Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2010 A591

REGULARITY FOR DEGENERATE NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Dissertation for the degree of Doctor of Science in Technology to be presented, with due permission of the Faculty of Information and Natural Sciences, for public examination and debate in auditorium E at Aalto University School of Science and Technology (Espoo, Finland) on the 29th of October 2010, at 12 o'clock noon.

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ISBN 978-952-60-3351-8 (print)
ISBN 978-952-60-3352-5 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)
Aalto University Mathematics, 2010

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Juhana Siljander: *Regularity for degenerate nonlinear parabolic partial differential equations*; Helsinki University of Technology Institute of Mathematics Research Reports A591 (2010).

Abstract: This dissertation studies regularity and existence questions related to nonlinear parabolic partial differential equations. The thesis consists of an overview and four research papers. The emphasis is on certain doubly nonlinear equations that are important in several applications. We study the Hölder continuity of weak solutions and the local boundedness of their gradients by modifying and extending known arguments for other similar equations.

We also consider an existence question for a parabolic obstacle problem. In particular, we show that the obstacle problem with a continuous obstacle admits a unique continuous solution up to the boundary, provided the domain is smooth enough.

AMS subject classifications: 35K65, 35K10, 35B65, 35D30

Keywords: Regularity, existence, higher regularity, Moser's iteration, reverse Hölder inequality, Caccioppoli inequality, Schwarz alternating method, obstacle problem

Juhana Siljander: *Säännollisyyskysymyksiä epälineaarisille parabolisille osittaisdifferentiaaliyhtälöille*

Tiivistelmä: Tässä väitöskirjassa tutkitaan säännöllisyys- ja olemassaolokysymyksiä epälineaarisille parabolisille osittaisdifferentiaaliyhtälöille. Työ koostuu yhteenvedosta ja neljästä tutkimusartikkelista. Työn pääpaino on sovellusten kannalta tärkeiden epälineaaristen yhtälöiden tutkimisessa. Artikkeleissa käsitellään ratkaisujen Hölder jatkuvuutta sekä korkeampaa säännöllisyyttä modifioimalla ja yleistämällä muille vastaaville yhtälöille tunnettuja argumentteja.

Tutkimme myös olemassaolokysymystä paraboliselle esteongelmalle. Esteen oletetaan olevan jatkuva funktio ja todistamme, että kyseisellä esteongelmalla on yksikäsitteinen jatkuva ratkaisu.

Avainsanat: Säännollisyys, olemassaolo, korkeampi säännöllisyys, Moserin iteraatio, käänteinen Hölderin epäyhtälö, Caccioppoli-epäyhtälö, Schwarzin alternoiva metodi, esteongelma

Acknowledgements

First of all, I would like to thank my advisor Juha Kinnunen for introducing me to the field and for his constant support during the process. His guidance has been essential for my mathematical development and I truly appreciate the work he has done in supervising me.

I would also like to present my gratitude to all my co-authors. They have patiently answered all the questions I have had during my studies. In particular, I would like to thank Tuomo Kuusi for his continuing mentoring which started already in 2006 when I was finishing my Master's thesis and has been going ever since. In addition, I am very grateful to Mathias Masson for the numerous discussions we have had and through which I have been able to build my understanding of the concepts considered in this thesis.

I am grateful to my parents for their never-ending support and advice. I also want to thank my wife, Helga, for bringing me so much joy outside mathematics.

For the financial support I am indebted to Alfred Kordelin Foundation, Finnish Fulbright Center and to Academy of Finland which supported my studies through the Finnish National Graduate School in Mathematics and Its Applications. I also acknowledge the support of the Institute of Mathematics at Aalto University where I have conducted most of the research.

Helsinki, September 2010,

Juhana Siljander

List of included articels

This dissertation studies regularity and existence questions for nonlinear parabolic and elliptic partial differential equations of divergence form. The work consists of an overview and of the following four papers.

1. R. Korte, T. Kuusi, J. Siljander. Obstacle problem for nonlinear parabolic equations. Journal of Differential Equations 246 (2009), no. 9, 3668-3680.

2. J. Siljander. Boundedness of the gradient for a doubly nonlinear parabolic equation. Journal of Mathematical Analysis and Applications 371 (2010), 158-167.

3. T. Kuusi, J. Siljander, J.M. Urbano. Local Hölder continuity for doubly nonlinear parabolic equations, arXiv:1006.0781v2.

4. J. Siljander. A note on the proof of Hölder continuity to weak solutions of elliptic equations, arXiv:1005.5080v1.

The work has mostly been done in 2009-2010 at Helsinki University of Technology. Paper (1) was partially written in 2007-2008 at Courant Institute of Mathematical Sciences of New York University.

Author's contribution

The author's role in preparing paper (1) was concentrated on the latter part where it is proved that continuous weak supersolutions are $\mathcal{A}$-superharmonic functions. Papers (2) and (4) are independent research by the author while in paper (3) the author has had a key role in both analysis and writing of the manuscript. In particular, deriving the required Caccioppoli estimates, analyzing the role of the measure and, for instance, introducing the Case I - Case II -division has been work of the author.

Contents

1 History and motivation

Regularity and existence questions for nonlinear partial differential equations have been under active research during the last decades. The questions date back to David Hilbert's famous list of 23 mathematical problems that turned out to shape the mathematics of the 20th century. In 19th of his questions, he asked whether minimizers to functionals of type

$$\mathcal{I}(u) = \int \mathcal{F}(x, u, \nabla u)\, dx \tag{1.1}$$

are necessarily analytic if the function $\mathcal{F} : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is assumed to satisfy certain regularity conditions.

Naturally, the problem was directly associated with the corresponding Euler-Lagrange equation. This resulted in the study of regularity for the solutions of the partial differential equation

$$-\operatorname{div} \mathcal{A}(x, u, \nabla u) = 0. \tag{1.2}$$

Related to the assumptions on the kernel $\mathcal{F}$ in (1.1), the operator $\mathcal{A}$ was assumed to satisfy certain structure conditions.

It was soon found out that if a solution to this problem is at least twice continuously differentiable it is, indeed, analytic as well. However, for proving existence for nonlinear equations assuming this kind of a priori regularity is too restrictive. Moreover, the statement of the minimization problem (1.1) does not require such a regularity assumption. This lead to the study of so called weak solutions which were defined in Sobolev spaces. Then the aim was to prove that the solutions would still have nice behavior, a posteriori. The key point turned out to be proving the local Hölder continuity of the weak solutions which would then yield the higher regularity through a bootstrap argument.

The problem was finally settled in 1950's when Ennio De Giorgi [7] and John Nash [42] proved the continuity of weak solutions independently. Their work then inspired a lot of other research in the field. In 1960's Jürgen Moser used an iteration method for showing that subsolutions are locally bounded and, moreover, he proved the Harnack inequality for the weak solutions [38]. This in turn, provided a new way to obtain the continuity result of De Giorgi and Nash.

Nash proved his results also for parabolic equations and further generalizations in this direction were also on the way. As the elliptic question was solved, the natural direction was to proceed to parabolic equations of type

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, u, \nabla u) = 0. \tag{1.3}$$

We call this equation the $\mathcal{A}$-parabolic equation.

Moser's and De Giorgi's methods, were both based on a successive use of Sobolev's inequality and Caccioppoli estimates. De Giorgi established a measure estimate for certain distribution sets and by combining this estimate with another measure theoretical result, which nowadays is often called the "De Giorgi lemma", the proof was almost complete.

Moser in turn used similar ideas than De Giorgi to prove a reverse Hölder inequality. This inequality could then be iterated to obtain an upper bound for the supremum of the solution. This was used to conclude Harnack's inequality which implies the Hölder regularity in a straightforward manner.

De Giorgi's argument turned out to be problematic for parabolic generalizations. But Moser's method, on the other hand, seemed to be more flexible. By introducing the parabolic BMO he was able to obtain the Harnack estimate in the quadratic case for parabolic equations [39], [40], [41], [49]. See also [30]. Nevertheless, the problem remained open for equations with more general growth conditions.

2 Parabolic equations

2.1 Evolution p-Laplace equation

A standard example of an equation of type (1.3) is the evolution p-Laplace, or p-parabolic, equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty. \tag{2.1}$$

When $p = 2$ this is just the usual heat equation, but for other values of p it is highly nonlinear. In particular, it does not have quadratic growth with respect to the gradient.

It is noteworthy that the equation behaves differently in the cases when $1 < p < 2$ and $p \geq 2$. In the first case the exponent $p - 2$ in the power of $|\nabla u|$ will be negative and thus the set where $|\nabla u| = 0$ might be problematic. In this work we only consider the latter case when $p \geq 2$. These two different type of equations are referred to as singular and degenerate, respectively.

The kind of nonlinear equations we are studying can be tried to understand through their Barenblatt solutions which play the role of the fundamental solutions [2]. For equation (2.1), the Barenblatt solution is

$$\mathcal{B}_{2,p}(x,t) = t^{-d/\lambda}\left(\frac{p-2}{p}\lambda^{-1/(p-1)}\left(C - \left(\frac{|x|}{t^{1/\lambda}}\right)^{p/(p-1)}\right)\right)_+^{(p-1)/(p-2)}$$

where $C > 0$ is a constant and $\lambda = d(p-2) + p$ with $p > 2$.

From this one can easily notice the important property of finite propagation speed of disturbancies, i.e. the set in which $\mathcal{B}_{2,p} > 0$ is bounded for every time level. This differs from the heat equation which has infinite propagation speed. Therefore the evolution p-Laplace equation provides in some sense a more realistic way to model the heat propagation. This is one of the greatest advantages, and challenges, of this equation. One can also observe that the function is spatially $C^{1,\alpha}_{loc}$, but not C^2. The irregularity occurs at the moving boundary, i.e. the boundary of the set $\{\mathcal{B}_{2,p}(\cdot,t) > 0\}$.

The finite propagation speed property produces also some problems. It corresponds to the fact that we cannot scale solutions like in the case of the heat equation. This kind of homogenuity was a crucial requirement for pushing through the original arguments of De Giorgi and Moser. The starting point in their reasoning is the Caccioppoli inequality, but for nonnegative solutions of equation (2.1) this energy estimate takes the form

$$\begin{aligned} &\operatorname*{ess\,sup}_t \int u^2\varphi^p \, d\mu + \int\int |\nabla u|^p \varphi^p \, d\mu \, dt \\ &\le C \int\int u^p |\nabla\varphi|^p \, d\mu \, dt + C\int\int u^2 \left|\frac{\partial\varphi}{\partial t}\right| \varphi^{p-1} \, d\mu \, dt \end{aligned} \tag{2.2}$$

which is not homogeneous. Indeed, the exponents of u are not the same in different terms and this produced difficulties with the known arguments.

2.2 Doubly nonlinear equation

As the parabolic theory encountered problems with these methods Trudinger presented a nonlinear parabolic equation satisfying nonquadratic growth conditions for which the Moser iteration scheme could be applied [49]. This equation

$$\frac{\partial(|u|^{p-2}u)}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \tag{2.3}$$

had the desirable property that solutions could be scaled by constants, i.e. if u is a solution also λu is a solution where $\lambda \in \mathbb{R}$. Also the Caccioppoli inequality will take a homogeneous form for this equation. In the sequel, we will call this equation "the doubly nonlinear equation" since the term in the time derivative is nonlinear as well.

We can again try to understand the behavior of this equation by considering its Barenblatt solution

$$\mathcal{B}_p(x,t) = Ct^{-\frac{d}{p(p-1)}} \exp\left(-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}\right), \qquad 1 < p < \infty.$$

Observe that this fundamental solution does not have a moving boundary. The function is everywhere positive, which indicates infinite propagation speed. The Barenblatt solution is even a smooth function, but this porperty does not hold for general solutions as can be seen already by the stationary theory for p-Laplace equation, see [32]. Nevertheless, it has been a common belief that the solutions for this equation should be $C^{1,\alpha}_{loc}$, but still even continuity results for weak solutions of equation (2.3) have been difficult to find in the litterature [51], [43], [23].

The problem with this equation is that we cannot add constants to solutions like in the linear case. De Giorgi's argument is based on estimating level sets by using energy estimates where instead of u we have $(u-k)_\pm$. However, this kind of Caccioppoli estimates are not clear for the doubly nonlinear equation. On the other hand, Moser's ideas won't work either, because even though Harnack's inequality can be proved by using Moser's method, see [49] and [26], it does not seem to directly imply the Hölder continuity. One of the main objectives of this thesis has been to reconsider these problems. The Hölder regularity and the spatial local Lipschitz continuity for the solutions of this equation have been studied in papers (3) and (2), respectively. We consider the weighted case where the Lebesgue measure is replaced by a more general Borel measure.

The doubly nonlinear equation (2.3) has some intrinsic properties which make it interesting. Namely, the solutions of the equation can be scaled and, moreover, it admits a scale and location invariant Harnack's inequality [26]. See also [14], [29], [4], [6], [5] and [19].

Grigor'yan and Saloff-Coste observed that the doubling condition and the Poincaré inequality, see chapter 3, are not only sufficient but also necessary conditions for a scale and location invariant parabolic Harnack principle for the heat equation on Riemannian manifolds, see [18], [46] and [45]. It would be interesting to know whether this result could be generalized for more general nonlinear parabolic equations as well. Due to its nice scaling properties the doubly nonlinear equation seems to be a good candidate for this purpose.

Moreover, recently Lewis and Nyström proved a boundary Harnack principle for the time-independent p-Laplace equation, [33]. See also [34]. They make heavy use of weighted regularity results in their argument. Generalizing this result for nonlinear parabolic equations would be an interesting problem. The equation under study might be a good candidate for studying this question, as well.

2.3 Intrinsic scaling

The breakthrough in the parabolic theory took place in 1980's when Emmanuele DiBenedetto represented the "intrinsic scaling" argument, [9], [10]. This method provided a way to handle the inhomogenuity in the evolution p-Laplace equation (2.1) by modifying the geometry to make the equation seem more like the doubly nonlinear equation (2.3).

The great insight of DiBenedetto was to use a scaling factor which depends on the solution itself [9]. As parabolic equations, like the p-parabolic equation, did not admit homogeneous energy estimates the aim was to modify the geometry so that the equations started to look homogeneous in this sense. The idea was to use a scaling argument to make the equation look like

$$|u|^{p-2}\frac{\partial u}{\partial t} - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$$

which looks very similar to the doubly nonlinear equation.

In constructing the Hölder continuity argument the idea was to use a De Giorgi type argument. De Giorgi's method was based on energy estimates on differences $(u-k)_\pm$ where k is suitably chosen so that by estimating u by its infimum or supremum we can bound the term by the oscillation of u in a given space-time cylinder. So the natural scaling factor turns out to be ω^{2-p} where ω denotes the oscillation in the given cylinder. This factor balances the powers of ω and the estimates become homogeneous.

In any case, the parabolic regularity problem was solved. Moreover, the $C^{1,\alpha}_{loc}$-regularity for the p-parabolic equation was proved at the same token, [11], [12], [13]. Naturally, these issues were also studied for nonlinear equations of more general type

$$\frac{\partial u}{\partial t} - \operatorname{div}(u^{m-1}|\nabla u|^{p-2}\nabla u) = 0, \quad m \in \mathbb{R}, 1 < p < \infty. \tag{2.4}$$

Observe that for nonnegative solutions of the doubly nonlinear equation substitution $v = u^{p-1}$ in the equation (2.3) yields

$$\frac{\partial v}{\partial t} - \operatorname{div}(u^{2-p}|\nabla u|^{p-2}\nabla u) = 0.$$

So equation (2.4) formally includes both of the examples we have considered as well as the porous medium equation [50]. For further aspects in the parabolic theory, see also [35] and [52].

3 Doubling measures and Poincaré inequality

Traditionally partial differential equations have been studied in subdomains of $\mathbb{R}^d$. However, this is not the whole story. The Euclidean space $\mathbb{R}^d$ can be replaced, for instance, by a Riemannian manifold [45]. Also the standard Lebesgue measure can be replaced by a more general Borel measure. Studying this generalization in the parabolic setting has been one of the main themes in our work. All of the arguments in (2), (3) and (4) are build in this kind of more general context. However, we still cannot take just any measure, but we need to assume some additional properties.

Now it turns out that a so called doubling condition and a weak Poincaré inequality are sufficient assumptions in our arguments. These together imply a Sobolev embedding which is an important tool for us.

Many of the arguments can, in fact, be done even in a general metric space. In a metric space it is not clear how to define directions and, consequently, a gradient. However, this problem can be overcome by introducing so called upper gradients, see [21], [3]. Nevertheless, we will not take this route and we only consider $\mathbb{R}^d$ in this work.

3.1 Doubling condition

Let us denote the standard open ball in $\mathbb{R}^d$ by

$$B(x,r) := \{y \in \mathbb{R}^d : d(x,y) < r\}.$$

A measure μ is said to be doubling if there is a constant $D_0 > 0$ such that for every point $x \in \mathbb{R}^d$ and for every $r > 0$ it is true that

$$\mu(B(x, 2r)) \leq D_0 \mu(B(x, r)).$$

Let $0 < r < R < \infty$. A simple iteration of this doubling condition implies

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C \left(\frac{R}{r}\right)^{d_\mu},$$

where $d_\mu = \log_2 D_0$ is the dimension related to the measure. For the Lebesgue measure $d_{\mathcal{L}} = d$.

This tells us that the measure of balls with the same center scale somehow boundedly when the radius is increased. Moreover, the scaling exponent is uniform in the whole space.

3.2 Poincaré inequality

Another rather standard assumption, in addition to the doubling condition, is that the measure μ supports a weak Poincaré inequality.

We say that a measure μ supports a weak $(1, p)$-Poincaré inequality if there are constants $P_0 > 0$ and $\tau \geq 1$ such that for every $x \in \mathbb{R}^d$ and every $r > 0$ the Poincaré inequality

$$\fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq P_0 r \left(\fint_{B(x,\tau r)} |\nabla u|^p \, d\mu \right)^{1/p} \tag{3.1}$$

holds for all Sobolev functions. Here we denoted

$$u_{B(x,r)} := \fint_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x, r))} \int_{B(x,r)} u \, d\mu.$$

The word "weak" is related to the factor $\tau \geq 1$. If the inequality holds when $\tau = 1$ we say the measure supports a $(1, p)$-Poincaré inequality. In $\mathbb{R}^d$ with a doubling measure, the weak $(1, p)$-Poincaré inequality with some $\tau \geq 1$ implies the $(1, p)$-Poincaré inequality with $\tau = 1$, see Theorem 3.4 in [20].

In standard real analysis we have the fundamental theorem of calculus which relates the oscillation of the function to its derivative. Poincaré inequality is a similar tool. From (3.1) it is easy to see that if the measure of the set where the gradient is large, i.e. the set where the function's values change a lot, is small then also the set where the function's values differ a lot from its average has to be small in measure.

3.3 Sobolev embeddings

The key idea behind the two main assumptions is the fact that they imply a Sobolev embedding [20]. Indeed, the Sobolev inequality is one of the most important tools in the theory of partial differential equations.

It says that for every $x \in \mathbb{R}^d$ and $r > 0$ there is a constant $C > 0$ such that

$$\left(\fint_{B(x,r)} |u - u_{B(x,r)}|^\kappa \, d\mu \right)^{1/\kappa} \leq Cr \left(\fint_{B(x,r)} |\nabla u|^p \, d\mu \right)^{1/p} \tag{3.2}$$

where

$$\kappa = \begin{cases} \dfrac{d_\mu p}{d_\mu - p}, & 1 < p < d_\mu, \\ 2p, & p \geq d_\mu. \end{cases}$$

The crucial fact here is that $\kappa > p$. This gives extra room to be used in the regularity arguments of (2), (3) and (4). Naturally, by Hölder's inequality, κ can be replaced by any smaller positive number, as well.

If u has zero boundary values in the Sobolev sense in $B(x, r)$, the above inequality can be written without the mean values as

$$\left(\fint_{B(x,r)} |u|^\kappa \, d\mu \right)^{1/\kappa} \leq Cr \left(\fint_{B(x,r)} |\nabla u|^p \, d\mu \right)^{1/p}.$$

This is the form that is most often used in our arguments. In general, we do not assume that the functions we consider have zero boundary values, but we introduce a cut-off function for enabling the use of this inequality.

In the next two chapters we will present some key ideas from the papers (1), (2), (3) and (4).

4 Regularity for parabolic equations

4.1 Local Hölder continuity for doubly nonlinear equations

In paper (3) we extend DiBenedetto's Hölder continuity proof to cover nonnegative solutions of the doubly nonlinear equation. However, we face several difficulties not present in the case of evolution p-Laplace equation with Lebesgue measure.

The original proof, see [10], studies the continuity in a neighborhood of a given point (x_0, t_0) by considering a space time cylinder $B(x_0, r) \times (t_0 - \eta r^p, t_0)$. Naturally, the intrinsic geometry plays an important role here. Indeed, the scaling factor $\eta > 0$ depends intrinsically on the solution itself.

The idea in the proof is to show that the oscillation in this intrinsic cylinder is reduced by a controlled factor when the cylinder is shrinked by another factor. The proof starts by taking any suitable subcylinder inside the initial cylinder with the same spatial center point and with the same spatial radius. In this setting, one gets a measure estimate, for different level sets, which after a suitable iteration implies that if the set where the solution u is small (large), is small enough portion of the subcylinder, then u is actually big (small) almost everywhere in a smaller cylinder. Now it only remains to show that this information can be forwarded in time. This is necessary since the subcylinder under study does not necessarily contain the point (x_0, t_0), but it can be positioned at an earlier time.

The proof is divided in two alternatives. In the first one it is assumed that the set where the solution is small is small and it only remains to show the forwarding argument.

In the second alternative, the set where u is small is assumed to be big. This is then used to show that also at later times the set where the solution is above some threshold level is not too big. Then by using a De Giorgi type lemma it can be shown that by increasing the threshold level the measure of the set where u is above that threshold can be chosen arbitrarily small. Finally, an iteration of the measure estimate proved in the beginning concludes the argument by deducing that, in fact, in a smaller cylinder the supremum of the solution has, indeed, strictly decreased.

4.1.1 Two cases

The doubly nonlinear equation behaves somewhat differently than the evolution p-Laplace equation and hence it requires a slightly different kind of consideration that is needed in the p-parabolic setting. This leads us to study two different cases which correspond to two different kinds of behavior.

In large scales, when the oscillation of the solution is big, the solution behaves like the solutions of the heat equation do. In this case, the scaling property and the consequent Harnack's inequality dominate and the reduction of oscillation follows easily, even in a non-intrinsic geometry, for all $1 < p < \infty$.

On the other hand, in small scales the oscillation is small and the supremum and infimum are close to each other. Correspondingly, in the time derivative term of (2.3), which formally looks like $u^{p-2}u_t$, the factor u^{p-2} behaves like a constant coefficient. Indeed, if the oscillation is very small, this factor is between two constants whose difference is negligible. This implies a p-parabolic type behavior and hence this case demands an argument of its own which respects this nature. Basically one needs to go through the DiBenedetto scheme.

So both of the cases require a full treatment, although the first one is significantly simpler. Neither of the arguments is, however, completely redundant to the known theory, but in any case we need to modify some key parts of DiBenedetto's proof.

4.1.2 Modified Caccioppoli inequality

The first problem in applying the DiBenedetto argument for the doubly nonlinear equation comes with the energy estimates which are used in proving a measure estimate for distribution sets. The argument is based on a successive use of Hölder's inequality, Sobolev embedding and an energy estimate, but now, as we cannot add constants to solutions, we do not have Caccioppoli inequalities for $(u-k)_\pm$ which are needed in DiBenedetto's proof. Usually, proving these energy estimates is done by substituting a suitable test function in the definition of the weak solution. For nonnegative solutions of the doubly nonlinear equation this weak formulation is

$$\int\int \left(\mathcal{A}(x,t,u,\nabla u) \cdot \nabla \phi - u^{p-1} \frac{\partial \phi}{\partial t} \right) d\mu \, dt = 0 \tag{4.1}$$

which has to hold true for every test function. Naturally, the solution is assumed to belong to a parabolic Sobolev space to guarantee the summability of this integral.

In the case of the evolution p-Laplace equation the exponent $p-1$ of u in the second term in this definition is replaced by the exponent one. In our case, the nonlinearity in this term causes problems in proving energy estimates.

We overcome this setback by modifying the Caccioppoli inequality similarly as done with the porous medium equation, see [12], [13], [8], [50]. More precisely, we introduce an integral term which absorbs the nonlinearity and for which we have suitable estimates. See also [23], [53], [51], [43].

The estimates we acquire are slightly different than in the original argument of DiBenedetto. Namely, the parabolic terms will have extra weights related to the homogenous nature of the equation. However, this does not cause any problems since in the first case of the argument we do not really need the full DiBenedetto theory, and in the second case these weights can be trivially estimated by infimum and supremum.

4.1.3 Measure estimate

DiBenedetto's proof for the measure estimate is based on De Giorgi type ideas with the intrinsic scaling argument included. The intrinsic scaling he uses is realized by a change of variable in the energy estimate. He also has to carefully study the measures which occur after the trivial estimates of u by its supremum which are done for constructing the oscillation terms. In particular, the knowledge, that Lebesgue measure of the ball is comparable to the radius in power of the dimension, is used. We are able to work with a more general measure satisfying the doubling condition and supporting a weak Poincaré inequality.

Our argument combines the De Giorgi type ideas used by DiBenedetto with Moser's iteration. We plug $(u-k)_\pm$ in the Moser iteration scheme and by doing so we deduce the same estimate as DiBenedetto concluded by a change of variable argument. Using Moser's method enables us to construct the whole argument on integral averages. The advantage is that we only need to use the doubling property of the measure, instead of the exact scaling behavior.

4.1.4 Forwarding in time

The forwarding in time argument has traditionally been done by logarithmic estimates and in the second case we use this method as well. Note, however, that the logarithmic estimates for this kind of doubly nonlinear equations have often been proved by a substitution of type $v = u^{p-1}$ which changes the form of the equation. It is not clear for us whether the definitions of weak solutions to these different equations are equivalent and hence we wanted to avoid this passage. Instead, our proof of the estimates is based on a similar type of integral term as we used to prove the energy estimate. This enables us to bypass the substitution still taking care of the nonlinearity in the time derivative term.

In the first case, when the infimum is small, we on the other hand use a completely different kind of idea which trivializes the whole question. Namely, we use the Harnack inequality

$$\operatorname*{ess\,sup}_{U^-} u \leq H_0 \operatorname*{ess\,inf}_{U^+} u.$$

Observe the "time lag", i.e. the sets U^- and U^+ are disjoint cylinders with the same spatial center and radius such that U^- lies below, or before, U^+. This is familiar already from the standard theory of the heat equation and it holds for the doubly nonlinear equation, [26], [4], [6], [5] and [19].

The fact that the infimum is taken at a later time than the supremum provides us a natural way to forward information in time. If we know that the solution is positive in a set of positive measure at an earlier time, then by this estimate we also have a lower bound for the solution at a later time level. This is all we need in the first case. We just need to consider two alternatives. In the first one, we assume that the measure of the set

$$\{(x,t) \in B(x_0,r) \times (t_0 - r^p, t_0 - \frac{r^p}{\lambda_2}) : u(x,t) > \operatorname{ess\,sup} u - \frac{\operatorname{ess\,osc} u}{2}\}$$

is not zero for a suitable $\lambda_2 > 1$. Then by the Harnack inequality we deduce that the infimum of u near (x_0, t_0) is rather big as well. To conclude the argument we just need to guarantee that the lower bound we get is not trivial in the sense that the infimum will, indeed, be increased also near the reference point (x_0, t_0). This can be done by demanding that the infimum in the original set is small enough. This requirement contributes to the assumption of the first case.

The second alternative consists of the case in which the measure of the above set is zero. But then we can choose the scaling factor η large enough to force the portion of the original set where u is big arbitrarily small. Indeed, this portion is at most $1/\eta$. The alternative, and the first case, is concluded by an iteration of the measure estimate which finally concludes the reduction of the oscillation.

After the energy estimate and the logarithmic lemmata have been proved the second case follows DiBenedetto's reasoning. There remains some details which have to be carefully analyzed, but morally the rest of the argument is identical to the case of the evolution p-Laplace equation. The key observation is to use an intrinsic geometry where the intrinsic scaling factor also includes the infimum of u in such a way that the weights appearing from the integral terms of the energy estimate and of the logarithmic lemmata will be handled by the geometrical context.

The final Hölder estimate follows from an iteration of the reduction of oscillation process. Consequently, we might end up jumping between the first and second cases in the interation. A priori, this might cause a problem since the cases are build on different kind of time geometries. However, quite remarkably the assumptions of the cases guarantee that everything works fine. Indeed, we only need to shrink the cylinders by a controllable amount also when we move from the first case to the second one. Observe that we can construct the argument so that this happens only once and we can also always start from the case one.

The DiBenedetto scheme was introduced in [9]. Recently, there have also been found other approaches for the regularity argument [15], [17]. These new ideas are based on the methods which were developed for Harnack estimates. In particular, the expansion of positivity type of ideas are used. It would be interesting to know whether these new ideas would provide less involved ways to prove these continuity results also for this kind of doubly nonlinear equations where the nonlinearity lies in the time derivative term.

4.2 DiBenedetto regularity argument and elliptic equations

In order to show the Hölder regularity for the doubly nonlinear equation, especially in the weighted case, we needed to modify DiBenedetto's original argument. His proof is based on the measure estimate which tells that if the set where u is, say, near the supremum is small enough then the oscillation can, in fact, be reduced by going to a smaller cylinder. Now by an easy application of Chebyshev's inequality we can get the upper bound

$$\begin{aligned}
&\mu(\{u > \operatorname{ess\,sup} u - \frac{\operatorname{ess\,osc} u}{2^\lambda}\}) \\
&= \mu(\{\frac{\operatorname{ess\,osc} u}{2^\lambda} > \operatorname{ess\,sup} u - u\}) \\
&\le \Big(\frac{\operatorname{ess\,osc} u}{2^\lambda}\Big)^\delta \int \Big(\frac{1}{\operatorname{ess\,sup} u - u}\Big)^\delta \, d\mu
\end{aligned} \tag{4.2}$$

for this set. Here $\delta > 0$ can be chosen as we please.

So basically, if we can show that the integral on the right hand side of the above inequality is bounded, choosing λ large enough gives the result directly. Of course, one should assume that the solution is not constant.

In paper (4) we use this kind of ideas to give some remarks on the Hölder continuity proof of elliptic equations. Indeed, DiBenedetto's continuity argument applies also in the elliptic setting. Consequently, it is enough to study the above integral which can be shown to be bounded for solutions of certain elliptic equations.

The modifications of DiBenedetto's argument we did for the proof for the doubly nonlinear equation turn out to be quite useful also in this case. Pluging $(u-k)_+$ in the Moser iteration scheme works also for elliptic equations and, consequently, we get a simple proof for the measure estimate. We also make the argument, as before, for general measures. See also [16].

Originally, Moser used the iteration scheme to prove the supremum estimate. Instead of that we get the measure estimate of DiBenedetto. Now Moser continued to prove weak Harnack's inequality to conclude the continuity from Harnack's principle. Our approach, however does not require the full Harnack theory. By using the above Chebyshev argument we are able to simplify this method and, consequently, we only need to prove the integrability of a solution to some negative power. This can be done by a cross-over lemma which is proved by BMO techniques. More precisely, one uses logarithmic Caccioppoli estimates together with the John-Nirenberg lemma. It is noteworthy that Moser's proof for the weak Harnack inequality follows the same outline, however, we are able to get the regularity result directly from the cross-over lemma, instead of using Harnack's inequality for the final conclusion.

Another way to conclude the Hölder regularity is De Giorgi's lemma which De Giorgi used in his regularity argument. Our approach of using the Chebyshev inequality seems to provide an alternative approach to these methods for deducing the result. Whether this kind of ideas could be used for proving the regularity result for parabolic equations, would be an interesting question for further study. For further aspects of the elliptic theory, see e.g. [31], [22] and [37].

4.3 Higher reqularity for doubly nonlinear equations

As the Barenblatt solution for the doubly nonlinear equation is smooth this raises the expectation that such regularity is true also for general solutions of the equation. However, as we already noted in the beginning already the standard stationary theory shows that C^∞-smoothness is too much to ask, [32]. Nevertheless, one would like to prove the $C^{1,\alpha}_{loc}$ -regularity for the solutions of the equation. In paper (2) we give a first step towards this goal. More precisely, we prove that the gradient of a positive solution is locally bounded and thus the solution itself is spatially locally Lipschitz continuous. Once again we construct the proof in the weighted case where the Lebesgue measure is replaced by a more general Borel measure.

Our proof is based on the argument by DiBenedetto and Friedman [11] which handles the evolution p-Laplace equation. See also [12], [13]. Their result was stated directly to systems of partial differential equations, but we only consider one equation.

DiBenedetto and Friedman start by differentiating the equation and then use standard techniques to prove Caccioppoli inequalities for this differentiated equation. Next they employ Moser's iteration to show that the gradient of the solution is locally integrable to any power. Finally, they conclude the boundedness of the gradient by a De Giorgi type argument.

The first difficulty with the doubly nonlinear equation comes again from the nonlinearity in the time derivative term. After differentiating the equation, we will have an extra factor of u^{p-2} in front of the time derivative. Recall that we only study positive solutions so that this factor makes sense even without absolute values. This assumption also allows us to freeze this nonlinear factor. Suppose we study the regularity in a neighborhood of $(x_0, t_0) \in \Omega_T \subset \mathbb{R}^{d+1}_+$ and assume further that our solution u is positive at this point. By continuity, as studied in the previous section, there is an open neighborhood of this point such that the solution is bounded away from zero in this set as well. Now we restrict our study to this set. By a standard covering argument any compact set can be handled this way.

For getting rid of the nonlinearity we use a freezing argument. We treat the nonlinear factor as a constant coefficient by freezing it at the point (x_0, t_0). Next we scale it to the time geometry and continue to prove the Caccioppoli inequalities as in [11]. Here we need to use an intrinsic scaling argument. After this trick proving the energy estimates turns out to be quite straightforward and we just need to follow DiBenedetto and Friedman.

Next DiBenedetto and Friedman proceed to prove that the gradient is locally integrable to any positive power. They use this result in their final De Giorgi argument which gives the theorem. We simplify this method by dropping off the De Giorgi argument. It was long thought that Moser's iteration cannot be used for non-homogeneous equations, like the equation for the gradient, to yield the local boundedness. This is, however, not the case. By a delicate analysis of Moser's method it can be pushed through to non-homogeneous equations, too. The reverse Hölder inequality which the Moser argument gives will not be homogeneous, but it still can be iterated. The common belief used to be that iterating this estimate would lead to the blow-up of the constants and other methods, like the De Giorgi type argument used in [11], were developed. However, this is not necessary as shown in (2).

The drawback in the theory is that, even though the argument can be generalized to equations like the doubly nonlinear example we are studying, it is rather intrinsic to the original p-parabolic equation. As a consequence, the final estimate we get is non-homogeneous although our original equation is homogeneous with respect to scaling. More precisely, we are only able to prove

$$\operatorname{ess\,sup} |\nabla u|^2 \leq C \left(\fint |\nabla u|^p \, d\nu + 1 \right).$$

We would like this estimate to resemble the homogenuity so that the exponents in both sides would be the same. Instead, we get the estimate related to the geometry of the evolution p-Laplace equation.

On the other hand, we do not use the scaling property at all. Consequently, our reasoning works also for other more general doubly nonlinear equations of type

$$\frac{\partial(u^m)}{\partial t} - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad m \geq 1, 2 \leq p < \infty.$$

However, for these equations the assumption that solutions are positive is not as natural as in the case of the doubly nonlinear equation. This is due to the fact that this kind of general doubly nonlinear equations do have the moving boundary outside of which the solution is not positive but merely nonnegative.

The crucial part where we need the positiveness assumption is the freezing argument. Observe that if we choose $m = 1$ in the above doubly nonlinear equation we get the evolution p-Laplace equation. In that case we do not need the freezing argument since we do not have the difficult nonlinearity anymore. Indeed, our proof also gives a simplified argument for the boundedness of the gradient of solutions to p-parabolic equation. We also study the weighted case where the Lebesgue measure is replaced by a more general Borel measure.

5 Questions on existence

5.1 Obstacle problem

In paper (1) we study the existence of solutions to an obstacle problem. Suppose we are given a continuous function ψ called the obstacle and we want to find a continuous function which in some sense solves a differential equation with the restriction that the solution has to lie above the obstacle ψ. In our paper the equation under study is the $\mathcal{A}$-parabolic equation

$$\frac{\partial u}{\partial t} - \operatorname{div} \mathcal{A}(x, t, u, \nabla u) = 0.$$

introduced in the first chapter.

We define the solution to the obstacle problem as the smallest weak supersolution of the equation which lies above the obstacle and which is $\mathcal{A}$-parabolic, i.e. solves the equation, in the set $\{u > \psi\}$. We prove that there is a unique continuous solution to this problem. We also show that if the boundary satisfies certain thickness condition, the solution attains the boundary values continuously. Furthermore, we prove that if the obstacle is Hölder continuous also the solution will have Hölder regularity.

The existence questions for parabolic obstacle problems has been studied via variational methods by Lions [36] and [25]. See also [1], [44], [47] and [48].

5.2 Schwarz method

Our argument is based on a modification of the Schwarz alternating method. See e.g. [24]. It starts with two overlapping domains in both of which one has a solution for a Dirichlet problem. In the overlapping domain one takes the boundary values from one of the solutions and solves the Dirichlet problem in the other domain with these boundary values. Consequently, one gets a new solution for the Dirichlet problem. Next this new solution is used to give boundary values for the other domain. By continuing this kind of iterative process one ends up with a bounded decreasing sequence of solutions.

Harnack's convergence theorem implies that this sequence converges to a solution in the union of the domains.

In our problem, just like in the Schwarz alternating method, we start from the obstacle and construct a sequence of functions using the previous function as boundary data. However, we do not consider union of two domains but instead we take all the cylinders with rational endpoints in the domain. Naturally, these cylinders can be enumerated as $Q^0, Q^1, \ldots$. We start the construction by defining $\varphi_0 = \psi$, i.e. the first function in the sequence is just the obstacle. Next we solve the boundary problem with boundary values φ_0 in Q^0 and thus we get a solution in Q^0 which we further extend as φ_0 to the whole domain. Now the next member φ_1 in the sequence is defined as the maximum of the function achieved by extending the solution and φ_0. We continue inductively through all the cylinders.

5.3 Limit function

By the construction the sequence will be increasing and bounded. All the functions in the sequence will also be continuous and the limit function will thus be lower semicontinuous. Moreover, in the set $\{\varphi_k > \psi\}$ the function φ_k is achieved as a maximum of subsolutions and thus in this set it will be a subsolution to the equation. So many of the desired properties we require from a solution of the obstacle problem are immediate consequences of our construction.

One of the key ideas in this method is that the limit function will, in fact, turn out to be $\mathcal{A}$-superparabolic, i.e. it will be finite in a dense subset of the original domain, lower semicontinuous and most importantly it will satisfy a comparison principle in any subset of the domain [27]. We are left to show that this function will be continuous and that continuous $\mathcal{A}$-superparabolic functions are weak supersolutions.

Clearly, by the continuity of the obstacle the limit function u will be continuous in the contact set $\{u = \psi\}$. On the other hand, in the set $\{u > \psi\}$ the function u turns out to be a solution to the equation and hence it is continuous there as well. So the problem is the boundary of the set $\{u > \psi\}$. This we handle by a modification of the obstacle function.

By the maximum principle we deduce that varying the obstacle by a small amount cannot change the solution too much. Next we dig a hole in the obstacle, i.e. we slightly lower it in a small neighborhood of the reference point we are studying. Now we are able to show that a small hole in the obstacle changes the solution by a small amount. The solution for the modified obstacle will also be continuous. Consequenty, the original function will be continuous at the point, too. We can apply the same argument for boundary points of the original domain, as well, provided solutions to the equation admit boundary values continuously. This is guaranteed by a thickness condition to the domain.

For proving that $\mathcal{A}$-superparabolic functions are weak supersolution we construct an increasing sequence of weak supersolutions which converges to the $\mathcal{A}$-superparabolic function u. The aim is to use a compactness result which tells us that the limit of increasing sequence of weak supersolutions is a weak supersolution, too [28].

The construction is again generated by the original $\mathcal{A}$-superparabolic function. We take a dyadic decomposition of the domain and solve the equation with the boundary values u in each cube of level k. Next we use a pasting lemma to conclude that the function u_k achieved by gluing these solutions of different cubes of the dyadic decomposition will, in fact, be a supersolution. Finally, we show that this sequence will be increasing and converges to u which proves the claim.

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ISBN 978-952-60-3351-8 (print)
ISBN 978-952-60-3352-5 (PDF)
ISSN 0784-3143 (print)
ISSN 1797-5867 (PDF)
Aalto University Mathematics, 2010