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On the uniqueness of nondecaying solutions for the Navier-Stokes equations

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Abstract

In this article, we obtain the uniqueness of solutions (u, p) of the Navier-Stokes equations in the class

$$u \in L^\infty((0,T)\times \mathbf{R}^n), \quad p \in L^1_{\rm loc}([0,T); BMO(\mathbf{R}^n))$$

for initial data in $L^\infty(\mathbf{R}^n)$. Although there are a few results which treats the uniqueness without decay assumption as $|x| \to \infty$ ([5], [15], [14]), our result gives the another characterization of condition on p.

1 Introduction and Main Result

We are concerned with the uniqueness of solutions for Navier-Stokes equations:

$$u_t - \Delta u + (u, \nabla)\, u + \nabla p = 0 \qquad \text{in } (0,\infty)\times \mathbf{R}^n, \tag{1.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } (0,\infty)\times \mathbf{R}^n, \tag{1.2}$$

with initial data $u|_{t=0} = u_0$, where $u = u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and its pressure respectively, while $u_0 = u_0(x) = (u_0^1(x), \cdots, u_0^n(x))$ is the given initial velocity vector field.

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It is by now well known that for initial data $u_0 \in L^\infty(\mathbf{R}^n)$ the equations (1.1), (1.2) admits a unique time-local (regular) solution u with

$$p = \sum_{i,j=1}^{n} R_i R_j u_i u_j, \tag{1.3}$$

where $R_j = (-\Delta)^{-1/2}\partial_j$ is the Riesz transform [1], [13], [2], [8] (Recently, it is shown in [9] that this solution can be extended globally in time when the space dimensions are two).

It is also well known that for L^r-initial data ($n \le r < \infty$) the equations (1.1), (1.2) admits a unique time-local solution u with some p [11], [12], [6], $\cdots$. Because of decay at the space infinity of u the relation (1.3) follow (up to constant) a posteriori for L^r-data ($n \le r < \infty$).

For L^∞-initial data the constructed solution u is bounded and may not decay at the space infinity. So even if u solves (1.1), (1.2) with some p the relation (1.3) may not follow. In fact, if we consider $u(t,x) = g(t)$ and $p(t,x) = -g'(t)\cdot x$, then (u,p) always solves (1.1), (1.2) no matter what function g is. Here $\cdot$ denotes the inner product in $\mathbf{R}^n$. This says the solution u with a constant initial data is not unique without assuming (1.3). This example suggests that contrary to L^r-case ($n \le r < \infty$) we need to impose some condition on p to derive uniqueness other than on u.

In [7] we announced that the uniqueness holds for L^∞-data under the assumption that u is bounded and p is of the form

$$p = \pi_0 + \sum_{i,j=1}^{n} R_i R_j \pi_{ij} \tag{1.4}$$

for some bounded functions π_0, π_{ij}. This result assures the uniqueness of solution (u,p) for L^∞-data with (1.3) under a priori assumption on p (1.4). This paper is based on the work [7] and gives an improvement of the condition (1.4).

In this paper we consider solutions in the following sense.

Definition 1.1. *We call (u,p) the solution of the Navier-Stokes equations (1.1), (1.2) on $(0,T)\times\mathbf{R}^n$ with initial data u_0 in the distribution sense if (u,p) satisfy* $\operatorname{div} u = 0$ *in $\mathcal{S}'$ for a. e. t and*

$$\begin{aligned}\int_0^T \big\{\langle u(s), \partial_s\Phi(s)\rangle + \langle u(s), \Delta\Phi(s)\rangle + \langle (u\otimes u)(s), \nabla\Phi(s)\rangle \\ + \langle p(s), \operatorname{div}\Phi(s)\rangle\big\} ds = -\langle u_0, \Phi(0)\rangle,\end{aligned} \tag{1.5}$$

for $\Phi \in C^1([0,T] \times \boldsymbol{R}^n)$ *satisfying* $\Phi(s,\cdot) \in \mathcal{S}(\mathbf{R}^n)$ *for* $0 \le s \le T$, *and* $\Phi(T,\cdot) \equiv 0$, *where* $\langle u \otimes u, \nabla \Phi \rangle = \sum_{i,j=1}^n \langle u_i u_j, \partial_i \Phi_j \rangle$.

Before stating our main result we prepare some notations. We denote by $BMO = BMO(\mathbf{R}^n)$ the space of functions of bounded mean oscillations. It is well known [16] that BMO strictly includes L^∞ and the Riesz transformation R_j is a bounded operator from L^∞ to BMO and from BMO to itself. We denote by $\mathcal{H}^1 = \mathcal{H}^1(\mathbf{R}^n)$ the Hardy space on $\mathbf{R}^n$. It is also known [16] that the Hardy space $\mathcal{H}^1$ is the dual space of BMO.

Now we are in a position to state our main result.

Theorem 1. *Let* $u_0 \in L^\infty$ *with* $\operatorname{div} u_0 = 0$. *Suppose that* (u,p) *is the solution of (1.1), (1.2) with initial data* u_0 *in the distribution sense satisfying*

$$u \in L^\infty((0,T) \times \mathbf{R}^n), \quad p \in L^1_{\mathrm{loc}}([0,T); BMO). \tag{1.6}$$

Then the solution $(u, \nabla p)$ *is unique.*

Moreover, we have $\nabla p = \sum_{i,j=1}^n \nabla R_i R_j u^i u^j$ in $\mathcal{S}'$, *for a. e.* t.

Remark 1.1. *The condition (1.6) on* p *involves (1.4), since the Riesz transformations are bounded.*

Let us mention a few known results closely related to our uniqueness results. It was shown in [5] that if u and ∇u are bounded in $(0,T) \times \mathbf{R}^3$, then the uniqueness of classical solutions holds provided that for some $C > 0$ and some $\varepsilon > 0$ the inequality

$$|p(t,x)| \le C(1 + |x|)^{1-\varepsilon} \tag{1.7}$$

holds. Later it was shown in [15], [14] that if $n = 2$, 3 and ∇u is bounded in $(0,T) \times \mathbf{R}^n$, then the uniqueness holds provided that (1.7) holds with $\varepsilon = n/2$. Our assumption (1.6) do not imply (1.7), so it is not comparable with those results.

To prove Theorem 1 we reduce the problem to the uniqueness of solutions to the integral equation corresponding to (1.1), (1.2). In fact, if (u,p) is the solution of (1.1), (1.2) in the distribution sense with (1.3), then we can observe that u is also the solution of the corresponding integral equation. Thus, our main task is to show that p has a representation such as (1.3). However, there are some difficulties to treat the Riesz transformations on L^∞, so we introduce the operators which approximates the Riesz transformations in suitable sense.

This paper is organized as follows. In section 2 we introduce operators which approximate the Riesz transformations. Its convergence properties are described in Theorem 2. In section 3 we prove Theorem 1. In section 4 we give a proof of Proposition 2.1, which is crucial to the proof of Theorem 2.

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2 Preliminaries

The essential part of the proof of Theorem 1 is to determine the condition on p which gives the unique representation (1.3) for merely bounded u. The difficulty comes from the fact that the symbol calculus for Fourier multipliers does not work well in L^∞. So we introduce the operator R_{ij}^ε which approximates the operator R_iR_j in suitable sense.

Let k denote the fundamental solution of $-\Delta$, i.e. $-\Delta k = \delta$. Its explicit form is

$$k(x) = \begin{cases} C_n|x|^{2-n}, & \text{for } n \geq 3, \\ C_2 \log|x|, & \text{for } n = 2, \end{cases}$$

where $1/C_n = (n-2)|S^{n-1}|$ for $n \geq 3$ and $1/C_2 = -2\pi$. Let $\psi \in C^\infty(\mathbf{R}^n)$ be a radial function with $0 \leq \psi \leq 1$, $\psi(x) = 0$ for $|x| \leq 1$, and $\psi(x) = 1$ for $|x| \geq 2$. We set $\lambda = 1 - \psi$. For $0 < \varepsilon < 1/2$ we define $\psi_\varepsilon(x) = \psi(x/\varepsilon)$, $\lambda_\varepsilon(x) = \lambda(\varepsilon x)$, and $k_\varepsilon = \psi_\varepsilon \lambda_\varepsilon k$ so that $\operatorname{supp} k_\varepsilon \subset \{x \,;\, \varepsilon \leq |x| \leq 2/\varepsilon\}$.

Definition 2.1. *For $f \in \mathcal{S}'$, $0 < \varepsilon < 1/4$, we define $R_{ij}^\varepsilon f$ by $R_{ij}^\varepsilon f = \partial_i\partial_j k_\varepsilon * f$.*

Since it is known that

$$R_iR_jf = (\text{p.v.}\, \partial_i\partial_j k) * f - \delta_{ij}f/n \tag{2.1}$$

for $f \in \mathcal{S}(\mathbf{R}^n)$, it is natural to expect that R_{ij}^ε approximates R_iR_j. We describe its convergence properties in the following theorem.

Remark 2.1. *The equality (2.1) is based on the fact that inverse Fourier transform for the symbol of R_iR_j is given by*

$$\mathcal{F}^{-1}\Big[-\frac{\xi_i\xi_j}{|\xi|}\Big] = \text{p.v.}\, \partial_i\partial_j k - \frac{\delta_{ij}}{n}\delta \quad \text{in } \mathcal{S}',$$

where δ is the Dirac's delta function.

Theorem 2. *Let $1 \le i, j, l \le n$.*
(1) *For $f \in L^\infty$, we have*

$$\lim_{\varepsilon \downarrow 0} \langle R^\varepsilon_{ij} f, \varphi \rangle = \langle R_i R_j f, \varphi \rangle$$

for all $\varphi \in \mathcal{S}$ with $\int \varphi = 0$. Moreover, we have

$$\lim_{\varepsilon \downarrow 0} R^\varepsilon_{ij} \partial_l f = \partial_l R_i R_j f \quad \text{in } \mathcal{S}'.$$

(2) *For $f \in \mathcal{S}'$ with $\operatorname{div} f = 0$, $0 < \varepsilon < 1/4$, we have*

$$\sum_{j=1}^n R^\varepsilon_{ij} f_j = 0 \quad \text{in } \mathcal{S}'.$$

(3) *For $f \in BMO$, we have*

$$\lim_{\varepsilon \downarrow 0} \sum_{j=1}^n R^\varepsilon_{ij} \partial_j f = -\partial_i f \quad \text{in } \mathcal{S}'.$$

Remark 2.2. *(1) For $f \in L^\infty$ we may define $R_i R_j f$ via the identity*

$$\langle R_i R_j f, \varphi \rangle = \langle f, R_i R_j \varphi \rangle \tag{2.2}$$

for $\varphi \in \mathcal{S}$ with $\int \varphi = 0$. (See [16, Chap. IV, §4] for details.) Notice that the right hand side of (2.2) makes sense, since $\varphi \in \mathcal{H}^1$ and the Riesz transformations are bounded from $\mathcal{H}^1$ to L^1 and from $\mathcal{H}^1$ to itself.
(2) If we set $\mathbf{P}_\varepsilon = (\delta_{ij} + R^\varepsilon_{ij})$, then the statements of Theorem 2 (2), (3) are rewritten as

$$\lim_{\varepsilon \downarrow 0} \mathbf{P}_\varepsilon u = u \quad \text{in } \mathcal{S}', \quad \text{if } u \in \mathcal{S}' \text{ with } \operatorname{div} u = 0,$$
$$\lim_{\varepsilon \downarrow 0} \mathbf{P}_\varepsilon \nabla p = 0 \quad \text{in } \mathcal{S}', \quad \text{if } p \in BMO,$$

respectively.

For the proof of Theorem 2, the following proposition is essentially used.

Proposition 2.1. *Let $1 \le i, j \le n$. We assume that $\varphi \in \mathcal{S}$. Then,*

(1) *$R^\varepsilon_{ij} \varphi$ converges to $R_i R_j \varphi$ uniformly in every compact subset in $\mathbf{R}^n$.*

(2) *If φ additionally satisfies $\int \varphi = 0$, then*

$$\lim_{\varepsilon \downarrow 0} R_{ij}^{\varepsilon} \varphi = R_i R_j \varphi \quad \text{in } \mathcal{H}^1. \tag{2.3}$$

In particular, we have $\lim_{\varepsilon \downarrow 0} \Delta k_\varepsilon * \varphi = \varphi$ in $\mathcal{H}^1$.

We postpone the proof of this proposition until section 4. Here we give a proof of Theorem 2 using Proposition 2.1.

Proof of Theorem 2. (1) For $\varphi \in \mathcal{S}$ with $\int \varphi = 0$ we have

$$\begin{aligned}
|\langle R_{ij}^{\varepsilon} f, \varphi \rangle - \langle R_i R_j f, \varphi \rangle| &= |\langle f, R_{ij}^{\varepsilon} \varphi - R_i R_j \varphi \rangle| \\
&\le \|f\|_{L^\infty} \|R_{ij}^{\varepsilon} \varphi - R_i R_j \varphi\|_{L^1} \\
&\le \|f\|_{L^\infty} \|R_{ij}^{\varepsilon} \varphi - R_i R_j \varphi\|_{\mathcal{H}^1} \\
&\to 0 \quad \text{as } \varepsilon \downarrow 0,
\end{aligned}$$

by Proposition 2.1 (2). The convergence of $R_{ij}^{\varepsilon} \partial_l f$ is similarly proved, since $\int \partial_l \varphi = 0$ for any $\varphi \in \mathcal{S}$.

(2) By the definition of R_{ij}^{ε} we obtain

$$\sum_{j=1}^{n} R_{ij}^{\varepsilon} f_j = \partial_i k_\varepsilon * \mathrm{div} f = 0 \quad \text{in } \mathcal{S}',$$

since $\mathrm{div} f = 0$.

(3) By the definition of R_{ij}^{ε} and Proposition 2.1 we have

$$\lim_{\varepsilon \downarrow 0} \Big\langle \sum_{j=1}^{n} R_{ij}^{\varepsilon} \partial_j f, \varphi \Big\rangle = \lim_{\varepsilon \downarrow 0} \langle f, \Delta k_\varepsilon * \partial_i \varphi \rangle = \langle f, \partial_i \varphi \rangle$$

for all $\varphi \in \mathcal{S}$, since $f \in BMO$ and BMO is the dual space of $\mathcal{H}^1$. □

3 Proof of Theorem 1

In this section we prove Theorem 1 by the following strategy. First, for a solution (u, p) of (1.1), (1.2) in the distribution sense we show that ∇p is represented by using the Riesz transforms and u. To derive such a representation of ∇p Theorem 2 is used. Next, using the above representation on ∇p, we observe that u is also a solution of the integral equation corresponding to (1.1), (1.2) with data u_0. Finally, by the uniqueness of bounded solutions to the integral equation, we obtain the uniqueness of u, and hence the uniqueness of ∇p follows.

Proof of Theorem 1. Let (u,p) be a solution of (1.1), (1.2) in the distribution sense such that

$$u \in L^\infty((0,T)\times\mathbf{R}^n), \quad p \in L^1_{\mathrm{loc}}([0,T);BMO).$$

Then we have

$$\begin{aligned}\int_0^T \{\langle u(s),\partial_s\Phi(s)\rangle + \langle u(s),\Delta\Phi(s)\rangle + \langle (u\otimes u)(s),\nabla\Phi(s)\rangle \\ + \langle p(s),\operatorname{div}\Phi(s)\rangle\}ds = -\langle u_0,\Phi(0)\rangle,\end{aligned} \tag{3.1}$$

for $\Phi \in C^1([0,T]\times\mathbf{R}^n)$ satisfying $\Phi(s,\cdot)\in\mathcal{S}$ for $0\le s\le T$, and $\Phi(T,\cdot)\equiv 0$.

Now, for $\varepsilon>0$, $1\le l\le n$, we take a test function Φ whose jth component is $R^\varepsilon_{lj}\tilde\varphi$, where $\tilde\varphi\in C^1([0,T]\times\mathbf{R}^n)$ satisfying $\tilde\varphi(s,\cdot)\in\mathcal{S}$ for $0\le s\le T$, and $\tilde\varphi(T,\cdot)\equiv 0$. Then, the first term on the left hand side of (3.1) equals to

$$\int_0^T\sum_{j=1}^n\langle R^\varepsilon_{lj}u_j(s),\partial_s\tilde\varphi(s)\rangle ds$$

and this turns out to be zero by Theorem 2 (2), since $\operatorname{div} u=0$. Similarly, the second term on the left hand side of (3.1) and the right hand side of (3.1) equal to zero. Thus we have

$$\int_0^T\Big\{\sum_{i,j=1}^n\langle\partial_i R^\varepsilon_{lj}u_i(s)u_j(s),\tilde\varphi(s)\rangle + \sum_{j=1}^n\langle\partial_j R^\varepsilon_{lj}p(s),\tilde\varphi(s)\rangle\Big\}ds = 0.$$

Letting ε to zero, we obtain

$$\int_0^T\sum_{j=1}^n\langle\partial_l p(s),\tilde\varphi(s)\rangle ds = \int_0^T\sum_{i,j=1}^n\langle\partial_i R_l R_j u_i(s)u_j(s),\tilde\varphi(s)\rangle ds \tag{3.2}$$

by Theorem 2 (1), (3). We notice that the above equality also holds if we change the order of the indices i, j, l of the derivatives and the Riesz transforms. By the arbitrary choice of $\tilde\varphi$, we observe that

$$\nabla p = \sum_{i,j=1}^n \nabla R_i R_j u_i u_j \quad \text{in } \mathcal{S}' \tag{3.3}$$

holds for a. e. t.

We next show that u satisfy the integral equation corresponding to (1.1), (1.2) with data u_0:

$$u(t) = e^{t\Delta}u_0 + \int_0^t \nabla\cdot e^{(t-s)\Delta}\mathbf{P}(u\otimes u)(s)ds \tag{3.4}$$

using the representation ∇p (3.3), where $\mathbf{P}=(\delta_{ij}+R_iR_j)$. To begin with, we refer to the following lemma.

Lemma 3.1 ([3], [10], [8]). *There exists a constant $C > 0$ such that*

$$\|\nabla e^{t\Delta}\mathbf{P}f\|_{L^\infty} \le Ct^{-1/2}\|f\|_{L^\infty}, \quad \text{for } t > 0,\ f \in L^\infty.$$

Combining (3.1) and (3.2), we have

$$\int_0^T \{\langle u(s), \partial_s\Phi(s)\rangle + \langle u(s), \Delta\Phi(s)\rangle - \langle \nabla\cdot\mathbf{P}(u\otimes u)(s), \Phi(s)\rangle\}ds = -\langle u_0, \Phi(0)\rangle.$$

Now, for $t \in (0,T)$, $\delta > 0$ with $t+\delta < T$, we take a test function of the form

$$\Phi(s,x) = \begin{cases} \eta(s)(e^{(t-s+\delta)\Delta}\varphi)(x), & 0 < s < t+\delta, \\ 0, & t+\delta \le s < T, \end{cases}$$

where $\eta \in C^1(\mathbf{R})$ with $\operatorname{supp}\eta \subset (-\infty, t+\delta)$, and $\varphi \in \mathcal{S}(\mathbf{R}^n)$. Then, we have

$$\begin{aligned} &\int_0^T \{\langle u(s), (\partial_s\eta)(s)e^{(t-s+\delta)\Delta}\varphi\rangle - \langle \nabla\cdot\mathbf{P}(u\otimes u)(s), \eta(s)e^{(t-s+\delta)\Delta}\varphi\rangle\}ds \\ &= -\langle u_0, e^{(t+\delta)\Delta}\varphi\rangle, \end{aligned} \tag{3.5}$$

since

$$\partial_s(\eta(s)e^{(t-s+\delta)\Delta}\varphi) = (\partial_s\eta)(s)e^{(t-s+\delta)\Delta}\varphi - \eta(s)\Delta e^{(t-s+\delta)\Delta}\varphi.$$

Now we further set

$$\eta(s) = \int_s^\infty \rho_\varepsilon(s'-t)ds',$$

where $\rho \in C(\mathbf{R})$ with $\rho \ge 0$, $\operatorname{supp}\rho \subset (-1,1)$, $\int\rho = 1$, and $\rho_\varepsilon(s) = \varepsilon^{-1}\rho(s/\varepsilon)$ for $0 < \varepsilon < \delta$. Then we have $\partial_s\eta(s) = -\rho_\varepsilon(s-t)$ and

$$\lim_{\varepsilon\downarrow 0}\int_s^\infty \rho_\varepsilon(s'-t)ds' = \chi_{(-\infty,t]}(s)$$

for $s \ne t$. For such η, the first term on the right hand side of (3.5) equals to

$$-\int_0^T \langle u(s), e^{(t-s+\delta)\Delta}\varphi\rangle\rho_\varepsilon(s-t)ds$$

and converges to $-\langle u(t), e^{\delta\Delta}\varphi\rangle$ as $\varepsilon \downarrow 0$ for a. e. t. In fact, for $t' > t$,

$$\begin{aligned} &\Big|\int_0^T \langle u(s), e^{(t-s+\delta)\Delta}\varphi\rangle\rho_\varepsilon(s-t)ds - \langle u(t), e^{\delta\Delta}\varphi\rangle\Big| \\ &\le \Big|\int_0^T \{\langle u(s), e^{(t-s+\delta)\Delta}\varphi\rangle - \langle u(s), e^{(t'-s+\delta)\Delta}\varphi\rangle\}\rho_\varepsilon(s-t)ds\Big| \\ &\quad + \Big|\int_0^T \langle u(s), e^{(t'-s+\delta)\Delta}\varphi\rangle\rho_\varepsilon(s-t)ds - \langle u(t), e^{(t'-t+\delta)\Delta}\varphi\rangle\Big| \\ &\quad + \big|\langle u(t), e^{(t'-t+\delta)\Delta}\varphi\rangle - \langle u(t), e^{\delta\Delta}\varphi\rangle\big|, \end{aligned} \tag{3.6}$$

and the second term on the right hand side of (3.6) converges to zero for a. e. t as $\varepsilon \to 0$. The first and the third term on the right hand side of (3.6) are easily bounded by

$$C\|u\|_{L^\infty}\|e^{(t'-t)\Delta}\varphi - \varphi\|_{L^1}$$

and converge to zero as $t' \to t$. Meanwhile, the second term on the right hand side of (3.5) converges to

$$\int_0^t \langle \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s), e^{\delta\Delta}\varphi \rangle ds$$

as $\varepsilon \downarrow 0$. Therefore, letting $\delta \downarrow 0$, we obtain

$$\left\langle u(t) - e^{t\Delta}u_0 + \int_0^t \nabla \cdot e^{(t-s)\Delta}\mathbf{P}(u \otimes u)(s)ds, \varphi \right\rangle = 0,$$

for a. e. t. By the arbitrary choice of $\varphi \in \mathcal{S}(\mathbf{R}^n)$, we observe that u satisfies the integral equation (3.4). We notice that u is identified with the $L^\infty(\mathbf{R}^n)$ valued continuous function on $(0, T)$.

Finally, the uniqueness of solutions of (3.4) follows by Lemma 3.1 and the Gronwall type inequality. In fact, if u and v are solutions of (3.4) in $L^\infty((0,T) \times \mathbf{R}^n)$ for the same initial data, then

$$\|u(t) - v(t)\|_{L^\infty} \le C \int_0^t (t-s)^{-1/2}(\|u(s)\|_{L^\infty} + \|v(s)\|_{L^\infty})\|u(s) - v(s)\|_{L^\infty}ds,$$

by Lemma 3.1. Thus, applying Gronwall type inequality, we obtain the desired result. (See [4, Lemma 8.1.1], for example.) This completes the proof of Theorem 1. □

4 Proof of Proposition 2.1

In this section, we give a proof of Proposition 2.1. In what follows, we repeatedly use the following properties on k, ψ, and λ which is defined in section 2.

Lemma 4.1. (1) *Let $1 \le i, j \le n$. We have*

$$\int (\partial_i\partial_j\psi)(x)k(x)dx = -\int (\partial_i\psi)(x)(\partial_j k)(x)dx = \frac{\delta_{ij}}{n}. \tag{4.1}$$

In the case $n = 2$, we especially have

$$\int (\partial_i\partial_j\psi)(x)k(\varepsilon x)dx = -\int (\partial_i\psi)(x)(\partial_j k)(x)dx = \frac{\delta_{ij}}{2}, \tag{4.2}$$

for $\varepsilon > 0$.

(2) *Let* $R > 2$ *and let* $|x| > R$. *We suppose* $\phi \in C^\infty(\mathbf{R}^n)$ *satisfies* $\mathrm{supp}\nabla\phi \subset \{1 \le |x| \le 2\}$ *and define* $\phi_\varepsilon(x) = \phi(\varepsilon x)$ *for* $\varepsilon > 0$. *Then we have*

$$|(\partial^\alpha \phi_\varepsilon)(x-y)(\partial^\beta k)(x-y) - (\partial^\alpha \phi_\varepsilon)(x)(\partial^\beta k)(x)| \le \begin{cases} C|y|\,|x|^{-3}\log|x|, & \text{if } n=2 \text{ and } \beta = 0, \\ C|y|\,|x|^{-n-1}, & \text{otherwise,} \end{cases}$$

for $|y| < |x|/2$, $0 < \varepsilon < 1/2$, $\alpha,\ \beta \in \mathbf{Z}^n_+$ *with* $|\alpha+\beta| = 2$.

Proof. (1) The first equality of (4.1), (4.2) is easily obtained using integration by parts. As for the first equality of (4.2), we notice that $\partial_i(k(\varepsilon x)) = (\partial_i k)(x)$ holds for $\varepsilon > 0$, since $n = 2$.

To obtain the second equality, we also apply integration by parts. Then we have

$$-\int (\partial_i\psi)(x)(\partial_j k)(x)dx = -\int_{|x|=2} \frac{x_i}{|x|}(\partial_j k)(x)dS_x + \int \psi(x)(\partial_i\partial_j k)(x)dx, \tag{4.3}$$

since $\psi(x) = 1$ if $|x| = 2$.

The second term of the right hand side of (4.3) is equal to zero, since ψ is a radial function and $\int_{S^{n-1}}(\partial_i\partial_j k)(\omega)dS_\omega = 0$.

The first term of the right hand side of (4.3) is equal to

$$-\int_{S^{n-1}} \omega_i(\partial_j k)(\omega)dS_\omega = |S^{n-1}|^{-1}\int \omega_i\omega_j dS_\omega = \frac{\delta_{ij}}{n}.$$

Therefore, we obtain (4.1) and (4.2).

(2) By mean value theorem, we have

$$\begin{aligned}&(\partial^\alpha\phi_\varepsilon)(x-y)(\partial^\beta k)(x-y) - (\partial^\alpha\phi_\varepsilon)(x)(\partial^\beta k)(x) \\ &= -\int_0^1 (\nabla\partial^\alpha\phi_\varepsilon)(x-\theta y)(\partial^\beta k)(x-\theta y)d\theta\cdot y \\ &\qquad - \int_0^1 (\partial^\alpha\phi_\varepsilon)(x-\theta y)(\nabla\partial^\beta k)(x-\theta y)d\theta\cdot y\ .\end{aligned}$$

Since we can estimate

$$|(\partial^{\alpha'}\phi_\varepsilon)(x-\theta y)| \le C|x|^{\alpha'}, \tag{4.4}$$

$$|(\partial^{\beta'} k)(x-\theta y)| \le \begin{cases} C|x|^{-3}\log|x|, & \text{if } n=2 \text{ and } \beta' = 0, \\ C|x|^{-n+2-|\beta'|}, & \text{otherwise,} \end{cases}$$

for $|x| > R$, $|y| < |x|/2$, $0 \le \theta \le 1$, we obtain the desired result.

The estimate (4.4) is obvious if $\alpha' = 0$. As for $|\alpha'| > 0$, we first notice that

$$(\partial^{\alpha'}\phi_\varepsilon)(x - \theta y) = 0, \quad \text{if } |x| > 4/\varepsilon.$$

In fact, the support of $(\partial^{\alpha'}\phi_\varepsilon)(x-\theta y)$ is contained in $\{1/\varepsilon < |x-\theta y| < 2/\varepsilon\}$ by assumption, and we have

$$|x - \theta y| > |x|/2 > 2/\varepsilon, \quad \text{if } 4/\varepsilon < |x| < |y|/2. \tag{4.5}$$

Thus, we can estimate

$$|(\partial^{\alpha'}\phi_\varepsilon)(x - \theta y)| \le C\varepsilon^{|\alpha'|} \le C|x|^{-|\alpha'|}.$$

The estimate on $(\partial^{\beta'}k)(x - \theta y)$ is obtained by using the monotonicity of $|(\partial^{\beta'}k)|$ and the first inequality of (4.5). □

Proof of Proposition 2.1 (1). We first derive the representation of $R_iR_j\varphi$ using k, the fundamental solution of $-\Delta$. Recall that (2.1), we have

$$R_iR_j\varphi(x) = \lim_{\varepsilon\downarrow 0}\int_{|x-y|>\varepsilon}(\partial_i\partial_jk)(x - y)\varphi(y)dy - \frac{\delta_{ij}}{n}\varphi(x)$$

for $x \in \mathbf{R}^n$. Then applying integration by parts, we obtain

$$R_iR_j\varphi(x) = \int(\partial_jk)(x - y)(\partial_i\varphi)(y)dy, \tag{4.6}$$

since the integration over $|x - y| = \varepsilon$ and $-\delta_{ij}\varphi(x)/n$ are canceled as $\varepsilon \downarrow 0$.

From the definition $R^\varepsilon_{ij}\varphi = (\partial_i\partial_jk_\varepsilon) * \varphi$ and $k_\varepsilon = \psi_\varepsilon\lambda_\varepsilon k$. Thus, by Leibnitz rule,

$$\begin{aligned} R^\varepsilon_{ij}\varphi = {} & \{\psi_\varepsilon\lambda_\varepsilon\partial_jk\} * (\partial_i\varphi) + \{(\partial_j\psi_\varepsilon)(\partial_ik)\} * \varphi \\ & + \{(\partial_i\partial_j\psi_\varepsilon)k\} * \varphi + \{(\partial_j\lambda_\varepsilon)(\partial_ik)\} * \varphi + \{(\partial_i\partial_j\lambda_\varepsilon)k\} * \varphi \end{aligned} \tag{4.7}$$

Applying Lemma 4.1 (1), we observe that the second term and the third term of the right hand side of (4.7) uniformly converges to $-\delta_{ij}\varphi/n$, $\delta_{ij}\varphi/n$, respectively. We also observe that the fourth term and the fifth term of the right hand side of (4.7) uniformly converges to zero over any compact subset in $\mathbf{R}^n$. Thus, it suffices to show that

$$\lim_{\varepsilon\downarrow 0}\sup_{|x|<R}\left|\int(1 - \psi_\varepsilon(y)\lambda_\varepsilon(y))(\partial_jk)(y)(\partial_i\varphi)(x - y)dy\right| = 0 \tag{4.8}$$

for $R>1$. For $|x|<R$, the above integral is bounded by

$$\int_{|y|<2R}(1-\psi_\varepsilon(y))|y|^{-n+1}dy+\int_{|y|>2R}(1-\lambda_\varepsilon(y))|y|^{-n-1}dy, \tag{4.9}$$

and (4.9) converges to zero as $\varepsilon\downarrow 0$. To obtain the bound (4.9) we used the estimate $|(\partial_i\varphi)(x-y)|\le C|y|^{-2}$ for $|y|>2R$, since $|x-y|\ge |y|/2$ in this range. Therefore, we obtain (4.8) and hence the proof is completed. □

To prove Proposition 2.1 (2), we prepare the following lemma.

Lemma 4.2. *Let $0<\alpha<1$. If a function f satisfies*

$$|x|^\alpha f\in L^1,\quad (1+|x|)^{n+\alpha}f\in L^\infty,\quad \textit{and}\ \int f=0, \tag{4.10}$$

then $f\in\mathcal{H}^1$. Moreover, there exists a constant $C>0$ such that

$$\|f\|_{\mathcal{H}^1}\le C(\,\||x|^\alpha f\|_{L^1}+\|(1+|x|)^{n+\alpha}f\|_{L^\infty}). \tag{4.11}$$

Remark 4.1. *The case $\alpha=1$ has been proved in [8] with additional assumption that the support of f is compact. The use of $\alpha\in(0,1)$ is a key point for the proof of Proposition 2.1 (2).*

Proof. We assume $\eta\in\mathcal{S}$ with $\int\eta\ne 0$ and set

$$\eta_t(x)\equiv t^{-n}\eta(x/t)\quad (t>0,\ x\in\mathbf{R}^n).$$

Then it is known that the norm of the Hardy space $\mathcal{H}^1$ is given by

$$\|f\|_{\mathcal{H}^1}\equiv\|\sup_{t>0}|\eta_t*f|\|_{L^1}.$$

We first prove

$$\||x|^{n+\alpha}\eta_t*f\|_{L^\infty}\le C(\,\||x|^\alpha f\|_{L^1}+\||x|^{n+\alpha}f\|_{L^\infty}) \tag{4.12}$$

for all $t>0$, f satisfying (4.10). To prove (4.12) we fix $x\in\mathbf{R}^n\setminus\{0\}$ and we divide the domain of integration as follows:

$$|x|^{n+\alpha}\,\eta_t*f(x)=\Big(\int_{|y|<\frac{|x|}{2}}+\int_{|y|>\frac{|x|}{2}}\Big)\,|x|^{n+\alpha}\,\eta_t(x-y)f(y)dy. \tag{4.13}$$

The second term of the right hand side of (4.13) is easily bounded by $C\||x|^{n+\alpha}f\|_{L^\infty}$, since we have $|x|^{n+\alpha} \le C|y|^{n+\alpha}$ in the domain of integration and $\|\eta_t\|_{L^1} = \|\eta\|_{L^1}$. We observe that the first term of the right hand side of (4.13) is equal to

$$
\begin{aligned}
|x|^{n+\alpha}\int_{|y|<\frac{|x|}{2}} \eta_t(x-y)f(y)dy &= \int_{|y|<\frac{|x|}{2}} (|x-y|^{n+\alpha}\eta_t(x-y) - |x|^{n+\alpha}\eta_t(x))f(y)dy \\
&\quad + \int_{|y|<\frac{|x|}{2}} (|x|^{n+\alpha} - |x-y|^{n+\alpha})\eta_t(x-y)f(y)dy \\
&\quad - \int_{|y|>\frac{|x|}{2}} |x|^{n+\alpha}\,\eta_t(x)f(y)dy \\
&\equiv I_1(x) + I_2(x) + I_3(x),
\end{aligned}
$$

since $\int f = 0$.

By mean value theorem we have

$$
\big||x-y|^{n+\alpha}\eta_t(x-y) - |x|^{n+\alpha}\eta_t(x)\big| \le C|y|^\alpha. \tag{4.14}
$$

In fact, the left hand side of (4.14) is bounded by

$$
\begin{aligned}
&C\int_0^1 \big(|x-\theta y|^{n+\alpha-1}|\eta_t(x-y)| + |x-\theta y|^{n+\alpha}|(\nabla\eta_t)(x-\theta y)|\big)d\theta\,|y| \\
&\le C\left(\||x|^n\eta_t\|_{L^\infty} + \||x|^{n+1}\nabla\eta_t\|_{L^\infty}\right)\int_0^1 |x-\theta y|^{-1+\alpha}d\theta\,|y|,
\end{aligned}
$$

and we can estimate $|x-\theta y| > |y|$ for $|y| < |x|/2$, $0 \le \theta \le 1$. Here, we notice that $\||x|^n\eta_t\|_{L^\infty} = \||x|^n\eta\|_{L^\infty}$, $\||x|^{n+1}\nabla\eta_t\|_{L^\infty} = \||x|^{n+1}\nabla\eta\|_{L^\infty}$. Thus, we obtain

$$
|I_1(x)| \le C\||x|^\alpha f\|_{L^1}.
$$

Similarly, we have

$$
\big||x|^{n+\alpha} - |x-y|^{n+\alpha}\big| \le C(|x-y|^n + |y|^n)|y|^\alpha
$$

for $|y| < |x|/2$ and hence we obtain

$$
\begin{aligned}
|I_2(x)| &\le C(\||x|^n\eta_t\|_{L^\infty}\||x|^\alpha f\|_{L^1} + \|\eta_t\|_{L^1}\||x|^{n+\alpha}f\|_{L^\infty}) \\
&\le C(\||x|^\alpha f\|_{L^1} + \||x|^{n+\alpha}f\|_{L^\infty}).
\end{aligned}
$$

Meanwhile, I_3 is bounded by $C\||x|^\alpha f\|_{L^1}$, since $|x|^\alpha \le |y|^\alpha$ in the domain of integration. Therefore we obtain (4.12).

Finally, (4.11) is obtained using (4.12). In fact,

$$\begin{aligned}
\|f\|_{\mathcal{H}^1} &= \|\sup_{t>0}|\eta_t * f|\|_{L^1} \\
&= \int_{|x|>1}|x|^{-n-\alpha}\sup_{t>0}|x|^{n+\alpha}|\eta_t * f(x)|dx + \int_{|x|\le 1}\sup_{t>0}|\eta_t * f(x)|dx \\
&\le C(\sup_{t>0}\||x|^{n+\alpha}\eta_t * f\|_{L^\infty} + \sup_{t>0}\|\eta_t * f\|_{L^\infty}) \\
&\le C(\||x|^\alpha f\|_{L^1} + \||x|^{n+\alpha} f\|_{L^\infty} + \|f\|_{L^\infty}). \qquad \square
\end{aligned}$$

Proof of Proposition 2.1 (2). The $\mathcal{H}^1$ convergence follows if we prove

$$\lim_{\varepsilon\downarrow 0}\|(1+|x|)^{n+\alpha}(R^\varepsilon_{ij}\varphi - R_iR_j\varphi)\|_{L^\infty} = 0 \tag{4.15}$$

for some $\alpha \in (0,1)$. In fact, applying Lemma 4.2 we have

$$\|R^\varepsilon_{ij}\varphi - R_iR_j\varphi\|_{\mathcal{H}^1} \le C\|(1+|x|)^{n+\alpha}(R^\varepsilon_{ij}\varphi - R_iR_j\varphi)\|_{L^\infty}, \tag{4.16}$$

and hence we obtain the desired result by (4.15). More precisely, to obtain (4.16) we apply Lemma 4.2 for α' which is less than α and each terms corresponding to the right hand side of (4.11) is bounded by the right hand side of (4.16).

By (4.6), (4.7), we observe that

$$\begin{aligned}
R^\varepsilon_{ij}\varphi - R_iR_j\varphi &= \{(\psi_\varepsilon\lambda_\varepsilon - 1)\partial_j k\} * \partial_i\varphi + \big(\{(\partial_j\psi_\varepsilon)\partial_i k\} * \varphi + \delta_{ij}\,\varphi/n\big) \\
&\quad + \big(\{(\partial_i\partial_j\psi_\varepsilon)k\} * \varphi - \delta_{ij}\,\varphi/n\big) + \{(\partial_j\lambda_\varepsilon)\partial_i k\} * \varphi + \{(\partial_i\partial_j\lambda_\varepsilon)k\} * \varphi,
\end{aligned} \tag{4.17}$$

and then we denote by I^ε_l the lth term of the right hand side of (4.17). To prove (4.15) it is sufficient to show that

$$\lim_{\varepsilon\downarrow 0}\||x|^{n+\alpha}I^\varepsilon_l\|_{L^\infty(\{|x|>R\})} = 0 \tag{4.18}$$

for some $\alpha \in (0,1)$, $l = 1, \cdots, 5$, since we have

$$\lim_{\varepsilon\downarrow 0}\|R^\varepsilon_{ij}\varphi - R_iR_j\varphi\|_{L^\infty(\{|x|\le R\})} = 0$$

by Proposition 2.1 (1), where $R > 2$.

To prove (4.18) we divide the domain of integration of $I^\varepsilon_1 = \{(\psi_\varepsilon\lambda_\varepsilon - 1)\partial_j k\} * \varphi$ into four parts

$$
\begin{aligned}
I_1^\varepsilon(x) = &\int_{|x-y|<\frac{|x|}{2}} (\lambda_\varepsilon(y)-1)(\partial_j k)(y)(\partial_i\varphi)(x-y)dy \\
&+\int_{|y|<\frac{R}{2}} (\psi_\varepsilon(y)-1)(\partial_j k)(y)(\partial_i\varphi)(x-y)dy \\
&+\int_{|y|>2|x|} (\lambda_\varepsilon(y)-1)(\partial_j k)(y)(\partial_i\varphi)(x-y)dy \\
&+\int_{|x-y|>\frac{|x|}{2},\ \frac{R}{2}<|y|<2|x|} (\lambda_\varepsilon(y)-1)(\partial_j k)(y)(\partial_i\varphi)(x-y)dy
\end{aligned}
\tag{4.19}
$$

for $|x|>R$ and we denote by J_l^ε the lth term of the right hand side of (4.19). As for J_2^ε, J_3^ε, and J_4^ε, we can use the decay of φ to eliminate the weight $|x|^{n+\alpha}$, since $|x-y|>|x|/2$ in each domain of integration. As for J_1^ε, instead of the decay of φ, we can make use of the decay of the integral kernel $(1-\lambda_\varepsilon)k$, since $\int\varphi=0$. We first observe that $J_1^\varepsilon(x)=0$ if $|x|<1/2\varepsilon$. In fact, although the domain of integration of $J_1^\varepsilon(x)$ is

$$
|x-y|<|x|/2, \quad |y|>1/\varepsilon,
$$

we have $|x-y|>|x|$ when $|x|<1/2\varepsilon$, $|y|>1/\varepsilon$. Thus, we may only consider $J_1^\varepsilon(x)$ for $|x|>1/2\varepsilon$.

Using integration by parts,

$$
\begin{aligned}
J_1^\varepsilon(x) = &\int_{|y|<\frac{|x|}{2}} (\lambda_\varepsilon(x-y)-1)(\partial_i\partial_j k)(x-y)\varphi(y)dy \\
&+\int_{|y|<\frac{|x|}{2}} (\partial_i\lambda_\varepsilon)(x-y)(\partial_j k)(x-y)\varphi(y)dy \\
&+\int_{|x-y|=\frac{|x|}{2}} \frac{x_i-y_i}{|x-y|}(\lambda_\varepsilon(y)-1)(\partial_j k)(y)\varphi(y)dS_y \\
\equiv\ & J_{1,1}^\varepsilon(x)+J_{1,2}^\varepsilon(x)+J_{1,3}^\varepsilon(x).
\end{aligned}
$$

Since $\int\varphi=0$,

$$
\begin{aligned}
J_{1,1}^\varepsilon(x) = &\int_{|y|<\frac{|x|}{2}} \big\{(\lambda_\varepsilon(x-y)-1)(\partial_i\partial_j k)(x-y)-(\lambda_\varepsilon(x)-1)(\partial_i\partial_j k)(x)\big\}\varphi(y)dy \\
&+\int_{|y|>\frac{|x|}{2}} (\lambda_\varepsilon(x)-1)(\partial_i\partial_j k)(x)\varphi(y)dy.
\end{aligned}
$$

Then applying Lemma 4.1 (2), we obtain

$$
|x|^{n+\alpha}|J_{1,1}^\varepsilon(x)| \le C|x|^{-1+\alpha}\int |y|\,|\varphi(y)|dy + C\varepsilon\int_{|y|>\frac{|x|}{2}} |y|^{1+\alpha}|\varphi(y)|dy, \tag{4.20}
$$

and the right hand side of (4.20) is bounded by $C\varepsilon^{1-\alpha}$, since $|x| > 1/2\varepsilon$. Similarly, we obtain $|x|^{n+\alpha}|J_{1,2}^{\varepsilon}(x)| \leq C\varepsilon^{1-\alpha}$.

Over the domain of integration of $J_{1,3}^{\varepsilon}(x)$, the estimate $|\varphi(x-y)| \leq C|x|^{-N}$ holds for any $N > 1$. Thus, taking $N > 2n + \alpha - 1$,

$$\begin{aligned} |x|^{n+\alpha}|J_{1,3}^{\varepsilon}(x)| &\leq C|x|^{n+\alpha} \int_{|x-y|=\frac{|x|}{2},\, |y|>\frac{1}{\varepsilon}} |y|^{-n+1}|\varphi(x-y)|dS_y \\ &\leq C\varepsilon^{n-1}|x|^{-N+2n+\alpha}. \end{aligned}$$

From the above arguments we obtain

$$\| |x|^{n+\alpha} J_1^{\varepsilon}\|_{L^\infty(\{|x|>R\})} \leq C\varepsilon^{1-\alpha}.$$

We use the following estimates on φ for J_2^{ε}, J_3^{ε}, and J_4^{ε}:

$$|(\partial_i\varphi)(x-y)| \leq \begin{cases} C|x|^{-N}, & \text{for } |y| < R/2 \ \text{ or } \ |x-y| > |x|/2, \\ C|x|^{-N}(1+|y|)^{-2}, & \text{for } |y| > 2|x|, \end{cases}$$

where $|x| > 1$ and $N > n + \alpha + 2$. Using the above estimates, we have

$$|x|^{n+\alpha}|J_2^{\varepsilon}(x)| \leq C|x|^{-N+n+\alpha} \int_{|y|<\frac{R}{2}} (1-\psi_\varepsilon(y))|y|^{-n+1}dy, \tag{4.21}$$

and

$$|x|^{n+\alpha}|J_3^{\varepsilon}(x)| \leq C|x|^{-N+n+\alpha} \int (1-\lambda_\varepsilon(y))(1+|y|)^{-2}|y|^{-n+1}dy, \tag{4.22}$$

and the right hand side of (4.21), (4.22) goes to zero as $\varepsilon \downarrow 0$ uniformly on $|x| > R$. Before the estimate of J_4^{ε}, we notice that $J_4^{\varepsilon}(x) = 0$ if $|x| < 1/2\varepsilon$, since the integrand is equal to zero when $|y| < 1/\varepsilon$. Thus, we may only consider $J_4^{\varepsilon}(x)$ for $|x| > 1/2\varepsilon$ and hence

$$\begin{aligned} |x|^{n+\alpha}|J_4^{\varepsilon}(x)| &\leq C|x|^{-N+n+\alpha} \int_{|y|<2|x|} |y|^{-n+1}dy \\ &\leq CR^{-N+n+\alpha+2}\,\varepsilon. \end{aligned}$$

Therefore, we obtain (4.18) for $l = 1$.

We can treat I_2^{ε} and I_3^{ε} in the same way, so we only prove about I_2^{ε} here.

By Lemma 4.1 (1) we have

$$\begin{aligned} |I_2^{\varepsilon}(x)| &= \left| \int (\partial_j\psi_\varepsilon)(y)(\partial_i k(y))\varphi(x-y)dy - \frac{\delta_{ij}}{n}\varphi(x) \right| \\ &\leq \int_{1<|y|<2} |(\partial_j\psi)(y)|\,|(\partial_i k)(y)|\,|\varphi(x-\varepsilon y) - \varphi(x)|dy. \end{aligned}$$

Since $\varphi \in \mathcal{S}$, we can estimate

$$|\varphi(x - \varepsilon y) - \varphi(x)| \leq C\varepsilon(1 + |x|)^{-n-\alpha}|y|$$

for $|y| < 2$ by mean value theorem. Thus, we obtain

$$\| |x|^{n+\alpha} I_2^\varepsilon \|_{L^\infty(\{|x|>R\})} \leq C\varepsilon.$$

We can also treat I_4^ε and I_5^ε in the same way, so we only prove about I_4^ε here.

For $|x| > 1$, we observe that

$$\begin{aligned}
I_4^\varepsilon(x) &= \int (\partial_j \lambda_\varepsilon)(x-y)(\partial_i k)(x-y)\varphi(y)dy \\
&= \int_{|y|<\frac{|x|}{2}} \{(\partial_j \lambda_\varepsilon)(x-y)(\partial_i k)(x-y) - (\partial_j \lambda_\varepsilon)(x)(\partial_i k)(x)\}\varphi(y)dy \\
&\quad + \int_{|y|>\frac{|x|}{2}} (\partial_j \lambda_\varepsilon)(x-y)(\partial_i k)(x-y)\varphi(y)dy \\
&\quad - \int_{|y|>\frac{|x|}{2}} (\partial_j \lambda_\varepsilon)(x)(\partial_i k)(x)\varphi(y)dy \\
&\equiv K_1^\varepsilon(x) + K_2^\varepsilon(x) + K_3^\varepsilon(x),
\end{aligned}$$

since $\int \varphi = 0$.

Here we notice that $K_1^\varepsilon(x) = 0$ if $|x| < 2/3\varepsilon$. In fact, $\partial_j \lambda_\varepsilon(x-y) = 0$ in this case, because

$$|x - y| < 3|x|/2 < 1/\varepsilon$$

for $|y| < |x|/2$. Of course $\partial_j \lambda_\varepsilon(x) = 0$, since $|x| < 1/\varepsilon$. Thus, we may only consider $K_1^\varepsilon(x)$ for $|x| > 2/3\varepsilon$, and hence we obtain

$$\begin{aligned}
|x|^{n+\alpha}|K_1^\varepsilon(x)| &\leq |x|^{n+\alpha} \int_{|y|<\frac{|x|}{2}} |(\partial_j \lambda_\varepsilon)(x-y)(\partial_i k)(x-y) - (\partial_j \lambda_\varepsilon)(x)(\partial_i k)(x)|\, |\varphi(y)|dy \\
&\leq C\,|x|^{-1+\alpha} \int |y|\,|\varphi(y)|dy \\
&\leq C\,\varepsilon^{1-\alpha},
\end{aligned}$$

by Lemma 4.1 (2). We also obtain

$$\begin{aligned}
|x|^{n+\alpha}|K_2^\varepsilon(x)| &\leq C\,\varepsilon|x|^{n+\alpha} \int_{|y|>\frac{|x|}{2},\ \frac{1}{\varepsilon}<|x-y|<\frac{2}{\varepsilon}} |x-y|^{-n+1}|\varphi(y)|dy \\
&\leq C\,\varepsilon^n \int |y|^{n+\alpha}|\varphi(y)|dy.
\end{aligned}$$

Similarly,

$$|x|^{n+\alpha}|K_3^\varepsilon(x)| \le C\,\varepsilon|x|^{2+\alpha}\int_{|y|>\frac{|x|}{2}}|\varphi(y)|dy$$
$$\le C\,\varepsilon\int|y|^{2+\alpha}|\varphi(y)|dy.$$

Thus, we conclude that

$$\||x|^{n+\alpha}I_4^\varepsilon\|_{L^\infty(\{|x|>R\})} \le C\,\varepsilon^{1-\alpha}.$$

Combining the above arguments, we obtain (4.15).

Finally, the $\mathcal{H}^1$ convergence of $\Delta k_\varepsilon * \varphi$ to φ is obtained by (2.3). In fact,

$$\lim_{\varepsilon\downarrow 0}\Delta k_\varepsilon * \varphi = \lim_{\varepsilon\downarrow 0}\sum_{j=1}^n R_{jj}^\varepsilon\varphi = \sum_{j=1}^n R_jR_j\varphi = \varphi \quad \text{in } \mathcal{H}^1.$$

This completes the proof. □

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