb{R}^n$, is studied.

For fluid bodies whose free boundary is governed by surface tension we obtain the following result.

Theorem 5: The free boundary problem (1.4) - (1.5) admits a unique solution $v \in C^{2+\mu}(\bar{\Omega})$, $p \in C^{1+\mu}(\bar{\Omega})$, $\Sigma \in C^{3+\mu}$, if the force density f is of class $C^{o+\mu}$ and satisfies (3.1).

One can easily extend this result to the case of two immiscible fluids of the same density, where the drop Ω is immersed in a second fluid that fills a fixed container.

Higher regularity $v \in C^{k+2+\mu}$, $p \in C^{k+1+\mu}$, $\Sigma \in C^{k+3+\mu}$ can be shown easily if the forces are more regular, too, like $f \in C^{k+\mu}$. Furthermore, in [BF] analyticity is proved.

Theorem 6: Let (v,p,Σ) be a solution to (1.4) - (1.6) and f an analytic force density. Then v,p and Σ are real analytic, provided $\|v\|_{C^{1+\mu}}$ is small.

Standard techniques for proving analyticity in free-boundary problems do not seem to apply and therefore the proof had to be based on Friedman's method to show analyticity for solutions of elliptic and parabolic equations, cf. [F1]. All derivatives are estimated successively, and this required the smallness of v .

In [B3] we investigated the problem of a drop Ω that falls down under its own weight in an unbounded reservoir of a second viscous fluid of smaller density. In this case the flow can be stationary only in a reference frame that is attached to Ω . Its speed γ relative to a fixed Galilean frame is a further unknown of the problem. The condition that in the moving frame the net weight of the drop Ω is balanced by the viscous forces determines γ uniquely.

If (v,p) , (u,q) denote the velocity and the pressure in Ω and its complement $\mathcal{E}$, resp., we obtain

Theorem 7. If the difference of the densities of the two fluids is small then exists a unique solution (v,p,u,q,Σ) to the problem of a falling drop. The regularity of the velocities v,u , the pressures

p,q and the free boundary Σ is the same as in Theorem 5. The solution is axially symmetric with respect to the direction of the (uniform) gravitational field.
The proof uses results of H.F.Weinberger [W] on the steady fall of a rigid body in a Navier-Stokes fluid; there a weak formulation is given by which also γ can be determined.

§ 5. A free boundary problem with a dynamic contact angle.

In the free boundary value problem (1.9) - (1.14) the core of the investigation by D.Kröner [K] lies in the estimates near the singular points of the boundary. As in the theorems of §§ 3,4 existence for the Navier-Stokes equations with boundary conditions rather than Dirichlet data in smooth domains poses no particular difficulty, also after the perturbation of the operators by the transformations onto a fixed domain.
In this context the first goal is to establish precise asymptotic estimates for the function g that represents the free boundary under the following hypotheses

(5.1) $$v \in H_2^1(\Omega) \quad \text{and} \quad v \text{ is smooth in } \bar{\Omega} \setminus M ,$$

where M denotes the set of singular boundary points,

(5.2) $$g \in C_1^{4+\upsilon}([0,1])$$

and

(5.3) $$\| g(y) - g'(0)y \|_{C_1^{4+\upsilon}([0,a])} \longrightarrow 0 ,$$

as a tends to zero.
The assumption on v means that the energy of the flow is finite. In addition to the weighted Hölder spaces defined in (2.10) we need to work also in Sobolev spaces with weights. They are defined as

(5.4) $$W_\mu^{k,p}(\Omega;M) = \left\{ u: \| u \|_{W_\mu^{k,p}(\Omega;M)} := \sum_{|\beta| \le k} \int_\Omega \rho^{p(\mu-k+p)} \, |D^\beta u|^p dx < \infty \right\}$$

where (ρ,φ) are polar coordinates with the singular point as its center. The main result on the regularity of ψ is contained in

Theorem 8: Let $\{\psi,g\}$ be a solution of (2.13) - (2.19) and let (5.1) - (5.3) be satisfied. Then there holds

(5.5) $$\psi \in C_{1+\delta}^{4+\upsilon}(\Omega)$$

$$(5.6) \qquad k \in C^4_{3-\mu_o}([o,1])$$

where $k(y) = g'(0)y - g(y)$ and

$$(5.7) \qquad \mu_o > \begin{cases} 3 - \Pi/\varphi_o & \text{if } \frac{\Pi}{2} < \varphi_o < \Pi \\ 1 & \text{if } 0 < \varphi_o < \Pi . \end{cases}$$

φ_o denotes the contact angle, i.e. $\varphi_o = \frac{\Pi}{2} - \arctan g'(0)$.

As a consequence one gets the following asymptotic expansion.

<u>Theorem 9:</u> Under the assumptions of Theorem 8 the stream function ψ is of the form $\psi = \psi_{as} + \psi_o$ with

$$(5.8) \qquad \psi_o \in W^4_\sigma(\Omega)$$

$$\text{with } \sigma > \begin{cases} \frac{3}{2} - \frac{\Pi}{2\varphi_o} & \text{if } 0 < \varphi_o \le \frac{\Pi}{2} \\ \frac{7}{2} - \frac{3\Pi}{2\varphi_o} & \text{if } \frac{\Pi}{2} < \varphi_o < \Pi , \end{cases}$$

and

$$(5.9) \qquad \psi_{as}(r,\varphi) = \begin{cases} u_3(r,\varphi) & \text{if } 0 < \varphi_o \le \frac{2\Pi}{5} \\ u_3(r,\varphi) + r^{\frac{\Pi}{\varphi_o}} \sum_{s=0}^{s_o} \log r\hat{P}_s(r \log^q r,\varphi) & \text{if } \frac{2\Pi}{5} < \varphi_o \le \frac{3\Pi}{5} \\ r^{\frac{\Pi}{\varphi_o}} \sum_{s=0}^{s_o} \log rP_s(r \log^q r,\varphi) & \text{if } \frac{3\Pi}{5} < \varphi_o < \Pi \end{cases}$$

with $r \in (0,1)$, $\varphi \in [0,\varphi_o]$ and an integer q . P_s and $\hat{P}_s$ are polynomials in $r \log^q r$ with smooth coefficients and $u_3(r,\varphi) = \frac{\gamma S}{4\upsilon} r^2 a(r,\varphi)$ with

$$a(r,\varphi) = \begin{cases} \frac{\varphi-\varphi_o}{\varphi_o} - \frac{\sin 2(\varphi-\varphi_o)}{\sin 2\varphi_o} , & \text{if } \varphi_o \ne \frac{\Pi}{2} \\ \frac{2}{\Pi} \{(\varphi-\varphi_o)(\cos 2(\varphi-\varphi_o)+1)+\sin 2(\varphi-\varphi_o)\log r , & \text{if } \varphi_o = \frac{\Pi}{2} \end{cases}$$

To show the result of Theorem 8 one improves on the regularity of ψ in several steps. The first estimate is $v \in C^o_{-1+\delta}$ for all $\delta \in (0,1)$. Because of the assumptions on g one can perform the transformation

$$w(z) = v(rz) , \; q(z) = rp(rz) , \; \gamma(y) = \frac{1}{\gamma} g(ry)$$

and then apply L_p-stimates in a fixed annular domain. Due to the regularity of g the transformed equations have smooth coefficients which finally gives $\psi \in C^{4+\upsilon}_{1-\delta}(\Omega)$. To improve on the regularity of ψ and to obtain (5.5) one applies a method used by V.A.Kondratev which allows to estimate solutions of linear elliptic equations in corners of the domain. As g and ψ are related by the boundary condition (2.18) this results in an improved estimate for g in the singular point as stated in (5.6).

The existence of a unique solution $\{\psi,g\}$ to (2.13) - (2.19) which has the properties stated in theorems 8 and 9 is shown to exist in a neighborhood of a hydrostatic capillary problem with a contact angle φ_s provided the data are small which means that the fluid is pushed through the tube such that $|\gamma S|$ in (2.15) is sufficiently small.

<u>Theorem 10:</u> Let g_o be the parametrization of a static configuration with contact angle $\frac{\Pi}{2}$-arctan $g_o'(0) = \varphi_s \in (0,\Pi)$. Let $\alpha = \Pi - \varphi_s > 0$, $0 < \delta < \min\{\frac{\alpha}{2\Pi-\alpha}\ ,\ 1\}$, and $\rho,\beta,\nu,\gamma,\gamma_o \in \mathbb{R}^+$. Then there exists an $\epsilon_o > 0$ such that for all values of γS with $|\gamma S| \le \epsilon_o$ the free boundary problem (2.13) - (2.19) is uniquely solvable. The solution $\{\psi,g\}$ satisfies $\psi \in \overline{C}^{4+\beta}_{1+\delta}(\Omega;M)$, $g \in \overline{C}^{4+\beta}_{1+\delta}([0,1],M)$ for all $\beta \in (0,1)$.

The proof is based on solving the equations of motion in fixed domains and the equation for the free boundary to given data, as outlined in §2. One starts with weak solutions to the linearization of (2.13) - (2.17) ; then estimates up to the boundary are given in weighted Hölder classes, as required in (2.20) such that eventually a fixed point argument gives the solution to the free boundary problem.

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