FINITE ELEMENT METHODS AND NAVIER-STOKES EQUATIONS

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1. INTRODUCTION.

The use of the finite element method (F.E.M.) for numerically solving nontrivial problems in Fluid Mechanics is a new and important step in the developement of the method. In the last few years, considerable efforts have been made concerning the finite element approximation of incompressible viscous flows and decisive progress have been obtained in this direction. The purpose of this paper is to discuss a variety of finite element methods for the stationary Navier-Stokes equations which have been successfully studied and tested particularly in the two-dimensional case.

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N = 2,3)$ with boundary Γ. The Navier-Stokes system of equations for flow of a viscous incompressible fluid may be written in the form

(1.1) $$-\nu\,\Delta\underline{u} + \sum_{i=1}^{N} u_i \frac{\partial \underline{u}}{\partial x_i} + \underline{\text{grad}}\, p = \underline{f} \quad \text{in} \quad \Omega ,$$

(1.2) $$\text{div}\, \underline{u} = 0 \quad \text{in} \quad \Omega ,$$

where $\underline{u} = (u_1, \ldots, u_N)$ is the velocity, p the pressure, $\underline{f}$ the body forces and ν the viscosity. The boundary conditions will be taken in the form

(1.3) $$\underline{u} = \underline{g} \quad \text{on} \quad \Gamma \quad \text{with} \quad \int_\Gamma g_n \, dS = 0 ,$$

where $g_n = \underline{g}\cdot\underline{n}$, $\underline{n}$ being the unit outward normal along Γ.

One main difficulty in solving problem (1.1)-(1.3) stems from the numerical treatment of the incompressibility condition (1.2). In fact, we cannot easily construct finite elements which satisfy exactly the constraint (1.2) together with the continuity of the velocity at the interelement boundaries. Therefore, we shall consider two class of F.E.M.; in the first class of methods based upon the classical variational formulation of the Navier-Stokes equations, the velocity is assumed to be continuous at the interelement boundaries but the incompressibility constraint is only approximatively satisfied.

On the other hand, the second class of methods is based upon a mixed variational formulation of the Navier-Stokes system : the divergence condition (1.2) is exactly satisfied at the expense however of relaxing the continuity of the tangential velocity at the interelement boundaries and introducing the vorticity as a dependent variable.

Therefore, an outline of the paper is as follows. In Section 2, we begin by considering the Stokes system of equations for a viscous incompressible creeping fluid flow

$$(1.4) \qquad \begin{cases} -\nu\,\Delta\underline{u} + \underline{\mathrm{grad}}\ p = \underline{f} \quad \text{in } \Omega\ , \\ \mathrm{div}\ \underline{u} = 0 \quad \text{in } \Omega\ , \\ \underline{u} = \underline{g} \quad \text{on } \Gamma\ . \end{cases}$$

We introduce the classical variational formulation of (1.4) and we analyze the associated F.E.M. The analysis is then extended to the full Navier-Stokes system. Section 3 is devoted to the derivation of a mixed F.E.M. for the Stokes system. In Section 4, using the mixed approach of § 3 , we introduce two discretization methods of the convective terms of the Navier-Stokes equations.

For the sake of conciseness, we have not discussed here the numerical algorithms for solving the linear and nonlinear discretized systems but some relevant papers are given in the references.

2. " CLASSICAL " F.E.M. FOR THE NAVIER-STOKES SYSTEM

Let us consider the Stokes system (1.4) and let us recall its classical variational formulation. For any integer $m \geq 1$, we introduce the usual Sobolev space

$$H^m(\Omega) = \{\varphi \in L^2(\Omega)\ ;\ \partial^\alpha\varphi = \frac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1}\dots\partial x_N^{\alpha_N}} \in L^2(\Omega)\ ,\ |\alpha| \leq m\}$$

provided with the norm

$$\|\varphi\|_{m,\Omega} = \Big(\sum_{|\alpha|\leq m} \|\partial^\alpha\varphi\|^2_{o,\Omega}\Big)^{1/2} \quad , \qquad \|\varphi\|_{o,\Omega} = \Big(\int_\Omega \varphi^2\,dx\Big)^{1/2}\ .$$

We set

$$H^1_o(\Omega) = \{\varphi \in H^1(\Omega) \; ; \; \varphi = 0 \quad \text{on} \quad \Gamma\} \quad .$$

Moreover, we introduce the quotient space $L^2(\Omega)/\mathbb{R}$ of functions $\varphi \in L^2(\Omega)$ which are defined up to an additive constant. We provide $L^2(\Omega)/\mathbb{R}$ with the quotient norm

$$\|\varphi\|_{L^2(\Omega)/\mathbb{R}} = \inf_{c \in \mathbb{R}} \|\varphi + c\|_{o,\Omega} \quad \|\varphi - \frac{1}{\text{meas}(\Omega)} \int_\Omega \varphi \, dx\|_{o,\Omega} \quad .$$

Now the precise mathematical formulation of problem (1.4) (cf. Ladyzhenskaya [21], Temam [26]) consists in finding a pair $(\underline{u},p) \in (H^1(\Omega))^N \times (L^2(\Omega)/\mathbb{R})$ such that for all $\underline{v} \in (H^1_o(\Omega))^N$

$$(2.1) \qquad \begin{cases} \displaystyle\int_\Omega \{\nu D_{ij}(\underline{u}) \, D_{ij}(\underline{v}) - p \,\text{div}\, \underline{v}\} \, dx = \int_\Omega f_i v_i \, dx \;, \\ \text{div}\, \underline{u} = 0 \quad \text{in} \quad \Omega \;, \\ \underline{u} = \underline{g} \quad \text{on} \quad \Gamma \;, \end{cases}$$

where $D_{ij}(\underline{u}) = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ (We have used the classical summation convention). Note that the first equation (2.1) expresses the principle of virtual powers, $\underline{v}$ playing the role of a virtual velocity.

Setting

$$(2.2) \qquad \begin{cases} a(\underline{u},\underline{v}) = \nu \displaystyle\int_\Omega D_{ij}(\underline{u}) \, D_{ij}(\underline{v}) \, dx \;, \\ (\cdot,\cdot) = \text{scalar product in } L^2(\Omega) \text{ or in } (L^2(\Omega))^N \;, \end{cases}$$

an equivalent form of (2.1) is given by

$$(2.3) \qquad \begin{cases} \forall\, \underline{v} \in (H^1_o(\Omega))^N \;, \; a(\underline{u},\underline{v}) - (p, \text{div}\, \underline{v}) = (\underline{f},\underline{u}) \;, \\ \forall\, q \in L^2(\Omega) \;, \; (q, \,\text{div}\, \underline{u}) = 0 \;, \\ \qquad \underline{u} = \underline{g} \quad \text{on} \quad \Gamma \;. \end{cases}$$

A first class of F.E.M. for solving problem (1.4) consists in using the characterization (2.3) of the solution $(\underline{u},p)$ of the Stokes system. We construct two finite dimensional spaces X_h and Q_h such that

$$X_h \in H^1(\Omega) \quad , \quad Q_h \subset L^2(\Omega) \quad ,$$

and we set

$$X_{h,o} = \{\varphi \in X_h \; ; \; \varphi = 0 \quad \text{on} \quad \Gamma\} = X_h \cap H^1_o(\Omega) \;,$$

$$V_h = X_h^N \quad , \quad V_{h,o} = X_{h,o}^N \quad .$$

In addition, we are given a function $\underline{g}_h$ defined on Γ and which approximates $\underline{g}$. We assume that $\underline{g}_h$ belongs to the space $V_{h|\Gamma}$ of the traces over Γ of all functions of V_h and satisfies

$$\int_\Gamma g_{h,n}\, dS = 0$$

Then the approximation scheme is defined as follows : To find a pair $(\underline{u}_h, p_h) \in V_h \times (Q_h/\mathbb{R})$ such that

$$(2.4) \quad \begin{cases} \forall\, \underline{v} \in V_{h,o} \quad , \quad a(\underline{u}_h, \underline{v}) - (p_h, \operatorname{div} \underline{v}) = (\underline{f}, \underline{v}) \ , \\ \forall\, q \in Q_h \quad , \quad (q, \operatorname{div} \underline{u}_h) = 0 \ , \\ \qquad \underline{u}_h = \underline{g}_h \quad \text{on} \quad \Gamma \ . \end{cases}$$

Note in general, the 2nd equation (2.4) <u>does not imply</u> $\operatorname{div} \underline{u}_h = 0$; this point will be made more precise in the examples. On the other hand, one cannot choose independently the finite-dimensional spaces V_h and Q_h in order that problem (2.4) has a unique solution. In fact, we have

<u>LEMMA 1</u>. <u>*The following compatibility condition between the spaces*</u> V_h <u>*and*</u> Q_h :

$$(2.5) \quad \left. \begin{array}{l} q \in Q_h \\ \forall\, v \in V_{h,o} \ , \ (q, \operatorname{div} v) = 0 \end{array} \right\} \Rightarrow q = \underline{\text{constant}}$$

<u>*is necessary and sufficient for problem (2.4) to have a unique solution.*</u>

<u>PROOF</u>. Since (2.4) is equivalent to a linear system of $(\dim V_h + \dim Q_h - 1)$ equations in the same number of unknowns, it is sufficient to prove the uniqueness of the solution. Hence assume that $(\underline{u}_h, p_h) \in V_{h,o} \times (Q_h/\mathbb{R})$ satisfies

$$\forall\, \underline{v} \in V_{h,o} \ , \ a(\underline{u}_h, \underline{v}) - (p_h, \operatorname{div} \underline{v}) = 0 \quad ,$$

$$\forall\, q \in Q_h \qquad (q, \operatorname{div} \underline{u}_h) = 0 \ .$$

Then taking $\underline{v} = \underline{u}_h$ in the first equation gives $a(\underline{u}_h, \underline{u}_h) = 0$ and therefore $\underline{u}_h = \underline{0}$. Now p_h is uniquely determined by

$$\forall\, \underline{v} \in V_{h,o} \ , \ (p_h \ , \operatorname{div} \underline{v}) = 0$$

up to an additive constant if and only if (2.5) holds.

Now, in order to establish good approximation properties of the scheme (2.4), we need a stronger version of this compatibility condition (2.5). A first result in this direction is the following consequence of a general result due to Brezzi [6] (cf. also Babuška [1]).

<u>*THEOREM 1*.</u> <u>*Assume that there exists a constant*</u> $\beta > 0$ <u>*such that*</u>

$$(2.6) \qquad \forall q \in Q_h \ , \quad \sup_{\underline{v} \in V_{h,o}} \frac{(q, \operatorname{div} \underline{v})}{\| \underline{v} \|_{1,\Omega}} \geqslant \beta \ \| q \|_{L^2(\Omega)/\mathbb{R}}$$

<u>*Then there exists a constant*</u> $C > 0$ <u>*which depends only on*</u> ν <u>*and*</u> β <u>*such that*</u>

$$(2.7) \qquad \begin{cases} \| \underline{u} - \underline{u}_h \|_{1,\Omega} + \| p - p_h \|_{L^2(\Omega)/\mathbb{R}} \leqslant \\ \leqslant C \, \{ \inf\limits_{\substack{\underline{v} \in V_h \\ \underline{v}|_\Gamma = \underline{g}_h}} \| \underline{u} - \underline{v} \|_{1,\Omega} + \inf\limits_{q \in Q_h} \| p - q \|_{L^2(\Omega)/\mathbb{R}} \} \end{cases}$$

The conditions (2.5) and (2.6) appear to be quite restrictive in practice. Hence it remains to construct effectively finite-dimensional spaces V_h and Q_h which satisfy (2.6). Let us give two examples of such a construction in the case of two-dimensional problems.

<u>EXAMPLE 1.</u> (cf. Fortin [12]). For simplicity, we assume that Ω is a polygonal plane domain and we consider a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ with triangles T. This means that :

(i) every side of T has length $\leqslant h$;

(ii) the angles of T are bounded by some fixed angle θ_o .

We define the spaces X_h and Q_h by :

X_h = space of continuous functions which are quadratic in each triangle T ;

Q_h = space of functions which are constant in each T .

Classically, the degrees of freedom of a function $\varphi \in X_h$ may be taken as its values at the vertices and the midpoints of the sides of the triangulation $\mathcal{T}_h$. On the other hand, one can choose the degrees of freedom of a function $q \in Q_h$ as its values at the centroids of the triangles T of $\mathcal{T}_h$. Then the incompressibility constraint is approximated by

$$\int_T \operatorname{div} \underline{u}_h \, dx = 0 \quad , \quad T \in \mathcal{T}_h \, .$$

Let us next define the approximate boundary data $\underline{g}_h$. Given an edge $[a_1, a_2]$ of $\mathcal{T}_h$ located on the boundary Γ , we set :

$$\underline{g}_h(a_i) = \underline{g}(a_i) \; , \; i = 1,2,$$
$$\int_{[a_1,a_2]} (\underline{g}_h - \underline{g}) \, d\sigma = 0 \; .$$

These conditions uniquely determine g_h on $[a_1, a_2]$. Now, one can prove that the compatibility condition (2.6) holds and, provided the solution (u,p) of the continuous problem is smooth enough $(\underline{u} \in (H^2(\Omega))^2 \; , \; \hat{p} \in H^1(\Omega)/\mathbb{R})$, we get :

$$\begin{cases} \|\underline{u}-\underline{u}_h\|_{1,\Omega} + \|p-p_h\|_{L^2(\Omega)/\mathbb{R}} = O(h) \; , \\ \|\underline{u}-\underline{u}_h\|_{o,\Omega} = O(h^2) \; . \end{cases} \qquad \blacksquare$$

<u>EXAMPLE 2</u>. (cf. Crouzeix & Raviart [11]). One can raise by one the asymptotic order of convergence of the previous method by slightly increasing the number of degrees of freedom. With each triangle $T \in \mathcal{T}_h$, we associate the space P_T of incomplete cubic polynomials spanned by

$$\lambda_1^2, \; \lambda_2^2, \; \lambda_3^2, \; \lambda_1\lambda_2, \; \lambda_2\lambda_3, \; \lambda_3\lambda_1, \; \lambda_1\lambda_2\lambda_3 \; ,$$

where the λ_i's are the barycentric coordinates with respect to the vertices of T . Then we define :

X_h = space of continuous functions which coincide with a polynomial of P_T in each $T \in \mathcal{T}_h$;

Q_h = space of functions which are affine in each $T \in \mathcal{T}_h$. Note that the functions of Q_h are generally discontinuous at the interelement boundaries.

Now the degrees of a function $\varphi \in X_h$ may be chosen at its values at the vertices, the midpoints of the sides and the centroids of the triangles of $\mathcal{T}_h$. The incompressibility constraint is approximated by

$$\int_T q \operatorname{div} \underline{u}_h \, dx = 0 \quad , \; q \in P_1 \quad , \quad T \in \mathcal{T}_h \quad ,$$

where, for any integer $\ell \geqslant 0$, P_ℓ denotes the space of all polynomials of degree $\leqslant \ell$ in the two variables x_1 , x_2 .

Here again, the condition (2.6) holds. Then, choosing g_h as in Example 1, we get as a consequence of Theorem 1 when the solution $(\underline{u},p)$ is smooth enough $(\underline{u} \in (H^3(\Omega))^2, p \in H^2(\Omega)/\mathbb{R})$:

$$\begin{cases} \| \underline{u}-\underline{u}_h \|_{1,\Omega} + \| p-p_h \|_{L^2(\Omega)/\mathbb{R}} = 0(h^2) \\ \| \underline{u}-\underline{u}_h \|_{o,\Omega} = 0(h^3). \quad \blacksquare \end{cases}$$

In the previous examples, the approximate pressure was discontinuous at the interelement boundaries. However, it is cheaper in practice to use continuous approximations for the pressure. This leads to a F.E.M. which is classically used by the Engineers (cf. [25],[14] for instance) but where the incompressibility constraint is poorly approximated.

We now assume that we have the inclusion

$$Q_h \subset H^1(\Omega) .$$

A variant of Theorem 1 due to Bercovier & Pironneau [4] is the following

<u>THEOREM 2.</u> <u>*Assume that there exists a constant*</u> $\beta > 0$ <u>*such that*</u>

(2.8) $$\forall\, q \in Q_h \;, \quad \sup_{v \in V_{h,o}} \frac{(q, \operatorname{div} \underline{v})}{\| v \|_{o,\Omega}} \geqslant \beta \; \| \underline{\operatorname{grad}}\, q \|_{o,\Omega} \;.$$

<u>*Then there exists a constant*</u> $C > 0$ <u>*which depends only on*</u> ν <u>*and*</u> β <u>*such that*</u>

(2.9) $$\begin{cases} \| \underline{u}-\underline{u}_h \|_{1,\Omega} \leqslant C \{ \inf\limits_{\substack{\underline{v} \in V_h \\ \underline{v}|_\Gamma = \underline{g}_h}} [\| \underline{u}-\underline{v} \|_{1,\Omega} + S(h) \| \underline{u}-\underline{v} \|_{o,\Omega}] + \\ \qquad + \inf\limits_{q \in Q_h} \| p-q \|_{L^2(\Omega)/R} \} , \end{cases}$$

(2.10) $$\| \underline{\operatorname{grad}}(p-p_h) \|_{o,\Omega} \leqslant C \{ S(h) \| \underline{u}-\underline{u}_h \|_{1,\Omega} + \inf_{q \in Q_h} \| \underline{\operatorname{grad}}\, (p-q) \|_{o,\Omega} \}$$

<u>*where*</u>

(2.11) $$S(h) = \sup_{\underline{v} \in V_{h,o}} \frac{\| \underline{v} \|_{1,\Omega}}{\| \underline{v} \|_{o,\Omega}} \;.$$

<u>EXAMPLE 3.</u> Again, we consider as in Examples 1 and 2 a regular triangulation $\mathcal{T}_h$ of a plane polygonal domain Ω . We set as in [25] , [14] :

X_h = space of continuous functions which are quadratic in each triangle $T \in \mathcal{T}_h$;

Q_h = space of <u>continuous</u> functions which are affine in each $T \in \mathcal{T}_h$.

Then one can prove (cf. [4]) that the condition (2.8) holds provided any triangle $T \in \mathcal{T}_h$ has no more than one side located on Γ . Moreover, if $\mathcal{T}_h$ is <u>uniformly regular</u> in the sense that there exists a constant $\sigma > 0$ such that

$$h_T = \text{diam }(T) \geqslant \sigma h \quad ,$$

we have $S(h) = 0(\frac{1}{h})$. Hence, choosing $\underline{g}_h$ as in Example 1 , we get if $\underline{u} \in (H^3(\Omega))^2$ and $p \in H^2(\Omega)/\mathbb{R}$:

$$\|\underline{u}-\underline{u}_h\|_{1,\Omega} = 0(h^2) \;,\quad \|\underline{\text{grad}}(p-p_h)\|_{o,\Omega} = 0(h) \qquad \blacksquare$$

Finally, let us briefly describe the application of this class of F.E.M. to the Navier-Stokes system of equations (1.1)-(1.3). We set for $\underline{u},\underline{v},\underline{w} \in (H^1(\Omega))^N$:

$$b(\underline{u},\underline{v},\underline{w}) = \frac{1}{2}\int_\Omega u_j \left(\frac{\partial v_i}{\partial x_j} w_i - \frac{\partial w_i}{\partial x_j} v_i\right) dx \;. \tag{2.12}$$

Then the classical variational formulation of problem (1.1)-(1.3) (cf.[21] ,[26]) consists in finding a pair $(\underline{u},p) \in (H^1(\Omega))^N \times (L^2(\Omega)/\mathbb{R})$ such that

$$\begin{cases} \forall\, \underline{v} \in (H^1_o(\Omega))^N \;, & a(\underline{u},\underline{v}) + b(\underline{u},\underline{u},\underline{v}) - (p,\text{div }\underline{v}) = (\underline{f},\underline{v}) \;, \\ \forall\, q \in L^2(\Omega) \;, & (q,\text{div }\underline{u}) = 0 \;, \\ & \underline{u} = \underline{g} \;\text{ on } \Gamma \;. \end{cases} \tag{2.13}$$

Introducing the finite-dimensional spaces V_h , $V_{h,o}$ and Q_h as above, the approximation scheme is defined as follows : To find a pair $(\underline{u}_h,p_h) \in V_h\times (Q_h/\mathbb{R})$ such that

$$\begin{cases} \forall\, \underline{v} \in V_{h,o} \;, & a(\underline{u}_h,\underline{v}) + b(\underline{u}_h,\underline{u}_h,\underline{v}) - (p_h,\text{ div }\underline{v}) = (\underline{f},\underline{v}), \\ \forall\, q \in Q_h \;, & (q,\text{ div }\underline{u}_h) = 0 \;, \\ & \underline{u}_h = \underline{g}_h \;\text{ on } \Gamma \;. \end{cases} \tag{2.14}$$

Under the same hypotheses as for the Stokes system, one can prove similar convergence results, at least for sufficiently small Reynolds numbers. For details, we refer to [20].

Various numerical algorithms of solution of the discretized nonlinear problem are available in the litterature. Most of them are based upon a fixed-point iteration method (cf. for instance Gartling & Becker [15]) together with a duality algorithm as in Temam [26] and Crouzeix [10] or with a penalty method as in Bercovier [3].

3. A MIXED F.E.M. FOR THE STOKES SYSTEM.

Let us next introduce another class F.E.M. for the Navier-Stokes system in which the incompressibility constraint is exactly satisfied. For simplicity, we restrict ourselves to the two-dimensional case and we begin by considering again the Stokes system.

By setting

(3.1) $$\omega = \operatorname{curl} \underline{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} ,$$

and

(3.2) $$\underline{\operatorname{curl}}\, \omega = \left(\frac{\partial \omega}{\partial x_2} , - \frac{\partial \omega}{\partial x_1} \right) ,$$

the equations (1.4) become

(3.3) $$\begin{cases} - \nu\, \underline{\operatorname{curl}}\, \omega + \operatorname{grad} p = f \quad \text{in} \quad \Omega , \\ \operatorname{div} \underline{u} = 0 \quad \text{in} \quad \Omega , \\ \underline{u} = \underline{g} \quad \text{on} \quad \Gamma . \end{cases}$$

Let us give a mixed variational formulation of the Stokes system using the velocity $\underline{u}$, the vorticity ω and the pressure p as dependent variables. We introduce the space

$$H(\operatorname{div} ; \Omega) = \{ \underline{v} \in (L^2(\Omega))^2 \; ; \; \operatorname{div} \underline{v} \in L^2(\Omega) \}$$

provided with the norm

$$\| \underline{v} \|_{H(\operatorname{div} ; \Omega)} = \left(\| \underline{v} \|^2_{o,\Omega} + \| \operatorname{div} \underline{v} \|^2_{o,\Omega} \right)^{1/2} ,$$

and the subspace

$$H_o(\operatorname{div} ; \Omega) = \{ \underline{v} \in H(\operatorname{div} ; \Omega) \; ; \; v_n = 0 \quad \text{on} \quad \Gamma \}$$

We set $g_t = \underline{g} \cdot \underline{t}$, where $\underline{t} = (- n_2, n_1)$ is the unit tangent along Γ . We are now able to state

THEOREM 3. Assume that $\omega = \text{curl } \underline{u}$ *belongs to the space* $H^1(\Omega)$. *Then the triple* $(\underline{u},\omega,p)$ *may be characterized as the unique solution in the product space* $H(\text{div} ; \Omega) \times H^1(\Omega) \times (L^2(\Omega)/\mathbb{R})$ *of the system of equations* :

$$(3.4) \quad \begin{cases} \forall\, \underline{v} \in H_0(\text{div} ; \Omega) \ , \ \nu(\underline{\text{curl}}\ \omega,\underline{v}) - (p,\text{div}\ \underline{v}) = (\underline{f},\underline{v}), \\ \forall\, \theta \in H^1(\Omega) \ , \ (\omega,\theta) - (\underline{u},\ \underline{\text{curl}}\ \theta) = \int_\Gamma g_t\ \theta\ dS \ , \\ \forall\, q \in L^2(\Omega) \ , \ (q,\text{div}\ \underline{u}) = 0 \ , \end{cases}$$

PROOF. Let $(\underline{u},p) \in (H^1(\Omega))^2 \times (L^2(\Omega)/\mathbb{R})$ be a solution of the Stokes system such that $\omega = \text{curl } \underline{u} \in H^1(\Omega)$. The 1st equation (3.4) follows immediately from the 1st equation (3.3). On the other hand, for all functions $\theta \in H^1(\Omega)$, we have by the Green's formula

$$(\omega,\theta) = (\text{curl } \underline{u},\theta) = (\underline{u},\underline{\text{curl}}\ \theta) + \int_\Gamma u_t \theta\ dS \quad ,$$

so that the 2nd equation (3.4) is satisfied. Hence $(\underline{u},\omega,p) \in H(\text{div} ; \Omega) \times H^1(\Omega) \times (L^2(\Omega)/\mathbb{R})$ is a solution (3.4). Conversely, it is readily seen that such a solution is necessarily unique. ■

Next, assume that the domain Ω is p-connected , $p \geq 0$, i.e., the boundary Γ has (p+1) connected components Γ_i , $0 \leq i \leq p$, and that

$$(3.5) \qquad \int_{\Gamma_i} g_n\ dS = 0 \quad , \quad 0 \leq i \leq p \ .$$

Then, in the formulation (3.4), we can eliminate the pressure p by introducing a stream-function $\psi \in H^2(\Omega)$ such that

$$(3.6) \qquad u = \underline{\text{curl}}\ \psi \ .$$

By the conditions (3.5), such a function ψ exists and is unique if we specify for instance that $\psi = 0$ at a given point $x_o \in \Gamma_o$ (say) . Hence, let χ be a function defined on Γ such that

$$(3.7) \qquad \frac{\partial \chi}{\partial t} = g_n \ \text{ on } \Gamma \ , \ \chi(x_o) = 0 \ ;$$

we have

$$\psi = \chi \ \text{ on } \Gamma_o \ , \ \psi = c_i + \chi \ \text{ on } \Gamma_i \ , \ 1 \leq i \leq p \ ,$$

where the c_i's are unknown constants.

On the other hand, let us introduce the space

$$\Phi = \{\varphi \in H^1(\Omega) \; ; \; \varphi = 0 \text{ on } \Gamma_i \; , \; \varphi = \text{constant} = d_i \text{ on } \Gamma_i \; , \; 1 \leq i \leq p\}$$

Then a function $\underline{v}$ belongs to the space $H_o(\text{div} \; ; \Omega)$ and satisfies the condition $\text{div}\, \underline{v} = 0$ if and only if there exists a (unique) function $\varphi \in \Phi$ such that $\underline{v} = \underline{\text{curl}}\, \varphi$. Hence, replacing in (3.4) $\underline{u}$ by $\underline{\text{curl}}\, \psi$ and $\underline{v}$ by $\underline{\text{curl}}\, \varphi$, $\varphi \in \Phi$, we obtain the following result.

THEOREM 4. *Assume that* $\omega = \text{curl}\, \underline{u} = - \Delta\psi$ *belongs to the space* $H^1(\Omega)$.

Then the pair $\{\psi, \omega\}$ *may be characterized as the unique solution in the product space* $H^1(\Omega) \times H^1(\Omega)$ *of the system of equations*

$$(3.8) \qquad \begin{cases} \forall \varphi \in \Phi \; , \; \nu(\underline{\text{curl}}\, \omega, \underline{\text{curl}}\, \varphi) = (\underline{f}, \underline{\text{curl}}\, \varphi) \; , \\ \forall \theta \in H^1(\Omega) \; , \; (\omega, \theta) - (\underline{\text{curl}}\, \psi \; , \; \underline{\text{curl}}\, \theta) = \int_\Gamma g_t \, \theta \, dS \; , \\ \psi = \chi \text{ on } \Gamma_o \; , \; \psi = c_i + \chi \text{ on } \Gamma_i \quad 1 \leq i \leq p \; . \end{cases}$$

We get here a mixed variational formulation of the Stokes system using the stream function ψ and the vorticity ω as dependent variables, i.e., of the system

$$(3.9) \qquad \begin{cases} - \Delta\psi = \omega \text{ in } \Omega \\ - \nu\Delta\omega = \text{curl}\, \underline{f} \text{ in } \Omega \; , \\ \psi = \chi \text{ on } \Gamma_o \; , \; \psi = c_i + \chi \text{ on } \Gamma_i \; , \quad 1 \leq i \leq p \; , \\ \dfrac{\partial\psi}{\partial n} = - g_t \text{ on } \Gamma \; , \\ \displaystyle\int_{\Gamma_i} (\nu \frac{\partial\omega}{\partial n} + f_t) \, dS = 0 \; , \quad 1 \leq i \leq p \; . \end{cases}$$

Let us go back to the mixed formulation (3.4) of the Stokes system. In fact, it is possible to practically construct finite-dimensional subspaces of $H(\text{div} \; ; \Omega)$ such that the incompressibility constraint can be <u>exactly</u> satisfied. We introduce three finite-dimensional spaces V_h, Θ_h and Q_h which satisfy the inclusions

$$V_h \subset H(\text{div} \; ; \Omega) \; , \; \Theta_h \subset H^1(\Omega) \; , \; Q_h \subset L^2(\Omega) \; ,$$

and we set

$$V_{h,o} = \{\underline{v} \in V_h \ ; \ v_n = 0 \ \text{ on } \ \Gamma\} = V_h \cap H_o(\text{div} \ ; \ \Omega).$$

In addition, we are given a function $g_{h,n}$ defined on Γ and which approximates g_n . We assume that $g_{h,n}$ belongs to the space $\{v_{n|\Gamma} \ ; \ \underline{v} \in V_h\}$ and satisfies the condition

$$\int_\Gamma g_{h,n} \, dS = 0 \ .$$

Then the approximation method based upon the mixed formulation (3.4) consists in finding a triple $(\underline{u}_h, \omega_h, p_h) \in V_h \times \Theta_h \times (Q_h/\mathbb{R})$ such that

$$(3.10) \begin{cases} \forall \ \underline{v} \in V_{h,o} \ , & \nu \ (\underline{\text{curl}} \ \omega_h, \underline{v}) - (p_h, \text{div} \ \underline{v}) = (\underline{f}, \underline{v}) \ , \\ \forall \ \theta \in \Theta_h \ , & (\omega_h, \theta) - (\underline{u}_h, \underline{\text{curl}} \ \theta) = \int_\Gamma g_t \ \theta \ dS \ , \\ \forall \ q \in Q_h \ , & (q, \text{div} \ \underline{u}_h) = 0 \ , \\ & u_{h,n} = g_{h,n} \ \text{ on } \ \Gamma \ . \end{cases}$$

Here again, we need to check some compatibility conditions between the spaces V_h , Θ_h and Q_h in order to ensure the existence and uniqueness of the solution $(\underline{u}_h \ , \ \omega_h \ , \ p_h)$ of problem (3.10). We set

$$Z_h = \{\underline{v} \in V_h \ ; \ \forall \ q \in Q_h \ , \ (q, \text{div} \ \underline{v}) = 0\} \ ,$$

$$Z_{h,o} = Z_h \cap V_{h,o} \ ,$$

and we assume that

$$(3.11) \qquad Z_h = \underline{\text{curl}} \ \Theta_h \ .$$

Hence <u>the functions</u> $\underline{v} \in Z_h$ <u>exactly satisfy the incompressibility constraint</u> div $\underline{v} = 0$. Therefore, setting

$$\Phi_h = \Theta_h \cap \Phi \ ,$$

we have

$$(3.12) \qquad Z_{h,o} = \underline{\text{curl}} \ \Phi_h \ .$$

Concerning the existence, uniqueness and approximation properties of the solution of (3.10), we can prove by using the techniques of Ciarlet-Raviart [9] and Brezzi-Raviart [7] the following result.

THEOREM 5. Assume that the condition (3.11) holds and there exists a constant $\beta > 0$ *such that*

(3.13) $$\forall q \in Q_h \ , \ \sup_{\underline{v} \in V_{h,o}} \frac{(q, \operatorname{div} \underline{v})}{\| \underline{v} \|_{H(\operatorname{div}\, ; \, \Omega)}} \geq \beta \, \| q \|_{L^2(\Omega)/\mathbb{R}} \ .$$

Then problem (3.10) has a unique solution $(\underline{u}_h , \omega_h , p_h) \in V_h \times \Theta_h \times (Q_h/\mathbb{R})$ *and there exists a constant* $C > 0$ *which depends only on* ν *and* β *such that*

(3.14) $$\begin{cases} \| \underline{u}-\underline{u}_h \|_{o,\Omega} + \| \omega-\omega_h \|_{o,\Omega} + \| p-p_h \|_{L^2(\Omega)/\mathbb{R}} \leq \\ \leq C \ \{(1 + S(h)) \displaystyle\inf_{\substack{\underline{v} \in Z_h \\ v_n = g_{h,n} \text{ on } \Gamma}} \| \underline{u}-\underline{v} \|_{o,\Omega} + \inf_{\theta \in \Theta_h} \| \omega-\theta \|_{1,\Omega} + \\ + \displaystyle\inf_{q \in Q_h} \| p-q \|_{L^2(\Omega)/\mathbb{R}} \} \end{cases}$$

where

(3.15) $$S(h) = \sup_{\theta \in \Theta_h} \frac{\| \theta \|_{1,\Omega}}{\| \theta \|_{o,\Omega}} \ .$$

Again, we can eliminate the pressure p_h by introducing the stream-function $\psi_h \in \Theta_h$ such that

(3.16) $$\underline{u}_h = \underline{\text{curl}}\, \psi_h \ , \ \psi_h(x_o) = 0 \ .$$

Let χ_h be a function of $\Theta_{h|\Gamma}$ such that

(3.17) $$\frac{\partial \chi_h}{\partial t} = g_{h,n} \ , \ \chi_h(x_o) = 0 \ .$$

Then the pair (ψ_h,ω_h) <u>may be characterized as the unique solution in the product space</u> $\Theta_h \times \Theta_h$ <u>of the system of equations</u> :

(3.18) $$\begin{cases} \forall \varphi \in \Phi_h \ , \ \nu(\underline{\text{curl}}\, \omega_h, \underline{\text{curl}}\, \varphi) = (f, \underline{\text{curl}}\, \varphi) \ , \\ \forall \theta \in \Theta_h \ , \ (\omega_h, \theta) - (\underline{\text{curl}}\, \psi_h \ , \underline{\text{curl}}\, \theta) = \displaystyle\int_\Gamma g_t \, \theta \, dS \ , \\ \psi_h = \chi_h \text{ on } \Gamma_o \quad \psi_h = c_{i,h} + \chi_h \text{ on } \Gamma_i \ , \ 1 \leq i \leq p \ . \end{cases}$$

Therefore the mixed F.E.M. has this nice property that <u>we can use equivalently the velocity-vorticity-pressure system of dependent variables or the stream-function-</u>

vorticity system , just as in the continuous case.

It remains to construct effectively finite-dimensional spaces V_h , Θ_h and Q_h which satisfy the conditions (3.11) and (3.13).

EXAMPLE 4. We consider as in Example 3 a uniformly regular triangulation $\mathcal{T}_h$ of a plane polygonal domain Ω with triangles T . We begin by constructing the space V_h . Given a triangle $T \in \mathcal{T}_h$, we denote by $\underline{n}_T$ the unit outward normal along the boundary ∂T of T . Then a function $\underline{v} \in (L^2(\Omega))^2$ which is smooth in each triangle $T \in \mathcal{T}_h$ belongs to the space $H(\mathrm{div}\ ;\ \Omega)$ if and only if the following reciprocity relation holds for any pair (T_1,T_2) of adjacent triangles :

$$\underline{v}_1 \cdot \underline{n}_{T_1} + \underline{v}_2 \cdot \underline{n}_{T_2} = 0 \quad \text{on} \quad \partial T_1 \cap \partial T_2 \quad ,$$

where $\underline{v}_i$ denotes the restriction of $\underline{v}$ to T_i , $i = 1,2$.

Given an integer $k \geq 0$, we associate with any triangle $T \in \mathcal{T}_h$ a space V_T of smooth vector-valued functions $\underline{v}$ such that :

(i) $\mathrm{div}\ \underline{v}$ is a polynomial of degree $\leq k$;

(ii) the restriction of $\underline{v} \cdot \underline{n}_T$ to any side T' of T is a polynomial of degree $\leq k$.

More precisely, we define V_T to be the space of functions $\underline{v} = (v_1,v_2)$ of the form

$$\begin{cases} v_1 = p_1 + \sum_{i=0}^{k} \alpha_i\, x_1^{k+1-i}\, x_2^{i} \quad , \\ v_2 = p_2 + \sum_{i=0}^{k} \alpha_i\, x_1^{k-i}\, x_2^{i+1} \quad , \end{cases}$$

where p_1 and p_2 belong to the space P_k . Then we define V_h to be the space of all functions $\underline{v} \in H(\mathrm{div}\ ;\ \Omega)$ whose restrictions to any triangle $T \in \mathcal{T}_h$ belong to V_T . The degrees of freedom of a function $\underline{v} \in V_h$ may be chosen as :

(i) the values of $\underline{v} \cdot n_{T'}$, at (k+1) distincts points of each edge T' of the triangulation $\mathcal{T}_h$;

(ii) the moments $\int_T v_i x_1^{\ell_1} x_2^{\ell_2}\, dx$, ℓ_1 , $\ell_2 \geq 0$, $\ell_1 + \ell_2 \leq k-1$, of order $k-1$ of $\underline{v}$ over each triangle $T \in \mathcal{T}_h$.

Next we choose :

Θ_h = space of continuous functions which are polynomials of degree $\leq k+1$ in each triangle $T \in \mathcal{T}_h$;

Q_h = space of functions which are polynomials of degree $\leq k$ in each $T \in \mathcal{T}_h$.

Again the functions of Q_h are generally discontinuous at the interelement boundaries.

Using the above definitions of the spaces V_h , Θ_h and Q_h , one can easily prove that the condition (3.11) holds. On the other hand, the condition (3.13) is more technical to check (cf. [24] , [27]). Hence, we may apply Theorem 5. Therefore assuming that χ_h is the classical interpolate of χ and provided the solution $(\underline{u},p)$ of the Stokes system is smooth enough $(\underline{u} \in (H^{k+2}(\Omega))^2,\ p \in H^{k+1}(\Omega)/\mathbb{R})$, we get :

$$\|\underline{u}-\underline{u}_h\|_{o,\Omega} + \|\omega-\omega_h\|_{o,\Omega} + \|p-p_h\|_{L^2(\Omega)/\mathbb{R}} = O(h^k) \ .$$

<u>In general</u>, the mixed method <u>does not converge</u> in the case $k = 0$. However the convergence result can be improved when the triangulation $\mathcal{T}_h$ is <u>uniform</u> (cf. [18], [16]). ∎

For an extensive discussion of algorithms of solution of the discretized problem (3.18) , we refer to Ciarlet & Glowinski [8] and Glowinski-Pironneau [19].

4. <u>TWO MIXED F.E.M. FOR THE NAVIER-STOKES SYSTEM</u>. <u>DISCRETIZATION OF THE CONVECTIVE TERMS</u>.

We now want to generalize the mixed method introduced in the previous section to the Navier-Stokes system. Since only the normal component of a function $\underline{v} \in V_h$ is continuous at the interelement boundaries, a new difficulty stems from the discretization of the convective terms $u_i \frac{\partial \underline{u}}{\partial x_i}$. We shall present here two methods for overcoming this difficulty.

4.1. <u>A first F.E.M.</u>

Setting again $\omega = \text{curl}\ \underline{u}$, the equation (1.1) may be written in the form

$$\nu\ \underline{\text{curl}}\ \omega + \sum_{i=1}^{2} u_i \frac{\partial \underline{u}}{\partial x_i} + \underline{\text{grad}}\ p = \underline{f} \ .$$

Now, in order to give a mixed vatiational formulation of the Navier-Stokes system, we notice that

$$u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} = \frac{1}{2} \frac{\partial}{\partial x_1}(u_1^2 + u_2^2) - \omega u_2 \quad ,$$

$$u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_2}{\partial x_2} = \frac{1}{2} \frac{\partial}{\partial x_2}(u_1^2 + u_2^2) + \omega u_1 \quad .$$

Then we can prove the following result which parallels Theorem 3.

THEOREM 6. *Assume that* $(\underline{u},p)$ *is a solution of the Navier-Stokes system* (1.1) - (1.3) *such that* $\omega = \text{curl}\ \underline{u}$ *belongs to the space* $H^1(\Omega)$. *Then the triple* $(\underline{u},\omega,p)$ *is a solution in the product space* $H(\text{div}\ ;\Omega) \times H^1(\Omega) \times (L^2(\Omega)/R)$ *of the system of equations :*

$$(4.1)\quad \begin{cases} \forall\ \underline{v} \in H_o(\text{div}\ ;\Omega)\ ,\ \nu(\underline{\text{curl}}\ \omega,\underline{v}) + \int_\Omega \omega(u_1v_2 - u_2v_1)\,dx - \\ \qquad\qquad - (p + \frac{1}{2}(u_1^2 + u_2^2), \text{div}\ \underline{v}) = (\underline{f},\underline{v})\ , \\ \forall\ \theta \in H^1(\Omega)\ ,\ (\omega,\theta) - (u, \underline{\text{curl}}\ \theta) = \int_\Gamma g_t\theta\ dS\ , \\ \forall\ q \in L^2(\Omega)\ ,\ (q, \text{div}\ \underline{u}) = 0\ , \\ \qquad u_n = g_n\ . \end{cases}$$

If we assume that the domain Ω is p-connected and that the conditions (3.5) hold, we may introduce a stream-function ψ. Then the pair (ψ,ω) is equivalently a solution in the product space $H^1(\Omega) \times H^1(\Omega)$ of the system of equations :

$$(4.2)\quad \begin{cases} \forall\ \varphi \in \Phi\ ,\ \nu(\underline{\text{curl}}\ \omega, \underline{\text{curl}}\ \varphi) + \int_\Omega \omega\left(\frac{\partial\psi}{\partial x_1}\frac{\partial\varphi}{\partial x_2} - \frac{\partial\psi}{\partial x_2}\frac{\partial\varphi}{\partial x_1}\right)dx = (\underline{f},\underline{\text{curl}}\ \varphi)\ , \\ \forall\ \theta \in H^1(\Omega)\ ,\ (\omega,\theta) - (\underline{\text{curl}}\ \psi, \underline{\text{curl}}\ \theta) = \int_\Gamma g_t\theta\ dS\ , \\ \psi = \chi\ \underline{\text{on}}\ \Gamma_o \quad \psi = c_i + \chi\ \underline{\text{on}}\ \Gamma_i\ ,\ 1 \leqslant i \leqslant p\ . \end{cases}$$

We obtain here a mixed variational formulation of the Navier-Stokes system using the dependent variables ψ and ω.

Next, using the finite-dimensional spaces V_h, Θ_h and Q_h introduced in Section 3, we can construct - exactly as for the Stokes-system - mixed F.E.M. based upon the variational formulations (4.1) or (4.2) of the Navier-Stokes system. Convergence results are obtained in Girault & Raviart [17]. For the practical implementation of this mixed method and for some numerical results corresponding to Reynolds numbers Re $\leqslant 400$, we refer to [5] .

4.2. An upwind discretization of the convective terms.

Another method for treating the convective terms has been introduced by Fortin [13] : he uses the techniques of discontinuous finite elements developed in Lesaint [22] and Lesaint & Raviart [23]. Since the velocity $\underline{u}_h$ is generally discontinuous in the mixed F.E.M., each convective term

$$(4.3)\qquad \sum_{j=1}^{2} u_j \frac{\partial u_i}{\partial x_j} = \sum_{j=1}^{2} \frac{\partial}{\partial x_j}(u_j u_i)$$

introduces at the interelement boundaries a superficial measure $u_{h,n}[u_{h,i}]\,dS$, where $[u_{h,i}]$ denotes the jump of the i th component $u_{h,i}$ of $\underline{u}_h$ at the interfaces. Now, in order to take into account the direction of the stream, we choose as the contribution of the triangular element T to the discretization of the convective term (4.3) :

$$(4.4)\qquad \sum_{j=1}^{2} \int_T \frac{\partial}{\partial x_j}(u_{h,j}\, u_{h,i}) v_i \, dx + \int_{\partial_- T} u_{h,n}(u_{h,i}^{ext} - u_{h,i}^{int}) v_i^{int}\, dS \;, \; \underline{v} \in V_{h,o} \;,$$

where $\partial_- T = \{x \in \partial T \;;\; u_{h,n}(x) < 0\}$ denotes the part of the boundary ∂T of T for which the flow is entering and where $u_{h,i}^{ext}$ (resp. $u_{h,i}^{int}$) denotes the outer value (resp. the inner value) of $u_{h,i}$ on ∂T . This discretization method may be viewed as the finite element analogue of the classical upwind differencing which is widely used in the finite difference method.

By setting $\partial_+ T = \{x \in \partial T \;;\; u_{h,n}(x) > 0\}$ and using Green's theorem , (4.4) may be put into the equivalent form

$$(4.5)\qquad \begin{cases} -\displaystyle\sum_{j=1}^{2} \int_T u_{h,j}\, u_{h,i} \frac{\partial v_i}{\partial x_j}\, dx + \int_{\partial_- T} u_{h,n}\, u_{h,i}^{ext}\, v_i^{int}\, dS \; + \\[2ex] + \displaystyle\int_{\partial_+ T} u_{h,n} \cdot u_{h,i}^{int}\, v_i^{int}\, dS \;. \end{cases}$$

A more symmetric treatment of the term (4.3) consists in using

$$(4.6)\qquad -\sum_{j=1}^{2} \int_T u_{h,i}\, u_{h,j} \frac{\partial v_i}{\partial x_j}\, dx + \frac{1}{2} \int_{\partial T} u_{h,n}(u_{h,i}^{ext} + u_{h,i}^{int}) v_i^{int}\, dS \;.$$

More generally, we can take a convex combination of (4.5) and (4.6). Hence the numerical scheme becomes : Find a triple $(\underline{u}_h, \omega_h, p_h) \in V_h \times \Theta_h \times (Q_h/\mathbb{R})$ such that

$$(4.7)\quad \begin{cases} \forall\, \underline{v} \in V_{h,o}\ ,\ \nu(\underline{\text{curl}}\ \omega_h, v) + \sum_{T \in \mathcal{T}_h} \{ - \sum_{i,j=1}^{2} \int_T u_{h,j}\, u_{h,i} \frac{\partial v_i}{\partial x_j}\, dx \\ \quad + \sum_{i=1}^{2} \int_{\partial \pm T} u_{h,n} ((\frac{1}{2} \pm \alpha) u_{h,i}^{int} + (\frac{1}{2} \mp \alpha) u_{h,i}^{ext}) v_i^{int}\, dS\} = (\underline{f}, \underline{v})\ , \\ \forall\, \theta \in \Theta_h\ ,\ (\omega_h, \theta) - (\underline{u}_h, \underline{\text{curl}}\ \theta) = \int_\Gamma g_t\, \theta\, dS\ , \\ \forall\, q \in Q_h\ ,\ (q, \text{div}\ \underline{u}_h) = 0\ , \\ \qquad u_{h,n} = g_{h,n} \quad \text{on } \Gamma\ . \end{cases}$$

One can easily check that the discretized convective terms are dissipative if and only if $0 \leq \alpha \leq \frac{1}{2}$ (the dissipation vanishes for $\alpha = 0$). Thus we restrict the range of the parameter α to the interval $[0, \frac{1}{2}]$: $\alpha = \frac{1}{2}$ corresponds to the upwind differencing.

For the implementation of this mixed method and numerical corresponding to Reynolds numbers $Re \leq 10.000$, we refer to [28] and [2]. In fact, this type of F.E.M. seems to be well adapted to the numerical solution of the Navier Stokes equations at high Reynolds numbers.

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